A note on degenerate Hermite-Fubini numbers and polynomials

Waseem A. Khan1, Idrees A. Khan2 and Nisar K S3

1, 2 Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, (India)

3 Department of Mathematics, College of Arts and Science-Wadi Al dawaser, Prince Sattam bin Abdulaziz University, Riyadh region 11991, Saudi Arabia

E-mail: waseem08_khan@rediffmail.com, idrees.maths@yahoo.com, n.sooppy@psau.edu.sa

Abstract. In this paper, we introduce a new class of degenerate Hermite-Fubini numbers and polynomials and investigate some properties of these polynomials. We establish summation formulas of these polynomials by summation techniques series. Furthermore, we derive symmetric identities of degenerate Hermite-Fubini numbers and polynomials by using generating functions.

Keywords: Hermite polynomials, degenerate Hermite polynomials, degenerate Fubini polynomials, degenerate Hermite-Fubini polynomials.

2010 Mathematics Subject Classification: 11B68, 11B83, 33C45, 33C99.

1. Introduction

The 2-variable Hermite Kampé de Féret polynomials (2VHKdFP) \( H_n(x, y) \) [1, 4] are defined as

\[
H_n(x, y) = n! \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{y^r x^{n-2r}}{r! (n-2r)!}.
\]

(1.1)

It is clear that

\[ H_n(2x, -1) = H_n(x, H_n(x, -\frac{1}{2})) = H_{e_n}(x), H_n(x, 0) = x^n, \]

where \( H_n(x) \) and \( H_{e_n}(x) \) being ordinary Hermite polynomials.

The Hermite polynomial \( H_n(x, y) \) (see [12, 13]) is defined by means of the following generating function as follows:

\[
e^{xt + yt^2} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}.
\]

(1.2)

Recently, Khan [7] introduced degenerate Hermite polynomials by means of the following generating function as follows:

\[
(1 + \lambda t)^\frac{1}{2} (1 + \lambda t^2)^\frac{1}{2} = \sum_{n=0}^{\infty} H_n(x, y; \lambda) \frac{t^n}{n!}.
\]

(1.3)

Note that

\[
\lim_{\lambda \to 0} (1 + \lambda t)^k = e^{xt}.
\]

It is evident that (1.3) reduces to (1.2). That is \( H_n(x, y) \) limiting case of \( H_n(x, y; \lambda) \), when \( \lim_{\lambda \to 0} \).
The explicit representation of degenerate Hermite polynomials \( H_n(x, y; \lambda) \) as follows:

\[
H_n(x, y; \lambda) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\lambda^{n-r} (x)_{n-2r} (y)_{r}}{r!(n-2r)!}.
\] (1.4)

For \( \lambda \in \mathbb{C} \), Carlitz introduced the degenerate Bernoulli polynomials given by the generating function

\[
\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}}} - 1 = \sum_{n=0}^{\infty} \beta_n(x; \lambda) \frac{t^n}{n!}, \quad \text{(see [3, 8, 9, 10, 11])}
\] (1.5)

so that

\[
\beta_n(x; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \beta_m(\lambda)(\frac{x}{\lambda})^{n-m}.
\] (1.6)

When \( x = 0 \), \( \beta_n(\lambda) = \beta_n(0; \lambda) \) are called the degenerate Bernoulli numbers.

From (1.5), we note that

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},
\] (1.7)

where \( B_n(x) \) are called the Bernoulli polynomials (see [1-15]).

Geometric polynomials (also known as Fubini polynomials) are defined as follows (see [2]):

\[
F_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k! x^k,
\] (1.8)

where \( \left\{ \begin{array}{c} n \\ k \end{array} \right\} \) is the Stirling number of the second kind (see [5]).

For \( x = 1 \) in (1.8), we get \( n^\text{th} \) Fubini number (ordered Bell number or geometric number) \( F_n \) [2, 5, 6, 15] is defined by

\[
F_n(1) = F_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!.
\] (1.9)

The exponential generating functions of geometric polynomials is given by (see [2]):

\[
\frac{1}{1 - x(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!},
\] (1.10)

and related to the geometric series (see [2]):

\[
\left( x \frac{d}{dx} \right)^m \frac{1}{1 - x} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1 - x} F_m(\frac{x}{1 - x}), \quad |x| < 1.
\]

Let us give a short list of these polynomials and numbers as follows:

\[
F_0(x) = 1, \quad F_1(x) = x, \quad F_2(x) = x + 2x^2, \quad F_3(x) = x + 6x^2 + 6x^3, \quad F_4(x) = x + 14x^2 + 36x^3 + 24x^4,
\]

and

\[
F_0 = 1, \quad F_1 = 1, \quad F_2 = 3, \quad F_3 = 13, \quad F_4 = 75.
\]

Geometric and exponential polynomials are connected by the relation (see [2]):

\[
F_n(x) = \int_{0}^{\infty} \phi_n(x)e^{-\lambda}d\lambda.
\] (1.11)
In (2016), Khan [7] introduced two variable degenerate Hermite-poly-Bernoulli polynomials is defined by means of the following generating function:

\[
\frac{\ln(1 - e^{-t})}{1 + \lambda t} (1 + \lambda t) \frac{x}{(1 + \lambda t^2) y} = \sum_{n=0}^{\infty} H_{n}^{(k)}(x, y; \lambda) \frac{t^n}{n!},
\]

(1.12)

so that

\[
H_{n}^{(k)}(x, y; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \beta_{n-m}^{(k)}(\lambda) H_{m}(x, y; \lambda).
\]

The object of this paper, we consider generating functions for degenerate Hermite-Fubini numbers and polynomials and give some properties of these numbers and polynomials. We derive summation formulas of degenerate Hermite-Fubini numbers and polynomials and we construct a symmetric identities of degenerate Hermite-Fubini numbers and polynomials by using generating functions.

2. Degenerate Hermite-Fubini numbers and polynomials

In this section, we define three-variable degenerate Hermite-Fubini polynomials and obtain some basic properties which gives us new formula for \( H_{n}^{(k)}(x, y; z) \) as follows:

We introduce 3-variable degenerate Hermite-Fubini polynomials by means of the following generating function:

\[
\frac{1}{1 - z((1 + \lambda t) \frac{x}{1 + \lambda t^2} - 1)} (1 + \lambda t) \frac{x}{(1 + \lambda t^2) y} = \sum_{n=0}^{\infty} H_{n}^{(k)}(x, y; z) \frac{t^n}{n!}.
\]

(2.1)

When \( x = y = 0 \), \( z = 1 \) in (2.1), we have

\[
H_{n}^{(k)}(0, 0; z) = F_{n}(z), \quad H_{n}^{(k)}(0, 0; 1) = F_{n}.
\]

Not that \( \lim_{\lambda \to 0} H_{n}^{(k)}(x, y; z) = H_{n}(x, y; z) \).

On setting \( y = 0 \) in (2.1), we obtain 2-variable Fubini polynomials which is defined by Kim et al. [9].

\[
\frac{1}{1 - z((1 + \lambda t) \frac{x}{1 + \lambda t^2} - 1)} (1 + \lambda t) \frac{x}{(1 + \lambda t^2) y} = \sum_{n=0}^{\infty} F_{n}(x; z) \frac{t^n}{n!}.
\]

(2.2)

**Theorem 2.1.** For \( n \geq 0 \), we have

\[
H_{n}^{(k)}(x, y; z) = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}(z) H_{m}(x, y; \lambda).
\]

(2.3)

**Proof.** Using definition (2.1), we have

\[
\sum_{n=0}^{\infty} H_{n}^{(k)}(x, y; z) \frac{t^n}{n!} = \frac{1}{1 - z((1 + \lambda t) \frac{x}{1 + \lambda t^2} - 1)} (1 + \lambda t) \frac{x}{(1 + \lambda t^2) y} \sum_{m=0}^{\infty} H_{m}(x, y; \lambda) \frac{t^m}{m!}
\]

\[
= \sum_{n=0}^{\infty} F_{n}(z) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_{m}(x, y; \lambda) \frac{t^m}{m!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} F_{n-m}(z) H_{m}(x, y; \lambda) \right) \frac{t^n}{n!}.
\]
Now, we observe that, by (2.6), we get

\[ \sum_{n=0}^{\infty} H_Fn,\lambda(x, y; z) \frac{t^n}{n!} = \frac{1}{1-z((1+\lambda t)^{\frac{1}{2}} - 1)} (1+\lambda t)^{\frac{1}{2}} (1+\lambda t^2)^{\frac{1}{2}} \]

\[ = (1+\lambda t)^{\frac{1}{2}} (1+\lambda t^2)^{\frac{1}{2}} \sum_{k=0}^{\infty} z^k((1+\lambda t)^{\frac{1}{2}} - 1) \]

\[ = (1+\lambda t)^{\frac{1}{2}} (1+\lambda t^2)^{\frac{1}{2}} \sum_{k=0}^{\infty} z^k \sum_{n=k}^{\infty} S_2(r, k) \frac{t^n}{r!} \]

\[ = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left( \sum_{k=0}^{r} z^k \right) S_2(r, k) \frac{t^n}{r!} \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} \left( \sum_{k=0}^{r} z^k \right) \frac{t^n}{r!} \right) \frac{t^n}{n!} \]

Equating the coefficients of \( \frac{t^n}{n!} \) in both sides, we get (2.4).

**Theorem 2.3.** For \( n \geq 0 \), the following formula for degenerate Hermite-Fubini polynomials holds true:

\[ \frac{1}{1-z} \sum_{m=0}^{n} \left( \frac{n!}{m!} \right) H_Fn,\lambda(x, y; z) \frac{t^n}{n!} = \sum_{r=0}^{n} \left( \sum_{k=0}^{r} z^k \right) \frac{t^n}{n!} \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} \left( \sum_{k=0}^{r} z^k \right) H_Fn,\lambda(x, y; z) \frac{t^n}{n!} \right) \frac{t^n}{n!} \]

**Proof.** We begin with the definition (2.1) and write

\[ \sum_{n=0}^{\infty} H_Fn,\lambda(x, y; z) \frac{t^n}{n!} = \frac{1}{1-z((1+\lambda t)^{\frac{1}{2}} - 1)} (1+\lambda t)^{\frac{1}{2}} (1+\lambda t^2)^{\frac{1}{2}} \]

Let

\[ \frac{1}{1-z} \left( \frac{1}{1-z((1+\lambda t)^{\frac{1}{2}} - 1)} \right) = \frac{1}{1-z((1+\lambda t)^{\frac{1}{2}} - 1)} \]

\[ = \sum_{r=0}^{\infty} \left( \sum_{k=0}^{\infty} z^k \right) \frac{t^n}{r!} \]

\[ \sum_{n=0}^{\infty} H_Fn,\lambda(x, y; z) \frac{t^n}{n!} = \sum_{r=0}^{\infty} \left( \sum_{k=0}^{\infty} z^k \right) \frac{t^n}{r!} \left( \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} \left( \sum_{k=0}^{r} z^k \right) H_Fn,\lambda(x, y; z) \frac{t^n}{n!} \right) \frac{t^n}{n!} \right) \frac{t^n}{n!} \]

Now, we observe that, by (2.6), we get

\[ \frac{1}{1-z} \left( \frac{1}{1-z((1+\lambda t)^{\frac{1}{2}} - 1)} \right) = \frac{1}{1-z} \sum_{n=0}^{\infty} F_Fn,\lambda \left( \frac{z}{1-z} \right) \frac{t^n}{n!} \]
Then, we have
\[
\sum_{n=0}^{\infty} H_{n,\lambda}(x, y; z) \frac{t^n}{n!} = \frac{1}{1-z} \sum_{m=0}^{\infty} F_{m,\lambda} \left( \frac{z}{1-z} \right) \frac{m^n}{m!} \left( \sum_{n=0}^{\infty} H_{n,\lambda}(x, y) \frac{t^n}{n!} \right)
\]
\[
= \frac{1}{1-z} \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} F_{m,\lambda} \left( \frac{z}{1-z} \right) H_{n-m,\lambda}(x, y) \right) \frac{t^n}{n!}.
\]
(2.8)

Comparing the coefficients of \( \frac{t^n}{n!} \) in equation (2.7) and (2.8), we get (2.5).

**Theorem 2.4.** For \( n \geq 0 \), the following formula for degenerate Hermite-Fubini polynomials holds true:
\[
H_{n,\lambda}(x, y) = H_{n}(x, y; z) - z_H F_{n,\lambda}(x + 1, y; z) + z_H F_{n,\lambda}(x, y; z).
\]
(2.9)

**Proof.** We begin with the definition (2.1) and write
\[
(1 + \lambda t)^{\frac{z}{1-z}} (1 + \lambda t^2)^{\frac{z}{1-z}} = \frac{1 - z((1 + \lambda)^{\frac{z}{1-z}} - 1)}{1 - z((1 + \lambda)^{\frac{z}{1-z}} - 1)} (1 + \lambda t)^{\frac{z}{1-z}} (1 + \lambda t^2)^{\frac{z}{1-z}}
\]
\[
= \frac{1 - z((1 + \lambda)^{\frac{z}{1-z}} - 1)}{1 - z((1 + \lambda)^{\frac{z}{1-z}} - 1)} (1 + \lambda t)^{\frac{z}{1-z}} (1 + \lambda t^2)^{\frac{z}{1-z}}.
\]
Then using the definition of Kampé de Fériet generalization of the degenerate Hermite polynomials \( H_{n,\lambda}(x, y) \) (1.3) and (2.1), we have
\[
\sum_{n=0}^{\infty} H_{n,\lambda}(x, y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} [H_{n}(x, y; z) - z_H F_{n,\lambda}(x + 1, y; z) + z_H F_{n,\lambda}(x, y; z)] \frac{t^n}{n!}.
\]
Finally, comparing the coefficients of \( \frac{t^n}{n!} \), we get (2.9).

**Theorem 2.5.** For \( n \geq 0 \) and \( z_1 \neq z_2 \), the following formula for degenerate Hermite-Fubini polynomials holds true:
\[
\sum_{k=0}^{n} \binom{n}{k} H_{n-k,\lambda}(x_1, y_1; z_1) H_{k,\lambda}(x_2, y_2; z_2)
\]
\[
= z_2 H_{n,\lambda}(x_1 + x_2, y_1 + y_2; z_2) - z_1 H_{n,\lambda}(x_1 + x_2, y_1 + y_2; z_1)
\]
\[
= z_2 H_{n,\lambda}(x_1 + x_2, y_1 + y_2; z_2) - z_1 H_{n,\lambda}(x_1 + x_2, y_1 + y_2; z_1)
\]
\[
\frac{t^n}{n!}.
\]
(2.10)

**Proof.** The products of (2.1) can be written as
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} H_{n}(x_1, y_1; z_1) \frac{t^n}{n!} H_{k}(x_2, y_2; z_2) \frac{t^k}{k!} = \frac{(1 + \lambda t)^{\frac{z_1 + z_2}{1-z_1} - 1}}{1 - z_1((1 + \lambda)^{\frac{z_1 + z_2}{1-z_1}} - 1)} \frac{(1 + \lambda t)^{\frac{z_1 + z_2}{1-z_1} - 1}}{1 - z_2((1 + \lambda)^{\frac{z_1 + z_2}{1-z_1}} - 1)}
\]
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(x_1, y_1; z_1) H_{k}(x_2, y_2; z_2) \right) \frac{t^n}{n!}
\]
\[
= \frac{z_2}{z_2 - z_1} \frac{(1 + \lambda t)^{\frac{z_1 + z_2}{1-z_1} - 1}}{1 - z_1((1 + \lambda)^{\frac{z_1 + z_2}{1-z_1}} - 1)} - \frac{z_1}{z_2 - z_1} \frac{(1 + \lambda t)^{\frac{z_1 + z_2}{1-z_1} - 1}}{1 - z_2((1 + \lambda)^{\frac{z_1 + z_2}{1-z_1}} - 1)}
\]
\[
= \left( z_2 H_{n}(x_1 + x_2, y_1 + y_2; z_2) - z_1 H_{n}(x_1 + x_2, y_1 + y_2; z_1) \right) \frac{t^n}{n!}.
\]
By equating the coefficients of \( \frac{t^n}{n!} \) on both sides, we get (2.10).
Theorem 2.6. For \( n \geq 0 \), the following formula for degenerate Hermite-Fubini polynomials holds true:

\[
z_H F_{n, \lambda}(x + 1, y; z) = (1 + z) H F_{n, \lambda}(x, y; z) - H_{n, \lambda}(x, y).
\]

(2.11)

Proof. From (2.1), we have

\[
\sum_{n=0}^{\infty} \left[ H F_{n, \lambda}(x + 1, y; z) - H F_{n, \lambda}(x, y; z) \right] \frac{t^n}{n!} = \frac{(1 + \lambda t) \frac{\hat{x}}{1 - z((1 + \lambda t)^{\frac{1}{\lambda}} - 1)}}{1 - z((1 + \lambda t)^{\frac{1}{\lambda}} - 1)} - (1 + \lambda t) \frac{\hat{x}}{1 - z((1 + \lambda t)^{\frac{1}{\lambda}} - 1)}
\]

\[
= \frac{1}{z} \sum_{n=0}^{\infty} \left[ H F_{n, \lambda}(x, y; z) - H_{n, \lambda}(x, y) \right] \frac{t^n}{n!}.
\]

Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides, we obtain (2.11).

Remark 2.3. On setting \( x = y = 0 \) and \( x = -1 \) in Theorem 2.6, we find

\[
z_H F_{n, \lambda}(1, 0; z) = (1 + z) H F_{n, \lambda}(0, 0; z),
\]

and

\[
z_H F_{n, \lambda}(0, 0; z) = (1 + z) H_{n, \lambda}(-1, 0; z) - (-\lambda)^n \frac{1}{\lambda}.
\]

(2.12)

(2.13)

Theorem 2.7. For \( n \geq 0 \), \( p, q \in \mathbb{R} \), the following formula for degenerate Hermite-Fubini polynomials holds true:

\[
H F_{n, \lambda}(px, qy; z)
\]

\[
= n! \sum_{k=0}^{n} \sum_{j=0}^{\left[ \frac{k}{2} \right]} \lambda^{k-j} H F_{n-k, \lambda}(x, y; z) \left( \frac{(p-1)x}{\lambda} \right)_{k-2j} \left( \frac{(q-1)y}{\lambda} \right)_{j} \frac{1}{(n-k-2j)!j!}.
\]

(2.14)

Proof. Rewrite the generating function (2.1), we have

\[
\sum_{n=0}^{\infty} H F_{n, \lambda}(px, qy; z) \frac{t^n}{n!} = \frac{(1 + \lambda t)^{\frac{qy}{\lambda}} (1 + \lambda t^{\frac{(n+1)y}{\lambda}} (1 + \lambda t)^{\frac{(n-1)y}{\lambda}})}{1 - z((1 + \lambda t)^{\frac{1}{\lambda}} - 1)}
\]

\[
= \left( \sum_{n=0}^{\infty} H F_{n, \lambda}(x, y; z) \frac{t^n}{n!} \right) \left( \sum_{k=0}^{\infty} \left( \frac{(p-1)x}{\lambda} \right)_{k} \frac{t^{k}}{k!} \right) \left( \sum_{j=0}^{\infty} \left( \frac{(q-1)y}{\lambda} \right)_{j} \frac{t^{j}}{j!} \right)
\]

\[
= \left( \sum_{n=0}^{\infty} H F_{n, \lambda}(x, y; z) \frac{t^n}{n!} \right) \left( \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{(p-1)x}{\lambda} \right)_{k} \left( \frac{(q-1)y}{\lambda} \right)_{j} \lambda^{k+j} \frac{t^{k+j}}{k!j!} \right)
\]

Replacing \( k \) by \( k - 2j \) in above equation, we have

\[
L.H.S. = \left( \sum_{n=0}^{\infty} H F_{n, \lambda}(x, y; z) \frac{t^n}{n!} \right) \left( \sum_{k=2j}^{\infty} \lambda^{k-j} \left( \frac{(p-1)x}{\lambda} \right)_{k-2j} \left( \frac{(q-1)y}{\lambda} \right)_{j} \frac{t^{k}}{(k-2j)!j!} \right).
\]

Again replacing \( n \) by \( n - k \) in above equation, we have

\[
L.H.S. = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\left[ \frac{k}{2} \right]} \lambda^{k-j} H F_{n-k, \lambda}(x, y; z) \left( \frac{(p-1)x}{\lambda} \right)_{k-2j} \left( \frac{(q-1)y}{\lambda} \right)_{j} \frac{t^{n}}{(n-k-2j)!j!k!}.
\]

Finally, equating the coefficients of \( t^n \) on both sides, we acquire the result (2.14).
Theorem 2.8. For \( n \geq 0 \), the following formula for degenerate Hermite-Fubini polynomials holds true:

\[
H_{F_{n, \lambda}}(x + r, y; z) \sum_{l=0}^{n} \binom{n}{l} H_{n-l, \lambda}(x, y) \sum_{k=0}^{l} z^{k} k! S_{2, \lambda}(l + r, k + r). \tag{2.15}
\]

Proof. Replacing \( x \) by \( x + r \) in (2.1), we have

\[
\sum_{n=0}^{\infty} H_{F_{n, \lambda}}(x + r, y; z) \frac{t^{n}}{n!} = \frac{(1 + \lambda t)^{x+y}}{1 - z((1 + \lambda T)^{x} - 1)}
\]

Replacing \( n \) by \( n - l \) in above equation, we get

\[
\sum_{n=0}^{\infty} H_{F_{n, \lambda}}(x + r, y; z) \frac{t^{n}}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} H_{n-l, \lambda}(x, y) \frac{z}{l} k! S_{2, \lambda}(l + r, k + r) \right) \frac{t^{n}}{n!}
\]

Comparing the coefficients of \( \frac{t^{n}}{n!} \) in both sides, we get (2.15).

3. Summation Formulae for degenerate Hermite-Fubini polynomials

First, we prove the following result involving the degenerate Hermite-Fubini polynomials \( H_{F_{n, \lambda}}(x, y; z) \) by using series rearrangement techniques and considered its special case:

Theorem 3.1. The following summation formula for degenerate Hermite-Fubini polynomials \( H_{F_{n}}(x, y; z) \) holds true:

\[
H_{F_{n, \lambda}}(u, v; z) H_{F_{m, \lambda}}(U, V; Z) = \sum_{r,k=0}^{m,n} \binom{n}{r} \binom{m}{k} H_{r, \lambda}(u - x, v - y) H_{n-r, \lambda}(x, y) \times H_{k, \lambda}(U - X, V - Y) H_{m-k, \lambda}(X, Y; Z). \tag{3.1}
\]

Proof. Consider the product of the degenerate Hermite-Fubini polynomials, we can be written as generating function (2.1) in the following form:

\[
\frac{1}{1 - z((1 + \lambda T)^{x} - 1)} \frac{1}{1 - z((1 + \lambda T)^{x} - 1)} (1 + \lambda T)^{x+y} Z((1 + \lambda T)^{x} - 1)
\]

Replacing \( x \) by \( u, y \) by \( v, X \) by \( U \) and \( Y \) by \( V \) in (3.2) and equating the resultant to itself,

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{F_{n, \lambda}}(u, v; z) H_{F_{m, \lambda}}(U, V; Z) \frac{t^{n}}{n!} \frac{T^{m}}{m!}
\]

\[
= (1 + \lambda t)^{x+y} (1 + \lambda T)^{x+y} (1 + \lambda T)^{x+y} (1 + \lambda T)^{x+y}
\]
The following summation formula for degenerate Hermite-Fubini polynomials $H_n(x, y; z)$ holds true:

$$H_n(x + w, y + u; z) = \sum_{s=0}^{n} \binom{n}{s} H_{n-s}(x, y; z) H_s(w, u). \quad (3.4)$$

Proof. We replace $x$ by $x + w$ and $y$ by $y + u$ in (2.1), use (1.3) and rewrite the generating function as:

$$\frac{1}{1 - z((1 + \lambda t)^z - 1)} (1 + \lambda t)^{x+w} (1 + \lambda^2 t^2)^{z+s} = \sum_{n=0}^{\infty} H_n(x, y; z) \frac{t^n}{n!} \sum_{s=0}^{\infty} H_s(w, u) \frac{t^s}{s!}$$

$$= \sum_{n=0}^{\infty} H_n(x + w, y + u; z) \frac{t^n}{n!}.$$

Now replacing $n$ by $n - s$ in l.h.s. and comparing the coefficients of $t^n$ on both sides, we get the result (3.4).

Theorem 3.3. The following summation formula for degenerate Hermite-Fubini polynomials $H_n(x, y; z)$ holds true:

$$H_n(x, y; z) = \sum_{r=0}^{n} \binom{n}{r} F_{n-r}(x - w; z) H_r(w, y). \quad (3.5)$$

Proof. By exploiting the generating function (1.3), we can write equation (2.1) as

$$\frac{1}{1 - z((1 + \lambda t)^z - 1)} (1 + \lambda t)^{x-w} (1 + \lambda^2 t^2)^{z+y} = \sum_{n=0}^{\infty} H_n(x-w; z) \frac{t^n}{n!} \sum_{r=0}^{\infty} H_r(w, y) \frac{t^r}{r!}.$$

On replacing $n$ by $n - r$ in above equation, we get

$$\sum_{n=0}^{\infty} H_n(x, y; z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} F_{n-r}(x - w; z) H_r(w, y) \frac{t^n}{(n-r)!r!}.$$

Equating the coefficients of the like powers of $t$ on both sides, we get (3.5).

Theorem 3.4. The following summation formula for degenerate Hermite-Fubini polynomials $H_n(x, y; z)$ holds true:

$$H_n(x+1, y; z) = \sum_{r=0}^{n} \binom{n}{r} H_{n-r}(x, y; z) \left(\frac{1}{\lambda}\right)_r \lambda^r. \quad (3.6)$$
Proof. Using the generating function (2.1), we have

\[
\begin{align*}
\sum_{n=0}^{\infty} H_{n,\lambda}(x+y+z) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} H_{n,\lambda}(x,y,z) \frac{t^n}{n!} \\
&= \left(\frac{1}{1 - z((1 + \lambda)t)^{1/2} - 1}\right) (1 + \lambda t)^{1/2} (1 + \lambda t^2)^{1/2} \\
&= \sum_{n=0}^{\infty} H_{n,\lambda}(x,y,z) \frac{t^n}{n!} \left( \sum_{r=0}^{\infty} \frac{\left(\frac{1}{\lambda}\right)_r \lambda^r t^r}{r!} - 1 \right) \\
&= \sum_{n=0}^{\infty} H_{n,\lambda}(x,y,z) \frac{t^n}{n!} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{\lambda}\right)_r \lambda^r t^r}{r!} - \sum_{n=0}^{\infty} H_{n,\lambda}(x,y,z) \frac{t^n}{n!}.
\end{align*}
\]

Finally, equating the coefficients of the like powers of \( t \) on both sides, we get (3.6).

4. Symmetric identities for degenerate Hermite-Fubini polynomials

In this section, we establish general symmetry identities for the degenerate Hermite-Fubini polynomials \( H_{n,\lambda}(x,y,z) \) by applying the generating function (2.1) and (2.2).

Theorem 4.1. Let \( x, y, z \in \mathbb{R} \) and \( n \geq 0 \), then the following identity holds true:

\[
\sum_{r=0}^{n} \binom{n}{r} b^r a^{n-r} H_{n-r,\lambda}(bx, b^2y; z) H_{r,\lambda}(ax, a^2y; z) = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} H_{n-r,\lambda}(ax, a^2y; z) H_{r,\lambda}(bx, b^2y; z). \tag{4.1}
\]

Proof. Start with

\[
A(t) = \frac{(1 + \lambda t)^{1/2} (1 + \lambda t^2)^{1/2}}{(1 - z((1 + \lambda t)^{1/2} - 1))(1 - z((1 + \lambda t)^{1/2} - 1))}.
\]

Then the expression for \( A(t) \) is symmetric in \( a \) and \( b \) and we can expand \( A(t) \) into series in two ways to obtain:

\[
\begin{align*}
A(t) &= \sum_{n=0}^{\infty} H_{n,\lambda}(bx, b^2y; z) \frac{(at)^n}{n!} \sum_{r=0}^{\infty} H_{r,\lambda}(ax, a^2y; z) \frac{(bt)^r}{r!} \\
&= \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} \binom{n}{r} b^r a^{n-r} H_{n-r,\lambda}(bx, b^2y; z) H_{r,\lambda}(ax, a^2y; z) \right) \frac{t^n}{n!}. \tag{4.2}
\end{align*}
\]

Similarly, we can show that

\[
\begin{align*}
A(t) &= \sum_{n=0}^{\infty} H_{n,\lambda}(ax, a^2y; z) \frac{(bt)^n}{n!} \sum_{r=0}^{\infty} H_{r,\lambda}(bx, b^2y; z) \frac{(at)^r}{r!} \\
&= \sum_{n=0}^{\infty} \left( \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} H_{n-r,\lambda}(ax, a^2y; z) H_{r,\lambda}(bx, b^2y; z) \right) \frac{t^n}{n!}. \tag{4.3}
\end{align*}
\]

By comparing the coefficients of \( \frac{t^n}{n!} \) on the right hand sides of the last two equations, we arrive at the desired result (4.1).
Theorem 4.2. For each pair of integers $a$ and $b$ and all integers and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k H F_{n-k, \lambda}(bx, b^2 y; z) \sum_{i=0}^{k} \binom{k}{i} \sigma_i \left( \frac{\lambda}{b}, a - 1 \right) F_{k-i, \lambda}(au; z)$$

$$\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k H F_{n-k, \lambda}(ax, a^2 y; z) \sum_{i=0}^{k} \binom{k}{i} \sigma_i \left( \frac{\lambda}{a}, b - 1 \right) F_{k-i, \lambda}(bu; z). \quad (4.4)$$

Proof. Let

$$B(t) = \frac{(1 + \lambda t)^{\frac{a(t-1)}{b}}}{(1 - z((1 + \lambda t)^{\frac{a}{b}} - 1))((1 + \lambda t)^{\frac{a}{b}} - 1)((1 + \lambda t)^{\frac{a}{b}} - 1)}$$

$$= \frac{(1 + \lambda t)^{\frac{a(t-1)}{b}}}{(1 - z((1 + \lambda t)^{\frac{a}{b}} - 1))((1 + \lambda t)^{\frac{a}{b}} - 1)}$$

$$B(t) = (1 + \lambda t)^{\frac{a(t-1)}{b}} \left( \sum_{k=0}^{n} F_{k, \lambda}(au; z) \frac{(bd)^k}{k!} \right) \left( \sum_{k=0}^{n} \frac{F_{k, \lambda}(au; z) \frac{(bd)^k}{k!}}{k!} \right)$$

$$= \frac{\sum_{n=0}^{\infty} F_{n, \lambda}(bx, b^2 y; z) \frac{(at)^n}{n!}}{n!}$$

$$= \frac{\sum_{n=0}^{\infty} \binom{n}{k} a^{n-k} b^k H F_{n-k, \lambda}(bx, b^2 y; z) \sum_{i=0}^{k} \binom{k}{i} \sigma_i \left( \frac{\lambda}{b}, a - 1 \right) F_{k-i, \lambda}(au; z) \frac{t^n}{n!}}{n!}.$$ \quad (4.5)

On the other hand, we have

$$B(t) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k H F_{n-k, \lambda}(ax, a^2 y; z) \sum_{i=0}^{k} \binom{k}{i} \sigma_i \left( \frac{\lambda}{a}, b - 1 \right) F_{k-i, \lambda}(bu; z) \right) \frac{t^n}{n!}.$$ \quad (4.6)

By comparing the coefficients of $t^n$ on the right hand sides of the last two equations, we arrive at the desired result.

References