A new class of Hermite-Fubini polynomials and its properties

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Abstract. In this paper, we introduce a new class of Hermite-Fubini numbers and polynomials and investigate some properties of these polynomials. We establish summation formulas of these polynomials by summation techniques series. Furthermore, we derive symmetric identities of Hermite-Fubini numbers and polynomials by using generating functions.

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1. Introduction

As is well known, the 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x, y)$ \cite{1, 3} are defined as

\begin{equation}
H_n(x, y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{y^r x^{n-2r}}{r!(n-2r)!}.
\end{equation}

It is clear that

\[H_n(2x, -1) = H_n(x, H_n(x, -\frac{1}{2})) = He_n(x), H_n(x, 0) = x^n,\]

where $H_n(x)$ and $He_n(x)$ being ordinary Hermite polynomials.

The Hermite polynomial $H_n(x,y)$ (see \cite{9, 10}) is defined by means of the following generating function as follows:

\begin{equation}
e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}.
\end{equation}

Geometric polynomials (also known as Fubini polynomials) are defined as follows (see \cite{2}):

\begin{equation}
F_n(x) = \sum_{k=0}^{n} \binom{n}{k} k! x^k,
\end{equation}

where $\binom{n}{k}$ is the Stirling number of the second kind (see \cite{5}).

For $x = 1$ in (1.3), we get $n^{th}$ Fubini number (ordered Bell number or geometric number) $F_n$ \cite{2, 4, 5, 6, 8, 12} is defined by

\begin{equation}
F_n(1) = F_n = \sum_{k=0}^{n} \binom{n}{k} k!.
\end{equation}
The exponential generating functions of geometric polynomials is given by (see [2]):

\[
\frac{1}{1 - x(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!},
\]

(1.5)

and related to the geometric series (see [2]):

\[
\left( x \frac{d}{dx} \right)^m \frac{1}{1 - x} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1 - x} F_m(\frac{x}{1 - x}), \quad |x| < 1.
\]

Let us give a short list of these polynomials and numbers as follows:

\[
F_0(x) = 1, \quad F_1(x) = x, \quad F_2(x) = x + 2x^2, \quad F_3(x) = x + 6x^2 + 6x^3, \quad F_4(x) = x + 14x^2 + 36x^3 + 24x^4,
\]

and

\[
F_0 = 1, \quad F_1 = 1, \quad F_2 = 3, \quad F_3 = 13, \quad F_4 = 75.
\]

Geometric and exponential polynomials are connected by the relation (see [2]):

\[
F_n(x) = \int_0^\infty \phi_n(x)e^{-\lambda}d\lambda.
\]

(1.6)

Recently, Pathan and Khan [9] introduced two variable Hermite-Bernoulli polynomials is defined by means of the following generating function:

\[
\left( \frac{e^t + y t^2}{e^t - 1} \right)^\alpha e^{xt + yt^2} = \sum_{n=0}^{\infty} H_{\alpha}^F(n, x, y) \frac{t^n}{n!}.
\]

(1.7)

On setting \( \alpha = 1 \) in (1.7), the result reduces to known result of Dattoli et al. [3].

The manuscript of this paper as follows: In section 2, we consider generating functions for Hermite-Fubini numbers and polynomials and give some properties of these numbers and polynomials. In section 3, we derive summation formulas of Hermite-Fubini numbers and polynomials. In Section 4, we construct a symmetric identities of Hermite-Fubini numbers and polynomials by using generating functions.

2. A new class of Hermite-Fubini numbers and polynomials

In this section, we define three-variable Hermite-Fubini polynomials and obtain some basic properties which give us new formula for \( H_{\alpha}^F_n(x, y, z) \). Moreover, we shall consider the sum of products of two Hermite-Fubini polynomials. The sum of products of various polynomials and numbers with or without binomial coefficients have been studied by (see [2, 4, 5, 6, 8]):

We introduce 3-variable Hermite-Fubini polynomials by means of the following generating function:

\[
\frac{e^{xt + yt^2}}{1 - z(e^t - 1)} = \sum_{n=0}^{\infty} H_{\alpha}^F_n(x, y, z) \frac{t^n}{n!}.
\]

(2.1)

It is easily seen from definition (2.1), we have

\[
H_{\alpha}^F_n(0, 0; z) = F_n(z), \quad H_{\alpha}^F_n(0, 0; 1) = F_n.
\]

For \( y = 0 \) in (2.1), we obtain 2-variable Fubini polynomials which is defined by Kargin [8].
When investigating the connection between Hermite polynomials \(H_n(x, y)\) and Fubini polynomials \(F_n(z)\) of importance is the following theorem.

**Theorem 2.1.** The following summation formula for Hermite-Fubini polynomials holds true:

\[
e^{-yt^2} \left[ \cos xt(z + 1) - z \cos(xt) \right] = \sum_{n=0}^{\infty} H_{2n}(x, y, z) \frac{(-1)^n t^{2n}}{(2n)!}, \tag{2.3}
\]

\[
e^{-yt^2} \left[ \sin xt(z + 1) + z \sin(xt) \right] = \sum_{n=0}^{\infty} H_{2n+1}(x, y, z) \frac{(-1)^n t^{2n+1}}{(2n+1)!}, \tag{2.4}
\]

where \(\Omega = [1 - z(\cos t - 1)]^2 + [z \sin t]^2\).

**Proof.** On separating the power series on r.h.s. of (2.1) into even and odd terms by using the elementary identity

\[
\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n) + \sum_{n=0}^{\infty} f(2n+1)
\]

and then replacing \(t\) by \(it\), where \(i^2 = -1\) and equating the real and imaginary parts in the resulting equation, we get the summation formulae (2.2) and (2.3).

**Remark 2.1.** On setting \(x = y = 0, z = 1\) in (2.3) and (2.4), we get the following well-known classical results involving Fubini numbers.

**Corollary 2.1.** The following summation formula for Hermite-Fubini polynomials holds true:

\[
\frac{2 - \cos t}{5 - 4 \cos t} = \sum_{n=0}^{\infty} F_{2n}(z) \frac{(-1)^n t^{2n}}{(2n)!}, \tag{2.5}
\]

\[
\frac{\sin t}{5 - 4 \cos t} = \sum_{n=0}^{\infty} F_{2n+1}(z) \frac{(-1)^n t^{2n+1}}{(2n+1)!}, \tag{2.6}
\]

**Theorem 2.2.** For \(n \geq 0\), the following formula for Hermite-Fubini polynomials holds true:

\[
HF_n(x, y, z) = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}(z) H_m(x, y). \tag{2.7}
\]

**Proof.** Using definition (2.1), we have

\[
\sum_{n=0}^{\infty} HF_n(x, y, z) \frac{t^n}{n!} = \frac{e^{xt+y^2}}{1 - z(e^t - 1)}
\]

\[
= \sum_{n=0}^{\infty} F_n(z) \frac{t^n}{n!} \sum_{m=0}^{n} H_m(x, y) \frac{t^m}{m!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} F_{n-m}(z) H_m(x, y) \right) \frac{t^n}{n!}.
\]

Comparing the coefficients of \(\frac{t^n}{n!}\) yields (2.7).
Theorem 2.2. For $n \geq 0$, the following formula for Hermite-Fubini polynomials holds true:

$$H_n(x, y) = H F_n(x, y; z) - z H F_n(x + 1, y; z) + z H F_n(x, y; z). \tag{2.8}$$

Proof. We begin with the definition (2.1) and write

$$e^{xt+yt^2} = \frac{1 - z(e^t - 1)}{1 - z(e^t - 1)} e^{xt+yt^2}$$

Then using the definition of Kampé de Fériet generalization of the Hermite polynomials $H_n(x, y)$ and (2.1), we have

$$\sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[ H F_n(x, y; z) - z H F_n(x + 1, y; z) + z H F_n(x, y; z) \right] \frac{t^n}{n!}.$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$, we get (2.8).

Theorem 2.3. For $n \geq 0$ and $z_1 \neq z_2$, the following formula for Hermite-Fubini polynomials holds true:

$$\sum_{k=0}^{n} \binom{n}{k} H F_{n-k}(x_1, y_1; z_1) H F_k(x_2, y_2; z_2) = \frac{z_2 H F_n(x_1 + x_2, y_1 + y_2; z_2) - z_1 H F_n(x_1 + x_2, y_1 + y_2; z_1)}{z_2 - z_1}. \tag{2.9}$$

Proof. The products of (2.1) can be written as

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} H F_n(x_1, y_1; z_1) \frac{t^n}{n!} H F_k(x_2, y_2; z_2) \frac{t^k}{k!} = \frac{e^{x_1+t+y_1t^2}}{1 - z_1(e^t - 1)} \frac{e^{x_2+t+y_2t^2}}{1 - z_2(e^t - 1)}$$

By equating the coefficients of $\frac{t^n}{n!}$ on both sides, we get (2.9).

Theorem 2.4. For $n \geq 0$, the following formula for Hermite-Fubini polynomials holds true:

$$z H F_n(x + 1, y; z) = (1 + z) H F_n(x, y; z) - H_n(x, y). \tag{2.10}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} \left[ H F_n(x + 1, y; z) - H F_n(x, y; z) \right] \frac{t^n}{n!} = \frac{e^{xt+yt^2}}{1 - z(e^t - 1)}(e^t - 1)$$

$$= \frac{1}{z} \left[ \frac{e^{xt+yt^2}}{1 - z(e^t - 1)} - e^{xt+yt^2} \right]$$

$$= \frac{1}{z} \sum_{n=0}^{\infty} \left[ H F_n(x, y; z) - H_n(x, y) \right] \frac{t^n}{n!}.$$
Comparing the coefficients of \( \frac{t^n}{n!} \) on both sides, we obtain (2.10).

**Remark 2.3.** On setting \( x = y = 0 \) and \( x = -1 \) in Theorem 2.4, we find
\[
z_H F_n(1,0; z) = (1 + z) H F_n(0,0; z),
\]
and
\[
z_H F_n(0,0; z) = (1 + z) H F_n(-1,0; z) - (-1)^n.
\]

**Theorem 2.5.** For \( n \geq 0, p, q \in \mathbb{R} \), the following formula for Hermite-Fubini polynomials holds true:
\[
H F_n(px, qy; z) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!} \frac{1}{(n-k-2j)!} (p-1)x^k ((q-1)y)^j.
\]

**Proof.** Rewrite the generating function (2.1), we have
\[
\sum_{n=0}^{\infty} H F_n(px, qy; z) \frac{t^n}{n!} = \frac{1}{1-z(e^t-1)} e^{xt+y^2} e^{(p-1)xt} e^{(q-1)y^2}.
\]
Replacing \( k \) by \( k-2j \) in above equation, we have
\[
L.H.S. = \sum_{n=0}^{\infty} H F_n(x, y; z) \frac{t^n}{n!} \sum_{k=2j}^{\infty} ((p-1)x)^k ((q-1)y)^j \frac{t^k}{(k-2j)!j!}.
\]
Again replacing \( n \) by \( n-k \) in above equation, we have
\[
L.H.S. = \sum_{n=0}^{\infty} \sum_{k=2j}^{\infty} H F_n-k(x, y; z) ((p-1)x)^k ((q-1)y)^j \frac{t^n}{(n-k-2j)!j!k!}.
\]
Finally, equating the coefficients of \( t^n \) on both sides, we acquire the result (2.13).

**Theorem 2.6.** For \( n \geq 0 \), the following formula for Hermite-Fubini polynomials holds true:
\[
H F_n(x, y; z) = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(x, y) \sum_{k=0}^{l} z^k k! S_2(l, k).
\]

**Proof.** From (2.1), we have
\[
\sum_{n=0}^{\infty} H F_n(x, y; z) \frac{t^n}{n!} = \frac{e^{xt+y^2}}{1-z(e^t-1)} e^{(p-1)xt} e^{(q-1)y^2} \sum_{k=0}^{\infty} k! S_2(l, k) \frac{t^l}{l!}.
\]
Replacing $n$ by $n - l$ in above equation, we get
\[
\sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(x, y) \sum_{k=0}^{l} z^k l! S_2(l, k) \right) \frac{t^n}{n!}.
\]

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.14).

**Theorem 2.7.** For $n \geq 0$, the following formula for Hermite-Fubini polynomials holds true:
\[
h_F_n(x + r, y, z) = \sum_{l=0}^{n} \binom{n}{l} H_n(l, y) \sum_{k=0}^{l} z^k l! S_2(l, k). 
\tag{2.15}
\]

**Proof.** Replacing $x$ by $x + r$ in (2.1), we have
\[
\sum_{n=0}^{\infty} H_n(x + r, y, z) \frac{t^n}{n!} = \frac{e^{(x+r)t + yt^2}}{1 - z(e^t - 1)}
\]
\[
= e^{xt + yt} e^{rt} \sum_{k=0}^{\infty} z^k (e^t - 1)^k = e^{xt + yt} e^{rt} \sum_{k=0}^{\infty} z^k \sum_{l=k}^{\infty} l! S_2(l, k) \frac{t^l}{l!}
\]
\[
= \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} \sum_{l=0}^{\infty} z^l \sum_{k=0}^{l} k! S_2(l, k) \frac{t^l}{l!}
\]

Replacing $n$ by $n - l$ in above equation, we get
\[
\sum_{n=0}^{\infty} H_n(x + r, y, z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(x, y) \sum_{k=0}^{l} z^k l! S_2(l, k) \right) \frac{t^n}{n!}.
\]

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.15).

### 3. Summation Formulas for Hermite-Fubini polynomials

First, we prove the following result involving the Hermite-Fubini polynomials $h_F_n(x, y; z)$ by using series rearrangement techniques and considered its special case:

**Theorem 3.1.** The following summation formula for Hermite-Fubini polynomials $h_F_n(x, y; z)$ holds true:
\[
h_F_{q+l}(w, y; z) = \sum_{n=0}^{q \cdot l} \binom{n}{l} \binom{l}{p} (w - y)^{n+p} h_F_{q+l-n-p}(x, y; z). 
\tag{3.1}
\]

**Proof.** Replacing $t$ by $t + u$ in (2.1) and then using the formula [11,p.52(2)]:
\[
\sum_{N=0}^{\infty} f(N) \frac{(x + y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n + m) \frac{x^n y^m}{n! m!},
\tag{3.2}
\]
in the resultant equation, we find the following generating function for the Hermite-Fubini polynomials $h_F_n(x, y; z)$:
\[
\frac{1}{1 - z(e^t + u - 1)} e^{y(t + u)^2} = e^{-z(t + u)^2} \sum_{q,l=0}^{\infty} H_{q+l}(x, y; z) \frac{t^q u^l}{q! l!}. 
\tag{3.3}
\]
Replacing $x$ by $w$ in the above equation and equating the resultant equation to the above equation, we find
\[
\exp((w - x)(t + u)) \sum_{q,l=0}^{\infty} H_{q+l}(x, y; z) \frac{t^q u^l}{q! l!} = \sum_{q,l=0}^{\infty} H_{q+l}(w, y; z) \frac{t^q u^l}{q! l!}.
\]

(3.4)

On expanding exponential function (3.4) gives
\[
\sum_{N=0}^{\infty} \frac{[(w - x)(t + u)]^N}{N!} \sum_{q,l=0}^{\infty} H_{q+l}(x, y; z) \frac{t^q u^l}{q! l!} = \sum_{q,l=0}^{\infty} H_{q+l}(w, y; z) \frac{t^q u^l}{q! l!},
\]

(3.5)

which on using formula (3.2) in the first summation on the left hand side becomes
\[
\sum_{N=0}^{\infty} \sum_{n,p=0}^{\infty} \frac{(w - x)^{n+p} u^p}{n! p!} \sum_{q,l=0}^{\infty} H_{q+l}(x, y; z) \frac{t^q u^l}{q! l!} = \sum_{q,l=0}^{\infty} H_{q+l}(w, y; z) \frac{t^q u^l}{q! l!}.
\]

(3.6)

Now replacing $q$ by $q - n$, $l$ by $l - p$ and using the lemma ([11, p.100(1)]):
\[
\sum_{k=0}^{\infty} \sum_{n=0}^{k} A(n, k) = \sum_{k=0}^{\infty} \sum_{n=0}^{k} A(n, k - n),
\]

(3.7)

in the l.h.s. of (3.6), we find
\[
\sum_{q,l=0}^{\infty} \sum_{n,p=0}^{q,l} \frac{(w - x)^{n+p}}{n! p!} H_{q+l-n-p}(x, y; z) \frac{t^q u^l}{(q - n)! (l - p)!} = \sum_{q,l=0}^{\infty} H_{q+l}(w, y; z) \frac{t^q u^l}{q! l!}.
\]

(3.8)

Finally, on equating the coefficients of the like powers of $t$ and $u$ in the above equation, we get the assertion (3.1) of Theorem 3.1.

Remark 3.1. Taking $l = 0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

Corollary 3.1. The following summation formula for Hermite-Fubini polynomials $H_{F_n}(x, y; z)$ holds true:
\[
H_{F_q}(w, y; z) = \sum_{n=0}^{q} \binom{q}{n} (w - x)^n H_{F_{q-n}}(x, y; z).
\]

(3.9)

Remark 3.2. Replacing $w$ by $w + x$ in (3.9), we obtain
\[
H_{F_q}(x + w, y; z) = \sum_{n=0}^{q} \binom{q}{n} w^n H_{F_{q-n}}(x, y; z).
\]

(3.10)

Theorem 3.2. The following summation formula for Hermite-Fubini polynomials $H_{F_n}(x, y; z)$ holds true:
\[
H_{F_n}(w, u; z) H_{F_m}(W, U; Z) = \sum_{r,k=0}^{n,m} \binom{n}{r} \binom{m}{k} H_{r}(w - x, u - y; z) H_{F_{n-r}}(x, y; z)
\times H_{k}(W - X, U - Y) H_{F_{m-k}}(X, Y; Z).
\]

(3.11)
\[ \frac{1}{1 - z(e^t - 1)} e^{xt + yt^2} = \frac{1}{1 - Z(e^t - 1)} e^{XT + YT^2} = \sum_{n=0}^{\infty} H_F_n(x, y; z) \frac{t^n}{n!} \sum_{m=0}^{\infty} H_F_m(X, Y; Z) \frac{T^m}{m!}. \]

(3.12)

Replacing \( x \) by \( w, y \) by \( u \), \( X \) by \( W \) and \( Y \) by \( U \) in (3.12) and equating the resultant to itself,

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_F_n(w, u; z) H_F_m(W, U; Z) \frac{t^n T^m}{n! m!} = \exp ((w - x)t + (u - y)t^2) \exp ((W - X)T + (U - Y)T^2) \times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_F_n(x, y; z) H_F_m(X, Y; Z) \frac{t^n T^m}{n! m!}, \]

which on using the generating function (3.7) in the exponential on the r.h.s., becomes

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_F_n(w, u; z) H_F_m(W, U; Z) \frac{t^n T^m}{n! m!} = \sum_{n, r=0}^{\infty} H_r(w-x, u-y) H_F_n(x, y; z) \frac{t^{n+r}}{n! r!} \sum_{m, k=0}^{\infty} H_k(W - X, U - Y) H_F_m(X, Y; Z) \frac{T^{m+k}}{m! k!}. \]

(3.13)

Finally, replacing \( n \) by \( n - r \) and \( m \) by \( m - k \) and using equation (3.7) in the r.h.s. of the above equation and then equating the coefficients of like powers of \( t \) and \( T \), we get assertion (3.11) of Theorem 3.2.

\[ \text{Remark 3.3.} \ \text{Replacing } u \text{ by } y \text{ and } U \text{ by } Y \text{ in assertion (3.11) of Theorem 3.2, we deduce the the following consequence of Theorem 3.2.} \]

\[ \text{Corollary 3.2.} \ \text{The following summation formula for } H_F_n(x, y; z) \text{ holds true:} \]

\[ H_F_n(w, u; z) H_F_m(W, U; Z) = \sum_{r, k=0}^{n, m} \binom{n}{r} \binom{m}{k} (w - x)^r H_F_{n-r}(x, y; z) \]

\[ \times (W - X)^k H_F_{m-k}(X, Y; Z). \]

(3.14)

\[ \text{Theorem 3.3.} \ \text{The following summation formula for } H_F_n(x, y; z) \text{ holds true:} \]

\[ H_F_n(x + w, y + u; z) = \sum_{s=0}^{n} \binom{n}{s} H_F_{n-s}(x, y; z) H_s(w, u). \]

(3.15)

\[ \text{Proof.} \ \text{We replace } x \text{ by } x + w \text{ and } y \text{ by } y + u \text{ in (2.1), use (1.2) and rewrite the generating function as:} \]

\[ \frac{1}{1 - z(e^t - 1)} \exp((x + w)t + (y + u)t^2) = \sum_{n=0}^{\infty} H_F_n(x, y; z) \frac{t^n}{n!} \sum_{s=0}^{\infty} H_s(w, u) \frac{t^s}{s!} \]

\[ = \sum_{n=0}^{\infty} H_F_n(x + w, y + u; z) \frac{t^n}{n!}. \]

(3.16)
Now replacing \( n \) by \( n - s \) in l.h.s. and comparing the coefficients of \( t^n \) on both sides, we get the result (3.15).

**Theorem 3.4.** The following summation formula for Hermite-Fubini polynomials \( H_F_n(x, y; z) \) holds true:

\[
H_F_n(y, x; z) = \sum_{s=0}^{[\frac{n}{2}]} F_{n-2s}(y; z) \frac{x^s}{(n-2s)!s!}.
\]  

(3.17)

**Proof.** We replace \( x \) by \( y \) and \( y \) by \( x \) in equation (2.1) to get

\[
\sum_{n=0}^{\infty} H_F_n(y, x; z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} F_n(y; z) \frac{t^n}{n!} \sum_{s=0}^{\infty} x^s t^{2s}.
\]  

(3.18)

Now replacing \( n \) by \( n - 2s \) in r.h.s. and comparing the coefficients of \( t \) on both sides, we arrive at the desired result (3.16).

**Theorem 3.5.** The following summation formula for Hermite-Fubini polynomials \( H_F_n(x, y; z) \) holds true:

\[
H_F_n(x, y; z) = \sum_{r=0}^{n} \binom{n}{r} F_{n-r}(x - w; z) H_r(w, y).
\]  

(3.19)

**Proof.** By exploiting the generating function (1.2), we can write equation (2.1) as

\[
\frac{1}{1-z(e^t-1)} e^{(x-w)t} e^{yt+yz^2} = \sum_{n=0}^{\infty} F_n(x-w; z) \frac{t^n}{n!} \sum_{r=0}^{\infty} H_r(w, y) \frac{t^r}{r!}.
\]  

(3.20)

On replacing \( n \) by \( n - r \) in above equation, we get

\[
\sum_{n=0}^{\infty} H_F_n(x, y; z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} F_{n-r}(x - w; z) H_r(w, y) \frac{t^n}{(n-r)!r!}.
\]

Equating the coefficients of the like powers of \( t \) on both sides, we get (3.19).

**Theorem 3.6.** The following summation formula for Hermite-Fubini polynomials \( H_F_n(x, y; z) \) holds true:

\[
H_F_n(x + 1, y; z) = \sum_{r=0}^{n} \binom{n}{r} H_F_{n-r}(x, y; z).
\]  

(3.21)

**Proof.** Using the generating function (2.1), we have

\[
\sum_{n=0}^{\infty} H_F_n(x + 1, y; z) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_F_n(x, y; z) \frac{t^n}{n!} + \frac{1}{1-z(e^t-1)} e^{xt+yt^2}
\]

\[
= \sum_{n=0}^{\infty} H_F_n(x, y; z) \frac{t^n}{n!} \left( \sum_{r=0}^{\infty} \frac{t^r}{r!} - 1 \right)
\]

\[
= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} H_F_{n-r}(x, y; z) \frac{t^n}{n!} \frac{t^r}{r!} - \sum_{n=0}^{\infty} H_F_n(x, y; z) \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} H_F_{n-r}(x, y; z) \frac{t^n}{n!} - \sum_{n=0}^{\infty} H_F_n(x, y; z) \frac{t^n}{n!}.
\]
Finally, equating the coefficients of the like powers of $t$ on both sides, we get (3.21).

## 4. Symmetric identities

Recently, Khan [7], Pathan and Khan [9, 10] have been introduced symmetric identities. In this section, we establish general symmetry identities for the generalized Hermite-Fubini polynomials $HF_n(x, y; z)$ by applying the generating function (2.1) and (2.2).

### Theorem 4.1

Let $x, y, z \in \mathbb{R}$ and $n \geq 0$, then the following identity holds true:

$$
\sum_{r=0}^{n} \binom{n}{r} b^r a^{n-r} HF_{n-r}(bx, b^2y; z) HF_r(ax, a^2y; z) = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} HF_{n-r}(ax, a^2y; z) HF_r(bx, b^2y; z). 
$$  

\[ (4.1) \]

**Proof.** Start with

$$
A(t) = \frac{1}{(1 - z(e^{at} - 1))(1 - z(e^{bt} - 1))} e^{abxt + a^2bt^2y^2}.
$$

Then the expression for $A(t)$ is symmetric in $a$ and $b$ and we can expand $A(t)$ into series in two ways to obtain:

$$
A(t) = \sum_{n=0}^{\infty} HF_n(ax, a^2y; z) \frac{(at)^n}{n!} \sum_{r=0}^{\infty} HF_r(bx, b^2y; z) \frac{(bt)^r}{r!}
$$

\[ (4.2) \]

Similarly, we can show that

$$
A(t) = \sum_{n=0}^{\infty} HF_n(ax, a^2y; z) \frac{(bt)^n}{n!} \sum_{r=0}^{\infty} HF_r(bx, b^2y; z) \frac{(at)^r}{r!}
$$

\[ (4.3) \]

By comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result (4.1).

### Theorem 4.2

For each pair of integers $a$ and $b$ and all integers and $n \geq 0$, the following identity holds true:

$$
\sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^{n-k-i} b^k HF_{n-k} \left( bx + \frac{b}{a} i + j, b^2y, z \right) F_k(au, z)
$$

$$
\sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{a-1} \sum_{i=0}^{b-1} b^{n-k-j} a^k HF_{n-k} \left( ax + \frac{a}{b} i + j, a^2y, z \right) F_k(bu, z).
$$  

\[ (4.4) \]

**Proof.** Let

$$
B(t) = \frac{e^{ab(x+u)+a^2y^2t^2}(e^{abt} - 1)^2}{(1 - z(e^{at} - 1))(1 - z(e^{bt} - 1))(e^{at} - 1)(e^{bt} - 1)}
$$

...
\[ B(t) = \frac{e^{abxt} + a^2 b^2 y t^2}{1 - z(e^{at} - 1)} e^{abt} - 1 \]

\[ \frac{e^{abt}}{1 - z(e^{bt} - 1)} \sum_{i=0}^{b-1} e^{abt i} \frac{b^i}{i!} \sum_{j=0}^{a} \sum_{k=0}^{b} F_k(au, z) \left( \frac{b t^j}{j!} \right) \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} a^{n-k} b^k H_{n-k} \left( bx + \frac{b}{a} i + j, b^2 y, z \right) F_k(au, z) \right) \frac{t^n}{n!} \quad (4.5) \]

On the other hand

\[ B(t) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} b^{n-k} a^k H_{n-k} \left( ax + \frac{a}{b} i + j, a^2 y, z \right) F_k(bu, z) \right) \frac{t^n}{n!} \quad (4.6) \]

By comparing the coefficients of \( t^n \) on the right hand sides of the last two equations, we arrive at the desired result.

**References**