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A new class of Laguerre-based Hermite-Fubini numbers and polynomials

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Abstract. In this paper, we introduce a new class of Laguerre-based Hermite-Fubini numbers and polynomials and investigate some properties of these polynomials. We establish summation formulas of these polynomials by summation techniques series. Furthermore, we derive symmetric identities of Laguerre-based Hermite-Fubini numbers and polynomials by using generating functions.

Keywords: Hermite polynomials, Fubini numbers and polynomials, Laguerre-based Hermite-Fubini polynomials, summation formulae, symmetric identities.
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1. Introduction

The generating function of the two variable Laguerre polynomials (2-VLP) $Ł_n(x,y)$ [5] is defined by

$$\frac{1}{(1-yt)}\exp\left(\frac{-xt}{1-yt}\right)=\sum_{n=0}^{\infty}Ł_n(x,y)t^n,(|\ yt\ |<1) \tag{1.1}$$

which is equivalently [6] given by

$$\exp(yt)C_0(xt)=\sum_{n=0}^{\infty}Ł_n(x,y)\frac{t^n}{n!}, \tag{1.2}$$

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined as:

$$C_n(x)=\sum_{n=0}^{\infty}\frac{(-1)^r x^r}{r!(n+r)!},(n\in\mathbb{N}_0) \tag{1.3}$$

with the following generating function:

$$\exp\left(t-\frac{x}{t}\right)=\sum_{n=0}^{\infty}C_n(x)t^n, \tag{1.4}$$

for $t\neq 0$ and for all finite x.

The Tricomi functions $C_n(x)$ are characterized by the following link with the Bessel function $J_n(x)$:

$$C_n(x)=x^{\frac{n}{2}}J_n(2\sqrt{x}). \tag{1.5}$$

From equations (1.2) and (1.3), we obtain

$$L_n(x,y)=n!\sum_{s=0}^{n}\frac{(-1)^s x^s y^{n-s}}{(s!)^2(n-s)!}=y^nL_n(x/y). \tag{1.6}$$

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Thus, we have

$$Ł_n(x,y) = \frac{(-1)^n x^n}{n!}, Ł_n(0,y) = y^n, Ł_n(x,1) = Ł_n(x), \tag{1.7}$$

where $L_n(x)$ are the ordinary Laguerre polynomials [1].

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [2, 4] are defined as:

$$H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}, \tag{1.8}$$

and is supported by the following generating function:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}. \tag{1.9}$$

When $y = -1$ and x is replaced by $2x$, (1.9) reduce to the ordinary Hermite polynomials $H_n(x)$ (see [2]).

Currently, Dattoli et al. [8, p.241] introduced the 3-variable Laguerre-Hermite polynomials (3VLHP) ${}_LH_n(x,y,z)$ which is defined as:

$${}_LH_n(x,y,z) = n! \sum_{k=0}^{[n/2]} \frac{z^k L_{n-2k}(x,y)}{k!(n-2k)!}. \tag{1.10}$$

The 3-variable Laguerre-Hermite polynomials (3VLHP) ${}_LH_n(x,y,z)$ of the following generating function:

$$\frac{1}{(1-zt)} \exp\left(\frac{-xt}{1-zt} + \frac{yt^2}{1-zt^2} \right) = \sum_{n=0}^{\infty} {}_LH_n(x,y,z) t^n, \tag{1.11}$$

equivalent to

$$\exp(yt + zt^2) C_0(xt) = \sum_{n=0}^{\infty} {}_LH_n(x,y,z) \frac{t^n}{n!}. \tag{1.12}$$

It is clear that

$${}_LH_n(x,y,-\frac{1}{2}) = {}_LH_n(x,y),$$
$${}_LH_n(x,1,-1) = {}_LH_n(x),$$

where ${}_LH_n(x,y)$ denotes the 2-variable Laguerre-Hermite polynomials (2VLHP) (see [6]) and ${}_LH_n(x)$ denotes the Laguerre-Hermite polynomials (LHP) (see [7]), respectively.

Geometric polynomials (also known as Fubini polynomials) are defined as follows (see [3]):

$$F_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k! x^k, \tag{1.13}$$

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is the Stirling number of the second kind (see [9]).
For $x = 1$ in (1.13), we get n^{th} Fubini number (ordered Bell number or geometric number) F_n [3, 9, 10, 12, 16] is defined by

$$F_n(1) = F_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!. \tag{1.14}$$

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The exponential generating functions of geometric polynomials is given by (see [3]):

$$\frac{1}{1-x(e^t-1)}=\sum_{n=0}^{\infty}F_n(x)\frac{t^n}{n!}, \tag{1.15}$$

and related to the geometric series (see [3]):

$$\left(x\frac{d}{dx}\right)^m\frac{1}{1-x}=\sum_{k=0}^{\infty}k^m x^k=\frac{1}{1-x}F_m(\frac{x}{1-x}), \mid x\mid<1.$$

Let us give a short list of these polynomials and numbers as follows:

$F_0(x)=1, F_1(x)=x, F_2(x)=x+2x^2, F_3(x)=x+6x^2+6x^3, F_4(x)=x+14x^2+36x^3+24x^4,$

and

$$F_0=1, F_1=1, F_2=3, F_3=13, F_4=75.$$

Geometric and exponential polynomials are connected by the relation (see [3]):

$$F_n(x)=\int_0^{\infty}\phi_n(x)e^{-\lambda}d\lambda. \tag{1.16}$$

Recently, Khan et al. [11] introduced three variable Laguerre-based Hermite-Bernoulli polynomials is defined by means of the following generating function:

$$\left(\frac{t^m}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}e^{yt+zt^2}C_0(xt)=\sum_{n=0}^{\infty}{}_LH^{B_n^{[\alpha,m-1]}}(x,y,z)\frac{t^n}{n!}. \tag{1.17}$$

On setting $\alpha=1$ in (1.13), the result reduces to the known result of Pathan and Khan [13] and further on taking $\alpha=1$, the result reduces to the known result of Dattoli et al. [4].

The manuscript of this paper as follows: In section 2, we consider generating functions for Laguerre-based Hermite-Fubini numbers and polynomials and give some properties of these numbers and polynomials. In section 3, we derive summation formulas of Laguerre-based Hermite-Fubini numbers and polynomials. In Section 4, we construct a symmetric identities of Laguerre-based Hermite-Fubini numbers and polynomials by using generating functions.

2. Laguerre-based Hermite-Fubini numbers and polynomials

In this section, we define three-variable Laguerre-based Hermite-Fubini polynomials and obtain some basic properties which gives us new formula for ${}_LH^{F_n}(x,y,z;w)$.

We introduce 4-variable Laguerre-based Hermite-Fubini polynomials by means of the following generating function:

$$\frac{e^{yt+zt^2}C_0(xt)}{1-w(e^t-1)}=\sum_{n=0}^{\infty}{}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!}. \tag{2.1}$$

Clearly from definition (2.1), we have

$${}_LH^{F_n}(0,0,0;w)=F_n(w), {}_LH^{F_n}(0,0,0;1)=F_n.$$

On setting $x=z=0$ in (2.1), we obtain 2-variable Fubini polynomials which is defined by Kargin [12].

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$$\frac{e^{yt}}{1-w(e^t-1)}=\sum_{n=0}^{\infty}F_n(x;z)\frac{t^n}{n!}. \tag{2.2}$$

Theorem 2.1. For $n \geq 0$, the following formula for Laguerre-based Hermite-Fubini polynomials holds true:

$${}_LH^{F_n}(x,y,z;w)=\sum_{m=0}^{n}\left(\begin{array}{c}n\\r\end{array}\right)F_{n-m}(z)_LH_m(x,y). \tag{2.3}$$

Proof. Using definition (2.1), we have

$$\sum_{n=0}^{\infty}{}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!}=\frac{e^{yt+zt^2}C_0(xt)}{1-w(e^t-1)}$$

$$=\sum_{n=0}^{\infty}F_n(z)\frac{t^n}{n!}\sum_{m=0}^{\infty}{}_LH_m(x,y)\frac{t^m}{m!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\begin{array}{c}n\\r\end{array}\right)F_{n-m}(z)_LH_m(x,y)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (2.3).

Theorem 2.2. For $n \geq 0$, the following formula for Laguerre-based Hermite-Fubini polynomials holds true:

$${}_LH_n(x,y,z)={}_LH^{F_n}(x,y,z;w)-w_LH^{F_n}(x,y,z;w)+w_LH^{F_n}(x,y,z;w). \tag{2.4}$$

Proof. We begin with the definition (2.1) and write

$$e^{yt+zt^2}C_0(xt)=\frac{1-w(e^t-1)}{1-w(e^t-1)}e^{yt+zt^2}C_0(xt)$$

$$=\frac{e^{yt+zt^2}C_0(xt)}{1-w(e^t-1)}-\frac{w(e^t-1)}{1-z(e^t-1)}e^{yt+zt^2}C_0(xt)$$

Then using the definition of Laguerre-based Hermite polynomials ${}_LH_n(x,y,z)$ and (2.1), we have

$$\sum_{n=0}^{\infty}{}_LH_n(x,y,z)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left[{}_LH^{F_n}(x,y,z;w)-w_LH^{F_n}(x,y,z;w)+w_LH^{F_n}(x,y,z;w)\right]\frac{t^n}{n!}.$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$, we get (2.4).

Theorem 2.3. For $n \geq 0$ and $w_1 \neq w_2$, we have

$$\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right){}_LH^{F_{n-k}}(x_1,y_1,z_1;w_1)_LH^{F_k}(x_2,y_2,z_2;w_2)$$

$$=\frac{w_{2L}H^{F_n}(x_1,y_1+y_2,z_1+z_2;w_1)-w_{1L}H^{F_n}(x_2,y_1+y_2,z_1+z_2;w_2)}{w_2-w_1}. \tag{2.5}$$

Proof. The products of (2.1) can be written as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_LH^{F_n}(x_1,y_1,z_1;w_1)\frac{t^n}{n!}{}_LH^{F_k}(x_2,y_2,z_2;w_2)\frac{t^k}{k!}=\frac{e^{y_1t+z_1t^2}C_0(x_1t)}{1-w_1(e^t-1)}\frac{e^{y_2t+z_2t^2}C_0(x_2t)}{1-w_2(e^t-1)}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right){}_LH^{F_{n-k}}(x_1,y_1,z_1;w_1)_LH^{F_k}(x_2,y_2,z_2;w_2)\right)\frac{t^n}{n!}$$

$$= \frac{w_2}{w_2-w_1}\frac{e^{(y_1+y_2)t+(z_1+z_2)t^2}C_0(x_1t)}{1-w_1(e^t-1)} - \frac{w_1}{w_2-w_1}\frac{e^{(y_1+y_2)t+(z_1+z_2)t^2}C_0(x_2t)}{1-w_2(e^t-1)}$$

$$= \left(\frac{w_2\,{}_LH^{F_n}(x_1,y_1+y_2,z_1+z_2;w_1) - w_1\,{}_LH^{F_n}(x_2,y_1+y_2,z_1+z_2;w_2)}{w_2-w_1}\right)\frac{t^n}{n!}.$$

By equating the coefficients of $\frac{t^n}{n!}$ on both sides, we get (2.5).

Theorem 2.4. For $n \geq 0$, the following formula for Laguerre-based Hermite-Fubini polynomials holds true:

$$w\,{}_LH^{F_n}(x,y+1,z;w) = (1+w)\,{}_LH^{F_n}(x,y,z;w) - {}_LH_n(x,y,z). \tag{2.6}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty}\left[{}_LH^{F_n}(x,y+1,z;w) - {}_LH^{F_n}(x,y+1,z;w)\right]\frac{t^n}{n!} = \frac{e^{yt+zt^2}C_0(xt)}{1-w(e^t-1)}(e^t-1)$$

$$= \frac{1}{w}\left[\frac{e^{yt+zt^2}C_0(xt)}{1-w(e^t-1)} - e^{yt+zt^2}C_0(xt)\right]$$

$$= \frac{1}{w}\sum_{n=0}^{\infty}\left[{}_LH^{F_n}(x,y,z;w) - {}_LH_n(x,y,z)\right]\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we obtain (2.6).

Remark 2.3. On setting $x=y=z=0$ and $y=-1$ in Theorem 2.4, we find

$$w\,{}_LH^{F_n}(0,1,0;w) = (1+w)\,{}_LH^{F_n}(0,0,0;w), \tag{2.7}$$

and

$$w\,{}_LH^{F_n}(0,0,0;w) = (1+w)\,{}_LH^{F_n}(0,-1,0;w) - (-1)^n. \tag{2.8}$$

Theorem 2.5. For $n \geq 0$, $p,q \in \mathbb{R}$, the following formula for Laguerre-based Hermite-Fubini polynomials holds true:

$${}_LH^{F_n}(x,py,qz;w) = n!\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]}{}_LH^{F_{n-k}}(x,y,z;w)((p-1)x)^k((q-1)y)^j\frac{1}{(n-k-2j)!j!}. \tag{2.9}$$

Proof. Rewrite the generating function (2.1), we have

$$\sum_{n=0}^{\infty}{}_LH^{F_n}(x,qy,qz;w)\frac{t^n}{n!} = \frac{1}{1-w(e^t-1)}e^{yt+zt^2}e^{(p-1)yt}e^{(q-1)zt^2}C_0(xt)$$

$$= \left(\sum_{n=0}^{\infty}{}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}((p-1)x)^k\frac{t^k}{k!}\right)\left(\sum_{j=0}^{\infty}((q-1)y)^j\frac{t^{2j}}{j!}\right)$$

$$= \left(\sum_{n=0}^{\infty}{}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}((p-1)x)^k((q-1)y)^j\frac{t^{k+2j}}{n!k!j!}\right)$$

Replacing k by $k-2j$ in above equation, we have

$$L.H.S. = \left(\sum_{n=0}^{\infty}{}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!}\right)\left(\sum_{k=2j}^{\infty}((p-1)x)^{k-2j}((q-1)y)^j\frac{t^k}{(k-2j)!j!}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{k=2j}^{\infty}{}_LH^{F_n}(x,y,z;w)((p-1)x)^{k-2j}((q-1)y)^j\frac{t^{n+k}}{(k-2j)!j!n!}$$

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Again replacing n by $n-k$ in above equation, we have

$$L.H.S. = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]} {}_LH^{F_{n-k}}(x,y,z;w)((p-1)x)^{k-2j}((q-1)y)^j \frac{t^n}{(n-k-2j)!j!k!}.$$

Finally, equating the coefficients of t^n on both sides, we acquire the result (2.9).

Theorem 2.6. For $n \geq 0$, the following formula for Laguerre-based Hermite-Fubini polynomials holds true:

$$ {}_LH^{F_n}(x,y,z;w) = \sum_{l=0}^{n} \binom{n}{l} {}_LH_{n-l}(x,y) \sum_{k=0}^{l} w^k k! S_2(l,k). \tag{2.10}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} {}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!} = \frac{e^{yt+zt^2}}{1-w(e^t-1)}C_0(xt)$$

$$= e^{yt+zt^2}C_0(xt)\sum_{k=0}^{\infty} w^k(e^t-1)^k = e^{xt+yt^2}\sum_{k=0}^{\infty} w^k \sum_{l=k}^{\infty} k!S_2(l,k)\frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} {}_LH_n(x,y,z)\frac{t^n}{n!}\sum_{l=0}^{\infty} w^k \sum_{k=0}^{l} k!S_2(l,k)\frac{t^l}{l!}.$$

Replacing n by $n-l$ in above equation, we get

$$\sum_{n=0}^{\infty} {}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} {}_LH_{n-l}(x,y,z)\sum_{k=0}^{l} w^k k!S_2(l,k)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.10).

Theorem 2.7. For $n \geq 0$, the following formula for Laguerre-based Hermite-Fubini polynomials holds true:

$$ {}_LH^{F_n}(x,y+r,z;w) = \sum_{l=0}^{n} \binom{n}{l} {}_LH_{n-l}(x,y,z) \sum_{k=0}^{l} w^k k! S_2(l+r,k+r). \tag{2.11}$$

Proof. Replacing y by $y+r$ in (2.1), we have

$$\sum_{n=0}^{\infty} {}_LH^{F_n}(x,y+r,z;w)\frac{t^n}{n!} = \frac{e^{(y+r)t+zt^2}}{1-w(e^t-1)}C_0(xt)$$

$$= e^{yt+zt^2}C_0(xt)e^{rt}\sum_{k=0}^{\infty} w^k(e^t-1)^k = e^{yt+zt^2}C_0(xt)e^{rt}\sum_{k=0}^{\infty} w^k \sum_{l=k}^{\infty} k!S_2(l,k)\frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} {}_LH_n(x,y,z)\frac{t^n}{n!}\sum_{l=0}^{\infty} w^k \sum_{k=0}^{l} k!S_2(l+r,k+r)\frac{t^l}{l!}.$$

Replacing n by $n-l$ in above equation, we get

$$\sum_{n=0}^{\infty} {}_LH^{F_n}(x,y+r,z;w)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} {}_LH_{n-l}(x,y,z)\sum_{k=0}^{l} w^k k!S_2(l+r,k+r)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.11).

3. Summation Formulae for Laguerre-based Hermite-Fubini polynomials

First, we prove the following result involving the Laguerre-based Hermite-Fubini polynomials ${}_LH^{F_n}(x,y,z;w)$ by using series rearrangement techniques and considered its special case:

Theorem 3.1. The following summation formula for Laguerre-based Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_LH^{F_{q+l}}(x,v,z;w) \;=\; \sum_{n,p=0}^{q,l} \binom{q}{n}\binom{l}{p}(v-y)^{n+p}\,{}_LH^{F_{q+l-n-p}}(x,y,z;w). \quad (3.1)$$

Proof. Replacing t by $t\;+\;u$ in (2.1) and then using the formula [15,p.52(2)]:

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \quad (3.2)$$

in the resultant equation, we find the following generating function for the Laguerre-based Hermite-Fubini polynomials ${}_LH^{F_n}(x,y,z;w)$:

$$\frac{1}{1-w(e^{t+u}-1)}e^{z(t+u)^2}C_0(xt) = e^{-y(t+u)}\sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(x,y,z;w)\frac{t^q}{q!}\frac{u^l}{l!}. \quad (3.3)$$

Replacing y by v in the above equation and equating the resultant equation to the above equation, we find

$$\exp((v-y)(t+u))\sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(x,y,z;w)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(x,v,z;w)\frac{t^q}{q!}\frac{u^l}{l!}. \quad (3.4)$$

On expanding exponential function (3.4) gives

$$\sum_{N=0}^{\infty}\frac{[(v-y)(t+u)]^N}{N!}\sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(x,y,z;w)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(x,v,z;w)\frac{t^q}{q!}\frac{u^l}{l!}, \quad (3.5)$$

which on using formula (3.2) in the first summation on the left hand side becomes

$$\sum_{n,p=0}^{\infty}\frac{(v-y)^{n+p}t^n u^p}{n!p!}\sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(x,y,z;w)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(x,v,z;w)\frac{t^q}{q!}\frac{u^l}{l!}. \quad (3.6)$$

Now replacing q by $q-n$, l by $l-p$ and using the lemma ([15, p.100(1)]):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A(n,k) = \sum_{k=0}^{\infty}\sum_{n=0}^{k} A(n,k-n), \quad (3.7)$$

in the l.h.s. of (3.6), we find

$$\sum_{q,l=0}^{\infty}\sum_{n,p=0}^{q,l}\frac{(v-y)^{n+p}}{n!p!}\,{}_LH^{F_{q+l-n-p}}(x,y,z;w)\frac{t^q}{(q-n)!}\frac{u^l}{(l-p)!}$$
$$= \sum_{q,l=0}^{\infty} {}_LH^{F_{q+l}}(x,v,z;w)\frac{t^q}{q!}\frac{u^l}{l!}. \quad (3.8)$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, we get the assertion (3.1) of Theorem 3.1.

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Remark 3.1. Taking $l=0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

Corollary 3.1. The following summation formula for Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_LH^{F_q}(x,v,z;w) = \sum_{n=0}^{q} \binom{q}{n} (v-y)^n {}_LH^{F_{q-n}}(x,y,z;w). \tag{3.9}$$

Remark 3.2. Replacing v by $v+y$ in (3.9), we obtain

$$ {}_LH^{F_q}(x,v+y,z;w) = \sum_{n=0}^{q} \binom{q}{n} v^n {}_LH^{F_{q-n}}(x,y,z;w). \tag{3.10}$$

Theorem 3.2. The following summation formula for Laguerre-based Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_LH^{F_n}(x,u,v;w){}_LH^{F_m}(X,U,V;W) = \sum_{r,k=0}^{n,m} \binom{n}{r}\binom{m}{k} H_r(u-y,v-z){}_LH^{F_{n-r}}(x,y,z;w)$$
$$\times H_k(U-Y,V-Z){}_LH^{F_{m-k}}(X,Y,Z;W). \tag{3.11}$$

Proof. Consider the product of the Laguerre-based Hermite-Fubini polynomials, we can be written as generating function (2.1) in the following form:

$$\frac{1}{1-w(e^t-1)}e^{yt+zt^2}C_0(xt)\frac{1}{1-W(e^T-1)}e^{YT+ZT^2}C_0(XT)$$
$$= \sum_{n=0}^{\infty} {}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!} \sum_{m=0}^{\infty} {}_LH^{F_m}(X,Y,Z;W)\frac{T^m}{m!}. \tag{3.12}$$

Replacing y by u, z by v, Y by U and Z by V in (3.12) and equating the resultant to itself,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_LH^{F_n}(x,u,v;w){}_LH^{F_m}(X,U,V;W)\frac{t^n}{n!}\frac{T^m}{m!}$$
$$= \exp\left((u-y)t+(v-z)t^2\right)\exp\left((U-Y)T+(V-Z)T^2\right)$$
$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_LH^{F_n}(x,y,z;w){}_LH^{F_m}(X,Y,Z;W)\frac{t^n}{n!}\frac{T^m}{m!},$$

which on using the generating function (3.7) in the exponential on the r.h.s., becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_LH^{F_n}(x,u,v;w){}_LH^{F_m}(X,U,V;W)\frac{t^n}{n!}\frac{T^m}{m!}$$
$$= \sum_{n,r=0}^{\infty} H_r(u-y,v-z){}_LH^{F_n}(x,y,z;w)\frac{t^{n+r}}{n!r!} \sum_{m,k=0}^{\infty} H_k(U-Y,V-Z){}_LH^{F_m}(X,Y,Z;W)\frac{T^{m+k}}{m!k!}. \tag{3.13}$$

Finally, replacing n by $n-r$ and m by $m-k$ and using equation (3.7) in the r.h.s. of the above equation and then equating the coefficients of like powers of t and T, we get assertion (3.11) of Theorem 3.2.

Remark 3.3. Replacing u by y and U by Y in assertion (3.11) of Theorem 3.2, we deduce the the following consequence of Theorem 3.2.

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Corollary 3.2. The following summation formula for Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_LH^{F_n}(x,y,v;w)\,{}_LH^{F_m}(X,Y,V;W)=\sum_{r,k=0}^{n,m}\binom{n}{r}\binom{m}{k}(v-z)^r\,{}_LH^{F_{n-r}}(x,y,z;w) $$
$$ \times(V-Z)^k\,{}_LH^{F_{m-k}}(X,Y,Z;W). \tag{3.14} $$

Theorem 3.3. The following summation formula for Laguerre-based Hermite-Fubini polynomials ${}_LH^{F_n}(x,y,z;w)$ holds true:

$$ {}_LH^{F_n}(x,y+u,z+v;w)=\sum_{s=0}^{n}\binom{n}{s}\,{}_LH^{F_{n-s}}(x,u,v;w)H_s(u,v). \tag{3.15} $$

Proof. We replace y by $y+u$ and z by $z+v$ in (2.1), use (1.9) and rewrite the generating function as:

$$ \frac{1}{1-w(e^t-1)}\exp((y+u)t+(z+v)t^2)C_0(xt)=\sum_{n=0}^{\infty}{}_LH^{F_n}(x,u,v;w)\frac{t^n}{n!}\sum_{s=0}^{\infty}H_s(u,v)\frac{t^s}{s!} $$
$$ =\sum_{n=0}^{\infty}{}_LH^{F_n}(x,y+u,z+v;w)\frac{t^n}{n!}. $$

Now replacing n by $n-s$ in l.h.s. and comparing the coefficients of t^n on both sides, we get the result (3.15).

Theorem 3.4. The following summation formula for Laguerre-based Hermite-Fubini polynomials ${}_LH^{F_n}(x,y,z;w)$ holds true:

$$ {}_LH^{F_n}(x,y,z;w)=\sum_{r=0}^{n}\binom{n}{r}\,{}_LF_{n-r}(x,y-u;w)H_r(u,z). \tag{3.16} $$

Proof. By exploiting the generating function (1.2), we can write equation (2.1) as

$$ \frac{1}{1-w(e^t-1)}e^{(y-u)t}e^{ut+zt^2}C_0(xt)=\sum_{n=0}^{\infty}{}_LF_n(x,y-u;w)\frac{t^n}{n!}\sum_{r=0}^{\infty}H_r(u,z)\frac{t^r}{r!}. \tag{3.17} $$

On replacing n by $n-r$ in above equation, we get

$$ \sum_{n=0}^{\infty}{}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_LF_{n-r}(x,y-u;w)H_r(u,z)\frac{t^n}{(n-r)!r!}. $$

Equating the coefficients of the like powers of t on both sides, we get (3.17).

Theorem 3.5. The following summation formula for Laguerre-based Hermite-Fubini polynomials ${}_LH^{F_n}(x,y,z;w)$ holds true:

$$ {}_LH^{F_n}(x,y+1,z;w)=\sum_{r=0}^{n}\binom{n}{r}\,{}_LH^{F_{n-r}}(x,y,z;w). \tag{3.18} $$

Proof. Using the generating function (2.1), we have

$$ \sum_{n=0}^{\infty}{}_LH^{F_n}(x,y+1,z;w)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!} $$
$$ =\left(\frac{1}{1-w(e^t-1)}\right)(e^t-1)e^{yt+zt^2}C_0(xt) $$

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$$= \sum_{n=0}^{\infty} {}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!}\left(\sum_{r=0}^{\infty}\frac{t^r}{r!}-1\right)$$

$$= \sum_{n=0}^{\infty} {}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!}\sum_{r=0}^{\infty}\frac{t^r}{r!} - \sum_{n=0}^{\infty} {}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r} {}_LH^{F_{n-r}}(x,y,z;w)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_LH^{F_n}(x,y,z;w)\frac{t^n}{n!}.$$

Finally, equating the coefficients of the like powers of t on both sides, we get (3.18).

4. General identities

Recently, Pathan and Khan [13, 14] have been introduced symmetric identities. In this section, we establish general symmetry identities for the generalized Laguerre-based Hermite-Fubini polynomials ${}_LH^{F_n}(x,y,z;w)$ by applying the generating function (2.1) and (2.2).

Theorem 4.1. Let $x,y,z \in \mathbb{R}$ and $n \geq 0$, then the following identity holds true:

$$\sum_{r=0}^{n}\binom{n}{r} b^r a^{n-r} {}_LH^{F_{n-r}}(bx,by,b^2z;w)\,{}_LH^{F_n}(ax,ay,a^2z;w)$$

$$= \sum_{r=0}^{n}\binom{n}{r} a^r b^{n-r} {}_LH^{F_{n-r}}(ax,ay,a^2z;w)\,{}_LH^{F_n}(bx,by,b^2z;w). \tag{4.1}$$

Proof. Start with

$$A(t) = \frac{1}{(1-w(e^{at}-1))(1-w(e^{bt}-1))} e^{abyt+a^2b^2zt^2} C_0(abxt).$$

Then the expression for $A(t)$ is symmetric in a and b and we can expand $A(t)$ into series in two ways to obtain:

$$A(t) = \sum_{n=0}^{\infty} {}_LH^{F_n}(bx,by,b^2z;w)\frac{(at)^n}{n!}\sum_{r=0}^{\infty} {}_LH^{F_n}(ax,ay,a^2z;w)\frac{(bt)^r}{r!}$$

$$A(t) = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r} b^r a^{n-r} {}_LH^{F_{n-r}}(bx,by,b^2z;w)\,{}_LH^{F_n}(ax,ay,a^2z;w)\right)\frac{t^n}{n!}. \tag{4.2}$$

Similarly, we can show that

$$A(t) = \sum_{n=0}^{\infty} {}_LH^{F_n}(ax,ay,a^2z;w)\frac{(bt)^n}{n!}\sum_{r=0}^{\infty} {}_LH^{F_n}(bx,by,b^2z;w)\frac{(at)^r}{r!}$$

$$A(t) = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r} a^r b^{n-r} {}_LH^{F_{n-r}}(ax,ay,a^2z;w)\,{}_LH^{F_n}(bx,by,b^2z;w)\right)\frac{t^n}{n!}. \tag{4.3}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result (4.1).

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Theorem 4.2. For each pair of integers a and b and all integers and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^{n-k} b^{k} {}_{L}H^{F_{n-k}} \left(by + \frac{b}{a} i + j, b^2 z, bx; w \right) F_k(au, w)$$

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{a-1} \sum_{i=0}^{b-1} b^{n-k} a^{k} {}_{L}H^{F_{n-k}} \left(ay + \frac{a}{b} i + j, a^2 z, ax; w \right) F_k(bu, w). \tag{4.4}$$

Proof. Let

$$B(t) = \frac{e^{ab(y+u)t + a^2 b^2 z t^2} (e^{abt} - 1)^2 C_0(abxt)}{(1 - w(e^{at} - 1))(1 - w(e^{bt} - 1))(e^{at} - 1)(e^{bt} - 1)}$$

$$= \frac{e^{abyt + a^2 b^2 z t^2} C_0(xt)}{1 - w(e^{at} - 1)} \frac{e^{abt} - 1}{e^{bt} - 1} \frac{e^{abut}}{1 - w(e^{bt} - 1)} \frac{e^{abt} - 1}{e^{at} - 1}$$

$$B(t) = \frac{e^{abyt + a^2 b^2 z t^2} C_0(abxt)}{1 - w(e^{at} - 1)} \sum_{i=0}^{a-1} e^{bti} \frac{e^{abut}}{1 - w(e^{bt} - 1)} \sum_{j=0}^{b-1} e^{atj}$$

$$= \frac{e^{a^2 b^2 z t^2} C_0(abxt)}{1 - w(e^{at} - 1)} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} e^{(by + \frac{b}{a} i + j)at} \sum_{k=0}^{\infty} F_k(au, w) \frac{(bt)^k}{k!}$$

$$B(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^{n-k} b^{k} {}_{L}H^{F_{n-k}} \left(by + \frac{b}{a} i + j, b^2 z, bx; w \right) F_k(au, w) \right) \frac{t^n}{n!}. \tag{4.5}$$

On the other hand

$$B(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{a-1} \sum_{i=0}^{b-1} b^{n-k} a^{k} {}_{L}H^{F_{n}} \left(ay + \frac{a}{b} i + j, a^2 z, ax; w \right) F_k(bu, w) \right) \frac{t^n}{n!}. \tag{4.6}$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

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