 doi:10.20944/preprints201904.0302.v1

Safe Reachability Verification of Nonlinear Switched Systems via a Barrier Density

Ayşegül Kıvılcım[1], Özkan Karabacak[1] and Rafael Wisniewski[1]

Abstract— **One of the notable temporal properties of dynamical systems is that a set of initial states leads the solutions to reach desired states avoiding a predetermined unsafe set. This property, that we call safe reachability has been studied in literature for autonomous systems using Barrier function and Barrier densities [1]. In this paper, we generalize a sufficient condition for safe reachability of autonomous system to switched systems under arbitrary switching signals. The condition relies upon the existence of a common Barrier density function for each subsystem. We apply the condition using the sum of squares method together with Putinar Positivstellensatz.**

I. Introduction

In many control applications, one needs to ensure that a system with an initial state reaches to a desired state (reachability) avoiding some undesired states (safety). One of the widely used method for the verification of the safety and the reachability is to certify via barrier functions which allows us to analyze a system without knowing the solutions as done in the analysis with Lyapunov functions. To mention a few applications, barrier functions are used for safety verification of unmanned aerial system to perform high speed in an environment with multiple obstacles [2], for model invalidation, i.e., checking the inconsistency of the measured data with the model [3], for detecting the faults in a system [4], for verification of safety and reachability of nonlinear autonomous systems and systems with disturbances [1], and for the computation of the reachable sets [5] (for more applications see [1], [4], [6], [7], [8], [9] and the references therein). As mentioned in the paper [1], one may not find a barrier function to certify safety and reachability due to the fact that the solution trajectory for some initial state, which is included in a set with measure zero, may enter an undesired set or may not reach a desired set. In [1], notions of "weak safety" and "weak reachability" are defined to indicate that the safety and reachability properties are satisfied except the set with Lebesgue measure zero. In the light of this, weak safe reachibility can be defined as follows: There exists a time such that for almost every initial state the solution enters a desired region without entering an undesired region. In [1], to certify the weak safety of nonlinear autonomous system Barrier density is utilized. Density functions are also used for stability analysis [10], [11], [12], [13], [14], [15], [16], [17], [18], [19]. Leaning upon the results of [1], our main goal in this paper is to obtain a sufficient condition for the weak safe reachibility of nonlinear switched system under arbitrary switching.

Nonlinear switched systems appear in various applications (for instance [20] and the references therein). In particular, nonlinear switched systems with time dependent switching can be used to model switched control systems where switching is generated by an external system [21] (for more applications, see the references in [20], [21], [22]) Recently, using common and multiple Lyapunov densities, we have obtained sufficient conditions for the almost global stability of switched systems [23]. In [24], safety verification of nonlinear switched systems is studied by utilizing Barrier functions and Barrier densities. In this paper, we extend the result of the paper [24] to the verification of safe reachability of switched systems.

Inspired by the common Lyapunov density approach in [23] and the dual Lyapunov analysis of weak safety and reachability in [1], we analyze the safe reachability of a nonlinear switched system. Leaning upon the existence of a common Barrier density, we present a sufficient condition for weak safe reachability under arbitrary switching.

The paper is structured as follows: In Section II, we present preliminaries about nonlinear switched systems under arbitrary switching. In Section III, we define safe reachability and weak safe reachability of a system and present a sufficient condition for the weak safe reachability of nonlinear switched systems with time-varying switching. In Section IV, we present an example by using the sum of squares algorithm to illustrate theoretical part of our paper and we present a brief summary about the usage of Sum of Squares (SoS) programming together with Putinar Positivstellensatz.

Notation. The following notations will be used in the remaining part of the paper.

- m denotes the Lebesgue measure on $\mathbb{R}^n$ and with the Borel measurable function ρ, $\mu_\rho(A) = \int_A \rho m(dx)$.
- For a function $f : \mathbb{R}^n \to \mathbb{R}^n$, and $\nabla \cdot f$ denotes the divergence of f and for a function $g : \mathbb{R}^n \to \mathbb{R}$, ∇g denotes the gradient of g.
- $\mathrm{Int}(A)$, A^c and $\overline{A}$ denote the interior, the complement and the closure of a set A, respectively.

The notions "almost every" and "almost all" are used to indicate that the given property is satisfied everywhere except for a set with Lebesgue measure zero.

II. Preliminaries

We have extended Corollary 3.11 [1] (rewritten as Proposition 1 below) to the nonlinear switched systems under

*This work has been supported by the Independent Research Fund Denmark (DeBaTe)

[1]Department of Electronic Systems, Automation and Control, Aalborg University, Fredrik Bajers Vej 7 C, 9220 Aalborg East, Denmark
ayk@es.aau.dk, ozk@es.aau.dk, raf@es.aau.dk

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2019 doi:10.20944/preprints201904.0302.v1

arbitrary switching signals. In the next section, the conditions of the proposition will be generalized to nonlinear switched systems by reformulating them to be compatible with searching for common barrier density with the sum of squares (SoS) algorithm together with Putinar positivstellensatz[25]. This is because it allows us to look for a polynomial which is non-negative on the semi algebraically defined sets.

For the sake of the completeness, let us restate Corollary 3.11 given in [1].

Proposition 1: [1, Corollary 3.11] Consider a system $\dot{x} = f(x)$, where f is a continuously differentiable function on $\mathbb{R}^n$. Let X be a bounded subset of R^n, and $X_0 \subset X$ be a set with positive Lebesgue measure. Assume that a desired set $X_r \subset X$ and an unsafe set $X_u \subset X$ are given. If there exists a continuously differentiable function $\rho(x) \in X$ satisfying

$$\rho(x) > 0, \quad \forall x \in X_0, \tag{1a}$$

$$\nabla \cdot (\rho f)(x) > 0, \quad \forall x \in \overline{(X \setminus X_r)}, \tag{1b}$$

$$\rho(x) \leq 0, \quad \forall x \in \overline{(\partial X \setminus \partial X_r)} \cup X_u, \tag{1c}$$

then the weak safe reachability property holds; i.e. for almost all initial states in $x_0 \in X_0$, the solution $x(t)$ starting at $x(0) = x_0$ satisfies, for some $T \geq 0$, $x(T) \in X_r$, $x(t) \notin X_u$, and $x(t) \in X$, for all $t \in [0, T]$.

In order to apply an SoS algorithm with Putinar positivstellensatz, the given sets should be defined semi algebraically. However, as seen in condition (1b) and (1c) of Proposition 1, we cannot convert it to the SoS algorithm since the sets cannot be defined semi-algebraically. For this reason, in the next section, we have generalized Proposition 1 to switched systems and reformulated the conditions to be compatible with the verification of safe reachability of nonlinear switched system via the SoS algorithm.

Consider a continuous-time nonlinear switched system of the following form

$$\dot{x}(t) = f_\sigma(x(t)), \quad \sigma \in \mathcal{S}_{\text{nonchat}} \tag{2}$$

where σ is the switching signal. The switching signal, $\sigma(t) : [0, \infty) \rightarrow \{1, 2, \ldots, N\}$, is a right continuous piece-wise constant function. The largest set for admissible switching signal is a set of switching signals which have finite number of discontinuities in a finite time interval and denoted by $\mathcal{S}_{\text{nonchat}}$. Each system given by $\dot{x}(t) = f_p(x(t))$, $p = 1, 2, \ldots, N$ is called the subsystem of the system (2). Assume that in the bounded domain (open and connected subset of $\mathbb{R}^n$) $X \subset \mathbb{R}^n$, each subsystem $f_p : X \rightarrow X$, $i = 1, 2, \ldots, N$, is continuously differentiable. Denote the constant value of the switching signal $\sigma(t)$ for $t \in [t_{i-1}, t_i)$ as p_i. By using these values switching signal can be defined as $\sigma(t) = \{(\Delta t_1, p_1), (\Delta t_2, p_2), \ldots\}$, where Δt_i is the operation time of the subsystem f_{p_i}. Assume that solutions of (2) with $\mathcal{S}_{\text{nonchat}}$ exist for all $t \in \mathbb{R}$. Denote a solution of system (2) for the switching signal $\sigma \in \mathcal{S}_{\text{nonchat}}$ and for the initial state x as $\phi_t^\sigma(x)$.

III. Main Result

The idea of the paper [1] about the safety and reachability verification by means of a barrier density is extended to the nonlinear switched systems under arbitrary switching signals and Proposition 1 is reformulated to be compatible with the analysis of nonlinear switched system by means of SoS algorithm with Putinar positivstellensatz. Moreover, in the sequel, we have also mentioned that under which conditions weak safe reachibility results can be used to certify safe reachability of nonlinear switched system. Now, let us define safe reachability and weak safe reachability.

Definition 1: ((Weak) safe reachability) We say that the system (2) is (weakly) safely reachable on a domain X from an initial set X_0 to a desired set X_r avoiding an unsafe set X_u if, for each switching signal $\sigma \in \mathcal{S}_{\text{nonchatt}}$, there exists a $T(x) > 0$ for (almost) every initial state $x \in X_0$ such that

$$\phi^\sigma_{T(x)}(x) \in X_r, \tag{3}$$

$$\phi^\sigma_t(x) \notin X_u, \text{ for all } t \in [0, T(x)], \tag{4}$$

$$\phi^\sigma_t(x) \in X \text{ for all } t \in [0, T(x)], \tag{5}$$

where $\phi^\sigma_t(x)$ denotes the solution of (2) for the switching signal $\sigma \in \mathcal{S}$ and for the initial state x.

(Weak) safe reachability from X_0 to X_r avoiding X_u is denoted as $X_0 \overset{(a.e.)}{\underset{X_u}{\frown}} X_r$.

The following lemma is needed for the proof of Lemma 2 which is used to certify the change of volume with the change of density along the solutions of a system.

Lemma 1: [12] Assume that a set $D \subset \mathbb{R}^n$ is open, $\phi_t(x)$ is a solution of the system $\dot{x} = f(x)$ with an initial state x, $f(x)$, $\rho(x)$ is continuously differentiable in D, and for a measurable set A, define $\phi_s(A) = \{\phi_s(x) | x \in A\}$. Assume that $\phi_s(A) \subset D$, for all $s \in [0, t]$. Then,

$$\int_{\phi_t(A)} \rho(x) dx - \int_A \rho(x) dx = \int_0^t \int_{\phi_\tau(A)} [\nabla \cdot (f\rho)(x)] dx d\tau. \tag{6}$$

One can see that $\frac{\partial \rho(t)}{\partial t} = -\nabla \cdot (f\rho)$ is the continuity equation which is widely used for explaining the conservation of mass. From (6), one can conclude that if density is constant along the set of solutions, i.e., $\frac{\partial \rho(t)}{\partial t} = -\nabla \cdot (f\rho) = 0$, then the vector field is volume preserving. Moreover, if the density decreases, i.e., $\frac{\partial \rho(t)}{\partial t} = -\nabla \cdot (f\rho) < 0$, we can say that the volume along the flow of the vector field increases [26].

Lemma 2: [27] Assume that a set $D \subset \mathbb{R}^n$ is open, $\phi_t(x)$ is a solution of the system $\dot{x} = f(x)$ with an initial state x, $f(x)$, $\rho(x)$ is continuously differentiable in D, and for a measurable set A, define $\phi_s(A) = \{\phi_s(x) | x \in A\}$. Let ρ is integrable in D and $\nabla \cdot (f\rho) > 0$. Assume that $\phi_s(A) \subset D$, for all $s \in [0, t]$. Then, for a fixed $t > 0$, $\mu_\rho(\phi_t(A)) > \mu_\rho(A)$.

Next, we will give a way for verification of safe reachability properties of nonlinear switched systems with the aid of a Barrier density which is common for each subsystems (a common barrier density).

Theorem 1: Let $X \subset \mathbb{R}^n$ be bounded and X_0 be a set with positive Lebesgue measure. If system (2) with the given

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2019 doi:10.20944/preprints201904.0302.v1

sets $X_0 \subset X$, $X_u \subset X$ and $X_r \subset X$, has a differentiable function ρ in X satisfying the following properties

$$\rho(x) > 0, \quad \forall x \in X_0, \tag{7a}$$
$$\nabla \cdot (\rho f_p)(x) > 0, \quad p = 1, 2, \dots, N, \forall x \in X \setminus X_r, \tag{7b}$$
$$\rho(x) \leq 0, \quad \forall x \in X^c \cup X_u, \tag{7c}$$

then system (2) is weakly safely reachable, $X_0 \overset{a.e}{\widehat{X_u}} X_r$.

Proof: Take an arbitrary switching signal $\sigma \in \mathcal{S}_{\text{nonchatt}}$. Let $\phi_t^\sigma(x)$ denotes the solution of (2) for the switching signal $\sigma \in \mathcal{S}$. Let $\phi_t^\sigma(A) = \{\phi_t^\sigma(x) | x \in A\}$ for $t \geq 0$ be the set of solutions starting from an arbitrary set $A \subset X_0$ with positive measure. Assume that for some $T(x) > 0$ $\phi_{T(x)}^\sigma(A) \subset X_u$ and $\phi_t^\sigma(A) \not\subset X_r$ and $\phi_t^\sigma(A) \subset X$ for all $t \in [0, T(x)]$. Utilizing Lemma 2 between each consecutive switching instants and considering condition (7a), we obtain $\mu_\rho(\phi_{T(x)}^\sigma(A)) > \mu_\rho(A) > 0$. Therefore, there exists no set $A \subset X_0$ with positive measure such that for some $T(x) > 0$, $\phi_{T(x)}^\sigma(A) \subset X_u$ and $\phi_t^\sigma(A) \subset X$ for all $t \in [0, T(x)]$.

Let us show reachability to the set X_r. Define $Z = \cup_{x \in X_0}\{x \in X_0 | \phi_t^\sigma(x) \subset X \setminus X_r, \forall t \geq 0\}$ and $\phi_t^\sigma(Z) \subset X \setminus X_r$, for all $t > 0$. By considering Lemma 2 together with $Z \subset X_0$ from 0 to t, we get $\mu_\rho(\phi_t^\sigma(Z)) > \mu_\rho(Z)$, for a fixed $t > 0$. The measure of all trajectories staring from Z and lying in $X \setminus X_r$ can be computed as $\mu_\rho(\phi_t^\sigma(Z)) = \sum_{i=1}^{\infty} \mu_\rho(\phi_{t_i}^\sigma(Z))$. Applying the condition (7b) together with Lemma 1 between each switching instants, we get $\mu_\rho(\phi_{t_i}^\sigma(Z)) > \mu_\rho(\phi_{t_{i-1}}^\sigma(Z))$, $i \in \mathbb{Z}_{>0}$. Applying this iteratively together with condition (7a), it is obtained that

$$\mu_\rho(\phi_{t_i}^\sigma(Z)) > \mu_\rho(Z), \quad i \in \mathbb{Z}_{>0}. \tag{8}$$

Combining the previous sum with (8), we get $\mu_\rho(\phi_t^\sigma(Z)) > \sum_{i=1}^{\infty} \mu_\rho(Z)$. If $\mu_\rho(Z) > 0$, we have $\mu_\rho(\phi_t^\sigma(Z)) = \infty$, which contradicts to the assumption of the integrability of ρ on X. Thus, the set Z is included in a set with measure zero. Any solution which stays inside of the region $X \setminus X_r$ for all $t \geq 0$ is included in a set with measure zero. For almost every initial state, the solution $\phi_t^\sigma(x)$ leaves the region $X \setminus X_r$, i.e., either it leaves the domain X or it enters the region X_r. From above discussion on safety, the set of solutions whose initial states are included in a set with positive measure cannot leave X since ρ is negative in X^c, so it reaches the set X_r in a finite time. To conclude, there exists a time $T(x) > 0$, for almost every initial state x in the set X_0 solution satisfy that $\phi_{T(x)}^\sigma(x) \in X_r$, $\phi_t^\sigma(x) \notin X_u$ and $\phi_t^\sigma(x) \in X$ for all $t \in [0, T(x)]$. ■

When the set of initial states and the set of undesired states satisfy a certain topological condition, weak safe reachability implies safe reachability as shown in the next corollary.

Corollary 1: Let $X \subset \mathbb{R}^n$ be bounded and X_0 be a set with positive Lebesgue measure. Assume that the sets X_0 and X_u satisfy the following property $\overline{\text{Int}(X_0)} = X_0$ and $\overline{\text{Int}(X_u)} = X_u$. If a system (2) with the given sets $X_0 \subset X$, $X_u \subset X$, and $X_r \subset X$, has a differentiable function ρ in X satisfying the conditions (7a)-(7c), then system (2) is safely reachable, $X_0 \widehat{X_u} X_r$.

Proof: Recall that if a non-empty set A satisfies $\overline{\text{Int}(A)} = A$, then for an arbitrary element $a \in A$, the intersection of any neighbourhood of a with A contains a non-empty open set. The conditions of (7a)-(7c) of Theorem 1 are satisfied with the given sets. Thus, (2) is weakly safely reachable. Let us assume that the system is not safely reachable, namely, for an arbitrary switching signal $\sigma \in \mathcal{S}_{\text{nonchat}}$, (2) has a solution $\phi_t^\sigma(x)$ starting with an initial state x from the set X_0 that reaches a point $\bar{x} := \phi_{T_1(x)}^\sigma(x) \in X_u$. Then, there exists a sufficiently small neighbourhood $U_{\bar{x}}$ of $\bar{x}$ and a non-empty open set W such that $W \subset U_{\bar{x}} \cap X_u$. Due to $\overline{\text{Int}(X_0)} = X_0$ and continuity of the flow map $\phi_t^\sigma(x)$, we can say that there exists a non-empty open set V such that $V \subset \left(\phi_{T_1(x)}^\sigma\right)^{-1}(W) \cap X_0$ and $\mu_\rho(V) > 0$, which contradicts to the weak safety. Thus, for the given sets X_0 and X_u the system is safe. Due to the fact that system is weak reachable, for almost every initial state x in X_0 the solution of (2) reaches the set X_r in a finite time. We can conclude that all solutions starting from the set X_0 reach the set X_r due to continuous dependence of solutions with the initial state. Thus, system is safely reachable. ■

Remark 1: Corollary 1 is also valid if we take the set of initial states X_0 and the set of avoided states X_u as open. We cannot use Putinar positivstellensatz to ensure the safe reachability, $X_0 \widehat{X_u} X_r$, since in order to use Putinar positivstellensatz the given sets should be defined semi algebraically. However, semi-algebraically defined sets are closed.

Remark 2: If there is a stable fixed point of the system in X, it should be removed from the domain. Otherwise, $\nabla \cdot (f_p \rho) > 0$ on X implies that ρ is not integrable on X.

Remark 3: If the condition (7b) is given as $\nabla \cdot (f_p \rho) \geq 0$, for all $p = 1, 2, \dots, N$, the safety property will be still valid since the solutions starting from X_0 will not reach the set X_u since positivity of ρ along the solutions starting in X_0 is still preserved. However, in this case, we cannot guarantee the reachability since under this condition the measure of the set either increases or stays constant. To ensure reachability, one should show that the measure of a set is strictly increasing along the solutions.

IV. Application of the Sum of Squares Algortihm to the Verification of Safe Reachibility

In this section, we will illustrate the theoretical results obtained in the previous section with the aid of an example and we will give a brief summary of SoS with Putinar positivstellensatz. The Putinar positivstellensatz can be interpreted as follows: On a compact semi algebraically defined set $K = \{x \in \mathbb{R}^n | p_1(x) \geq 0, p_2(x) \geq 0, \dots, p_n(x) \geq 0\}$ to certify the non-negativity of a polynomial q is same as finding a sum of representation of q in the form $q = s_0 + s_1 p_1 + s_2 p_2 + \dots + s_n p_n$ for some sum of square polynomials $s_0, s_1, \dots, s_n$. Then, we can guarantee that the polynomial q is non-negative on the given sets. For this reason, in the

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 April 2019 doi:10.20944/preprints201904.0302.v1

example, we will use SoS together with Putinar Positivstellensatz [25] to search for a common Barrier density satisfying some non-negativity and non-positivity properties on the given sets since the given sets are semi algebraically defined and the vector fields are polynomials. The main advantage of using SoS with Putinar positivstellensatz is that it allows us to look for non-negative polynomials over a region which can be defined semi algebraically.

In the example, the search of a Barrier density is done by means of SOSTOOLS, a sum of squares programming solver and SDPT3, a semi-definite programming solver.

In the following, we use a periodic switching signal as $\{(\Delta t_1, p_1), \ldots, (\Delta t_n, p_n)\}$, that is $\{(\Delta t_1, p_1), \ldots, (\Delta t_n, p_n), (\Delta t_1, p_1), \ldots\}$. Thus, the system has a switching signal with a period $\Delta t_1 + \ldots + \Delta t_N$.

Example 1: (Example for a common Barrier density) Let us consider a nonlinear switched system (2) with the following subsystems:

$$f_1(x) = \begin{bmatrix} x_2 \\ -x_1 - x_2 + \frac{x_2^3}{27} \end{bmatrix},$$

$$f_2(x) = \begin{bmatrix} x_2 \\ -\frac{3x_1}{2} - x_2 + \frac{x_2^3}{9} \end{bmatrix}.$$

A set of initial state, a set of undesired states (unsafe set), a set of desired states (reachable set) and a domain can be defined as

$$X_0 := \{x \in \mathbb{R}^2 | p_1(x) := -(x_1 - 1.5)^2 - x_2^2 + 0.5^2 \geq 0\},$$
$$X_u := \{x \in \mathbb{R}^2 | p_2(x) := -(x_1 + 0.7)^2 - (x_2 + 1)^2 + 0.5^2 \geq 0\},$$
$$X_r := \{x \in \mathbb{R}^2 | p_3(x) := -x_1^2 - x_2^2 - 0.5^2 \geq 0, \},$$
$$X := \{x \in \mathbb{R}^2 | p_4(x) := -x_1^2 - x_2^2 + 3^2 \geq 0\},$$

respectively. To ensure that all solutions of the system lie in X, the following set is needed

$$X_d = \{x \in \mathbb{R}^2 | p_5(x) := -x_1^2 - x_2^2 + 3.5^2 \geq 0\}.$$

Moreover, the set $X_d \setminus X$ can be implemented to the conditions of Theorem 1 as a set of undesired states; i.e., by replacing the set X^c in condition (7b) with $X_d \setminus X$.

Considering the above given sets, the conditions (7a)-(7c) of Theorem 1 can be converted to the following form:

- $\rho(x) > 0$, for all $x \in X_0$,
- $\nabla \cdot (\rho(x) f_i(x)) > 0$, $i = 1, 2$, for all $x \in X \setminus X_r$, and
- $\rho(x) \leq 0$ for all $x \in (X_d \setminus X) \cup X_u$.

The problem of finding such a function ρ as a polynomial can be interpreted as looking for an SoS representation by using Putinar positivstellensatz on the semi algebraically defined given sets with the polynomial vector fields.

By searching ρ, as a polynomial and using SoS algorithm together with Putinar Positivstellensatz, we will verify the safely reachability of the system by using common Barrier density. The conditions (7a)-(7c) of Theorem 1 can be converted to SoS algorithm with the Putinar positivstellensatz as:

- $\text{sos}(\rho - s_1 p_1 - \varepsilon)$,
- $\text{sos}(-\rho - s_2 p_2)$,
- $\text{sos}(-\rho - s_3 p_5 + p_4 s_4)$,
- $\text{sos}(\nabla \cdot (f_1)\rho + \nabla\rho \cdot f_1 - s_5 p_4 + s_6 p_3 - \varepsilon)$ and
- $\text{sos}(\nabla \cdot (f_2)\rho + \nabla\rho \cdot f_2 - s_7 p_4 + s_8 p_3 - \varepsilon)$,

where s_i, $i = 1, 2, \ldots, 8$ are sum of square polynomials and ε is a sufficiently small positive number. We have obtained a tenth degree polynomial ρ by applying the above SOS algorithm with a tolerance $\varepsilon = 0.005$. We use such tolerance to guarantee that the density is positive on X_0 and $\nabla \cdot (f_p \rho) > 0$ on $X \setminus X_r$. Thus, the existence of barrier density proves that the system is weak safely reachable from X_0 to X_r avoiding X_u. Additionally, given sets X_0 and X_u satisfy the condition $\overline{\text{Int } (X_0)} = X_0$ and $\overline{\text{Int } (X_u)} = X_u$ of Corollary 1, then we can say that the system is safely reachable from X_0 to X_r avoiding X_u.

Figure 1 is obtained by using the given subsystems with a switching signal $\{(0.4, 1), (0.3, 2)\}$. The dashed curve in Figure 1 is drawn to indicate the places where $\rho = 0$. In the inside of the dashed curve where the measure of the set of solutions starting from X_0 is positive and at the outside of the curve where the undesired states present, the density ρ takes negative values. We can say that all the states satisfying $\rho(x) = 0$ can be seen as a barrier between the undesired states and desired states. Moreover, the inside of the dashed curve, where $\rho(x) = 0$, is an invariant region for the solutions which are initiated from X_0. The undesired set is taken as close as possible to the barrier $\rho(x) = 0$ by checking feasibility of the SoS algorithm given above.

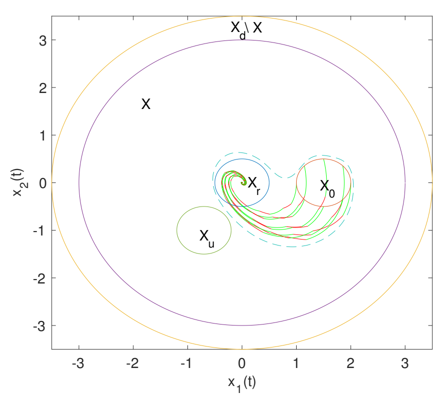

Fig. 1. A set of solutions of the system (2) with $\mathcal{S} = \mathcal{S}_{0.3}$ defined by subsystems given in Example 1 which start from the boundary of the set of initial states X_0 and reach X_r without arriving the unsafe set X_u is depicted. The conditions of Theorem 1 are satisfied. Thus, the system is safely reachable for each switching signal $\sigma \in \mathcal{S}_{\text{nonchatt}}$. In the region between purple circle and yellow circle, $X_d \setminus X$, ρ is taken to be negative to ensure that no solution starting from X_0 leave the region X.

V. Conclusion

We have extended the idea of verification of safe reachability to switched nonlinear systems with time dependent switching via a common barrier density. We have shown that the safely reachability analysis can be carried out by means of SoS programming together with Putinar positivstellensatz.

 doi:10.20944/preprints201904.0302.v1

As a further work, the proposed method for the verification of the safe reachability of nonlinear switched systems can be extended for switched systems with state dependent switching. Furthermore, common barrier density approach can be used for model invalidation and for detection of the faults in a system.

REFERENCES

[1] S. Prajna and A. Rantzer, "Convex programs for temporal verification of nonlinear dynamical systems," *Siam Journal on Control and Optimization*, vol. 46, no. 3, pp. 999–1021, 2007.

[2] A. J. Barry, A. Majumdar, and R. Tedrake, "Safety verification of reactive controllers for uav flight in cluttered environments using barrier certificates," in *2012 IEEE International Conference on Robotics and Automation*, May 2012, pp. 484–490.

[3] J. Anderson and A. Papachristodoulou, "On validation and invalidation of biological models," *BMC Bioinformatics*, vol. 10, no. 1, p. 132, May 2009. [Online]. Available: https://doi.org/10.1186/1471-2105-10-132

[4] R. L. Christensen, R. Wisniewski, and K. K. Srensen, "Safety verification of refrigeration containers using barrier certificates," in *2016 IEEE Conference on Computer Aided Control System Design (CACSD)*, Sep. 2016, pp. 635–640.

[5] O. Bouissou, A. Chapoutot, A. Djaballah, and M. Kieffer, "Computation of parametric barrier functions for dynamical systems using interval analysis," in *53rd IEEE Conference on Decision and Control*, Dec 2014, pp. 753–758.

[6] A. Rantzer and S. Prajna, "On analysis and synthesis of safe control laws," in *Proceedings of the Allerton Conference on Communication, Control, and Computing*, 2004.

[7] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, Aug 2017.

[8] C. Sloth and R. Wisniewski, "Safety analysis of stochastic dynamical systems," *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 62 – 67, 2015, analysis and Design of Hybrid Systems ADHS.

[9] S. Prajna, A. Jadbabaie, and G. J. Pappas, "A framework for worst-case and stochastic safety verification using barrier certificates," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1415–1428, Aug 2007.

[10] U. Vaidya and P. G. Mehta, "Lyapunov measure for almost everywhere stability," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 307–323, 2008.

[11] U. Vaidya, P. G. Mehta, and U. V. Shanbhag, "Nonlinear stabilization via control lyapunov measure," *IEEE Transactions on Automatic Control*, vol. 55, no. 6, pp. 1314–1328, 2010.

[12] A. Rantzer, "A dual to Lyapunov's stability theorem," *Systems & Control Letters*, vol. 42, pp. 1–17, 2001.

[13] A. Rantzer and F. Ceragioli, "Smooth blending of nonlinear controllers using density functions," in *Proceedings of the 2001 European Control Conference*, 2001, pp. 2851–2853.

[14] P. Monzon, "Almost global attraction in planar systems," *Systems & Control Letters*, vol. 54, pp. 753–758, 2005.

[15] D. Angeli, "An almost global notion of input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 49, no. 6, pp. 866–874, JUN 2004.

[16] Ö. Karabacak, R. Wisniewski, and A. Kıvılcım, "On the almost global stability of invariant sets," in *Proceedings of 57th IEEE Conference on Decision and Control*, 2018, pp. ..–..

[17] I. Masubuchi and T. Kikuchi, "Lyapunov density for almost attraction of nonlinear time-varying systems: A condition without assuming local stability," in *2017 25th Mediterranean Conference on Control and Automation (MED)*, July 2017, pp. 169–173.

[18] I. Masubuchi and Y. Ohta, "Analysis of almost-everywhere stability of a class of discontinuous systems via lyapunov densities," in *2016 European Control Conference (ECC)*, June 2016, pp. 567–574.

[19] S. Prajna, P. A. Parrilo, and A. Rantzer, "Nonlinear control synthesis by convex optimization," *IEEE Transactions on Automatic Control*, vol. 49, no. 2, pp. 310–314, 2004.

[20] D. Liberzon, *Switching in systems and control*. Springer Science & Business Media, 2003.

[21] D. Hristu Varsakelis, W. S. Levine, R. Alur, K.-E. Arzen, J. Baillieul, and T. A. Henzinger, *Handbook of Networked and Embedded Control Systems (Control Engineering)*. Birkhauser, 2005.

[22] M. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, 1998.

[23] Ö. Karabacak, A. Kıvılcım, and R. Wisniewski, "Almost global stability of nonlinear switched systems with time-dependent switching," *Preprints*, no. 2018090514, 2018.

[24] A. Kıvılcım, Ö. Karabacak, and R. Wisniewski, "Safety verification of nonlinear switched systems via barrier functions and barrier densities," in *Proceedings of the 2019 European Control Conference (accepted)*, 2019.

[25] J. B. Lasserre, *Moments, Positive Polynomials and Their Applications*, ser. Imperial College Press Optimization Series. Imperial College Press, London, 2010, vol. 1.

[26] V. I. Arnold, *Mathematical Methods of Classical Mechanics*. New York: Springer-Verlag, 1978.

[27] Ö. Karabacak, R. Wisniewski, and J. Leth, "On the almost global stability of invariant sets," in *Proceedings of the 2018 European Control Conference*, 2018.