A new class of Laguerre-based Hermite-Fubini numbers and polynomials

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Abstract. In this paper, we introduce a new class of Laguerre-based Hermite-Fubini numbers and polynomials and investigate some properties of these polynomials. We establish summation formulas of these polynomials by summation techniques series. Furthermore, we derive symmetric identities of Laguerre-based Hermite-Fubini numbers and polynomials by using generating functions.

Keywords: Hermite polynomials, Fubini numbers and polynomials, Laguerre-based Hermite-Fubini polynomials, summation formulae, symmetric identities.

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1. Introduction

The generating function of the two variable Laguerre polynomials (2-VLP) \(L_n(x, y)\) \([5]\) is defined by

\[
\frac{1}{1-yt} \exp \left( \frac{-xt}{1-yt} \right) = \sum_{n=0}^{\infty} L_n(x, y)t^n, \quad (|yt| < 1) \tag{1.1}
\]

which is equivalently \([6]\) given by

\[
\exp(yt)C_0(xt) = \sum_{n=0}^{\infty} L_n(x, y) \frac{t^n}{n!}, \quad (1.2)
\]

where \(C_0(x)\) denotes the 0\(^{th}\) order Tricomi function. The \(n^{th}\) order Tricomi functions \(C_n(x)\) are defined as:

\[
C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r! (n+r)!}, \quad (n \in \mathbb{N}_0) \tag{1.3}
\]

with the following generating function:

\[
\exp \left( t - \frac{x}{t} \right) = \sum_{n=0}^{\infty} C_n(x) t^n, \quad (1.4)
\]

for \(t \neq 0\) and for all finite \(x\).

The Tricomi functions \(C_n(x)\) are characterized by the following link with the Bessel function \(J_n(x)\):

\[
C_n(x) = x^{\frac{n}{2}} J_n(2\sqrt{x}). \quad (1.5)
\]

From equations (1.2) and (1.3), we obtain

\[
L_n(x, y) = y^n \sum_{s=0}^{n} \frac{(-1)^s x^s y^{n-s}}{(s)! (n-s)!} = y^n L_n(x/y). \tag{1.6}
\]
Thus, we have
\[ L_n(x, y) = \frac{(-1)^n x^n}{n!}, \quad L_n(0, y) = y^n, \quad L_n(x, 1) = L_n(x), \] (1.7)
where \( L_n(x) \) are the ordinary Laguerre polynomials [1].

The 2-variable Hermite Kampe de Fériet polynomials (2VHKdFP) \( H_n(x, y) \) [2, 4] are defined as:
\[ H_n(x, y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}, \] (1.8)
and is supported by the following generating function:
\[ e^{xt + yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}. \] (1.9)
When \( y = -1 \) and \( x \) is replaced by \( 2x \), (1.9) reduce to the ordinary Hermite polynomials \( H_n(x) \) (see [2]).

Currently, Dattoli et al. [8, p.241] introduced the 3-variable Laguerre-Hermite polynomials (3VLHP) \( L_H n(x, y, z) \) which is defined as:
\[ L_H n(x, y, z) = n! \sum_{k=0}^{[n/2]} \frac{z^k L_{n-2k}(x, y)}{k!(n-2k)!}. \] (1.10)
The 3-variable Laguerre-Hermite polynomials (3VLHP) \( L_H n(x, y, z) \) of the following generating function:
\[ \frac{1}{(1-zt)} \exp \left( \frac{-xt}{1-zt} + \frac{yt^2}{1-zt^2} \right) = \sum_{n=0}^{\infty} L_H n(x, y, z) t^n, \] (1.11)
equivalent to
\[ \exp(yt + zt^2)C_0(xt) = \sum_{n=0}^{\infty} L_H n(x, y, z) \frac{t^n}{n!}. \] (1.12)
It is clear that
\[ L_H n(x, y, -\frac{1}{2}) = L_H n(x, y), \]
\[ L_H n(x, 1, -1) = L_H n(x), \]
where \( L_H n(x, y) \) denotes the 2-variable Laguerre-Hermite polynomials (2VLHP) (see [6]) and \( L_H n(x) \) denotes the Laguerre-Hermite polynomials (LHP) (see [7]), respectively.

Geometric polynomials (also known as Fubini polynomials) are defined as follows (see [3]):
\[ F_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!x^k, \] (1.13)
where \( \left\{ \begin{array}{c} n \\ k \end{array} \right\} \) is the Stirling number of the second kind (see [9]).
For \( x = 1 \) in (1.13), we get \( n^{th} \) Fubini number (ordered Bell number or geometric number) \( F_n [3, 9, 10, 12, 16] \) is defined by
\[ F_n(1) = F_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!. \] (1.14)
The exponential generating functions of geometric polynomials is given by (see [3]):

\[
\frac{1}{1 - x(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!},
\]

(1.15)

and related to the geometric series (see [3]):

\[
\left( x \frac{d}{dx} \right)^m \frac{1}{1 - x} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1 - x} F_m \left( \frac{x}{1 - x} \right), \quad |x| < 1.
\]

Let us give a short list of these polynomials and numbers as follows:

\[
F_0(x) = 1, \quad F_1(x) = x, \quad F_2(x) = x + 2x^2, \quad F_3(x) = x + 6x^2 + 6x^3, \quad F_4(x) = x + 14x^2 + 36x^3 + 24x^4,
\]

and

\[
F_0 = 1, \quad F_1 = 1, \quad F_2 = 3, \quad F_3 = 13, \quad F_4 = 75.
\]

Geometric and exponential polynomials are connected by the relation (see [3]):

\[
F_n(x) = \int_0^\infty \phi(x) e^{-\lambda} d\lambda.
\]

(1.16)

Recently, Khan et al. [11] introduced three variable Laguerre-based Hermite-Bernoulli polynomials is defined by means of the following generating function:

\[
\left( \frac{t^m}{e^t - \sum_{h=0}^{m-1} \frac{t^h}{h!}} \right)^\alpha e^{yt+z^2} C_0(x) = \sum_{n=0}^{\infty} LHF_{n,m-1}^{(n,m)}(x,y,z) \frac{t^n}{n!}.
\]

(1.17)

On setting \( \alpha = 1 \) in (1.13), the result reduces to the known result of Pathan and Khan [13] and further on taking \( \alpha = 1 \), the result reduces to the known result of Dattoli et al. [4].

The manuscript of this paper as follows: In section 2, we consider generating functions for Laguerre-based Hermite-Fubini numbers and polynomials and give some properties of these numbers and polynomials. In section 3, we derive summation formulas of Laguerre-based Hermite-Fubini numbers and polynomials. In Section 4, we construct a symmetric identities of Laguerre-based Hermite-Fubini numbers and polynomials by using generating functions.

2. Laguerre-based Hermite-Fubini numbers and polynomials

In this section, we define three-variable Laguerre-based Hermite-Fubini polynomials and obtain some basic properties which gives us new formula for \( LHF_n(x, y, z; w) \).

We introduce 4-variable Laguerre-based Hermite-Fubini polynomials by means of the following generating function:

\[
\frac{e^{yt+z^2} C_0(x)}{1 - w(e^t - 1)} = \sum_{n=0}^{\infty} LHF_{n}(x, y, z; w) \frac{t^n}{n!}.
\]

(2.1)

Clearly from definition (2.1), we have

\[
LHF_n(0, 0, 0; w) = F_n(w), \quad LHF_n(0, 0, 0; 1) = F_n.
\]

On setting \( x = z = 0 \) in (2.1), we obtain 2-variable Fubini polynomials which is defined by Kargin [12].
\[
\frac{e^{yt}}{1 - w(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x; z) \frac{t^n}{n!}.
\]

**Theorem 2.1.** For \( n \geq 0 \), the following formula for Laguerre-based Hermite-Fubini polynomials holds true:

\[
L H^F_n(x, y, z; w) = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}(z) L H_m(x, y).
\]

**Proof.** Using definition (2.1), we have

\[
\sum_{n=0}^{\infty} L H^F_n(x, y, z; w) \frac{t^n}{n!} = \frac{e^{yt+z} C_0(xt)}{1 - w(e^t - 1)}
\]

\[
= \sum_{n=0}^{\infty} F_n(z) \frac{t^n}{n!} \sum_{m=0}^{\infty} L H_m(x, y) \frac{t^m}{m!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} F_{n-m}(z) L H_m(x, y) \right) \frac{t^n}{n!}.
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) yields (2.3).

**Theorem 2.2.** For \( n \geq 0 \), the following formula for Laguerre-based Hermite-Fubini polynomials holds true:

\[
L H_n(x, y, z) = L H^F_n(x, y, z; w) - w L H^F_n(x, y, z; w) + w L H^F_n(x, y, z; w).
\]

**Proof.** We begin with the definition (2.1) and write

\[
e^{yt+z} C_0(xt) = \frac{1 - w(e^t - 1)}{1 - w(e^t - 1)} \frac{e^{yt+z} C_0(xt)}{1 - w(e^t - 1)}
\]

Then using the definition of Laguerre-based Hermite polynomials \( L H_n(x, y, z) \) and (2.1), we have

\[
\sum_{n=0}^{\infty} L H_n(x, y, z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[ L H^F_n(x, y, z; w) - w L H^F_n(x, y, z; w) + w L H^F_n(x, y, z; w) \right] \frac{t^n}{n!}.
\]

Finally, comparing the coefficients of \( \frac{t^n}{n!} \), we get (2.4).

**Theorem 2.3.** For \( n \geq 0 \) and \( w_1 \neq w_2 \), we have

\[
\sum_{k=0}^{n} \binom{n}{k} L H^{F_{n-k}}(x_1, y_1, z_1; w_1) L H^F_k(x_2, y_2, z_2; w_2) = \frac{w_2 L H^F_n(x_1, y_1 + y_2, z_1 + z_2; w_1) - w_1 L H^F_n(x_2, y_1 + y_2, z_1 + z_2; w_2)}{w_2 - w_1}.
\]

**Proof.** The products of (2.1) can be written as

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} L H^F_n(x_1, y_1, z_1; w_1) \frac{t^n}{n!} L H^F_k(x_2, y_2, z_2; w_2) \frac{t^k}{k!} = \frac{e^{yt+z} C_0(x_1t)}{1 - w_1(e^t - 1)} \frac{e^{yt+z} C_0(x_2t)}{1 - w_2(e^t - 1)}
\]

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} L H^{F_{n-k}}(x_1, y_1, z_1; w_1) L H^F_k(x_2, y_2, z_2; w_2) \frac{t^n}{n!}.
\]
Replacing Proof.

L.Fubini polynomials holds true:

For On setting Remark 2.3.

Comparing the coefficients of \(t\)

\[ L.H.S. = \sum_{n=0}^{\infty} \frac{L H_{F_n} (x, y_1 + y_2, z_1 + z_2; w_1) - w_1 L H_{F_n} (x_2, y_1 + y_2, z_1 + z_2; w_2)}{w_2 - w_1} t^n \]

By equating the coefficients of \( \frac{t^n}{n!} \) on both sides, we get (2.5).

**Theorem 2.4.** For \( n \geq 0 \), the following formula for Laguerre-based Hermite-Fubini polynomials holds true:

\[ w_L H_{F_n} (x, y + 1, z; w) = (1 + w)_L H_{F_n} (x, y, z; w) - L H_n (x, y, z). \] (2.6)

**Proof.** From (2.1), we have

\[ \sum_{n=0}^{\infty} \left[ L H_{F_n} (x, y + 1, z; w) - L H_{F_n} (x, y + 1, z; w) \right] \frac{t^n}{n!} = \frac{e^{yt+zt^2} C_0 (xt)}{1 - w(e^{t} - 1)} \]

\[ = \frac{1}{w} \left[ \frac{e^{yt+zt^2} C_0 (xt)}{1 - w(e^{t} - 1)} - e^{yt+zt^2} C_0 (xt) \right] \]

\[ = \frac{1}{w} \sum_{n=0}^{\infty} \left[ L H_{F_n} (x, y, z; w) - L H_n (x, y, z) \right] \frac{t^n}{n!}. \]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides, we obtain (2.5).

**Remark 2.3.** On setting \( x = y = z = 0 \) and \( y = -1 \) in Theorem 2.4, we find

\[ w_L H_{F_n} (0, 1, 0; w) = (1 + w)_L H_{F_n} (0, 0, 0; w), \]

and

\[ w_L H_{F_n} (0, 0, 0; w) = (1 + w)_L H_{F_n} (0, -1, 0; w) - (1)^n. \] (2.8)

**Theorem 2.5.** For \( n \geq 0 \), \( p, q \in \mathbb{R} \), the following formula for Laguerre-based Hermite-Fubini polynomials holds true:

\[ L H_{F_n} (x, pq, qz; w) = n! \sum_{k=0}^{n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} L H_{F_{n-k}} (x, y, z; w)((p-1)x)^{(p-1)j}((q-1)y)^j \frac{1}{(n-k-2j)!j!}. \] (2.9)

**Proof.** Rewrite the generating function (2.1), we have

\[ \sum_{n=0}^{\infty} \frac{L H_{F_n} (x, qy, qz; w) t^n}{n!} = \frac{1}{1 - w(e^{t} - 1)} e^{yt+zt^2} e^{(p-1)yt} e^{(q-1)zt^2} C_0 (xt) \]

\[ = \left( \sum_{n=0}^{\infty} \frac{L H_{F_n} (x, y, z; w) t^n}{n!} \right) \left( \sum_{k=0}^{\infty} \frac{((p-1)x)^{k}l^{k}}{k!} \right) \left( \sum_{j=0}^{\infty} \frac{((q-1)y)^{j}l^{2j}}{j!} \right) \]

\[ = \left( \sum_{n=0}^{\infty} \frac{L H_{F_n} (x, y, z; w) t^n}{n!} \right) \left( \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{((p-1)x)^{k}((q-1)y)^{j}l^{k+2j}}{n!k!j!} \right) \]

Replacing \( k \) by \( k - 2j \) in above equation, we have

\[ L.H.S. = \left( \sum_{n=0}^{\infty} \frac{L H_{F_n} (x, y, z; w) t^n}{n!} \right) \left( \sum_{k=2j}^{\infty} \frac{((p-1)x)^{k-2j}((q-1)y)^{j}l^{k}}{(k-2j)!j!n!} \right) \]

\[ = \sum_{n=0}^{\infty} \sum_{k=2j}^{\infty} L H_{F_n} (x, y, z; w)((p-1)x)^{k-2j}((q-1)y)^{j} \frac{t^{n+k}}{(k-2j)!j!n!} \]
Again replacing \( n \) by \( n - k \) in above equation, we have
\[
L.H.S. = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{[\frac{k}{2}]} L H_{n-k}(x, y, z; w)((p - 1)x)^{k-2j}((q - 1)y)^{j} \frac{t^n}{(n-k-2j)!j!k!}.
\]

Finally, equating the coefficients of \( t^n \) on both sides, we acquire the result (2.9).

**Theorem 2.6.** For \( n \geq 0 \), the following formula for Laguerre-based Hermite-Fubini polynomials holds true:
\[
L H_{n}(x, y, z; w) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) L H_{n-l}(x, y) \sum_{k=0}^{l} w^{k} k! S_{2}(l, k). \tag{2.10}
\]

**Proof.** From (2.1), we have
\[
\sum_{n=0}^{\infty} L H_{n}(x, y, z; w) \frac{t^n}{n!} = \frac{e^{yt+z} t^{n}}{1 - w(e^{t} - 1)} C_{0}(xt)
\]
\[
= e^{yt+z} C_{0}(xt) \sum_{k=0}^{\infty} \frac{w^{k} (e^{t} - 1)^{k}}{k!} = e^{xt+yt} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{w^{k} l!}{l!} k! S_{2}(l, k) \frac{t^{l}}{l!}
\]
\[
= \sum_{n=0}^{\infty} L H_{n}(x, y, z) \frac{t^{n}}{n!} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) L H_{n-l}(x, y, z) \sum_{k=0}^{l} w^{k} k! S_{2}(l, k) \frac{t^{l}}{l!}
\]
\[
= \sum_{n=0}^{\infty} L H_{n}(x, y, z) \frac{t^{n}}{n!} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) L H_{n-l}(x, y, z) \sum_{k=0}^{l} w^{k} k! S_{2}(l, k) \frac{t^{l}}{l!}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) L H_{n-l}(x, y, z) \sum_{k=0}^{l} w^{k} k! S_{2}(l, k) \right) \frac{t^{n}}{n!}
\]

Comparing the coefficients of \( \frac{t^{n}}{n!} \) in both sides, we get (2.10).

**Theorem 2.7.** For \( n \geq 0 \), the following formula for Laguerre-based Hermite-Fubini polynomials holds true:
\[
L H_{n}(x, y + r, z; w) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) L H_{n-l}(x, y, z) \sum_{k=0}^{l} w^{k} k! S_{2}(l + r, k + r). \tag{2.11}
\]

**Proof.** Replacing \( y \) by \( y + r \) in (2.1), we have
\[
\sum_{n=0}^{\infty} L H_{n}(x, y + r, z; w) \frac{t^{n}}{n!} = \frac{e^{(y+r)t+zt} t^{n}}{1 - w(e^{t} - 1)} C_{0}(xt)
\]
\[
= e^{yt+z} e^{rt} C_{0}(xt) t^{n} \sum_{k=0}^{\infty} \frac{w^{k} (e^{t} - 1)^{k}}{k!} = e^{xt+yt+rt} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{w^{k} l!}{l!} k! S_{2}(l, k) \frac{t^{l}}{l!}
\]
\[
= \sum_{n=0}^{\infty} L H_{n}(x, y, z) \frac{t^{n}}{n!} \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) L H_{n-l}(x, y, z) \sum_{k=0}^{l} w^{k} k! S_{2}(l + r, k + r) \frac{t^{l}}{l!}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) L H_{n-l}(x, y, z) \sum_{k=0}^{l} w^{k} k! S_{2}(l + r, k + r) \right) \frac{t^{n}}{n!}
\]

Comparing the coefficients of \( \frac{t^{n}}{n!} \) in both sides, we get (2.11).

3. Summation Formulae for Laguerre-based Hermite-Fubini polynomials
First, we prove the following result involving the Laguerre-based Hermite-Fubini polynomials $LH^n(x,y,z,w)$ by using series rearrangement techniques and considered its special case:

**Theorem 3.1.** The following summation formula for Laguerre-based Hermite-Fubini polynomials $H_n(x,y,z)$ holds true:

$$LH^{F_{q+l}}(x,v,z,w) = \sum_{n,p=0}^{\infty} \binom{q}{n} \binom{l}{p} (v-y)^n u^p LH^{F_{q+l-n-p}}(x,y,z,w). \quad (3.1)$$

**Proof.** Replacing $t$ by $t + u$ in (2.1) and then using the formula [15, p.52(2)]:

$$\sum_{N=0}^{\infty} f(N)(x+y)^N N! = \sum_{n,m=0}^{\infty} f(n+m) \frac{x^n y^m}{n! m!}, \quad (3.2)$$

in the resultant equation, we find the following generating function for the Laguerre-based Hermite-Fubini polynomials $LH^n(x,y,z,w)$:

$$\frac{1}{1-w(e^{x+y}-1)} e^{x(t+u)} C_0(xt) = e^{-y(t+u)} \sum_{q,l=0}^{\infty} LH^{F_{q+l}}(x,y,z,w) \frac{t^q u^l}{q! \cdot l!}. \quad (3.3)$$

Replacing $y$ by $v$ in the above equation and equating the resultant equation to the above equation, we find

$$\exp((v-y)(t+u)) \sum_{q,l=0}^{\infty} LH^{F_{q+l}}(x,y,z,w) \frac{t^q u^l}{q! \cdot l!} = \sum_{q,l=0}^{\infty} LH^{F_{q+l}}(x,v,z,w) \frac{t^q u^l}{q! \cdot l!}. \quad (3.4)$$

On expanding exponential function (3.4) gives

$$\sum_{N=0}^{\infty} \frac{(v-y)(t+u)^N}{N!} \sum_{q,l=0}^{\infty} LH^{F_{q+l}}(x,y,z,w) \frac{t^q u^l}{q! \cdot l!} = \sum_{q,l=0}^{\infty} LH^{F_{q+l}}(x,v,z,w) \frac{t^q u^l}{q! \cdot l!}, \quad (3.5)$$

which on using formula (3.2) in the first summation on the left hand side becomes

$$\sum_{n,p=0}^{\infty} \frac{(v-y)^n u^p}{n! \cdot p!} \sum_{q,l=0}^{\infty} LH^{F_{q+l}}(x,y,z,w) \frac{t^q u^l}{q! \cdot l!} = \sum_{q,l=0}^{\infty} LH^{F_{q+l}}(x,v,z,w) \frac{t^q u^l}{q! \cdot l!}. \quad (3.6)$$

Now replacing $q$ by $q - n$, $l$ by $l - p$ and using the lemma ([15, p.100(1)]):

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A(n,k) = \sum_{k=0}^{\infty} A(n,k - n), \quad (3.7)$$

in the l.h.s. of (3.6), we find

$$\sum_{q,l=0}^{\infty} \sum_{n,p=0}^{\infty} \frac{(v-y)^n u^p}{n! \cdot p!} LH^{F_{q+l-n-p}}(x,y,z,w) \frac{t^q u^l}{(q-n)! \cdot (l-p)!} \cdot (3.8)$$

Finally, on equating the coefficients of the like powers of $t$ and $u$ in the above equation, we get the assertion (3.1) of Theorem 3.1.
Remark 3.1. Taking \( l = 0 \) in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

Corollary 3.1. The following summation formula for Hermite-Fubini polynomials \( H_n(x, y; z) \) holds true:

\[
LH_n(x, v, z; w) = \sum_{n=0}^{\infty} \left( \frac{q^n}{n!} \right) (v - y)^n L H_{n-r}(x, y; z; w). \tag{3.9}
\]

Remark 3.2. Replacing \( v \) by \( v + y \) in (3.9), we obtain

\[
LH_n(x, v + y, z; w) = \sum_{n=0}^{\infty} \left( \frac{q^n}{n!} \right) v^n L H_{n-r}(x, y; z; w). \tag{3.10}
\]

Theorem 3.2. The following summation formula for Laguerre-based Hermite-Fubini polynomials \( H_n(x, y; z) \) holds true:

\[
LH_n(x, u, v; w) LH_n(x, U, V; W) = \sum_{r,k=0}^{n,m} \binom{n}{r} \binom{m}{k} H_r(u-y, v-z)LH_{n-r}(x, y; z; w)
\times H_k(U - Y, V - Z)LH_{n-k}(X, Y, Z; W). \tag{3.11}
\]

Proof. Consider the product of the Laguerre-based Hermite-Fubini polynomials, we can be written as generating function (2.1) in the following form:

\[
\frac{1}{1 - w(e^t - 1)} e^{yt + zt^2} C_0(xt) \frac{1}{1 - W(e^t - 1)} e^{YT + ZT^2} C_0(XT)
= \sum_{n=0}^{\infty} L H_n(x, y, z; w) \sum_{m=0}^{\infty} L H_m(X, Y, Z; W) \frac{T^n}{n!} \frac{T^m}{m!}. \tag{3.12}
\]

Replacing \( y \) by \( u, z \) by \( v \), \( Y \) by \( U \) and \( Z \) by \( V \) in (3.12) and equating the resultant to itself,

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} L H_n(x, u, v; w) LH_m(x, U, V; W) \frac{t^n}{n!} \frac{T^m}{m!}
= \exp \left( (u-y)t + (v-z)t^2 \right) \exp \left( (U - Y)T + (V - Z)T^2 \right)
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} L H_n(x, y, z; w) LH_m(x, Y, Z; W) \frac{t^n}{n!} \frac{T^m}{m!},
\]

which on using the generating function (3.7) in the exponential on the r.h.s., becomes

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} L H_n(x, u, v; w) LH_m(x, U, V; W) \frac{t^n}{n!} \frac{T^m}{m!}
= \sum_{n, r=0}^{\infty} H_r(u-y, v-z)LH_n(x, y, z; w) \frac{t^{n+r}}{n!r!} \sum_{m, k=0}^{\infty} H_k(U - Y, V - Z)LH_m(x, Y, Z; W) \frac{T^{m+k}}{m!k!}. \tag{3.13}
\]

Finally, replacing \( n \) by \( n - r \) and \( m \) by \( m - k \) and using equation (3.7) in the r.h.s. of the above equation and then equating the coefficients of like powers of \( t \) and \( T \), we get assertion (3.11) of Theorem 3.2.

Remark 3.3. Replacing \( u \) by \( y \) and \( U \) by \( Y \) in assertion (3.11) of Theorem 3.2, we deduce the following consequence of Theorem 3.2.
Corollary 3.2. The following summation formula for Hermite-Fubini polynomials $H_F(x, y; z)$ holds true:

$$L H_F^n(x, y, w) = \sum_{r=0}^{n} \binom{n}{r} L H_n(x, y; w)^r L H^{n-r}(x, y, z; w)$$

$$\times (1 - w)^r L H^{n-r}(x, y, z; w).$$

Theorem 3.3. The following summation formula for Laguerre-based Hermite-Fubini polynomials $L H_F^n(x, y, z; w)$ holds true:

$$L H_F^n(x, y + u, z + v; w) = \sum_{s=0}^{n} \binom{n}{s} L H^{n-s}(x, u, v; w) H_s(u, v).$$

Proof. We replace $y$ by $y + u$ and $z$ by $z + v$ in (2.1), use (1.9) and rewrite the generating function as:

$$\frac{1}{1 - w(e^t - 1)} \exp((y + u)t + (z + v)t^2) C_0(x t) = \sum_{n=0}^{\infty} L H_F^n(x, u, v; w) \frac{t^n}{n!} \sum_{s=0}^{\infty} H_s(u, v) \frac{t^s}{s!}$$

$$= \sum_{n=0}^{\infty} L H_F^n(x, y + u, z + v; w) \frac{t^n}{n!}.$$

Now replacing $n$ by $n - s$ in l.h.s. and comparing the coefficients of $t^n$ on both sides, we get the result (3.15).

Theorem 3.4. The following summation formula for Laguerre-based Hermite-Fubini polynomials $L H_F^n(x, y, z; w)$ holds true:

$$L H_F^n(x, y, z; w) = \sum_{r=0}^{n} \binom{n}{r} L F_{n-r}(x, y - u; w) H_r(u, z).$$

Proof. By exploiting the generating function (1.2), we can write equation (2.1) as

$$\frac{1}{1 - w(e^t - 1)} e^{(y-u)t} e^{ut + zt^2} C_0(x t) = \sum_{n=0}^{\infty} L F_n(x, y - u; w) \frac{t^n}{n!} \sum_{r=0}^{\infty} H_r(u, z) \frac{t^r}{r!}.$$

On replacing $n$ by $n - r$ in above equation, we get

$$\sum_{n=0}^{\infty} L H_F^n(x, y, z; w) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} L F_n(x, y - u; w) H_r(u, z) \frac{t^n}{(n - r)!r!}.$$

Equating the coefficients of the like powers of $t$ on both sides, we get (3.17).

Theorem 3.5. The following summation formula for Laguerre-based Hermite-Fubini polynomials $L H_F^n(x, y, z; w)$ holds true:

$$L H_F^n(x, y + 1, z; w) = \sum_{r=0}^{n} \binom{n}{r} L H^{n-r}(x, y, z; w).$$

Proof. Using the generating function (2.1), we have

$$\sum_{n=0}^{\infty} L H_F^n(x, y + 1, z; w) \frac{t^n}{n!} - \sum_{n=0}^{\infty} L H_F^n(x, y, z; w) \frac{t^n}{n!}$$

$$= \left(\frac{1}{1 - w(e^t - 1)} \right) (e^t - 1) e^{yt + zt^2} C_0(x t)$$
\[ = \sum_{n=0}^{\infty} L H^{F_n}(x, y, z; w) \frac{t^n}{n!} \left( \sum_{r=0}^{n} t^r - 1 \right) \]
\[ = \sum_{n=0}^{\infty} L H^{F_n}(x, y, z; w) \frac{t^n}{n!} \sum_{r=0}^{\infty} \frac{t^r}{r!} - \sum_{n=0}^{\infty} L H^{F_n}(x, y, z; w) \frac{t^n}{n!} \]
\[ = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) L H^{F_{n-r}}(x, y, z; w) \frac{t^n}{n!} - \sum_{n=0}^{\infty} L H^{F_n}(x, y, z; w) \frac{t^n}{n!}. \]

Finally, equating the coefficients of the like powers of \( t \) on both sides, we get (3.18).

4. General identities

Recently, Pathan and Khan [13, 14] have been introduced symmetric identities. In this section, we establish general symmetry identities for the generalized Laguerre-based Hermite-Fubini polynomials \( L H^{F_n}(x, y, z; w) \) by applying the generating function (2.1) and (2.2).

**Theorem 4.1.** Let \( x, y, z \in \mathbb{R} \) and \( n \geq 0 \), then the following identity holds true:
\[ \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) b^r a^{n-r} L H^{F_{n-r}}(x, by, b^2 z; w) L H^{F_n}(ax, ay, a^2 z; w) \]
\[ = \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) a^r b^{n-r} L H^{F_{n-r}}(ax, ay, a^2 z; w) L H^{F_n}(bx, by, b^2 z; w). \quad (4.1) \]

**Proof.** Start with
\[ A(t) = \frac{1}{(1 - w(e^{ax} - 1))(1 - w(e^{by} - 1))} e^{b^2 z t^2} C_0(abxt). \]
Then the expression for \( A(t) \) is symmetric in \( a \) and \( b \) and we can expand \( A(t) \) into series in two ways to obtain:
\[ A(t) = \sum_{n=0}^{\infty} L H^{F_n}(bx, by, b^2 z; w) \frac{(at)^n}{n!} \sum_{r=0}^{\infty} L H^{F_n}(ax, ay, a^2 z; w) \frac{(bt)^r}{r!} \]
\[ = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) b^r a^{n-r} L H^{F_{n-r}}(bx, by, b^2 z; w) L H^{F_n}(ax, ay, a^2 z; w) \frac{t^n}{n!}. \quad (4.2) \]

Similarly, we can show that
\[ A(t) = \sum_{n=0}^{\infty} L H^{F_n}(ax, ay, a^2 z; w) \frac{(bt)^n}{n!} \sum_{r=0}^{\infty} L H^{F_n}(bx, by, b^2 z; w) \frac{(at)^r}{r!} \]
\[ = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) a^r b^{n-r} L H^{F_{n-r}}(ax, ay, a^2 z; w) L H^{F_n}(bx, by, b^2 z; w) \frac{t^n}{n!}. \quad (4.3) \]

By comparing the coefficients of \( \frac{t^n}{n!} \) on the right hand sides of the last two equations, we arrive at the desired result (4.1).
Theorem 4.2. For each pair of integers $a$ and $b$ and all integers and $n \geq 0$, the following identity holds true:

$$
\sum_{k=0}^{n} \binom{n}{k} (\sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^{n-k} b^{k} L_{H}^{F_{a-k}} \left( b y + \frac{b}{a} i + j, b^{2} z, b x; w \right) F_{k}(a u, w))
$$

$$
\sum_{k=0}^{n} \binom{n}{k} (\sum_{j=0}^{a-1} \sum_{i=0}^{b-1} b^{n-k} a^{k} L_{H}^{F_{a-k}} \left( a y + \frac{a}{b} i + j, a^{2} z, a x; w \right) F_{k}(b u, w)). \quad (4.4)
$$

Proof. Let

$$
B(t) = \frac{e^{(y+w)t+abz^2t^2}(e^{abt} - 1)^2C_0(abzt)}{(1-w(e^{at}-1))(1-w(e^{bt}-1))(e^{abt} - 1)(e^{bt} - 1)}
$$

$$
= \frac{e^{abzt+a^2b^2z^2}C_0(abzt)}{1-w(e^{at}-1)} \frac{e^{abt} - 1}{e^{abt} - 1} \frac{e^{abt} - 1}{e^{abt} - 1}
$$

$$
B(t) = \frac{e^{abzt+a^2b^2z^2}C_0(abzt)\sum_{i=0}^{a-1} e^{abt} \sum_{j=0}^{b-1} e^{abt}}{1-w(e^{at}-1)} \frac{\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} F_{k}(a u, w) (bt)^k}{k!}
$$

$$
B(t) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} (\sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^{n-k} b^{k} L_{H}^{F_{a-k}} \left( b y + \frac{b}{a} i + j, b^{2} z, b x; w \right) F_{k}(a u, w)) \right) \frac{t^n}{n!}.
$$

On the other hand

$$
B(t) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} (\sum_{j=0}^{a-1} \sum_{i=0}^{b-1} b^{n-k} a^{k} L_{H}^{F_{a-k}} \left( a y + \frac{a}{b} i + j, a^{2} z, a x; w \right) F_{k}(b u, w)) \right) \frac{t^n}{n!}.
$$

By comparing the coefficients of $t^n$ on the right hand sides of the last two equations, we arrive at the desired result.

References


