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A new class of Hermite-Fubini polynomials and its properties

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Abstract. In this paper, we introduce a new class of Hermite-Fubini numbers and polynomials and investigate some properties of these polynomials. We establish summation formulas of these polynomials by summation techniques series. Furthermore, we derive symmetric identities of Hermite-Fubini numbers and polynomials by using generating functions.

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1. Introduction

As is well known, the 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [1, 3] are defined as

$$H_n(x,y) = n!\sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.1}$$

It is clear that

$$H_n(2x,-1) = H_n(x, H_n(x,-\frac{1}{2}) = He_n(x), H_n(x,0) = x^n,$$

where $H_n(x)$ and $He_n(x)$ being ordinary Hermite polynomials.

The Hermite polynomial H_n(x,y) (see ([9, 10]) is defined by means of the following generating function as follows:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.2}$$

Geometric polynomials (also known as Fubini polynomials) are defined as follows (see [2]):

$$F_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!x^k, \tag{1.3}$$

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is the Stirling number of the second kind (see [5]).
For $x = 1$ in (1.3), we get n^{th} Fubini number (ordered Bell number or geometric number) F_n [2, 4, 5, 6, 8, 12] is defined by

$$F_n(1) = F_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!. \tag{1.4}$$

The exponential generating functions of geometric polynomials is given by (see [2]):

$$\frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}, \tag{1.5}$$

and related to the geometric series (see [2]):

$$\left(x\frac{d}{dx}\right)^m \frac{1}{1-x} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1-x}F_m(\frac{x}{1-x}), \mid x \mid < 1.$$

Let us give a short list of these polynomials and numbers as follows:

$F_0(x) = 1, F_1(x) = x, F_2(x) = x+2x^2, F_3(x) = x+6x^2+6x^3, F_4(x) = x+14x^2+36x^3+24x^4$,

and

$$F_0 = 1, F_1 = 1, F_2 = 3, F_3 = 13, F_4 = 75.$$

Geometric and exponential polynomials are connected by the relation (see [2]):

$$F_n(x) = \int_0^{\infty} \phi_n(x)e^{-\lambda}d\lambda. \tag{1.6}$$

Recently, Pathan and Khan [9] introduced two variable Hermite-Bernoulli polynomials is defined by means of the following generating function:

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!}. \tag{1.7}$$

On setting $\alpha = 1$ in (1.7), the result reduces to known result of Dattoli et al. [3].

The manuscript of this paper as follows: In section 2, we consider generating functions for Hermite-Fubini numbers and polynomials and give some properties of these numbers and polynomials. In section 3, we derive summation formulas of Hermite-Fubini numbers and polynomials. In Section 4, we construct a symmetric identities of Hermite-Fubini numbers and polynomials by using generating functions.

2. A new class of Hermite-Fubini numbers and polynomials

In this section, we define three-variable Hermite-Fubini polynomials and obtain some basic properties which gives us new formula for ${}_HF_n(x,y,z)$. Moreover, we shall consider the sum of products of two Hermite-Fubini polynomials. The sum of products of various polynomials and numbers with or without binomial coefficients have been studied by (see [2, 4, 5, 6, 8]):

We introduce 3-variable Hermite-Fubini polynomials by means of the following generating function:

$$\frac{e^{xt+yt^2}}{1-z(e^t-1)} = \sum_{n=0}^{\infty} {}_HF_n(x,y;z)\frac{t^n}{n!}. \tag{2.1}$$

It is easily seen from definition (2.1), we have

$${}_HF_n(0,0;z) = F_n(z), {}_HF_n(0,0;1) = F_n.$$

For $y = 0$ in (2.1), we obtain 2-variable Fubini polynomials which is defined by Kargin [8].

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$$\frac{e^{xt}}{1-z(e^t-1)} = \sum_{n=0}^{\infty} F_n(x;z)\frac{t^n}{n!}. \tag{2.2}$$

When investigating the connection between Hermite polynomials $H_n(x,y)$ and Fubini polynomials $F_n(z)$ of importance is the following theorem.

Theorem 2.1. The following summation formula for Hermite-Fubini polynomials holds true:

$$\frac{e^{-yt^2}}{\Omega}\left[\cos xt(z+1) - z\cos(t-xt)\right] = \sum_{n=0}^{\infty} {}_HF_{2n}(x,y,z)\frac{(-1)^n t^{2n}}{(2n)!} \tag{2.3}$$

$$\frac{e^{-yt^2}}{\Omega}\left[\sin xt(z+1) + z\sin(t-xt)\right] = \sum_{n=0}^{\infty} {}_HF_{2n+1}(x,y,z)\frac{(-1)^n t^{2n+1}}{(2n+1)!}, \tag{2.4}$$

where $\Omega = [1-z(\cos t-1)]^2 + [z\sin t]^2$.

Proof. On separating the power series on r.h.s. of (2.1) in to their even and odd terms by using the elementary identity

$$\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n) + \sum_{n=0}^{\infty} f(2n+1)$$

and then replacing t by it where $i^2 = -1$ and equating the real and imaginary parts in the resulting equation, we get the summation formulae (2.2) and (2.3).

Remark 2.1. On setting $x = y = 0$, $z = 1$ in (2.3) and (2.4), we get the following well-known classical results involving Fubini numebrs.

Corollary 2.1. The following summation formula for Hermite-Fubini polynomials holds true:

$$\frac{2-\cos t}{5-4\cos t} = \sum_{n=0}^{\infty} F_{2n}\frac{(-1)^n t^{2n}}{(2n)!} \tag{2.5}$$

$$\frac{\sin t}{5-4\cos t} = \sum_{n=0}^{\infty} F_{2n+1}\frac{(-1)^n t^{2n+1}}{(2n+1)!}. \tag{2.6}$$

Theorem 2.2. For $n \geq 0$, the following formula for Hermite-Fubini polynomials holds true:

$${}_HF_n(x,y,z) = \sum_{m=0}^{n} \binom{n}{r} F_{n-m}(z)H_m(x,y). \tag{2.7}$$

Proof. Using definition (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HF_n(x,y,z)\frac{t^n}{n!} &= \frac{e^{xt+yt^2}}{1-z(e^t-1)} \\ &= \sum_{n=0}^{\infty} F_n(z)\frac{t^n}{n!}\sum_{m=0}^{\infty} H_m(x,y)\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{r} F_{n-m}(z)H_m(x,y)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (2.7).

Theorem 2.2. For $n \geq 0$, the following formula for Hermite-Fubini polynomials holds true:

$$H_n(x,y) = {}_HF_n(x,y;z) - z\,{}_HF_n(x+1,y;z) + z\,{}_HF_n(x,y;z). \tag{2.8}$$

Proof. We begin with the definition (2.1) and write

$$\begin{aligned} e^{xt+yt^2} &= \frac{1-z(e^t-1)}{1-z(e^t-1)}e^{xt+yt^2} \\ &= \frac{e^{xt+yt^2}}{1-z(e^t-1)} - \frac{z(e^t-1)}{1-z(e^t-1)}e^{xt+yt^2} \end{aligned}$$

Then using the definition of Kampé de Fériet generalization of the Hermite polynomials $H_n(x,y)$ and (2.1), we have

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[{}_HF_n(x,y;z) - z\,{}_HF_n(x+1,y;z) + z\,{}_HF_n(x,y;z)\right]\frac{t^n}{n!}.$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$, we get (2.8).

Theorem 2.3. For $n \geq 0$ and $z_1 \neq z_2$, the following formula for Hermite-Fubini polynomials holds true:

$$\begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}{}_HF_{n-k}(x_1,y_1;z_1)\,{}_HF_k(x_2,y_2;z_2) \\ &= \frac{z_2\,{}_HF_n(x_1+x_2,y_1+y_2;z_2) - z_1\,{}_HF_n(x_1+x_2,y_1+y_2;z_1)}{z_2-z_1}. \end{aligned} \tag{2.9}$$

Proof. The products of (2.1) can be written as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}{}_HF_n(x_1,y_1;z_1)\frac{t^n}{n!}\,{}_HF_k(x_2,y_2;z_2)\frac{t^k}{k!} = \frac{e^{x_1t+y_1t^2}}{1-z_1(e^t-1)}\frac{e^{x_2t+y_2t^2}}{1-z_2(e^t-1)}$$

$$\begin{aligned} &\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}{}_HF_{n-k}(x_1,y_1;z_1)\,{}_HF_k(x_2,y_2;z_2)\right)\frac{t^n}{n!} \\ &= \frac{z_2}{z_2-z_1}\frac{e^{(x_1+x_2)t+(y_1+y_2)t^2}}{1-z_1(e^t-1)} - \frac{z_1}{z_2-z_1}\frac{e^{(x_1+x_2)t+(y_1+y_2)t^2}}{1-z_2(e^t-1)} \\ &= \left(\frac{z_2\,{}_HF_n(x_1+x_2,y_1+y_2;z_2) - z_1\,{}_HF_n(x_1+x_2,y_1+y_2;z_1)}{z_2-z_1}\right)\frac{t^n}{n!}. \end{aligned}$$

By equating the coefficients of $\frac{t^n}{n!}$ on both sides, we get (2.9).

Theorem 2.4. For $n \geq 0$, the following formula for Hermite-Fubini polynomials holds true:

$$z\,{}_HF_n(x+1,y;z) = (1+z)\,{}_HF_n(x,y;z) - H_n(x,y). \tag{2.10}$$

Proof. From (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty}\left[{}_HF_n(x+1,y;z) - {}_HF_n(x,y;z)\right]\frac{t^n}{n!} &= \frac{e^{xt+yt^2}}{1-z(e^t-1)}(e^t-1) \\ &= \frac{1}{z}\left[\frac{e^{xt+yt^2}}{1-z(e^t-1)} - e^{xt+yt^2}\right] \\ &= \frac{1}{z}\sum_{n=0}^{\infty}\left[{}_HF_n(x,y;z) - H_n(x,y)\right]\frac{t^n}{n!}. \end{aligned}$$

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Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we obtain (2.10).

Remark 2.3. On setting $x = y = 0$ and $x = -1$ in Theorem 2.4, we find
$$z_H F_n(1, 0; z) = (1 + z)_H F_n(0, 0; z), \tag{2.11}$$
and
$$z_H F_n(0, 0; z) = (1 + z)_H F_n(-1, 0; z) - (-1)^n. \tag{2.12}$$
Theorem 2.5. For $n \geq 0$, $p, q \in \mathbb{R}$, the following formula for Hermite-Fubini polynomials holds true:
$${}_H F_n(px, qy; z) = n! \sum_{k=0}^{n} \sum_{j=0}^{[\frac{k}{2}]} {}_H F_{n-k}(x, y; z)((p-1)x)^k((q-1)y)^j \frac{1}{(n-k-2j)!j!}. \tag{2.13}$$
Proof. Rewrite the generating function (2.1), we have
$$\sum_{n=0}^{\infty} {}_H F_n(px, qy; z)\frac{t^n}{n!} = \frac{1}{1 - z(e^t - 1)} e^{xt+yt^2} e^{(p-1)xt} e^{(q-1)yt^2}$$
$$= \left(\sum_{n=0}^{\infty} {}_H F_n(x, y; z)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}((p-1)x)^k \frac{t^k}{k!}\right)\left(\sum_{j=0}^{\infty}((q-1)y)^j \frac{t^{2j}}{j!}\right)$$
$$= \left(\sum_{n=0}^{\infty} {}_H F_n(x, y; z)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}((p-1)x)^k((q-1)y)^j \frac{t^{k+2j}}{n!k!j!}\right)$$
Replacing k by $k - 2j$ in above equation, we have
$$L.H.S. = \left(\sum_{n=0}^{\infty} {}_H F_n(x, y; z)\frac{t^n}{n!}\right)\left(\sum_{k=2j}^{\infty}((p-1)x)^{k-2j}((q-1)y)^j \frac{t^k}{(k-2j)!j!}\right)$$
$$= \sum_{n=0}^{\infty}\sum_{k=2j}^{\infty} {}_H F_n(x, y; z)((p-1)x)^{k-2j}((q-1)y)^j \frac{t^{n+k}}{(k-2j)!j!n!}$$
Again replacing n by $n - k$ in above equation, we have
$$L.H.S. = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]} {}_H F_{n-k}(x, y; z)((p-1)x)^{k-2j}((q-1)y)^j \frac{t^n}{(n-k-2j)!j!k!}.$$
Finally, equating the coefficients of t^n on both sides, we acquire the result (2.13).

Theorem 2.6. For $n \geq 0$, the following formula for Hermite-Fubini polynomials holds true:
$${}_H F_n(x, y, z) = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(x, y) \sum_{k=0}^{l} z^k k! S_2(l, k). \tag{2.14}$$
Proof. From (2.1), we have
$$\sum_{n=0}^{\infty} {}_H F_n(x, y, z)\frac{t^n}{n!} = \frac{e^{xt+yt^2}}{1 - z(e^t - 1)}$$
$$= e^{xt+yt^2}\sum_{k=0}^{\infty} z^k(e^t - 1)^k = e^{xt+yt^2}\sum_{k=0}^{\infty} z^k \sum_{l=k}^{\infty} k! S_2(l, k)\frac{t^l}{l!}$$
$$= \sum_{n=0}^{\infty} H_n(x, y)\frac{t^n}{n!}\sum_{l=0}^{\infty} z^k \sum_{k=0}^{l} k! S_2(l, k)\frac{t^l}{l!}$$

Replacing n by $n-l$ in above equation, we get

$$\sum_{n=0}^{\infty} {}_HF_n(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} H_{n-l}(x,y)\sum_{k=0}^{l} z^k k! S_2(l,k)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.14).

Theorem 2.7. For $n \geq 0$, the following formula for Hermite-Fubini polynomials holds true:

$$ {}_HF_n(x+r,y,z) = \sum_{l=0}^{n}\binom{n}{l} H_{n-l}(x,y)\sum_{k=0}^{l} z^k k! S_2(l+r,k+r). \tag{2.15}$$

Proof. Replacing x by $x+r$ in (2.1), we have

$$\sum_{n=0}^{\infty} {}_HF_n(x+r,y,z)\frac{t^n}{n!} = \frac{e^{(x+r)t+yt^2}}{1-z(e^t-1)}$$

$$= e^{xt+yt^2}e^{rt}\sum_{k=0}^{\infty} z^k(e^t-1)^k = e^{xt+yt^2}e^{rt}\sum_{k=0}^{\infty} z^k \sum_{l=k}^{\infty} k! S_2(l,k)\frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}\sum_{l=0}^{\infty} z^k \sum_{k=0}^{l} k! S_2(l+r,k+r)\frac{t^l}{l!}$$

Replacing n by $n-l$ in above equation, we get

$$\sum_{n=0}^{\infty} {}_HF_n(x+r,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} H_{n-l}(x,y)\sum_{k=0}^{l} z^k k! S_2(l+r,k+r)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.15).

3. Summation Formulae for Hermite-Fubini polynomials

First, we prove the following result involving the Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ by using series rearrangement techniques and considered its special case:

Theorem 3.1. The following summation formula for Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_HF_{q+l}(w,y;z) \;=\; \sum_{n,p=0}^{q,l}\binom{q}{n}\binom{l}{p}(w-y)^{n+p}\, {}_HF_{q+l-n-p}(x,y;z). \tag{3.1}$$

Proof. Replacing t by $t + u$ in (2.1) and then using the formula [11,p.52(2)]:

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.2}$$

in the resultant equation, we find the following generating function for the Hermite-Fubini polynomials ${}_HF_n(x,y;z)$:

$$\frac{1}{1-z(e^{t+u}-1)}e^{y(t+u)^2} = e^{-x(t+u)}\sum_{q,l=0}^{\infty} {}_HF_{q+l}(x,y;z)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.3}$$

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Replacing x by w in the above equation and equating the resultant equation to the above equation, we find

$$\exp((w-x)(t+u)) \sum_{q,l=0}^{\infty} {}_HF_{q+l}(x,y;z)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_HF_{q+l}(w,y;z)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.4}$$

On expanding exponential function (3.4) gives

$$\sum_{N=0}^{\infty} \frac{[(w-x)(t+u)]^N}{N!} \sum_{q,l=0}^{\infty} {}_HF_{q+l}(x,y;z)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_HF_{q+l}(w,y;z)\frac{t^q}{q!}\frac{u^l}{l!}, \tag{3.5}$$

which on using formula (3.2) in the first summation on the left hand side becomes

$$\sum_{n,p=0}^{\infty} \frac{(w-x)^{n+p}t^n u^p}{n!p!} \sum_{q,l=0}^{\infty} {}_HF_{q+l}(x,y;z)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_HF_{q+l}(w,y;z)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.6}$$

Now replacing q by $q-n$, l by $l-p$ and using the lemma ([11, p.100(1)]):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A(n,k) = \sum_{k=0}^{\infty}\sum_{n=0}^{k} A(n,k-n), \tag{3.7}$$

in the l.h.s. of (3.6), we find

$$\sum_{q,l=0}^{\infty}\sum_{n,p=0}^{q,l} \frac{(w-x)^{n+p}}{n!p!} {}_HF_{q+l-n-p}(x,y;z)\frac{t^q}{(q-n)!}\frac{u^l}{(l-p)!}$$
$$= \sum_{q,l=0}^{\infty} {}_HF_{q+l}(w,y;z)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.8}$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, we get the assertion (3.1) of Theorem 3.1.

Remark 3.1. Taking $l=0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

Corollary 3.1. The following summation formula for Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$${}_HF_q(w,y;z) = \sum_{n=0}^{q} \binom{q}{n} (w-x)^n {}_HF_{q-n}(x,y;z). \tag{3.9}$$

Remark 3.2. Replacing w by $w+x$ in (3.9), we obtain

$${}_HF_q(x+w,y;z) = \sum_{n=0}^{q} \binom{q}{n} w^n {}_HF_{q-n}(x,y;z). \tag{3.10}$$

Theorem 3.2. The following summation formula for Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$${}_HF_n(w,u;z)\,{}_HF_m(W,U;Z) = \sum_{r,k=0}^{n,m} \binom{n}{r}\binom{m}{k} H_r(w-x,u-y)\,{}_HF_{n-r}(x,y;z)$$
$$\times H_k(W-X,U-Y)\,{}_HF_{m-k}(X,Y;Z). \tag{3.11}$$

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Proof. Consider the product of the Hermite-Fubini polynomials, we can be written as generating function (2.1) in the following form:

$$\frac{1}{1-z(e^t-1)}e^{xt+yt^2}\frac{1}{1-Z(e^T-1)}e^{XT+YT^2}=\sum_{n=0}^{\infty}{}_HF_n(x,y;z)\frac{t^n}{n!}\sum_{m=0}^{\infty}{}_HF_m(X,Y;Z)\frac{T^m}{m!}. \tag{3.12}$$

Replacing x by w, y by u, X by W and Y by U in (3.12) and equating the resultant to itself,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HF_n(w,u;z)_HF_m(W,U;Z)\frac{t^n}{n!}\frac{T^m}{m!}$$
$$=\exp\left((w-x)t+(u-y)t^2\right)\exp\left((W-X)T+(U-Y)T^2\right)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HF_n(x,y;z)_HF_m(X,Y;Z)\frac{t^n}{n!}\frac{T^m}{m!},$$

which on using the generating function (3.7) in the exponential on the r.h.s., becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_HF_n(w,u;z)_HF_m(W,U;Z)\frac{t^n}{n!}\frac{T^m}{m!}$$
$$=\sum_{n,r=0}^{\infty}H_r(w-x,u-y)_HF_n(x,y;z)\frac{t^{n+r}}{n!r!}\sum_{m,k=0}^{\infty}H_k(W-X,U-Y)_HF_m(X,Y;Z)\frac{T^{m+k}}{m!k!}. \tag{3.13}$$

Finally, replacing n by $n-r$ and m by $m-k$ and using equation (3.7) in the r.h.s. of the above equation and then equating the coefficients of like powers of t and T, we get assertion (3.11) of Theorem 3.2.

Remark 3.3. Replacing u by y and U by Y in assertion (3.11) of Theorem 3.2, we deduce the the following consequence of Theorem 3.2.

Corollary 3.2. The following summation formula for Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_HF_n(w,u;z)_HF_m(W,U;Z)=\sum_{r,k=0}^{n,m}\binom{n}{r}\binom{m}{k}(w-x)^r{}_HF_{n-r}(x,y;z)$$
$$\times(W-X)^k{}_HF_{m-k}(X,Y;Z). \tag{3.14}$$

Theorem 3.3. The following summation formula for Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_HF_n(x+w,y+u;z)=\sum_{s=0}^{n}\binom{n}{s}{}_HF_{n-s}(x,y;z)H_s(w,u). \tag{3.15}$$

Proof. We replace x by $x+w$ and y by $y+u$ in (2.1), use (1.2) and rewrite the generating function as:

$$\frac{1}{1-z(e^t-1)}\exp((x+w)t+(y+u)t^2)=\sum_{n=0}^{\infty}{}_HF_n(x,y;z)\frac{t^n}{n!}\sum_{s=0}^{\infty}H_s(w,u)\frac{t^s}{s!}$$
$$=\sum_{n=0}^{\infty}{}_HF_n(x+w,y+u;z)\frac{t^n}{n!} \tag{3.16}$$

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Now replacing n by $n-s$ in l.h.s. and comparing the coefficients of t^n on both sides, we get the result (3.15).

Theorem 3.4. The following summation formula for Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_HF_n(y,x;z) = \sum_{s=0}^{[\frac{n}{2}]} F_{n-2s}(y;z)\frac{x^s}{(n-2s)!s!}. \tag{3.17}$$

Proof. We replace x by y and y by x in equation (2.1) to get

$$ \sum_{n=0}^{\infty} {}_HF_n(y,x;z)\frac{t^n}{n!} = \sum_{n=0}^{\infty} F_n(y;z)\frac{t^n}{n!}\sum_{s=0}^{\infty}\frac{x^s t^{2s}}{k!}. \tag{3.18}$$

Now replacing n by $n-2s$ in r.h.s. and comparing the coefficients of t on both sides, we arrive at the desired result (3.16).

Theorem 3.5. The following summation formula for Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_HF_n(x,y;z) = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right) F_{n-r}(x-w;z)H_r(w,y). \tag{3.19}$$

Proof. By exploiting the generating function (1.2), we can write equation (2.1) as

$$ \frac{1}{1-z(e^t-1)}e^{(x-w)t}e^{wt+yt^2} = \sum_{n=0}^{\infty} F_n(x-w;z)\frac{t^n}{n!}\sum_{r=0}^{\infty} H_r(w,y)\frac{t^r}{r!}. \tag{3.20}$$

On replacing n by $n-r$ in above equation, we get

$$ \sum_{n=0}^{\infty} {}_HF_n(x,y;z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{r=0}^{n} F_{n-r}(x-w;z)H_r(w,y)\frac{t^n}{(n-r)!r!}.$$

Equating the coefficients of the like powers of t on both sides, we get (3.19).

Theorem 3.6. The following summation formula for Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_HF_n(x+1,y;z) = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right) {}_HF_{n-r}(x,y;z). \tag{3.21}$$

Proof. Using the generating function (2.1), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_HF_n(x+1,y;z)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HF_n(x,y;z)\frac{t^n}{n!} \\
&= \left(\frac{1}{1-z(e^t-1)}\right)(e^t-1)e^{xt+yt^2} \\
&= \sum_{n=0}^{\infty} {}_HF_n(x,y;z)\frac{t^n}{n!}\left(\sum_{r=0}^{\infty}\frac{t^r}{r!}-1\right) \\
&= \sum_{n=0}^{\infty} {}_HF_n(x,y;z)\frac{t^n}{n!}\sum_{r=0}^{\infty}\frac{t^r}{r!} - \sum_{n=0}^{\infty} {}_HF_n(x,y;z)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right) {}_HF_{n-r}(x,y;z)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HF_n(x,y;z)\frac{t^n}{n!}.
\end{aligned}$$

Finally, equating the coefficients of the like powers of t on both sides, we get (3.21).

4. Symmetric identities

Recently, Khan [7], Pathan and Khan [9, 10] have been introduced symmetric identities. In this section, we establish general symmetry identities for the generalized Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ by applying the generating function (2.1) and (2.2).

Theorem 4.1. Let $x, y, z \in \mathbb{R}$ and $n \geq 0$, then the following identity holds true:

$$\sum_{r=0}^{n} \binom{n}{r} b^r a^{n-r} {}_HF_{n-r}(bx, b^2y; z)\, {}_HF_r(ax, a^2y; z)$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} {}_HF_{n-r}(ax, a^2y; z)\, {}_HF_r(bx, b^2y; z). \tag{4.1}$$

Proof. Start with

$$A(t) = \frac{1}{(1-z(e^{at}-1))(1-z(e^{bt}-1))} e^{abxt+a^2b^2yt^2}.$$

Then the expression for $A(t)$ is symmetric in a and b and we can expand $A(t)$ into series in two ways to obtain:

$$A(t) = \sum_{n=0}^{\infty} {}_HF_n(bx, b^2y; z) \frac{(at)^n}{n!} \sum_{r=0}^{\infty} {}_HF_r(ax, a^2y; z) \frac{(bt)^r}{r!}$$

$$A(t) = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \binom{n}{r} b^r a^{n-r} {}_HF_{n-r}(bx, b^2y; z)\, {}_HF_r(ax, a^2y; z) \right) \frac{t^n}{n!}. \tag{4.2}$$

Similarly, we can show that

$$A(t) = \sum_{n=0}^{\infty} {}_HF_n(ax, a^2y; z) \frac{(bt)^n}{n!} \sum_{r=0}^{\infty} {}_HF_r(bx, b^2y; z) \frac{(at)^r}{r!}$$

$$A(t) = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} {}_HF_{n-r}(ax, a^2y; z)\, {}_HF_r(bx, b^2y; z) \right) \frac{t^n}{n!}. \tag{4.3}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result (4.1).

Theorem 4.2. For each pair of integers a and b and all integers and $n \geq 0$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^{n-k} b^k {}_HF_{n-k} \left(bx + \frac{b}{a} i + j, b^2y, z \right) F_k(au, z)$$

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{a-1} \sum_{i=0}^{b-1} b^{n-k} a^k {}_HF_{n-k} \left(ax + \frac{a}{b} i + j, a^2y, z \right) F_k(bu, z). \tag{4.4}$$

Proof. Let

$$B(t) = \frac{e^{ab(x+u)t+a^2b^2yt^2}(e^{abt}-1)^2}{(1-z(e^{at}-1))(1-z(e^{bt}-1))(e^{at}-1)(e^{bt}-1)}$$

$$= \frac{e^{abxt+a^2b^2yt^2}}{1-z(e^{at}-1)} \frac{e^{abt}-1}{e^{bt}-1} \frac{e^{abut}}{1-z(e^{bt}-1)} \frac{e^{abt}-1}{e^{at}-1}$$

$$B(t) = \frac{e^{abxt+a^2b^2yt^2}}{1-z(e^{at}-1)} \sum_{i=0}^{a-1} e^{bti} \frac{e^{abut}}{1-z(e^{bt}-1)} \sum_{j=0}^{b-1} e^{atj}$$

$$= \frac{e^{a^2b^2yt^2}}{1-z(e^{at}-1)} \sum_{i=0}^{a-1}\sum_{j=0}^{b-1} e^{(bx+\frac{b}{a}i+j)at} \sum_{k=0}^{\infty} F_k(au,z)\frac{(bt)^k}{k!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} a^{n-k}b^k\,{}_HF_{n-k}\left(bx+\frac{b}{a}i+j, b^2y, z\right)F_k(au,z)\right)\frac{t^n}{n!} \quad (4.5)$$

On the other hand

$$B(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{a-1}\sum_{i=0}^{b-1} b^{n-k}a^k\,{}_HF_{n-k}\left(ax+\frac{a}{b}i+j, a^2y, z\right)F_k(bu,z)\right)\frac{t^n}{n!}. \quad (4.6)$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

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