SOME DISTORTION THEOREMS FOR NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS

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Abstract. In the present paper, we introduced and study a new class of harmonic univalent functions on unit disc $\mathbb{U}$. also we obtain coefficient conditions, extreme points, convolution condition for the above class of harmonic univalent functions.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic. In any simply connected domain $B \subset \mathbb{C}$, we can write $f = h+g$, where h and g are analytic in B. We call h and g are analytic part and co-analytic part of f respectively. Clunie and Sheil-Small [5] observed that a necessary and sufficient condition for the harmonic functions $f = h + g$ to be locally univalent and sense-preserving in B is that $|h'(z)| > |g'(z)|$ for all $z \in B$. Denote by $\mathcal{S}_H$ the class of functions $f = h+\overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$.

In 1984, Clunie and Sheil-Small [5] investigated the class $\mathcal{S}_H$ as well as its geometric subclasses and obtained some coefficient bounds. They proved that although $\mathcal{S}_H$ is not compact, it is normal with respect to the topology of uniform convergence on compact subsets of $\mathbb{U}$. Meanwhile the subclass $\mathcal{S}_H^0$ of $\mathcal{S}_H$ consisting of the functions having the property $f_{\bar{z}}(0) = 0$ is compact.

Some of the following subcalsses has been studied as mentioned:

(i) The classes $\mathcal{S}_H(1, 0; 0, 0) = \mathcal{S}_H$ and $\mathcal{S}_H(2, 1; 0, 0) = \mathcal{C}_H$ were studied by Avci and Zlotkiewicz in [4].
(ii) The classes $\mathcal{S}_H(1, 0; \alpha, 0) = \mathcal{S}_H(\alpha)$ and $\mathcal{S}_H(2, 1; \alpha, 0) = \mathcal{C}_H(\alpha)$ were studied by Ozturk and Yalcin in [9].

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(iii) The class $\mathcal{S}_H(m,n;\alpha,0) = \mathcal{S}_H(m,n,\alpha)$ was studied by Dixit *et al.* *in* [6].
(iv) The class $\mathcal{S}_H(1,0;\alpha,\beta) = \mathcal{S}_H(\alpha,\beta)$ *was studied by Seoudy in* [12].
(v) The class $\mathcal{S}_H(n+1,n;\alpha,0) = \mathcal{S}_H(\alpha,n)$ *was studied by Aouf et al. in* [11].

In secion 2, we denote some fundamental defenitions, theorems and lemmas and in section 3, we investigate several properties of the classes $\mathcal{S}_H(m,n;\alpha,\beta)$ and $\mathcal{S}_H^0(m,n;\alpha,\beta)$. Also, we generalize, improve and correct some results of Ozturk and Yalcin [9]. More recent works in this area can be found in [1] and [2].

2. Prilliminaries and Definitions

We begin with the basic definition on harmonic univalent functions.

Definition 2.1. A harmonic, complex-valued, orientation preserving, univalent mapping f defined on $\mathbb{U}$ can be written as:

$$f = h + \overline{g}, \tag{1}$$

where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{2}$$

We call h the analytic part and g the co-analytic part of f.

Denote by $\mathcal{S}_H(m,n;\alpha,\beta)$ the class of all functions of the form (1) that satisfy the following inequality:

$$\sum_{k=2}^{\infty} (k^m - \alpha k^n)(1-\beta+\beta k)(|a_k| + |b_k|) \le (1-\alpha)(1-|b_1|) \tag{3}$$

Where $m \in \mathbb{N}, n \in \mathbb{N}_0, m > n, 0 \le \alpha < 1, \beta \ge 0$ and $0 \le |b_1| < 1$.
The class $\mathcal{S}_H(m,n;\alpha,\beta)$ with $|b_1| = 0$ will be denoted by $\mathcal{S}_H^0(m,n;\alpha,\beta)$.

Definition 2.2. If h, g are the form (2) and if $f = h + \overline{g}, F = H + \overline{G}$, where the functions H and G are given by

$$H(z) = z + \sum_{k=2}^{\infty} A_k z^k, \quad G(z) = \sum_{k=1}^{\infty} B_k z^k. \tag{4}$$

Then the convolution of f and F is defined to be the function

$$(f * F)(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k}, \tag{5}$$

while the Integral convolution is defined by:

$$(f \Diamond F)(z) = z + \sum_{k=2}^{\infty} \frac{a_k A_k}{k} z^k + \overline{\sum_{k=1}^{\infty} \frac{b_k B_k}{k} z^k}. \tag{6}$$

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In Ozturk and Yalcin [9] defined the generalized $\delta-$neighborhood of f to be the set:

$$N(f)=\left\{F:\sum_{k=2}^{\infty}(k-\alpha)(|a_k-A_k|+|b_k-B_k|)+(1-\alpha)|b_1-B_1|\leq\delta(1-\alpha)\right\},$$

where $F=H+\overline{G}$, and H,G are the form (4).

Theorem 2.3. *For harmonic univalent mapping f as above, $|a_1|\leq 2$ and for all $n\geq 2$,*

$$|a_n|=|b_n|\leq 2/n.$$

Complete proof is explained in theorem 4.7 of [7].

3. Main results

We start this section with the most important following theorem.

Theorem 3.1. *For $0\leq\alpha_1\leq\alpha_2<1$, we have*

$$\mathcal{S}_H(m,n;\alpha_2,\beta)\subseteq\mathcal{S}_H(m,n;\alpha_1,\beta)$$

Consequently

$$\mathcal{S}^0_H(m,n;\alpha_2,\beta)\subseteq\mathcal{S}^0_H(m,n;\alpha_1,\beta)$$

In particular

$$\mathcal{S}_H(m,n;\alpha,\beta)\subseteq\mathcal{S}_H(m,n;0,\beta)$$

and

$$\mathcal{S}^0_H(m,n;\alpha,\beta)\subseteq\mathcal{S}^0_H(m,n;0,\beta)$$

Where $m\in\mathbb{N},n\in\mathbb{N}_0,m>n$ and $\beta\geq 0$.

Proof. Let $f\in\mathcal{S}_H(m,n;\alpha_2,\beta)$, Thus we have

$$\sum_{k=2}^{\infty}\frac{(k^m-\alpha_2k^n)(1-\beta+\beta k)}{1-\alpha_2}(|a_k|+|b_k|)\leq 1-|b_1| \tag{7}$$

Now, using (7), we have

$$\sum_{k=2}^{\infty}\frac{(k^m-\alpha_1k^n)(1-\beta+\beta k)}{1-\alpha_1}(|a_k|+|b_k|)\leq\sum_{k=2}^{\infty}\frac{(k^m-\alpha_2k^n)(1-\beta+\beta k)}{1-\alpha_2}(|a_k|+|b_k|)$$
$$\leq 1-|b_1|$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *$\mathcal{S}_H(m,n;\alpha,\beta)\subseteq\mathcal{S}_H(\alpha)$ for $m\in\mathbb{N},n\in\mathbb{N}_0,m>n,0\leq\alpha<1,\beta\geq 0$ and $\mathcal{S}_H(m,n;\alpha,\beta)\subseteq\mathcal{C}_H(\alpha)$ if $\beta\geq 1$ or $m\geq 2,n\geq 1$ and $m>n$.*

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Proof. Let $f \in \mathcal{S}_H(m,n;\alpha,\beta)$, Then

$$\sum_{k=2}^{\infty} \frac{(k^m-\alpha k^n)(1-\beta+\beta k)}{1-\alpha}(|a_k|+|b_k|) \leq 1-|b_1| \tag{8}$$

Now, using (8), we obtain

$$\sum_{k=2}^{\infty} \frac{k-\alpha}{1-\alpha}(|a_k|+|b_k|) \leq \sum_{k=2}^{\infty} \frac{(k^m-\alpha k^n)(1-\beta+\beta k)}{1-\alpha}(|a_k|+|b_k|) \leq 1-|b_1|,$$

Thus, $f \in \mathcal{S}_H(\alpha)$ and we get $\mathcal{S}_H(m,n;\alpha,\beta) \subseteq \mathcal{S}_H(\alpha)$.
We have to show that $\mathcal{S}_H(m,n;\alpha,\beta) \subseteq \mathcal{C}_H(\alpha)$. By using the inequality (8), we have

$$\sum_{k=2}^{\infty} \frac{k(k-\alpha)}{1-\alpha}(|a_k|+|b_k|) \leq \sum_{k=2}^{\infty} \frac{(k^m-\alpha k^n)(1-\beta+\beta k)}{1-\alpha}(|a_k|+|b_k|) \leq 1-|b_1|,$$

where $\beta \geq 1$ or $m \geq 2, n \geq 1$ and $m > n$.
Thus, $f \in \mathcal{C}_H(\alpha)$ and we get $\mathcal{S}_H(m,n;\alpha,\beta) \subseteq \mathcal{C}_H(\alpha)$. □

Theorem 3.3. *The class $\mathcal{S}_H(m,n;\alpha,\beta)$ consists of univalent sense preserving harmonic mappings.*

Proof. If $z_1 \neq z_2$ then

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\
&= 1-\left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum_{k=2}^{\infty} a_k(z_1^k-z_2^k)}\right| \\
&> 1-\left|\frac{\sum_{k=1}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|}\right| \geq 1-\frac{\sum_{k=1}^{\infty}\frac{(k^m-\alpha k^n)(1-\beta+\beta k)}{1-\alpha}|b_k|}{1-\sum_{k=2}^{\infty}\frac{(k^m-\alpha k^n)(1-\beta+\beta k)}{1-\alpha}|a_k|} \geq 0.
\end{aligned}$$

which proves univalence. Note that f is sense preserving in $\mathbb{U}$ because

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{k=2}^{\infty} k|a_k||z|^k > 1-\sum_{k=2}^{\infty}\frac{(k^m-\alpha k^n)(1-\beta+\beta k)}{1-\alpha}|a_k| \\
&\geq \sum_{k=2}^{\infty}\frac{(k^m-\alpha k^n)(1-\beta+\beta k)}{1-\alpha}|b_k| > \sum_{k=2}^{\infty}\frac{(k^m-\alpha k^n)(1-\beta+\beta k)}{1-\alpha}|b_k||z|^{k-1} \\
&\geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|.
\end{aligned}$$

So, the proof of Theorem 3.3 is completed. □

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Theorem 3.4. *If $f \in \mathcal{S}_H(m,n;\alpha,\beta)$, then*

$$(1-|b_1|)(|z|-\psi|z|^2) \leq |f(z)| \leq (1+|b_1|)|z| + \psi(1-|b_1|)|z|^2. \tag{9}$$

Where $\psi = \dfrac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)}$, *and equalities are attained by the functions:*

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + (1-|b_1|)\psi z^2 \tag{10}$$

and

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + (1-|b_1|)\psi \bar{z}^2 \tag{11}$$

For properly chosen real θ.

Proof. We have

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)|z| + |z|^2 \sum_{k=2}^{\infty}(|a_k|+|b_k|) \\
&\leq (1+|b_1|)|z| + |z|^2 \frac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)} \sum_{k=2}^{\infty} \frac{(1-\beta+\beta k)(k^m-k^n\alpha)}{1-\alpha}(|a_k|+|b_k|) \\
&\leq (1+|b_1|)|z| + |z|^2 \frac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)}(1-|b_1|). \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)|z| - \sum_{k=2}^{\infty}(|a_k|+|b_k|)|z|^k \geq (1-|b_1|)|z| - |z|^2 \sum_{k=2}^{\infty}(|a_k|+|b_k|) \\
&\geq (1-|b_1|)|z| - |z|^2 \frac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)} \sum_{k=2}^{\infty} \frac{(1-\beta+\beta k)(k^m-k^n\alpha)}{1-\alpha}(|a_k|+|b_k|) \\
&\geq (1-|b_1|)|z| - |z|^2 \frac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)}(1-|b_1|) \\
&= (1-|b_1|)\Big(|z| - |z|^2 \frac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)}\Big). \qquad (13)
\end{aligned}$$

We define $\psi := \dfrac{1-\alpha}{(1+\beta)(2^m-2^n\alpha)}$ in inequalities (12) and (13), and so attained (9). It can be easily seen that the functions $f_\theta(z)$ defined by (10) and (11) are extremal for Theorem 3.4.
Thus the class $\mathcal{S}_H(m,n;\alpha,\beta)$ is uniformly bounded, and hence it is normal by Montel's Theorem. □

Putting $m=1, n=0$ and $\beta=0$ in Theorem 3.4, we obtain the following result which correct the result of Ozturk and Yalcin [9, Theorem 3.6].

corollary 3.5. *If $f \in \mathcal{S}_H(\alpha)$, then*

$$(1-|b_1|)(|z|-\frac{1-\alpha}{2-\alpha}|z|^2) \leq |f(z)| \leq (1+|b_1|)|z| + \frac{(1-\alpha)(1-|b_1|)}{2-\alpha}|z|^2. \tag{14}$$

Equalities are attained by the functions:

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{(1-\alpha)(1-|b_1|)}{2-\alpha}z^2 \tag{15}$$

and

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{(1-\alpha)(1-|b_1|)}{2-\alpha}\bar{z}^2 \tag{16}$$

For properly chosen real θ.

Remark 3.6. The above result is different from that of Ozturk and Yalcin [9, Theorem 3.6]. Also, our result gives a better estimate than that of [9] because

$$|f(z)| \leq (1+|b_1|)|z| + \frac{(1-\alpha)(1-|b_1|)}{2-\alpha}|z|^2 \leq (1+|b_1|)|z| + \frac{(1-\alpha^2)(1-|b_1|)}{2}|z|^2$$

and

$$|f(z)| \geq (1-|b_1|)(|z| - \frac{1-\alpha}{2-\alpha}|z|^2) \geq (1-|b_1|)\Big(|z| - \frac{1-\alpha^2}{2}|z|^2\Big).$$

Although, zturk and Yalcin [9] state that the result is sharp for the function

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{(1-\alpha^2)(1-|b_1|)}{2}\bar{z}^2$$

it can be easily seen that the function $f_\theta(z)$ does not satisfy the coefficient condition for the class $f \in \mathcal{S}_H(\alpha)$ defined by them. Hence, the function $f_\theta(z)$ does not belong to the class $f \in \mathcal{S}_H(\alpha)$. Therefore the result of Ozturk and Yalcin [9, Theorem 3.6] is incorrect. The correct result is mentioned in (14) and the result is sharp for functions given by (15) and (16), respectively.

Putting $m = 2, n = 1$ and $\beta = 0$ in Theorem 3.4, we obtain the following result which correct the result of Ozturk and Yalcin [9, Theorem 3.6].

corollary 3.7. *If $f \in \mathcal{C}_H(\alpha)$, then*

$$(1-|b_1|)(|z| - \frac{1-\alpha}{2(2-\alpha)}|z|^2) \leq |f(z)| \leq (1+|b_1|)|z| + \frac{(1-\alpha)(1-|b_1|)}{2(2-\alpha)}|z|^2. \tag{17}$$

Equalities are attained by the functions:

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{(1-\alpha)(1-|b_1|)}{2(2-\alpha)}z^2 \tag{18}$$

and

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{(1-\alpha)(1-|b_1|)}{2(2-\alpha)}\bar{z}^2 \tag{19}$$

For properly chosen real θ.

Remark 3.8. This result is different from the result of Ozturk and Yalcin [9, Theorem 3.8], and it can be easily seen that our result gives a better

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estimate. Also, it can be easily verified that the sharp result for [9, Theorem 3.8] is given by the function

$$f_\theta(z) = z + |b_1|e^{i\theta}\bar{z} + \frac{3-\alpha-2\alpha}{2\alpha}\bar{z}^2$$

does not belong to the class $f \in \mathcal{C}_H(\alpha)$. Hence the result of Ozturk and Yalcin [9] is incorrect. The correct result is given by corollary 3.7.

Theorem 3.9. *Let $f = h + \bar{g}$, where h and g are of the form (2). Then $f \in \mathcal{S}_H^0(m,n;\alpha,\beta)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty}[x_k h_k(z) + y_k g_k(z)],$$

where

$$h_1(z) = z, \quad h_k(z) = z + \frac{1-\alpha}{(1+\beta)(2^m - 2^n\alpha)}z^k \quad (k = 2, 3, \dots)$$

and

$$g_k(z) = z + \frac{1-\alpha}{(1+\beta)(2^m - 2^n\alpha)}\bar{z}^k \quad (k = 2, 3, \dots),$$

$$x_k, y_k \geq 0, y_1 = 0, x_1 = 1 - \sum_{k=2}^{\infty}(x_k + y_k).$$

In particular, the extreme points of the class $\mathcal{S}_H^0(m,n;\alpha,\beta)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}(x_k h_k(z) + y_k g_k(z)) \\ &= z + \sum_{k=2}^{\infty}\Big(\frac{1-\alpha}{(1-\beta+\beta z)(k^m - k^n\alpha)}x_k z^k + \frac{1-\alpha}{(1-\beta+\beta k)(k^m - k^n\alpha)}y_k \bar{z}^k\Big) \end{aligned}$$

Than

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{(1-\beta+\beta z)(k^m - k^n\alpha)}{1-\alpha}\Big(\frac{1-\alpha}{(1-\beta+\beta z)(k^m - k^n\alpha)}x_k z^k \\ &\quad + \frac{1-\alpha}{(1-\beta+\beta k)(k^m - k^n\alpha)}y_k \bar{z}^k\Big) = \sum_{k=2}^{\infty}x_k + \sum_{k=2}^{\infty}y_k = 1 - x_1 \leq 1 \end{aligned}$$

and so $f \in clco\mathcal{S}_H^0(m,n;\alpha,\beta)$.
conversely, if $f \in clco\mathcal{S}_H^0(m,n;\alpha,\beta)$. Set

$$x_k = \frac{1-\alpha}{(1-\beta+\beta k)(k^m - k^n\alpha)}|a_k|, y_k = \frac{1-\alpha}{(1-\beta+\beta k)(k^m - k^n\alpha)}|b_k|, \quad (k = 2, 3, 4, \dots).$$

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Then note that $0 \leq x_k, y_k \leq 1, \quad (k = 2,3,4,\dots)$. We define $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=2}^{\infty} y_k, x_1 \geq 0$ and $y_1 = 0$. Consequently, we obtain required representation, since

$$\begin{aligned} f(z) &= z + \sum_{k=2}^{\infty} (|a_k| z^k + |b_k| \bar{z}^k) \\ &= z + \sum_{k=2}^{\infty} \Big(\frac{1-\alpha}{(1+\beta)(2^m - 2^n\alpha)} x_k z^k + \frac{1-\alpha}{(1+\beta)(2^m - 2^n\alpha)} y_k \bar{z}^k \Big) \\ &= z + \sum_{k=2}^{\infty} \Big((h_k(z) - z) x_k + (g_k(z) - z) y_k \Big) \\ &= \Big(1 - \sum_{k=2}^{\infty} x_k - \sum_{k=2}^{\infty} y_k \Big) z + \sum_{k=2}^{\infty} h_k(z) x_k + \sum_{k=2}^{\infty} g_k(z) y_k \\ &= \sum_{k=1}^{\infty} [x_k h_k(z) + y_k g_k(z)]. \end{aligned}$$

This completes the proof of Theorem 3.9. □

Remark 3.10. (i). Putting $m = 1, n = 0$ and $\beta = 0$ in Theorem 3.9, we obtain the extreme points of the class $\mathcal{S}_H^0(\alpha)$.
(ii). Putting $m = 2, n = 1$ and $\beta = 0$ in Theorem 3.9, we obtain the extreme points of the class $\mathcal{C}_H^0(\alpha)$.

Let $\mathcal{K}_H^0$ denote the class of harmonic univalent functions of the form (1) with $b_1 = 0$ that map $\mathbb{U}$ on to convex domains. It is known [5, Theorem 5.10], that the sharp inequalities $|A_k| \leq \frac{k+1}{2}, |B_k| \leq \frac{k-1}{2}$ are true. These results will be used in the next theorem.

Theorem 3.11. *Suppose that* $F(z) = z + \sum_{k=1}^{\infty} (A_k z^k + \overline{B_k z^k})$ *belongs to the class* $\mathcal{K}_H^0$*. If* $f \in \mathcal{S}_H^0(m,n;\alpha,\beta)$*, then* $f * F \in \mathcal{S}_H^0(m-1,n-1;\alpha,\beta)$ *if* $n > 1$ *and* $f \Diamond F \in \mathcal{S}_H^0(m,n;\alpha,\beta)$*.*

Proof. Since $f \in \mathcal{S}_H^0(m,n;\alpha,\beta)$, then

$$\sum_{k=2}^{\infty} (k^m - \alpha k^n)(1 - \beta + \beta k)(|a_k| + |b_k|) \leq (1-\alpha), \tag{20}$$

Using (20), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}(k^{m-1}-\alpha k^{n-1})(1-\beta+\beta k)(|a_kA_k|+|b_kB_k|)\\
&\quad=\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)\Big(|a_k|\Big|\frac{A_k}{k}\Big|+|b_k|\Big|\frac{B_k}{k}\Big|\Big)\\
&\quad\le\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)\Big(|a_k|\Big|\frac{k+1}{2k}\Big|+|b_k|\Big|\frac{k-1}{2k}\Big|\Big)\\
&\quad\le\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)(|a_k|+|b_k|)\le(1-\alpha).
\end{aligned}$$

It fallows that $f*F\in\mathcal{S}_H^0(m-1,n-1;\alpha,\beta)$. Next, aggain using (20), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)\Big(\Big|\frac{a_kA_k}{k}\Big|+\Big|\frac{b_kB_k}{k}\Big|\Big)\\
&\quad\le\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)\Big(|a_k|\Big|\frac{k+1}{2k}\Big|+|b_k|\Big|\frac{k-1}{2k}\Big|\Big)\\
&\quad\le\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)(|a_k|+|b_k|)\le(1-\alpha).
\end{aligned}$$

Thus we have $f\lozenge F\in\mathcal{S}_H^0(m,n;\alpha,\beta)$. This completes the proof of Theorem 3.11. □

Let $\mathcal{S}$ denote the class of analytic univalent functions of the form $F(z)=z+\sum_{k=2}^{\infty}A_kz^k$. It is well known that the sharp inequality $|A_k|\le k$ is true. It is needed in the next theorem.

Theorem 3.12. *If* $f\in\mathcal{S}_H^0(m,n;\alpha,\beta)$ *and* $F\in\mathcal{S}$, *then for* $|\zeta|\le1, f*(F+\zeta\overline{F})\in\mathcal{S}_H^0(m-1,n-1;\alpha,\beta)$ *if* $n>1$.

Proof. Since $f\in\mathcal{S}_H^0(m,n;\alpha,\beta)$, then

$$\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)(|a_k|+|b_k|)\le(1-\alpha), \tag{21}$$

Using (21), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}(k^{m-1}-\alpha k^{n-1})(1-\beta+\beta k)(|a_kA_k|+|\zeta b_k\overline{A_k}|)\\
&\qquad\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)(|a_k|+|b_k|)\le(1-\alpha).
\end{aligned}$$

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It follows that $f*(F+\zeta\overline{F})\in\mathcal{S}_H^0(m-1,n-1;\alpha,\beta)$ if $n>1$. □

Let $\mathcal{P}_H^0$ denote the class of complex and harmonic functions F in $\mathbb{U}$, $F=H+\overline{G}$ such that $Re\{F(z)\}>0, z\in\mathbb{U}$ and

$$H(z)=1+\sum_{k=1}^{\infty}A_kz^k, G(z)=\sum_{k=2}^{\infty}B_kz^k.$$

It is known [8, Theorem 3] that the sharp inequalities $|A_k|<k+1, |B_k|<k-1$ are true.

Theorem 3.13. *Suppse that* $F(z)=1+\sum\limits_{k=2}^{\infty}(A_kz^k+\overline{B_kz^k})$ *belong to class* $\mathcal{P}_H^0$. *If* $f\in\mathcal{S}_H^0(m,n;\alpha,\beta)$ *and for* $\frac{2}{3}\leq|A_1|\leq 2$, *then* $\frac{1}{A_1}f*F\in\mathcal{S}_H^0(m-1,n-1;\alpha,\beta)$ *if* $n\geq 1$ *and* $\frac{1}{A_1}f\Diamond F\in\mathcal{S}_H^0(m,n;\alpha,\beta)$.

Proof. Since $f\in\mathcal{S}_H^0(m,n;\alpha,\beta)$, then

$$\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)(|a_k|+|b_k|)\leq(1-\alpha), \tag{22}$$

Using (22), we have

$$\begin{aligned}&\sum_{k=2}^{\infty}(k^{m-1}-\alpha k^{n-1})(1-\beta+\beta k)\Big(\Big|\frac{a_kA_k}{A_1}\Big|+\Big|\frac{b_kB_k}{A_1}\Big|\Big)\\ &\leq\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)\Big(\Big|\frac{a_k}{A_1}\Big|\frac{k+1}{k}+\Big|\frac{b_k}{A_1}\Big|\frac{k-1}{k}\Big)\\ &\leq\sum_{k=2}^{\infty}(k^m-\alpha k^n)(1-\beta+\beta k)(|a_k|+|b_k|)\leq(1-\alpha).\end{aligned}$$

Thus $\frac{1}{A_1}f*F\in\mathcal{S}_H^0(m-1,n-1;\alpha,\beta)$ if $n\geq 1$.
Similarly, we can show that $\frac{1}{A_1}f\Diamond F\in\mathcal{S}_H^0(m,n;\alpha,\beta)$.This completes the proof of Theorem 3.13. □

Theorem 3.14. *Let*

$$f(z)=z+\overline{b_1z}+\sum_{k=2}^{\infty}(a_kz^k+\overline{b_kz^k})$$

be a member of the class $\mathcal{S}_H(m,n;\alpha,\beta)$. *If*

$$\delta\leq\frac{(1-|b_1|)((2^m-2^n\alpha)(1+\beta)-1)}{(2^m-2^n\alpha)(1+\beta)}$$

then $N(f)\subset\mathcal{S}_H(\alpha)$.

Proof. Let $f \in \mathcal{S}_H(m,n;\alpha,\beta)$ and $F(z) = z + \overline{B_1 z} + \sum_{k=2}^{\infty}(A_k z^k + \overline{B_k z^k})$ belong to $N(f)$. We have

$$(1-\alpha)|B_1| + \sum_{k=2}^{\infty}(k-\alpha)(|A_k|+|B_k|)$$

$$\leq (1-\alpha)|B_1 - b_1| + \sum_{k=2}^{\infty}(k-\alpha)(|A_k - a_k| + |B_k - b_k|)$$

$$+ (1-\alpha)|b_1| + \sum_{k=2}^{\infty}(k-\alpha)(|a_k|+|b_k|)$$

$$\leq (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{1}{(1+\beta)(2^m - 2^n\alpha)}\sum_{k=2}^{\infty}(1-\beta+\beta k)(k^m - k^n\alpha)(|a_k|+|b_k|)$$

$$\leq (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{(1-\alpha)(1-|b_1|)}{(1+\beta)(2^m - 2^n\alpha)} \leq 1-\alpha.$$

If $\delta \leq \dfrac{(1-|b_1|)((2^m-2^n\alpha)(1+\beta)-1)}{(2^m-2^n\alpha)(1+\beta)}$. Thus $F(z) \in \mathcal{S}_H(\alpha)$. □

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