A note on degenerate Hermite-Fubini numbers and polynomials

Waseem A. Khan[1], Idrees A. Khan[2] and Nisar K S[3]

[1,2]Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, (India)

[3]Department of Mathematics, College of Arts and Science-Wadi Al dawaser, Prince Sattam bin Abdulaziz University, Riyadh region 11991, Saudi Arabia

E-mail: waseem08_khan@rediffmail.com, idrees_maths@yahoo.com, n.sooppy@psau.edu.sa

Abstract. In this paper, we introduce a new class of degenerate Hermite-Fubini numbers and polynomials and investigate some properties of these polynomials. We establish summation formulas of these polynomials by summation techniques series. Furthermore, we derive symmetric identities of degenerate Hermite-Fubini numbers and polynomials by using generating functions.

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1. Introduction

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [1, 4] are defined as

$$H_n(x,y) = n!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.1}$$

It is clear that

$$H_n(2x,-1) = H_n(x, H_n(x,-\frac{1}{2}) = He_n(x), H_n(x,0) = x^n,$$

where $H_n(x)$ and $He_n(x)$ being ordinary Hermite polynomials.

The Hermite polynomial H_n(x,y) (see ([12, 13]) is defined by means of the following generating function as follows:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.2}$$

Recently, Khan [7] introduced degenerate Hermite polynomials by means of the following generating function as follows:

$$(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} H_n(x,y;\lambda)\frac{t^n}{n!}. \tag{1.3}$$

Note that

$$\lim_{\lambda\longrightarrow 0}(1+\lambda t)^{\frac{x}{\lambda}} = e^{xt}.$$

It is evident that (1.3) reduces to (1.2). That is $H_n(x,y)$ limiting case of $H_n(x,y;\lambda)$, when $\lim_{\lambda\longrightarrow 0}$.

The explicit representation of degenerate Hermite polynomials $H_n(x,y;\lambda)$ as follows:

$$H_n(x,y;\lambda) = n!\sum_{r=0}^{[\frac{n}{2}]} \frac{\lambda^{n-r}(\frac{x}{\lambda})_{n-2r}(\frac{y}{\lambda})_r}{r!(n-2r)!}. \tag{1.4}$$

For $\lambda \in \mathbb{C}$, Carlitz introduced the degenerate Bernoulli polynomials given by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x;\lambda)\frac{t^n}{n!}, \quad \text{(see [3, 8, 9, 10, 11])} \tag{1.5}$$

so that

$$\beta_n(x;\lambda) = \sum_{n=0}^{m} \left(\begin{array}{c} n \\ m \end{array} \right) \beta_m(\lambda)(\frac{x}{\lambda})_{n-m}. \tag{1.6}$$

When $x=0$, $\beta_n(\lambda) = \beta_n(0;\lambda)$ are called the degenerate Bernoulli numbers.

From (1.5), we note that

$$\sum_{n=0}^{\infty} \lim_{\lambda\longrightarrow 0} \beta_n(x;\lambda)\frac{t^n}{n!} = \lim_{\lambda\longrightarrow 0} \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}$$

$$= \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \tag{1.7}$$

where $B_n(x)$ are called the Bernoulli polynomials (see [1-15]).

Geometric polynomials (also known as Fubini polynomials) are defined as follows (see [2]):

$$F_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!x^k, \tag{1.8}$$

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is the Stirling number of the second kind (see [5]).

For $x=1$ in (1.8), we get n^{th} Fubini number (ordered Bell number or geometric number) F_n [2, 5, 6, 15] is defined by

$$F_n(1) = F_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!. \tag{1.9}$$

The exponential generating functions of geometric polynomials is given by (see [2]):

$$\frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}, \tag{1.10}$$

and related to the geometric series (see [2]):

$$\left(x\frac{d}{dx}\right)^m \frac{1}{1-x} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1-x}F_m(\frac{x}{1-x}), |\, x \,|< 1.$$

Let us give a short list of these polynomials and numbers as follows:

$F_0(x) = 1, F_1(x) = x, F_2(x) = x+2x^2, F_3(x) = x+6x^2+6x^3, F_4(x) = x+14x^2+36x^3+24x^4,$

and

$$F_0 = 1, F_1 = 1, F_2 = 3, F_3 = 13, F_4 = 75.$$

Geometric and exponential polynomials are connected by the relation (see [2]):

$$F_n(x) = \int_0^{\infty} \phi_n(x)e^{-\lambda}d\lambda. \tag{1.11}$$

In (2016), Khan [7] introduced two variable degenerate Hermite-poly-Bernoulli polynomials is defined by means of the following generating function:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}=\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!},\tag{1.12}$$

so that

$${}_H\beta_n^{(k)}(x,y;\lambda)=\sum_{m=0}^{n}\binom{n}{m}\beta_{n-m}^{(k)}(\lambda)H_m(x,y;\lambda).$$

The object of this paper, we consider generating functions for degenerate Hermite-Fubini numbers and polynomials and give some properties of these numbers and polynomials. We derive summation formulas of degenerate Hermite-Fubini numbers and polynomials and we construct a symmetric identities of degenerate Hermite-Fubini numbers and polynomials by using generating functions.

2. Degenerate Hermite-Fubini numbers and polynomials

In this section, we define three-variable degenerate Hermite-Fubini polynomials and obtain some basic properties which gives us new formula for ${}_HF_{n,\lambda}(x,y;z)$ as follows:

We introduce 3-variable degenerate Hermite-Fubini polynomials by means of the following generating function:

$$\frac{1}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}=\sum_{n=0}^{\infty}{}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!}.\tag{2.1}$$

When $x=y=0$. $z=1$ in (2.1), we have

$${}_HF_{n,\lambda}(0,0;z)=F_{n,\lambda}(z),\ {}_HF_{n,\lambda}(0,0;1)=F_{n,\lambda}.$$

Not that $\lim_{\lambda\longrightarrow 0}{}_HF_{n,\lambda}(x,y;z)={}_HF_n(x,y;z)$.

On setting $y=0$ in (2.1), we obtain 2-variable Fubini polynomials which is defined by Kim et al. [9].

$$\frac{1}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}F_{n,\lambda}(x;z)\frac{t^n}{n!}.\tag{2.2}$$

Theorem 2.1. For $n\geq 0$, we have

$${}_HF_{n,\lambda}(x,y;z)=\sum_{m=0}^{n}\binom{n}{r}F_{n-m,\lambda}(z)H_m(x,y;\lambda).\tag{2.3}$$

Proof. Using definition (2.1), we have

$$\sum_{n=0}^{\infty}{}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!}=\frac{1}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$=\sum_{n=0}^{\infty}F_{n,\lambda}(z)\frac{t^n}{n!}\sum_{m=0}^{\infty}H_m(x,y;\lambda)\frac{t^m}{m!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{r}F_{n-m,\lambda}(z)H_m(x,y;\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (2.3).

Theorem 2.2. For $n \geq 0$, we have

$$ {}_HF_{n,\lambda}(x,y;z) = \sum_{r=0}^{n} \binom{n}{r} H_{n-r}(x,y;\lambda) \sum_{k=0}^{r} z^k k! S_2(r,k). \tag{2.4} $$

Proof. Using definition (2.1), we have

$$ \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!} = \frac{1}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} $$
$$ = (1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} \sum_{k=0}^{\infty} z^k((1+\lambda t)^{\frac{1}{\lambda}}-1)^k $$
$$ = (1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} \sum_{k=0}^{\infty} z^k \sum_{n=k}^{\infty} S_2(r,k)\frac{t^r}{r!} $$
$$ = \sum_{n=0}^{\infty} H_n(x,y;\lambda)\frac{t^n}{n!}\left(\sum_{r=0}^{\infty}\sum_{k=0}^{r} z^k k! S_2(r,k)\frac{t^r}{r!}\right) $$
$$ L.H.S = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r} H_{n-r}(x,y;\lambda)\sum_{k=0}^{r} z^k k! S_2(r,k)\right)\frac{t^n}{n!}. $$

Equating the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.4).

Theorem 2.3. For $n \geq 0$, the following formula for degenerate Hermite-Fubini polynomials holds true:

$$ \frac{1}{1-z}\sum_{m=0}^{n}\binom{n}{m} F_{m,\lambda}\left(\frac{z}{1-z}\right) H_{n-m,\lambda}(x,y) = \sum_{r=0}^{n}\binom{n}{r}\sum_{k=0}^{\infty} z^k (k)_{r,\lambda} H_{n-r,\lambda}(x,y). \tag{2.5} $$

Proof. We begin with the definition (2.1) and write

$$ \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!} = \frac{1}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} $$

Let

$$ \frac{1}{1-z}\left(\frac{1}{1-\frac{z}{1-z}(1+\lambda t)^{\frac{1}{\lambda}}-1}\right) = \frac{1}{1-z(1+\lambda t)^{\frac{1}{\lambda}}} = \sum_{k=0}^{\infty} z^k (1+\lambda t)^{\frac{k}{\lambda}} \tag{2.6} $$
$$ = \sum_{r=0}^{\infty}\left(\sum_{k=0}^{\infty} z^k (k)_{r,\lambda}\right)\frac{t^r}{r!} $$

$$ \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!} = \sum_{r=0}^{\infty}\left(\sum_{k=0}^{\infty} z^k (k)_{r,\lambda}\right)\frac{t^r}{r!}\left(\sum_{n=0}^{\infty} H_{n,\lambda}(x,y)\frac{t^n}{n!}\right) $$
$$ = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}\sum_{k=0}^{\infty} z^k (k)_{r,\lambda} H_{n-r,\lambda}(x,y)\right)\frac{t^n}{n!}. \tag{2.7} $$

Now, we observe that, by (2.6), we get

$$ \frac{1}{1-z}\left(\frac{1}{1-\frac{z}{1-z}(1+\lambda t)^{\frac{1}{\lambda}}-1}\right) = \frac{1}{1-z}\sum_{n=0}^{\infty} F_{n,\lambda}\left(\frac{z}{1-z}\right)\frac{t^n}{n!} $$

Then, we have

$$\sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!} = \frac{1}{1-z}\sum_{m=0}^{\infty} F_{m,\lambda}\left(\frac{z}{1-z}\right)\frac{t^m}{m!}\left(\sum_{n=0}^{\infty} H_{n,\lambda}(x,y)\frac{t^n}{n!}\right)$$
$$= \frac{1}{1-z}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} F_{m,\lambda}\left(\frac{z}{1-z}\right)H_{n-m,\lambda}(x,y)\right)\frac{t^n}{n!}. \tag{2.8}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in equation (2.7) and (2.8), we get (2.5).

Theorem 2.4. For $n \geq 0$, the following formula for degenerate Hermite-Fubini polynomials holds true:

$$H_{n,\lambda}(x,y) = {}_HF_{n,\lambda}(x,y;z) - z\,{}_HF_{n,\lambda}(x+1,y;z) + z\,{}_HF_{n,\lambda}(x,y;z). \tag{2.9}$$

Proof. We begin with the definition (2.1) and write

$$(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \frac{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$
$$= \frac{(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)} - \frac{z((1+\lambda t)^{\frac{1}{\lambda}}-1)}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}.$$

Then using the definition of Kampé de Fériet generalization of the degenerate Hermite polynomials $H_{n,\lambda}(x,y)$ (1.3) and (2.1), we have

$$\sum_{n=0}^{\infty} H_{n,\lambda}(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[{}_HF_{n,\lambda}(x,y;z) - z\,{}_HF_{n,\lambda}(x+1,y;z) + z\,{}_HF_{n,\lambda}(x,y;z)\right]\frac{t^n}{n!}.$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$, we get (2.9).

Theorem 2.5. For $n \geq 0$ and $z_1 \neq z_2$, the following formula for degenerate Hermite-Fubini polynomials holds true:

$$\sum_{k=0}^{n}\binom{n}{k} {}_HF_{n-k,\lambda}(x_1,y_1;z_1)\,{}_HF_{k,\lambda}(x_2,y_2;z_2)$$
$$= \frac{z_2\,{}_HF_{n,\lambda}(x_1+x_2,y_1+y_2;z_2) - z_1\,{}_HF_{n,\lambda}(x_1+x_2,y_1+y_2;z_1)}{z_2-z_1}. \tag{2.10}$$

Proof. The products of (2.1) can be written as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_HF_n(x_1,y_1;z_1)\frac{t^n}{n!}\,{}_HF_k(x_2,y_2;z_2)\frac{t^k}{k!} = \frac{(1+\lambda t)^{\frac{x_1}{\lambda}}(1+\lambda t^2)^{\frac{y_1}{\lambda}}}{1-z_1((1+\lambda t)^{\frac{1}{\lambda}}-1)}\,\frac{(1+\lambda t)^{\frac{x_2}{\lambda}}(1+\lambda t^2)^{\frac{y_2}{\lambda}}}{1-z_2((1+\lambda t)^{\frac{1}{\lambda}}-1)}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} {}_HF_{n-k}(x_1,y_1;z_1)\,{}_HF_k(x_2,y_2;z_2)\right)\frac{t^n}{n!}$$
$$= \frac{z_2}{z_2-z_1}\,\frac{(1+\lambda t)^{\frac{x_1+x_2}{\lambda}}(1+\lambda t^2)^{\frac{y_1+y_2}{\lambda}}}{1-z_1((1+\lambda t)^{\frac{1}{\lambda}}-1)} - \frac{z_1}{z_2-z_1}\,\frac{(1+\lambda t)^{\frac{x_1+x_2}{\lambda}}(1+\lambda t^2)^{\frac{y_1+y_2}{\lambda}}}{1-z_2((1+\lambda t)^{\frac{1}{\lambda}}-1)}$$
$$= \left(\frac{z_2\,{}_HF_n(x_1+x_2,y_1+y_2;z_2) - z_1\,{}_HF_n(x_1+x_2,y_1+y_2;z_1)}{z_2-z_1}\right)\frac{t^n}{n!}.$$

By equating the coefficients of $\frac{t^n}{n!}$ on both sides, we get (2.10).

Theorem 2.6. For $n \geq 0$, the following formula for degenerate Hermite-Fubini polynomials holds true:

$$z_H F_{n,\lambda}(x+1,y;z) = (1+z)_H F_{n,\lambda}(x,y;z) - H_{n,\lambda}(x,y). \tag{2.11}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} [_H F_{n,\lambda}(x+1,y;z) - {_H F_{n,\lambda}}(x,y;z)] \frac{t^n}{n!} = \frac{(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}((1+\lambda t)^{\frac{1}{\lambda}}-1)$$

$$= \frac{1}{z}\left[\frac{(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)} - (1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}\right]$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} [_H F_n(x,y;z) - H_n(x,y)] \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we obtain (2.11).

Remark 2.3. On setting $x = y = 0$ and $x = -1$ in Theorem 2.6, we find

$$z_H F_{n,\lambda}(1,0;z) = (1+z)_H F_{n,\lambda}(0,0;z), \tag{2.12}$$

and

$$z_H F_{n\lambda}(0,0;z) = (1+z)_H F_{n,\lambda}(-1,0;z) - (-\lambda)^n \left(\frac{1}{\lambda}\right)_n. \tag{2.13}$$

Theorem 2.7. For $n \geq 0$, $p, q \in \mathbb{R}$, the following formula for degenerate Hermite-Fubini polynomials holds true:

$$_H F_{n,\lambda}(px,qy;z)$$

$$= n!\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]} \lambda^{k-j}{_H F_{n-k,\lambda}}(x,y;z)\left(\frac{(p-1)x}{\lambda}\right)_{k-2j}\left(\frac{(q-1)y}{\lambda}\right)_j \frac{1}{(n-k-2j)!j!}. \tag{2.14}$$

Proof. Rewrite the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {_H F_n}(px,qy;z)\frac{t^n}{n!} = \frac{1}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}(1+\lambda t)^{\frac{(p-1)x}{\lambda}}(1+\lambda t^2)^{\frac{(q-1)y}{\lambda}}$$

$$= \left(\sum_{n=0}^{\infty} {_H F_{n,\lambda}}(x,y;z)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\left(\frac{(p-1)x}{\lambda}\right)_k \lambda^k \frac{t^k}{k!}\right)\left(\sum_{j=0}^{\infty}\left(\frac{(q-1)y}{\lambda}\right)_j \lambda^j \frac{t^{2j}}{j!}\right)$$

$$= \left(\sum_{n=0}^{\infty} {_H F_{n,\lambda}}(x,y;z)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{(p-1)x}{\lambda}\right)_k\left(\frac{(q-1)y}{\lambda}\right)_j \lambda^{k+j}\frac{t^{k+2j}}{k!j!}\right)$$

Replacing k by $k-2j$ in above equation, we have

$$L.H.S. = \left(\sum_{n=0}^{\infty} {_H F_n}(x,y;z)\frac{t^n}{n!}\right)\left(\sum_{k=2j}^{\infty} \lambda^{k-j}\left(\frac{(p-1)x}{\lambda}\right)_{k-2j}\left(\frac{(q-1)y}{\lambda}\right)_j \frac{t^k}{(k-2j)!j!}\right).$$

Again replacing n by $n-k$ in above equation, we have

$$L.H.S. = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]} \lambda^{k-j}{_H F_{n-k,\lambda}}(x,y;z)\left(\frac{(p-1)x}{\lambda}\right)_{k-2j}\left(\frac{(q-1)y}{\lambda}\right)_j \frac{t^n}{(n-k-2j)!j!k!}.$$

Finally, equating the coefficients of t^n on both sides, we acquire the result (2.14).

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Theorem 2.8. For $n \geq 0$, the following formula for degenerate Hermite-Fubini polynomials holds true:

$$ {}_HF_{n,\lambda}(x+r,y;z) = \sum_{l=0}^{n} \binom{n}{l} H_{n-l,\lambda}(x,y) \sum_{k=0}^{l} z^k k! S_{2,\lambda}(l+r,k+r). \tag{2.15} $$

Proof. Replacing x by $x+r$ in (2.1), we have

$$ \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x+r,y;z)\frac{t^n}{n!} = \frac{(1+\lambda t)^{\frac{x+r}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)} $$

$$ = (1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}(1+\lambda t)^{\frac{r}{\lambda}} \sum_{k=0}^{\infty} z^k((1+\lambda t)^{\frac{1}{\lambda}}-1)^k $$

$$ = (1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}(1+\lambda t)^{\frac{r}{\lambda}} \sum_{k=0}^{\infty} z^k \sum_{l=k}^{\infty} k! S_2(l,k)\frac{t^l}{l!} $$

$$ = \sum_{n=0}^{\infty} H_{n,\lambda}(x,y)\frac{t^n}{n!} \sum_{l=0}^{\infty} z^k \sum_{k=0}^{l} k! S_{2,\lambda}(l+r,k+r)\frac{t^l}{l!}. $$

Replacing n by $n-l$ in above equation, we get

$$ \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x+r,y;z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} H_{n-l,\lambda}(x,y)\sum_{k=0}^{l} z^k k! S_{2,\lambda}(l+r,k+r)\right)\frac{t^n}{n!}. $$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.15).

3. Summation Formulae for degenerate Hermite-Fubini polynomials

First, we prove the following result involving the degenerate Hermite-Fubini polynomials ${}_HF_{n,\lambda}(x,y;z)$ by using series rearrangement techniques and considered its special case:

Theorem 3.1. The following summation formula for degenerate Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_HF_{n,\lambda}(u,v;z)\,{}_HF_{m,\lambda}(U,V;Z) = \sum_{r,k=0}^{n,m}\binom{n}{r}\binom{m}{k} H_{r,\lambda}(u-x,v-y)\,{}_HF_{n-r,\lambda}(x,y;z) $$

$$ \times H_{k,\lambda}(U-X,V-Y)\,{}_HF_{m-k,\lambda}(X,Y;Z). \tag{3.1} $$

Proof. Consider the product of the degenerate Hermite-Fubini polynomials, we can be written as generating function (2.1) in the following form:

$$ \frac{1}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}\frac{1}{1-Z((1+\lambda T)^{\frac{1}{\lambda}}-1)}(1+\lambda T)^{\frac{x}{\lambda}}(1+\lambda T^2)^{\frac{y}{\lambda}} $$

$$ = \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!}\sum_{m=0}^{\infty} {}_HF_{m,\lambda}(X,Y;Z)\frac{T^m}{m!}. \tag{3.2} $$

Replacing x by u, y by v, X by U and Y by V in (3.2) and equating the resultant to itself,

$$ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_HF_{n,\lambda}(u,v;z)\,{}_HF_{m,\lambda}(U,V;Z)\frac{t^n}{n!}\frac{T^m}{m!} $$

$$ = (1+\lambda t)^{\frac{u-x}{\lambda}}(1+\lambda t^2)^{\frac{v-y}{\lambda}}(1+\lambda T)^{\frac{U-X}{\lambda}}(1+\lambda T^2)^{\frac{V-Y}{\lambda}} $$

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$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_HF_{n,\lambda}(x,y;z){}_HF_{m,\lambda}(X,Y;Z)\frac{t^n}{n!}\frac{T^m}{m!},$$

which on using the generating function [14] in the r.h.s., becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} {}_HF_{n,\lambda}(u,v;z){}_HF_{m,\lambda}(U,V;Z)\frac{t^n}{n!}\frac{T^m}{m!}$$

$$= \sum_{n,r=0}^{\infty} H_{r,\lambda}(u-x,v-y){}_HF_{n,\lambda}(x,y;z)\frac{t^{n+r}}{n!r!}\sum_{m,k=0}^{\infty} H_{k,\lambda}(U-X,V-Y){}_HF_{m,\lambda}(X,Y;Z)\frac{T^{m+k}}{m!k!}. \tag{3.3}$$

Finally, replacing n by $n-r$ and m by $m-k$ and using the lemma [14] in the r.h.s. of the above equation and then equating the coefficients of like powers of t and T, we get assertion (3.1) of Theorem 3.1.

Theorem 3.2. The following summation formula for degenerate Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_HF_{n,\lambda}(x+w,y+u;z) = \sum_{s=0}^{n}\binom{n}{s} {}_HF_{n-s,\lambda}(x,y;z)H_{s,\lambda}(w,u). \tag{3.4}$$

Proof. We replace x by $x+w$ and y by $y+u$ in (2.1), use (1.3) and rewrite the generating function as:

$$\frac{1}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x+w}{\lambda}}(1+\lambda t^2)^{\frac{y+u}{\lambda}} = \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!}\sum_{s=0}^{\infty} H_{s,\lambda}(w,u)\frac{t^s}{s!}$$

$$= \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x+w,y+u;z)\frac{t^n}{n!}.$$

Now replacing n by $n-s$ in l.h.s. and comparing the coefficients of t^n on both sides, we get the result (3.4).

Theorem 3.3. The following summation formula for degenerate Hermite-Fubini polynomials ${}_HF_{n,\lambda}(x,y;z)$ holds true:

$$ {}_HF_{n,\lambda}(x,y;z) = \sum_{r=0}^{n}\binom{n}{r} F_{n-r,\lambda}(x-w;z)H_{r,\lambda}(w,y). \tag{3.5}$$

Proof. By exploiting the generating function (1.3), we can write equation (2.1) as

$$\frac{1}{1-z((1+\lambda t)^{\frac{1}{\lambda}}-1)}(1+\lambda t)^{\frac{x-w}{\lambda}}(1+\lambda t)^{\frac{w}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} F_{n,\lambda}(x-w;z)\frac{t^n}{n!}\sum_{r=0}^{\infty} H_{r,\lambda}(w,y)\frac{t^r}{r!}.$$

On replacing n by $n-r$ in above equation, we get

$$\sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} F_{n-r,\lambda}(x-w;z)H_{r,\lambda}(w,y)\frac{t^n}{(n-r)!r!}.$$

Equating the coefficients of the like powers of t on both sides, we get (3.5).

Theorem 3.4. The following summation formula for degenerate Hermite-Fubini polynomials ${}_HF_{n,\lambda}(x,y;z)$ holds true:

$$ {}_HF_{n,\lambda}(x+1,y;z) = \sum_{r=0}^{n}\binom{n}{r} {}_HF_{n-r,\lambda}(x,y;z)\left(\frac{1}{\lambda}\right)_r \lambda^r. \tag{3.6}$$

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Proof. Using the generating function (2.1), we have

$$\sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x+1,y;z)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!}$$
$$= \left(\frac{1}{1-1-z((1+\lambda t)^{\frac{1}{\lambda}}-1}\right)((1+\lambda t)^{\frac{1}{\lambda}}-1)(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$
$$= \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!}\left(\sum_{r=0}^{\infty}\frac{\left(\frac{1}{\lambda}\right)_r \lambda^r t^r}{r!}-1\right)$$
$$= \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{\lambda}\right)_r \lambda^r t^r}{r!} - \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r} {}_HF_{n-r,\lambda}(x,y;z)\left(\frac{1}{\lambda}\right)_r \lambda^r\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(x,y;z)\frac{t^n}{n!}.$$

Finally, equating the coefficients of the like powers of t on both sides, we get (3.6).

4. Symmetric identities for degenerate Hermite-Fubini polynomials

In this section, we establish general symmetry identities for the degenerate Hermite-Fubini polynomials ${}_HF_{n,\lambda}(x,y;z)$ by applying the generating function (2.1) and (2.2).

Theorem 4.1. Let $x, y, z \in \mathbb{R}$ and $n \geq 0$, then the following identity holds true:

$$\sum_{r=0}^{n}\binom{n}{r} b^r a^{n-r} {}_HF_{n-r,\lambda}(bx,b^2y;z) {}_HF_{r,\lambda}(ax,a^2y;z)$$
$$= \sum_{r=0}^{n}\binom{n}{r} a^r b^{n-r} {}_HF_{n-r,\lambda}(ax,a^2y;z) {}_HF_{r,\lambda}(bx,b^2y;z). \quad (4.1)$$

Proof. Start with

$$A(t) = \frac{(1+\lambda t)^{\frac{abx}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2y}{\lambda}}}{(1-z((1+\lambda t)^{\frac{a}{\lambda}}-1))((1-z((1+\lambda t)^{\frac{b}{\lambda}}-1)}.$$

Then the expression for $A(t)$ is symmetric in a and b and we can expand $A(t)$ into series in two ways to obtain:

$$A(t) = \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(bx,b^2y;z)\frac{(at)^n}{n!}\sum_{r=0}^{\infty} {}_HF_{r,\lambda}(ax,a^2y;z)\frac{(bt)^r}{r!}$$
$$A(t) = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r} b^r a^{n-r} {}_HF_{n-r,\lambda}(bx,b^2y;z) {}_HF_{r,\lambda}(ax,a^2y;z)\right)\frac{t^n}{n!}. \quad (4.2)$$

Similarly, we can show that

$$A(t) = \sum_{n=0}^{\infty} {}_HF_{n,\lambda}(ax,a^2y;z)\frac{(bt)^n}{n!}\sum_{r=0}^{\infty} {}_HF_{r,\lambda}(bx,b^2y;z)\frac{(at)^r}{r!}$$
$$A(t) = \sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r} a^r b^{n-r} {}_HF_{n-r,\lambda}(ax,a^2y;z) {}_HF_{r,\lambda}(bx,b^2y;z)\right)\frac{t^n}{n!}. \quad (4.3)$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result (4.1).

Theorem 4.2. For each pair of integers a and b and all integers and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k} {}_{H}F_{n-k,\lambda}(bx, b^2y; z) \sum_{i=0}^{k} \binom{k}{i} \sigma_i\left(\frac{\lambda}{b}, a-1\right) F_{k-i,\lambda}(au; z)$$

$$\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^{k} {}_{H}F_{n-k,\lambda}(ax, a^2y; z) \sum_{i=0}^{k} \binom{k}{i} \sigma_i\left(\frac{\lambda}{a}, b-1\right) F_{k-i,\lambda}(bu; z). \quad (4.4)$$

Proof. Let

$$B(t) = \frac{(1+\lambda t)^{\frac{ab(x+u)}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2y}{\lambda}}((1+\lambda t)^{\frac{ab}{\lambda}}-1)}{(1-z((1+\lambda t)^{\frac{a}{\lambda}}-1))(1-z((1+\lambda t)^{\frac{b}{\lambda}}-1)((1+\lambda t)^{\frac{a}{\lambda}}-1)((1+\lambda t)^{\frac{b}{\lambda}}-1)}$$

$$= \frac{(1+\lambda t)^{\frac{abx}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2y}{\lambda}}}{(1-z((1+\lambda t)^{\frac{a}{\lambda}}-1))} \frac{((1+\lambda t)^{\frac{ab}{\lambda}}-1)}{(1+\lambda t)^{\frac{b}{\lambda}}-1)} \frac{(1+\lambda t)^{\frac{abu}{\lambda}}}{(1-z((1+\lambda t)^{\frac{b}{\lambda}}-1))}$$

$$B(t) = \frac{(1+\lambda t)^{\frac{abx}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2y}{\lambda}}}{(1-z((1+\lambda t)^{\frac{a}{\lambda}}-1))} \left(\sum_{i=0}^{\infty} \sigma_i\left(\frac{\lambda}{b}, a-1\right) \frac{(bt)^i}{i!}\right) \left(\sum_{k=0}^{\infty} F_{k,\lambda}(au; z) \frac{(bt)^k}{k!}\right)$$

$$\left(\sum_{n=0}^{\infty} {}_{H}F_{n,\lambda}(bx, b^2y; z) \frac{(at)^n}{n!}\right) \left(\sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{k}{i} b^k \sigma_i\left(\frac{\lambda}{b}, a-1\right) F_{k-i,\lambda}(au; z) \frac{t^k}{k!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k {}_{H}F_{n-k,\lambda}(bx, b^2y; z) \sum_{i=0}^{k} \binom{k}{i} \sigma_i\left(\frac{\lambda}{b}, a-1\right) F_{k-i,\lambda}(au; z)\right) \frac{t^n}{n!}. \quad (4.5)$$

On the other hand, we have

$$B(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k {}_{H}F_{n-k,\lambda}(ax, a^2y; z) \sum_{i=0}^{k} \binom{k}{i} \sigma_i\left(\frac{\lambda}{a}, b-1\right) F_{k-i,\lambda}(bu; z)\right) \frac{t^n}{n!}. \quad (4.6)$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

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