Some properties of $q$-Hermite Fubini numbers and polynomials

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Abstract. The main purpose of this paper is to introduce a new class of $q$-Hermite-Fubini numbers and polynomials by combining the $q$-Hermite polynomials and $q$-Fubini polynomials. By using generating functions for these numbers and polynomials, we derive some alternative summation formulas including powers of consecutive $q$-integers. Also, we establish some relationships for $q$-Hermite-Fubini polynomials associated with $q$-Bernoulli polynomials, $q$-Euler polynomials and $q$-Genocchi polynomials and $q$-Stirling numbers of the second kind.

Keywords: $q$-Hermite polynomials, $q$-Hermite-Fubini polynomials, $q$-Bernoulli polynomials, $q$-Euler polynomials, $q$-Genocchi polynomials, Stirling numbers of the second kind.

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1. Introduction

The subject of $q$-calculus started appearing in the nineteenth century due to its applications in various fields of mathematics, physics and engineering. The definitions and notations of $q$-calculus reviewed here are taken from (see [1]):

The $q$-analogue of the shifted factorial $(a)_n$ is given by

$$(a; q)_0 = 1, (a; q)_n = \prod_{m=0}^{n-1} (1 - q^m a), n \in \mathbb{N}.$$ 

The $q$-analogue of a complex number $a$ and of the factorial function are given by

$$[a]_q = \frac{1 - q^a}{1 - q}, q \in \mathbb{C} - \{1\}; a \in \mathbb{C},$$

$$[n]_q! = \prod_{m=1}^{n} [m]_q = [1]_q[2]_q \cdots [n]_q = \frac{(q; q)_n}{(1 - q)_n}, q \neq 1; n \in \mathbb{N},$$

$$[0]_q! = 1, q \in \mathbb{C}; 0 < q < 1.$$ 

The Gauss $q$-binomial coefficient $\binom{n}{k}_q$ is given by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, k = 0, 1, \ldots, n.$$ 

The $q$-analogue of the function $(x + y)^n_q$ is given by

$$(x + y)^n_q = \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-1)/2} x^{n-k} y^k, n \in \mathbb{N}_0. \quad (1.1)$$

The $q$-analogue of exponential function are given by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \lim_{q \to 1} \frac{1}{((1 - q)x; q)_\infty}, 0 < |q| < 1; |x| < 1 - q^{-1}. \quad (1.2)$$
\[ E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (-1 - q)x; q) \infty, 0 < q < 1; x \in \mathbb{C}. \]  

(1.3)

Moreover, the functions \( e_q(x) \) and \( E_q(x) \) satisfy the following properties:

\[ D_q e_q(x) = e_q(x), \quad D_q E_q(x) = E_q(qx), \]

where the \( q \)-derivative \( D_q f \) of a function \( f \) at a point \( 0 \neq z \in \mathbb{C} \) is defined as follows:

\[ D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < q < 1. \]

For any two arbitrary functions \( f(z) \) and \( g(z) \), the \( q \)-derivative operator \( D_q \) satisfies the following product and quotient relations:

\[
D_{q,z} (f(z)g(z)) = f(z)D_{q,z} g(z) + g(qz)D_{q,z} f(z), \tag{1.5}
\]

\[
D_{q,z} \left( \frac{f(z)}{g(z)} \right) = \frac{g(qz)D_{q,z} f(z) - f(qz)D_{q,z} g(z)}{g(z)g(qz)}. \]

The \( q \)-Hermite polynomials are special or limiting case of the orthogonal polynomials as they contain no parameter other than \( q \) and appears to be at the bottom of a hierarchy of the classical polynomials [2]. The \( q \)-Hermite polynomials constitute a 1-parameter family of orthogonal polynomials, which for \( q = 1 \) reduce to the well known Hermite polynomials. We recall that the \( q \)-Hermite polynomials \( H_{n,q}(x) \) is defined by means of the following generating function (see [9]):

\[
F_q(x, t) = F_q(t)e_q(xt) = \sum_{n=0}^{\infty} H_{n,q}(x) \frac{t^n}{[n]_q!, \tag{1.6}}
\]

\[
F_q(t) = \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} \frac{t^n}{[n]_q!!} [2n]_q!! = [2n][2n - 2] \cdots [2],
\]

The \( q \)-Bernoulli polynomials \( B^{(\alpha)}_{n,q}(x, y) \) of order \( \alpha \), the \( q \)-Euler polynomials \( E^{(\alpha)}_{n,q}(x, y) \) of order \( \alpha \) and the \( q \)-Genocchi polynomials \( G^{(\alpha)}_{n,q}(x, y) \) of order \( \alpha \) are defined by means of the following generating function (see [1-2, 8-11]):

\[
\left( \frac{t}{e_q(t) - 1} \right)^\alpha e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} B^{(\alpha)}_{n,q}(x, y) \frac{t^n}{[n]_q!}, \quad |t| < 2\pi, 1^\alpha = 1, \tag{1.7}
\]

\[
\left( \frac{2}{e_q(t) + 1} \right)^\alpha e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} E^{(\alpha)}_{n,q}(x, y; \lambda) \frac{t^n}{[n]_q!}, \quad |t| < \pi, 1^\alpha = 1, \tag{1.8}
\]

\[
\left( \frac{2t}{e_q(t) + 1} \right)^\alpha e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} G^{(\alpha)}_{n,q}(x, y) \frac{t^n}{[n]_q!}, \quad |t| < \pi, 1^\alpha = 1. \tag{1.9}
\]

Clearly, we have

\[ B^{(\alpha)}_{n,q} = B^{(\alpha)}_{n,q}(0, 0), \quad E^{(\alpha)}_{n,q} = E^{(\alpha)}_{n,q}, \quad G^{(\alpha)}_{n,q} = G^{(\alpha)}_{n,q}. \]

Geometric polynomials (also known as Fubini polynomials) are defined as follows (see [3]):

\[
F_n(x) = \sum_{k=0}^{n} \binom{n}{k} k! x^k, \tag{1.10}
\]
where \( \left\{ \begin{array}{c} n \\ k \end{array} \right\} \) is the Stirling number of the second kind (see [5]).

For \( x = 1 \) in (1.10), we get \( n \)th Fubini number (ordered Bell number or geometric number) \( F_n \) [4, 6, 7, 13] is defined by

\[
F_n(1) = F_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!.
\]

(1.11)

The exponential generating functions of geometric polynomials is given by (see [3]):

\[
\frac{1}{1-x(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!},
\]

(1.12)

and related to the geometric series (see [3]):

\[
\left( x \frac{d}{dx} \right)^m \frac{1}{1-x} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1-x} F_m(x) \left( \frac{x}{1-x} \right), |x| < 1.
\]

Let us give a short list of these polynomials and numbers as follows:

\( F_0(x) = 1, F_1(x) = x, F_2(x) = x^2 + 2x^2, F_3(x) = x^3 + 6x^2 + 6x, F_4(x) = x^4 + 14x^3 + 36x^2 + 24x^3 + 24 \),

and \( F_0 = 1, F_1 = 1, F_2 = 3, F_3 = 13, F_4 = 75 \).

Geometric and exponential polynomials are connected by the relation (see [3]):

\[
F_n(x) = \int_{0}^{\infty} \phi_n(x)e^{-\lambda}d\lambda.
\]

(1.13)

The manuscript of this paper as follows: In section 2, we consider generating functions for \( q \)-Hermite-Fubini numbers and polynomials and give some properties of these numbers and polynomials. In section 3, we derive summation formulas of \( q \)-Hermite-Fubini numbers and polynomials and some relationships between \( q \)-Bernoulli polynomials, \( q \)-Euler polynomials and \( q \)-Genocchi polynomials and Stirling numbers of the second kind.

2. A \( q \)-analogue type of Hermite-Fubini numbers and polynomials

In this section, we define \( q \)-analogue type of Hermite-Fubini polynomials and obtain some basic properties which gives us new formula for \( H_{F_n,q}(x; y) \). Moreover, we shall consider the sum of products of two \( q \)-analogue type of Hermite-Fubini polynomials. The sum of products of various polynomials and numbers with or without binomial coefficients have been studied by (see [4, 6, 7, 13]):

We introduce \( q \)-Hermite-based Fubini polynomials in two variables by means of the following generating function:

\[
\frac{1}{1 - y(e_q(t) - 1)} F_q(t)e_q(xt) = \sum_{n=0}^{\infty} H_{F_n,q}(x; y) \frac{t^n}{[n]_q!}.
\]

(2.1)

Taking \( x = 0, y = 1 \) in (2.1), we get

\( H_{F_n,q}(0; 1) = H_{F_n,q} \),

where \( H_{F_n,q} \) are the \( q \)-Hermite-based Fubini numbers.

When investigating the connection between \( q \)-Hermite polynomials \( H_{n,q}(x) \) and \( q \)-Fubini polynomials \( F_{n,q}(y) \) of importance is the following theorem.
Theorem 2.1. The following formula for $q$-Hermite-based Fubini polynomials holds true:

$$HF_{n,q}(x; y) = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) q_{n-m,q}(y)H_{m,q}(x). \tag{2.2}$$

$$HF_{n,q}(x; y) = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) HF_{m,q}(y)x^{n-m}. \tag{2.3}$$

Proof. Using definition (2.1), we have

$$\sum_{n=0}^{\infty} HF_{n,q}(x; y) \frac{t^n}{[n]_q!} = \frac{1}{1-y(e_q(t)-1)} F_q(t)e_q(xt)$$

$$= \sum_{n=0}^{\infty} F_{n,q}(y) \frac{t^n}{[n]_q!} \sum_{m=0}^{\infty} H_{m,q}(x) \frac{t^m}{[m]_q!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) q_{n-m,q}(y)H_{m,q}(x) \frac{t^n}{[n]_q!}.$$ 

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ yields (2.2).

Utilizing equation (1.6) in the l.h.s. of generating function (2.1), it follows that

$$\sum_{n=0}^{\infty} HF_{n,q}(x; y) \frac{t^n}{[n]_q!} = \sum_{m=0}^{\infty} HF_{n,q}(y) \frac{t^n}{[m]_q!} \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!},$$

which on applying the Cauchy product rule in the l.h.s. and then comparing the coefficients of same powers of $t$ in both sides of resultant equation yield assertion (2.3).

Proposition 2.1. The following formula for $q$-Hermite-based Fubini polynomials holds true:

$$D_{q,t}e_q(xt) = xe_q(xt)$$

$$D_{q,x} (HF_{n,q}(x; y)) = [n]_q HF_{n-1,q}(x; y). \tag{2.4}$$

Theorem 2.2. For $n \geq 0$, the following formula for $q$-Hermite-based Fubini polynomials holds true:

$$H_{n,q}(x) = HF_{n,q}(x; y) - yHF_{n,q}(x+1; y) + yHF_{n,q}(x; y). \tag{2.5}$$

Proof. We begin with the definition (2.1) and write

$$F_q(t)e_q(xt) = \frac{1-y(e_q(t)-1)}{1-y(e_q(t)-1)} F_q(t)e_q(xt)$$

$$= \frac{1-y(e_q(t)-1)}{1-y(e_q(t)-1)} F_q(t)e_q(xt).$$

Then using the definition of $q$-Hermite polynomials $H_{n,q}(x)$ and (2.1), we have

$$\sum_{n=0}^{\infty} H_{n,q}(x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} [HF_{n,q}(x; y) - yHF_{n,q}(x+1; y) + yHF_{n,q}(x; y)] \frac{t^n}{[n]_q!}.$$ 

Finally, comparing the coefficients of $\frac{t^n}{m!}$, we get (2.5).

\qed
From (2.1), we have
\[ q \text{ case. Also we obtain some relationships for } H \text{ Bernoulli polynomials, } \text{Euler polynomials and } \text{Genocchi polynomials and Stirling numbers of the second kind in Theorems 4.1, 4.2, 4.3, 4.4, 4.5.} \]

\[ \sum_{k=0}^{n} \binom{n}{k} q F_{n-k,q}(x_1; y_1) H_{k,q}(x_2; y_2) = y_2 H_{n,q}(x_1 + x_2; y_1) - y_1 H_{n,q}(x_1 + x_2; y_2), \]  
\quad (2.6)

**Proof.** The products of (2.1) can be written as
\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} H_{n,q}(x_1; y_1) t^n \frac{F_{k}(x_2; y_2)}{[k]_q!} = \frac{F_q(t)e_q(x_1 t) - F_q(t)e_q((x_1 + x_2) t)}{1 - y_1 (e_q(t) - 1)} 1 - y_2 (e_q(t) - 1) \]
\[ = \frac{y_2}{y_2 - y_1} \frac{F_q(t)e_q((x_1 + x_2) t)}{1 - y_1 (e_q(t) - 1)} - \frac{y_1}{y_2 - y_1} \frac{F_q(t)e_q((x_1 + x_2) t)}{1 - y_2 (e_q(t) - 1)} \]
\[ = \left(y_2 H_{n,q}(x_1 + x_2; y_1) - y_1 H_{n,q}(x_1 + x_2; y_2)\right) \frac{t^n}{[n]_q!}. \]

By equating the coefficients of \(\frac{t^n}{[n]_q!}\) on both sides, we get (2.6). \(\square\)

**Theorem 2.4.** For \(n \geq 0\), the following formula for \(q\)-Hermite-based Fubini polynomials holds true:
\[ y H_{n,q}(x + 1; y) = (1 + y) H_{n,q}(x; y) - H_{n,q}(x). \]  
\quad (2.7)

**Proof.** From (2.1), we have
\[ \sum_{n=0}^{\infty} [y H_{n,q}(x + 1; y) - H_{n,q}(x; y)] \frac{t^n}{[n]_q!} = \frac{F_q(t)e_q(x t)}{1 - y (e_q(t) - 1)} (e_q(t) - 1) \]
\[ = \frac{1}{y} \left[ \frac{F_q(t)e_q(x t)}{1 - y (e_q(t) - 1)} - F_q(t)e_q(x t) \right] \]
\[ = \frac{1}{y} \sum_{n=0}^{\infty} [y H_{n,q}(x; y) - H_{n,q}(x)] \frac{t^n}{[n]_q!}. \]

Comparing the coefficients of \(\frac{t^n}{[n]_q!}\) on both sides, we obtain (2.7). \(\square\)

**Remark 2.1.** On setting \(x = 0\) and \(x = -1\) in Theorem 2.4, we find
\[ y H_{n,q}(1; y) = (1 + y) H_{n,q}(y) - (-1)^n q^{n(n-1)/2} \frac{[n]_q}{[2n]_q!}, \]  
\quad (2.8)

and
\[ y H_{n,q}(-1; y) = (1 + y) H_{n,q}(-1; y) - (-1)^n q^{n(n-1)/2} \frac{[n]_q}{[2n]_q!}. \]  
\quad (2.9)

### 3. Main results

In this section, we prove the following result involving \(q\)-Hermite-Fubini polynomials \(H_{n,q}(x; y)\) by using series rearrangement techniques and considered its special case. Also we obtain some relationships for \(q\)-Hermite Fubini polynomials related to \(q\)-Bernoulli polynomials, \(q\)-Euler polynomials and \(q\)-Genocchi polynomials and Stirling numbers of the second kind in Theorems 4.1, 4.2, 4.3, 4.4, 4.5.
Theorem 3.1. The following formula for \( q \)-Hermite-based Fubini polynomials holds true:

\[
H F_{k+l,q}(w; y) = \sum_{n,p=0}^{k,l} \binom{k}{n} \binom{l}{p} q^n (w-y)^n p^n H F_{k+l-n-p,q}(x; y).
\]

(3.1)

Proof. Replacing \( t \) by \( t + u \) in (2.1) and then using the formula [12, p.52(2)]:

\[
\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m) \frac{x^n y^m}{n! m!},
\]

(3.2)

in the resultant equation, we find the following generating function for the Hermite-Fubini polynomials \( H F_n(x; y; z) \):

\[
\frac{1}{1 - y e_q(t + u) - 1} F_q(t + u) = e_q(-x(t+u)) \sum_{k,l=0}^{\infty} H F_{k+l,q}(x; y) \frac{t^k u^l}{[k]_q! [l]_q!).
\]

(3.3)

Replacing \( x \) by \( w \) in the above equation and equating the resultant equation to the above equation, we find

\[
e_q((w-x) (t+u)) \sum_{k,l=0}^{\infty} H F_{k+l,q}(x; y) \frac{t^k u^l}{[k]_q! [l]_q!) = \sum_{k,l=0}^{\infty} H F_{k+l,q}(w; y) \frac{t^k u^l}{[k]_q! [l]_q!).
\]

(3.4)

On expanding exponential function (3.4) gives

\[
\sum_{N=0}^{\infty} \frac{[(w-x)(t+u)]^N}{[N]_q!} \sum_{k,l=0}^{\infty} H F_{k+l,q}(x; y) \frac{t^k u^l}{[k]_q! [l]_q!) = \sum_{k,l=0}^{\infty} H F_{k+l,q}(w; y) \frac{t^k u^l}{[k]_q! [l]_q!).
\]

(3.5)

which on using formula (3.2) in the first summation on the left hand side becomes

\[
\sum_{n,p=0}^{\infty} \frac{(w-x)^n t^p u^p}{[n]_q! [p]_q!} \sum_{k,l=0}^{\infty} H F_{k+l,q}(x; y) \frac{t^k u^l}{[k]_q! [l]_q!) = \sum_{k,l=0}^{\infty} H F_{k+l,q}(w; y) \frac{t^k u^l}{[k]_q! [l]_q!).
\]

(3.6)

Now replacing \( q \) by \( q-n \), \( l \) by \( l-p \) and using the lemma ([12, p.100(1)]):

\[
\sum_{k=0}^{\infty} \sum_{n=0}^{k} A(n,k) = \sum_{k=0}^{\infty} \sum_{n=0}^{k} A(n,k-n),
\]

(3.7)

in the l.h.s. of (3.6), we find

\[
\sum_{k,l=0}^{\infty} \sum_{n,p=0}^{k,l} \frac{(w-x)^n t^p u^p}{[n]_q! [p]_q!} H F_{k+l-n-p,q}(x; y) \frac{t^k u^l}{(k-n)_q! (l-p)_q!} = \sum_{q,l=0}^{\infty} H F_{q+l}(w; y; z) \frac{t^q u^l}{q! l!}.
\]

(3.8)

Finally, on equating the coefficients of the like powers of \( t \) and \( u \) in the above equation, we get the assertion (3.1) of Theorem 3.1.

\[\square\]

Remark 3.1. Taking \( l = 0 \) in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.
Corollary 3.1. The following summation formula for Hermite-Fubini polynomials $H_{F_n}(x, y; z)$ holds true:

$$H_{F_k}(w; y) = \sum_{n=0}^{k} \binom{k}{n} q^{n+k} H_{F_{k-n,q}}(x, y).$$  \hspace{1cm} (3.9)

Remark 3.2. Replacing $w$ by $w + x$ in (3.9), we obtain

$$H_{F_q}(x + w; y) = \sum_{n=0}^{k} \binom{k}{n} w^n H_{F_{k-n,q}}(x, y).$$ \hspace{1cm} (3.10)

Theorem 3.2. The following formula for $q$-Hermite-based Fubini polynomials holds true:

$$H_{F_n}(x + 1; y) = \sum_{r=0}^{n} \binom{n}{r} H_{F_{n-r,q}}(x; y).$$ \hspace{1cm} (3.11)

Proof. Using the generating function (2.1), we have

$$\sum_{n=0}^{\infty} H_{F_n,q}(x + 1; y) \frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} H_{F_n,q}(x; y) \frac{t^n}{[n]_q!} = \left( \frac{1}{1 - y e_q(t) - 1} \right) (e_q(t) - 1) F_q(t) e_q(x t)$$

$$= \sum_{n=0}^{\infty} H_{F_n,q}(x; y) \frac{t^n}{[n]_q!} \left( \sum_{r=0}^{\infty} \frac{t^r}{[r]_q!} - 1 \right)$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} H_{F_{n-r,q}}(x; y) \frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} H_{F_n,q}(x; y) \frac{t^n}{[n]_q!}.$$

Finally, equating the coefficients of the like powers of $t$ on both sides, we get (3.11). \qed

Theorem 3.3. Each of the following relationships holds true:

$$H_{F_n,q}(x; y)$$

$$= \sum_{s=0}^{n+1} \binom{n+1}{s} q \sum_{k=0}^{s} \binom{s}{k} B_{s-k,q}(x) - B_{s,q}(x) \frac{H_{F_{k+n-s,q}}(y)}{[n+1]_q},$$ \hspace{1cm} (3.12)

where $B_{n,q}(x)$ is $q$-Bernoulli polynomials.
Proof. By using definition (2.1), we have
\[
\left(\frac{1}{1 - y(e_q(t) - 1)}\right) \frac{t}{e_q(t) - 1} F_q(t)e_q(x) = \left(\frac{1}{1 - y(e_q(t) - 1)}\right) \frac{t}{e_q(t) - 1} F_q(t)e_q(x) \\
= \frac{1}{t} \sum_{s=0}^{\infty} \left( \sum_{k=0}^{s} \binom{s}{k} B_{s-k,q}(x) \right) t^s \sum_{n=0}^{\infty} hF_{n,q}(y) \frac{t^n}{[n]_q!} \\
= \frac{1}{t} \sum_{s=0}^{\infty} B_{s,q}(x) \frac{t^s}{[s]_q!} \sum_{n=0}^{\infty} hF_{n,q}(y) \frac{t^n}{[n]_q!} \\
= \frac{1}{t} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \binom{n}{s} \binom{s}{k} B_{s-k,q}(x) hF_{n-s,q}(y) \frac{t^n}{[n]_q!} \\
= \frac{1}{t} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \binom{n}{s} B_{s,q}(x) hF_{n-s,q}(y) \frac{t^n}{[n]_q!}
\]
By using Cauchy product and comparing the coefficients of \(\frac{t^n}{[n]_q!}\), we arrive at the required result (3.12).

Theorem 3.4. Each of the following relationships holds true:
\[
hF_{n,q}(x; y) = \sum_{s=0}^{n} \binom{n}{s} \sum_{k=0}^{s} \binom{s}{k} E_{s-k,q}(x) + E_{s,q}(x) \ \frac{hF_{n-s,q}(y)}{[2]^q}, \quad (3.13)
\]
where \(E_{n,q}(x)\) is the \(q\)-Euler polynomials.

Proof. By using definition (2.1), we have
\[
\left(\frac{1}{1 - y(e_q(t) - 1)}\right) \frac{t}{e_q(t) - 1} F_q(t)e_q(x) \\
= \left(\frac{1}{1 - y(e_q(t) - 1)}\right) \frac{t}{e_q(t) - 1} F_q(t)e_q(x) \\
= \frac{1}{[2]^q} \left[ \sum_{n=0}^{\infty} \binom{n}{k} E_{n-k,q}(x) \right] t^n \left[ \sum_{n=0}^{\infty} hF_{n,q}(y) \frac{t^n}{[n]_q!} \right] \\
\times \sum_{n=0}^{\infty} hF_{n-s,q}(y) \frac{t^n}{[n]_q!} \\
= \frac{1}{[2]^q} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \binom{n}{s} \sum_{k=0}^{s} \binom{s}{k} E_{s-k,q}(x) + \sum_{s=0}^{n} \binom{n}{s} E_s(x; p, q) \\
\times hF_{n-s,q}(y) \frac{t^n}{[n]_q!}
\]
Comparing the coefficients of \(\frac{t^n}{[n]_q!}\), we arrive at the desired result (3.13).

Theorem 3.5. Each of the following relationships holds true:
\[
hF_{n,q}(x; y) = \sum_{s=0}^{n} \binom{n+1}{s} \sum_{k=0}^{s} \binom{s}{k} G_{s-k,q}(x) + G_{s,q}(x) \ \frac{hF_{n-t-s,q}(y)}{[2]^q[n+1]_q}, \quad (3.14)
\]
where \(G_{n,q}(x)\) is the \(q\)-Genocchi polynomials.
Proof. By using definition (2.1), we have
\[
\left( \frac{1}{1 - y(e_q(t) - 1)} \right) F_q(t)e_{p,q}(xt) = \left( \frac{1}{1 - y(e_q(t) - 1)} \right) F_q(t)e_{p,q}(xt) - \frac{1}{2}e_q(t) + 1 \frac{F_q(t) + 1}{2} F_q(t)e_{p,q}(xt)
\]
\[
= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} q G_{n-k,q}(x) t^n \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} \frac{t^n}{n!}
\]
\[
= H_{F_n.q}(x; y) \frac{t^n}{n!},
\]
Comparing the coefficients of \( \frac{t^n}{n!} \), then we have the asserted result (3.14).

Theorem 3.6. For \( n \geq 0 \), the following formula for \( q \)-Hermite-Fubini polynomials holds true:
\[
h_{F_n.q}(x; y) = \sum_{n=0}^{\infty} \binom{n}{l} q H_{n-l,q}(x) \sum_{k=0}^{l} y^k k! S_{2,q}(l, k).
\]

Proof. From (2.1), we have
\[
\sum_{n=0}^{\infty} \frac{H_{F_n.q}(n; y) t^n}{n!} = \frac{F_q(t)e_q(x)}{1 - y(e_q(t) - 1)}
\]
\[
= F_q(t)e_q(x) \sum_{k=0}^{\infty} y^k (e_q(t) - 1)^k = F_q(t)e_q(x) \sum_{k=0}^{\infty} y^k \sum_{l=k}^{\infty} k! S_{2,q}(l, k) \frac{t^l}{l!}
\]
\[
= \sum_{n=0}^{\infty} H_{n,q}(x) \frac{t^n}{n!} \sum_{k=0}^{\infty} y^k \sum_{l=k}^{\infty} k! S_{2,q}(l, k) \frac{t^l}{l!}.
\]
Replacing \( n \) by \( n - l \) in above equation, we get
\[
\sum_{n=0}^{\infty} \frac{H_{F_n.q}(n; y) t^n}{n!} = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} q H_{n-l,q}(x) \sum_{k=0}^{l} y^k k! S_{2,q}(l, k) \frac{t^n}{n!}.
\]
Comparing the coefficients of \( \frac{t^n}{n!} \) in both sides, we get (3.15).

Theorem 3.7. For \( n \geq 0 \), the following formula for \( q \)-Hermite-Fubini polynomials holds true:
\[
h_{F_n.q}(x + r; y) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} q H_{n-l,q}(x) \sum_{k=0}^{l} y^k k! S_{2,q}(l + r, k + r).
\]

Proof. Replacing \( x \) by \( x + r \) in (2.1), we have
\[
\sum_{n=0}^{\infty} \frac{H_{F_n.q}(x + r; y) t^n}{n!} = \frac{F_q(t)e_q(x + r)t}{1 - y(e_q(t) - 1)}
\]
\[
= F_q(t)e_q(x) \sum_{k=0}^{\infty} y^k (e_q(t) - 1)^k = F_q(t)e_q(x) \sum_{k=0}^{\infty} y^k \sum_{l=k}^{\infty} k! S_{2,q}(l, k) \frac{t^l}{l!}.
\]
Replacing \( n \) by \( n - l \) in above equation, we get

\[
\frac{\sum_{n=0}^{\infty} H_{n,q}(x) \frac{t^n}{[n]_q!} \sum_{l=0}^{\infty} y^k \sum_{k=0}^{l} k! S_{2,q}(l + r, k + r) \frac{t^l}{[l]_q!}}{[n]_q!} = \frac{\sum_{n=0}^{\infty} H_{n-1,q}(x) \sum_{l=0}^{\infty} y^k \sum_{k=0}^{l} k! S_{2,q}(l + r, k + r) \frac{t^l}{[l]_q!}}{[n]_q!}.
\]

Comparing the coefficients of \( \frac{t^n}{[n]_q!} \) in both sides, we get (3.16).

References