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Some properties of q-Hermite Fubini numbers and polynomials

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Abstract. The main purpose of this paper is to introduce a new class of q-Hermite-Fubini numbers and polynomials by combining the q-Hermite polynomials and q-Fubini polynomials. By using generating functions for these numbers and polynomials, we derive some alternative summation formulas including powers of consecutive q-integers. Also, we establish some relationships for q-Hermite-Fubini polynomials associated with q-Bernoulli polynomials, q-Euler polynomials and q-Genocchi polynomials and q-Stirling numbers of the second kind.

Keywords: q-Hermite polynomials, q-Hermite-Fubini polynomials, q-Bernoulli polynomials, q-Euler polynomials, q-Genocchi polynomials, Stirling numbers of the second kind.
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1. Introduction

The subject of q-calculus started appearing in the nineteenth century due to its applications in various fields of mathematics, physics and engineering. The definitions and notations of q-calculus reviewed here are taken from (see [1]):

The q-analogue of the shifted factorial $(a)_n$ is given by

$$(a;q)_0 = 1, (a;q)_n = \prod_{m=0}^{n-1}(1-q^m a), n \in \mathbb{N}.$$

The q-analogue of a complex number a and of the factorial function are given by

$$[a]_q = \frac{1-q^a}{1-q}, q \in \mathbb{C} - \{1\}; a \in \mathbb{C},$$

$$[n]_q! = \prod_{m=1}^{n}[m]_q = [1]_q[2]_q \cdots [n]_q = \frac{(q;q)_n}{(1-q)^n}, q \neq 1; n \in \mathbb{N},$$

$$[0]_q! = 1, q \in \mathbb{C}; 0 < q < 1.$$

The Gauss q-binomial coefficient $\left(\begin{array}{c} n \\ k \end{array}\right)_q$ is given by

$$\left(\begin{array}{c} n \\ k \end{array}\right)_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, k = 0, 1, \cdots, n.$$

The q-analogue of the function $(x+y)_q^n$ is given by

$$(x+y)_q^n = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q q^{k(k-1)/2}x^{n-k}y^k, n \in \mathbb{N}_0. \tag{1.1}$$

The q-analogue of exponential function are given by

$$e_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!} = \frac{1}{((1-q)x;q)_\infty}, 0 <| q |< 1; | x |<| 1-q |^{-1}, \tag{1.2}$$

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$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (-(1-q)x; q)_\infty, 0 <| q < 1; x \in \mathbb{C}. \tag{1.3}$$

Moreover, the functions $e_q(x)$ and $E_q(x)$ satisfy the following properties:

$$D_q e_q(x) = e_q(x), D_q E_q(x) = E_q(qx), \tag{1.4}$$

where the q-derivative $D_q f$ of a function f at a point $0 \neq z \in \mathbb{C}$ is defined as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, 0 <| q |< 1.$$

For any two arbitrary functions $f(z)$ and $g(z)$, the q-derivative operator D_q satisfies the following product and quotient relations:

$$D_{q,z}(f(z)g(z)) = f(z)D_{q,z}g(z) + g(qz)D_{q,z}f(z), \tag{1.5}$$

$$D_{q,z}\left(\frac{f(z)}{g(z)}\right) = \frac{g(qz)D_{q,z}f(z) - f(qz)D_{q,z}g(z)}{g(z)g(qz)}.$$

The q-Hermite polynomials are special or limiting case of the orthogonal polynomials as they contain no parameter other than q and appears to be at the bottom of a hierarchy of the classical polynomials [2]. The q-Hermite polynomials constitute a 1-parameter family of orthogonal polynomials, which for $q = 1$ reduce to the well known Hermite polynomials. We recall that the q-Hermite polynomials $H_{n,q}(x)$ is defined by means of the following generating function (see [9]):

$$F_q(x,t) = F_q(t)e_q(xt) = \sum_{n=0}^{\infty} H_{n,q}(x)\frac{t^n}{[n]_q!}, \tag{1.6}$$

$$F_q(t) = \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} \frac{t^n}{[2n]_q!!}, [2n]_q!! = [2n]_q[2n-2]_q \cdots [2]_q.$$

The q-Bernoulli polynomials $B_{n,q}^{(\alpha)}(x,y)$ of order α, the q-Euler polynomials $E_{n,q}^{(\alpha)}(x,y)$ of order α and the q-Genocchi polynomials $G_{n,q}^{(\alpha)}(x,y)$ of order α are defined by means of the following generating function (see [1-2, 8-11]):

$$\left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}, | t |< 2\pi, 1^\alpha = 1, \tag{1.7}$$

$$\left(\frac{2}{e_q(t)+1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} E_{n,q}^{(\alpha)}(x,y;\lambda)\frac{t^n}{[n]_q!}, | t |< \pi, 1^\alpha = 1, \tag{1.8}$$

$$\left(\frac{2t}{e_q(t)+1}\right)^{\alpha} e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}, | t |< \pi, 1^\alpha = 1. \tag{1.9}$$

Clearly, we have

$$B_{n,q}^{(\alpha)} = B_{n,q}^{(\alpha)}(0,0), E_{n,q}^{(\alpha)} = E_{n,q}^{(\alpha)}, G_{n,q}^{(\alpha)} = G_{n,q}^{(\alpha)}.$$

Geometric polynomials (also known as Fubini polynomials) are defined as follows (see [3]):

$$F_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k! x^k, \tag{1.10}$$

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where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is the Stirling number of the second kind (see [5]).
For $x = 1$ in (1.10), we get n^{th} Fubini number (ordered Bell number or geometric number) F_n [4, 6, 7, 13] is defined by

$$F_n(1) = F_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!. \tag{1.11}$$

The exponential generating functions of geometric polynomials is given by (see [3]):

$$\frac{1}{1 - x(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!}, \tag{1.12}$$

and related to the geometric series (see [3]):

$$\left(x \frac{d}{dx} \right)^m \frac{1}{1-x} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1-x} F_m\left(\frac{x}{1-x}\right), \mid x \mid < 1.$$

Let us give a short list of these polynomials and numbers as follows:
$F_0(x) = 1, F_1(x) = x, F_2(x) = x+2x^2, F_3(x) = x+6x^2+6x^3, F_4(x) = x+14x^2+36x^3+24x^4,$
and

$$F_0 = 1, F_1 = 1, F_2 = 3, F_3 = 13, F_4 = 75.$$

Geometric and exponential polynomials are connected by the relation (see [3]):

$$F_n(x) = \int_0^{\infty} \phi_n(x) e^{-\lambda} d\lambda. \tag{1.13}$$

The manuscript of this paper as follows: In section 2, we consider generating functions for q-Hermite-Fubini numbers and polynomials and give some properties of these numbers and polynomials. In section 3, we derive summation formulas of q-Hermite-Fubini numbers and polynomials and some relationships between q-Bernoulli polynomials, q-Euler polynomials and q-Genocchi polynomials and Stirling numbers of the second kind.

2. A q-analogue type of Hermite-Fubini numbers and polynomials

In this section, we define q-analogue type of Hermite-Fubini polynomials and obtain some basic properties which gives us new formula for ${}_HF_{n,q}(x;y)$. Moreover, we shall consider the sum of products of two q-analogue type of Hermite-Fubini polynomials. The sum of products of various polynomials and numbers with or without binomial coefficients have been studied by (see [4, 6, 7, 13]):

We introduce q-Hermite-based Fubini polynomials in two variables by means of the following generating function:

$$\frac{1}{1 - y(e_q(t) - 1)} F_q(t) e_q(xt) = \sum_{n=0}^{\infty} {}_HF_{n,q}(x;y) \frac{t^n}{[n]_q!}. \tag{2.1}$$

Taking $x = 0, y = 1$ in (2.1), we get

$${}_HF_{n,q}(0;1) = {}_HF_{n,q},$$

where ${}_HF_{n,q}$ are the q-Hermite-based Fubini numbers.

When investigating the connection between q-Hermite polynomials $H_{n,q}(x)$ and q-Fubini polynomials $F_{n,q}(y)$ of importance is the following theorem.

Theorem 2.1. The following formula for q-Hermite-based Fubini polynomials holds true:

$$_HF_{n,q}(x;y) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right)_q F_{n-m,q}(y) H_{m,q}(x). \tag{2.2}$$

$$_HF_{n,q}(x;y) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right)_q {}_HF_{m,q}(y) x^{n-m}. \tag{2.3}$$

Proof. Using definition (2.1), we have

$$\sum_{n=0}^{\infty} {}_HF_{n,q}(x;y)\frac{t^n}{[n]_q!} = \frac{1}{1-y(e_q(t)-1)}F_q(t)e_q(xt)$$
$$= \sum_{n=0}^{\infty} F_{n,q}(y)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty} H_{m,q}(x)\frac{t^m}{[m]_q!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\left(\begin{array}{c} n \\ r \end{array} \right)_q F_{n-m,q}(y)H_{m,q}(x)\right)\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ yields (2.2).
Utilizing equation (1.6) in the l.h.s. of generating function (2.1), it follows that

$$\sum_{n=0}^{\infty} {}_HF_{n,q}(x;y)\frac{t^n}{[n]_q!} = \sum_{m=0}^{\infty} {}_HF_{n,q}(y)\frac{t^m}{[m]_q!}\sum_{n=0}^{\infty} x^n\frac{t^n}{[n]_q!},$$

which on applying the Cauchy product rule in the l.h.s. and then comparing the coefficients of same powers of t in both sides of resultant equation yield assertion (2.3).

□

Proposition 2.1. The following formula for q-Hermite-based Fubini polynomials holds true:

$$D_{q,t}e_q(xt) = xe_q(xt)$$
$$D_{q,x}\left({}_HF_{n,q}(x;y)\right) = [n]_q {}_HF_{n-1,q}(x;y). \tag{2.4}$$

Theorem 2.2. For $n \geq 0$, the following formula for q-Hermite-based Fubini polynomials holds true:

$$H_{n,q}(x) = {}_HF_{n,q}(x;y) - y {}_HF_{n,q}(x+1;y) + y {}_HF_{n,q}(x;y). \tag{2.5}$$

Proof. We begin with the definition (2.1) and write

$$F_q(t)e_q(xt) = \frac{1-y(e_q(t)-1)}{1-y(e_q(t)-1)}F_q(t)e_q(xt)$$
$$= \frac{F_q(t)e_q(xt)}{1-y(e_q(t)-1)} - \frac{y(e_q(t)-1)}{1-y(e_q(t)-1)}F_q(t)e_q(xt).$$

Then using the definition of q-Hermite polynomials $H_{n,q}(x)$ and (2.1), we have

$$\sum_{n=0}^{\infty} H_{n,q}(x)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\left[{}_HF_{n,q}(x;y) - y {}_HF_{n,q}(x+1;y) + y {}_HF_{n,q}(x;y)\right]\frac{t^n}{[n]_q!}.$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$, we get (2.5).

□

Theorem 2.3. For $n \geq 0$, the following formula for q-Hermite-based Fubini polynomials holds true:

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)_q {}_HF_{n-k,q}(x_1; y_1)\, {}_HF_{k,q}(x_2; y_2)$$
$$= \frac{y_2\, {}_HF_{n,q}(x_1 + x_2; y_1) - y_1\, {}_HF_{n,q}(x_1 + x_2; y_2)}{y_2 - y_1}. \tag{2.6}$$

Proof. The products of (2.1) can be written as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_HF_{n,q}(x_1; y_1) \frac{t^n}{[n]_q!} {}_HF_k(x_2; y_2) \frac{t^k}{[k]_q!} = \frac{F_q(t)e_q(x_1 t)}{1 - y_1(e_q(t) - 1)} \frac{F_q(t)e_q(x_2 t)}{1 - y_2(e_q(t) - 1)}$$

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)_q {}_HF_{n-k,q}(x_1; y_1)\, {}_HF_{k,q}(x_2; y_2) \right) \frac{t^n}{[n]_q!}$$
$$= \frac{y_2}{y_2 - y_1} \frac{F_q(t)e_q((x_1 + x_2)t)}{1 - y_1(e_q(t) - 1)} - \frac{y_1}{y_2 - y_1} \frac{F_q(t)e_q((x_1 + x_2)t)}{1 - y_2(e_q(t) - 1)}$$
$$= \left(\frac{y_2\, {}_HF_{n,q}(x_1 + x_2; y_1) - y_1\, {}_HF_{n,q}(x_1 + x_2; y_2)}{y_2 - y_1} \right) \frac{t^n}{[n]_q!}.$$

By equating the coefficients of $\frac{t^n}{[n]_q!}$ on both sides, we get (2.6). □

Theorem 2.4. For $n \geq 0$, the following formula for q-Hermite-based Fubini polynomials holds true:

$$y\, {}_HF_{n,q}(x + 1; y) = (1 + y)\, {}_HF_{n,q}(x; y) - H_{n,q}(x). \tag{2.7}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} \left[{}_HF_{n,q}(x + 1; y) - {}_HF_{n,q}(x; y) \right] \frac{t^n}{[n]_q!} = \frac{F_q(t)e_q(xt)}{1 - y(e_q(t) - 1)} (e_q(t) - 1)$$
$$= \frac{1}{y} \left[\frac{F_q(t)e_q(xt)}{1 - y(e_q(t) - 1)} - F_q(t)e_q(xt) \right]$$
$$= \frac{1}{y} \sum_{n=0}^{\infty} \left[{}_HF_{n,q}(x; y) - H_{n,q}(x) \right] \frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ on both sides, we obtain (2.7). □

Remark 2.1. On setting $x = 0$ and $x = -1$ in Theorem 2.4, we find

$$y\, {}_HF_{n,q}(1; y) = (1 + y)\, {}_HF_{n,q}(y) - (-1)^n q^{n(n-1)/2} \frac{[n]_q!}{[2n]_q!!}, \tag{2.8}$$

and

$$y\, {}_HF_{n,q}(0; y) = (1 + y)\, {}_HF_{n,q}(-1; y) - (-1)^n q^{n(n-1)/2} \frac{[n]_q!}{[2n]_q!!}. \tag{2.9}$$

3. Main results

In this section,, we prove the following result involving q-Hermite-Fubini polynomials ${}_HF_{n,q}(x; y)$ by using series rearrangement techniques and considered its special case. Also we obtain some relationships for q-Hermite Fubini polynomials related to q-Bernoulli polynomials, q-Euler polynomials and q-Genocchi polynomials and Stirling numbers of the second kind in Theorems 4.1, 4.2, 4.3, 4.4, 4.5.

Theorem 3.1. The following formula for q-Hermite-based Fubini polynomials holds true:

$$_HF_{k+l,q}(w;y) = \sum_{n,p=0}^{k,l} \left(\begin{array}{c} k \\ n \end{array}\right)_q \left(\begin{array}{c} l \\ p \end{array}\right)_q (w-y)^{n+p}\, _HF_{k+l-n-p,q}(x;y). \tag{3.1}$$

Proof. Replacing t by $t + u$ in (2.1) and then using the formula [12,p.52(2)]:

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.2}$$

in the resultant equation, we find the following generating function for the Hermite-Fubini polynomials $_HF_n(x,y;z)$:

$$\frac{1}{1-y(e_q(t+u)-1)}F_q(t+u) = e_q(-x(t+u))\sum_{k,l=0}^{\infty} {}_HF_{k+l,q}(x;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!}. \tag{3.3}$$

Replacing x by w in the above equation and equating the resultant equation to the above equation, we find

$$e_q((w-x)(t+u))\sum_{k,l=0}^{\infty} {}_HF_{k+l,q}(x;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!} = \sum_{k,l=0}^{\infty} {}_HF_{k+l,q}(w;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!}. \tag{3.4}$$

On expanding exponential function (3.4) gives

$$\sum_{N=0}^{\infty}\frac{[(w-x)(t+u)]^N}{[N]_q!}\sum_{k,l=0}^{\infty} {}_HF_{k+l,q}(x;y)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_HF_{k+l,q}(w;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!}, \tag{3.5}$$

which on using formula (3.2) in the first summation on the left hand side becomes

$$\sum_{n,p=0}^{\infty}\frac{(w-x)^{n+p}t^n u^p}{[n]_q![p]_q!}\sum_{k,l=0}^{\infty} {}_HF_{k+l,q}(x;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!} = \sum_{k,l=0}^{\infty} {}_HF_{k+l}(w;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!}. \tag{3.6}$$

Now replacing q by $q-n$, l by $l-p$ and using the lemma ([12, p.100(1)]):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A(n,k) = \sum_{k=0}^{\infty}\sum_{n=0}^{k} A(n,k-n), \tag{3.7}$$

in the l.h.s. of (3.6), we find

$$\sum_{k,l=0}^{\infty}\sum_{n,p=0}^{k,l}\frac{(w-x)^{n+p}}{[n]_q![p]_q!}\, _HF_{k+l-n-p}(x;y)\frac{t^k}{(k-n)_q!}\frac{u^l}{(l-p)_q!}$$

$$= \sum_{q,l=0}^{\infty} {}_HF_{q+l}(w,y;z)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.8}$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, we get the assertion (3.1) of Theorem 3.1.

□

Remark 3.1. Taking $l = 0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

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Corollary 3.1. The following summation formula for Hermite-Fubini polynomials ${}_HF_n(x,y;z)$ holds true:

$$ {}_HF_{k,q}(w;y) = \sum_{n=0}^{k} \left(\begin{array}{c} k \\ n \end{array}\right)_q (w-y)^{n+p}\, {}_HF_{k-n,q}(x;y). \tag{3.9} $$

Remark 3.2. Replacing w by $w+x$ in (3.9), we obtain

$$ {}_HF_q(x+w;y) = \sum_{n=0}^{k} \left(\begin{array}{c} k \\ n \end{array}\right) w^n\, {}_HF_{k-n,q}(x;y). \tag{3.10} $$

Theorem 3.2. The following formula for q-Hermite-based Fubini polynomials holds true:

$$ {}_HF_{n,q}(x+1;y) = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right)_q {}_HF_{n-r,q}(x;y). \tag{3.11} $$

Proof. Using the generating function (2.1), we have

$$ \sum_{n=0}^{\infty} {}_HF_{n.q}(x+1;y)\frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} {}_HF_{n,q}(x;y)\frac{t^n}{[n]_q!} $$

$$ = \left(\frac{1}{1-y(e_q(t)-1)}\right)(e_q(t)-1)F_q(t)e_q(xt) $$

$$ = \sum_{n=0}^{\infty} {}_HF_{n,q}(x;y)\frac{t^n}{[n]_q!}\left(\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}-1\right) $$

$$ = \sum_{n=0}^{\infty} {}_HF_{n,q}(x;y)\frac{t^n}{[n]_q!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!} - \sum_{n=0}^{\infty} {}_HF_{n,q}(x;y)\frac{t^n}{[n]_q!} $$

$$ = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right)_q {}_HF_{n-r,q}(x;y)\frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} {}_HF_{n,q}(x;y)\frac{t^n}{[n]_q!}. $$

Finally, equating the coefficients of the like powers of t on both sides, we get (3.11). □

Theorem 3.3. Each of the following relationships holds true:

$$ {}_HF_{n,q}(x;y) = \sum_{s=0}^{n+1} \left(\begin{array}{c} n+1 \\ s \end{array}\right)_q \left[\sum_{k=0}^{s} \left(\begin{array}{c} s \\ k \end{array}\right)_q B_{s-k.q}(x) - B_{s,q}(x)\right] \frac{{}_HF_{n+1-s,q}(y)}{[n+1]_q}, \tag{3.12} $$

where $B_{n,q}(x)$ is q-Bernoulli polynomials.

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Proof. By using definition (2.1), we have

$$\left(\frac{1}{1-y(e_q(t)-1)}\right)F_q(t)te_q(xt)$$
$$=\left(\frac{1}{1-y(e_q(t)-1)}\right)\frac{t}{e_q(t)-1}\frac{e_q(t)-1}{t}F_q(t)te_q(xt)$$
$$=\frac{1}{t}\sum_{s=0}^{\infty}\left(\sum_{k=0}^{s}\left(\begin{array}{c}s\\k\end{array}\right)_q B_{s-k,q}(x)\right)\frac{t^s}{[s]_q!}\sum_{n=0}^{\infty}{}_HF_{n,q}(y)\frac{t^n}{[n]_q!}$$
$$-\frac{1}{t}\sum_{s=0}^{\infty}B_{s,q}(x)\frac{t^s}{[s]_q!}\sum_{n=0}^{\infty}{}_HF_{n,q}(y)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{t}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c}n\\s\end{array}\right)_q\sum_{k=0}^{s}\left(\begin{array}{c}s\\k\end{array}\right)_q B_{s-k,q}(x)\right]{}_HF_{n-s,q}(y)\frac{t^n}{[n]_q!}$$
$$-\frac{1}{t}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c}n\\s\end{array}\right)_q B_{s,q}(x)\right]{}_HF_{n-s,q}(y)\frac{t^n}{[n]_q!}.$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{[n]_q!}$, we arrive at the required result (3.12). □

Theorem 3.4. Each of the following relationships holds true:

$$ {}_HF_{n,q}(x;y)$$
$$=\sum_{s=0}^{n}\left(\begin{array}{c}n\\s\end{array}\right)_q\left[\sum_{k=0}^{s}\left(\begin{array}{c}s\\k\end{array}\right)_q E_{s-k,q}(x)+E_{s,q}(x)\right]\frac{{}_HF_{n-s,q}(y)}{[2]_q}, \qquad (3.13)$$

where $E_{n,q}(x)$ is the q-Euler polynomials.

Proof. By using definition (2.1), we have

$$\left(\frac{1}{1-y(e_q(t)-1)}\right)F_q(t)e_q(xt)$$
$$=\left(\frac{1}{1-z(e_q(t)-1)}\right)\frac{[2]_q}{e_q(t)+1}\frac{e_q(t)+1}{[2]_q}F_q(t)e_q(xt)$$
$$=\frac{1}{[2]_q}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)_q E_{n-k,q}(x)\right)\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{[n]_q!}\right]$$
$$\times\sum_{n=0}^{\infty}{}_HF_{n,q}(y)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{[2]_q}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c}n\\s\end{array}\right)_q\sum_{k=0}^{s}\left(\begin{array}{c}s\\k\end{array}\right)_q E_{s-k,q}(x)+\sum_{s=0}^{n}\left(\begin{array}{c}n\\s\end{array}\right)_{p,q}E_s(x;p,q)\right]$$
$$\times{}_HF_{n-s,q}(y)\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, we arrive at the desired result (3.13). □

Theorem 3.5. Each of the following relationships holds true:

$$ {}_HF_{n,q}(x;y)$$
$$=\sum_{s=0}^{n}\left(\begin{array}{c}n+1\\s\end{array}\right)_q\left[\sum_{k=0}^{s}\left(\begin{array}{c}s\\k\end{array}\right)_q G_{s-k,q}(x)+G_{s,q}(x)\right]\frac{{}_HF_{n+1-s,q}(y)}{[2]_q[n+1]_q}, \qquad (3.14)$$

where $G_{n,q}(x)$ is the q-Genocchi polynomials.

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Proof. By using definition (2.1), we have

$$\left(\frac{1}{1-y(e_q(t)-1)}\right)F_q(t)e_{p,q}(xt)$$
$$=\left(\frac{1}{1-y(e_q(t)-1)}\right)F_q(t)e_{p,q}(xt)\frac{[2]_qt}{e_q(t)+1}\frac{e_q(t)+1}{[2]_qt}F_q(t)e_{p,q}(xt)$$
$$=\frac{1}{[2]_qt}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)_q G_{n-k,q}(x)\right)\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{[n]_q!}\right]$$
$$\times\sum_{n=0}^{\infty}{}_HF_{n,q}(x;y)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{[2]_q}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c}n\\s\end{array}\right)_q\sum_{k=0}^{s}\left(\begin{array}{c}s\\k\end{array}\right)_q G_{s-k,q}(x)+\sum_{s=0}^{n}\left(\begin{array}{c}n\\s\end{array}\right)_q G_{s,q}(x)\right]$$
$$\times{}_HF_{n+1-s,q}(y)\frac{t^n}{[n+1]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, then we have the asserted result (3.14). □

Theorem 3.6. For $n\geq 0$, the following formula for q-Hermite-Fubini polynomials holds true:

$${}_HF_{n,q}(x;y)=\sum_{l=0}^{n}\left(\begin{array}{c}n\\l\end{array}\right)_q H_{n-l,q}(x)\sum_{k=0}^{l}y^kk!S_{2,q}(l,k). \tag{3.15}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty}{}_HF_{n,q}(x;y)\frac{t^n}{[n]_q!}=\frac{F_q(t)e_q(xt)}{1-y(e_q(t)-1)}$$
$$=F_q(t)e_q(xt)\sum_{k=0}^{\infty}y^k(e_q(t)-1)^k=F_q(t)e_q(xt)\sum_{k=0}^{\infty}y^k\sum_{l=k}^{\infty}k!S_{2,q}(l,k)\frac{t^l}{[l]_q!}$$
$$=\sum_{n=0}^{\infty}H_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{l=0}^{\infty}y^k\sum_{k=0}^{l}k!S_{2,q}(l,k)\frac{t^l}{[l]_q!}.$$

Replacing n by $n-l$ in above equation, we get

$$\sum_{n=0}^{\infty}{}_HF_{n,q}(x;y)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\left(\begin{array}{c}n\\l\end{array}\right)_q H_{n-l,q}(x)\sum_{k=0}^{l}y^kk!S_{2,q}(l,k)\right)\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in both sides, we get (3.15).

□

Theorem 3.7. For $n\geq 0$, the following formula for q-Hermite-Fubini polynomials holds true:

$${}_HF_{n,q}(x+r;y)=\sum_{l=0}^{n}\left(\begin{array}{c}n\\l\end{array}\right)_q H_{n-l,q}(x)\sum_{k=0}^{l}y^kk!S_{2,q}(l+r,k+r). \tag{3.16}$$

Proof. Replacing x by $x+r$ in (2.1), we have

$$\sum_{n=0}^{\infty}{}_HF_{n,q}(x+r;y)\frac{t^n}{[n]_q!}=\frac{F_q(t)e_q(x+r)t}{1-y(e_q(t)-1)}$$
$$=F_q(t)e_q(xt)e_q(rt)\sum_{k=0}^{\infty}y^k(e_q(t)-1)^k=F_q(t)e_q(xt)e_q(rt)\sum_{k=0}^{\infty}y^k\sum_{l=k}^{\infty}k!S_{2,q}(l,k)\frac{t^l}{[l]_q!}$$

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$$= \sum_{n=0}^{\infty} H_{n,q}(x) \frac{t^n}{[n]_q!} \sum_{l=0}^{\infty} y^k \sum_{k=0}^{l} k! S_{2,q}(l+r, k+r) \frac{t^l}{[l]_q!}.$$

Replacing n by $n-l$ in above equation, we get

$$\sum_{n=0}^{\infty} {}_H F_{n,q}(x+r; y) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l}_q H_{n-l,q}(x) \sum_{k=0}^{l} y^k k! S_{2,q}(l+r, k+r) \right) \frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in both sides, we get (3.16).

□

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