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Article

Riemannian Geometry Framed as a Generalized Heisenberg Lie Algebra

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Abstract: The Heisenberg Lie algebra (HA) plays an important role in mathematics with Fourier transforms, as well as for the foundations of quantum theory where it expresses the operators of space-time, X, and their commutation rules with the momentum operators, D, that execute infinitesimal translations in X. Yet it is known that space-time is curved and thus the D operators must interfere thus giving "structure constants" that vary with location which suggests a mathematical generalization of the concept of a Lie algebra to allow for "structure constants" that are functions of X. We here investigate the mathematics of such a "generalized Heisenberg algebra" (GHA) which has "structure constants" that are functions of X and thus are in the enveloping algebra rather than constants. As expected, the Jacobi identity no longer holds globally but only in small regions of space-time where the [D, X] commutator can be considered locally constant and thus where one has a true Lie algebra. We show that one is able to reframe Riemannian geometry in this GHA. As an example, it is then shown that one can express the Einstein equations of general relativity as commutation rules. If one requires that the GHA commutator reduces to the HA of quantum theory in the limit of no curvature, then there are observable effects for quantum theory in this curved space time.

Keywords: Riemannian Geometry, Lie algebra, Quantum Theory, Metric, Heisenberg General Relativity

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1. Introduction

Lie algebras and the Lie groups which they generate have played a central role in both mathematics and theoretical physics since their introduction by Sophius Lie in 1888 [1]. Both relativistic quantum theory (QT) and the phenomenological standard model (SM) of particles and their interactions are framed in terms of observables which form Lie algebras and are firmly established [2,3,4,5]. A prime example is the Heisenberg Lie algebra (HA) among position operators, X, and operators, D, which translate one in the space of the X operators. The HA has applications also in mathematics in studies related to Fourier transforms and harmonic analysis [6,7,8,9]. But the theory of gravitation as expressed in Einstein's general theory of relativity (GR), although also firmly established, is formulated in terms of a Riemannian geometry (RG) of a curved space-time where the metric is determined by nonlinear differential equations from the distribution of matter and energy [10,11]. Since the space-time has a curvature depending upon one's position, it follows that the actions of the infinitesimal translation D operators will interfere with each other and the commutators may vary depending upon position. This suggests the generalization of a Lie algebra to allow for structure constants that are functions of the X operators in the algebra and thus are no longer constants except approximately in small neighborhoods.

This paper reframes the mathematics of RG [13] in terms of such a generalized Heisenberg Lie algebra, (GHA). We show that the fundamental concepts in RG such as the coordinate transformations, contravariant and covariant tensors, Christoffel symbols, Riemann and Ricci tensors, and the covariant derivative can be expressed in terms of commutators in a GHA. This framework is reminiscent of contractions of Lie algebras where the structure constants are modified to vary smoothly among different algebras based upon certain parameters [14, 15, 16, 17, 18, 19] which are non-spatial variables. In a similar way, our generalized Lie algebra allows the structure constants to be dependent upon the X operators in the algebra so that RG is retrieved as a representation of the algebra as one moves over the Riemann space. We then are able to frame the equations of general relativity as commutators in such a GHA.

2. Materials and Methods

Consider a set of n independent linear self adjoint operators, X^μ, which form an Abelian Lie algebra of order n, where

$$[X^\mu , X^\nu] = 0 \text{ and where } \mu, \nu = 0, 1, 2, \dots (n\text{-}1). \tag{1}$$

Consider a Hilbert space of square integrable complex functions $|\Psi>$ as a representation space for this algebra where a scalar product is used to normalize the vectors to unity: $<\Psi | \Psi> =1$. The simultaneous eigenvectors of the Abelian Lie algebra, which is to serve as a complete basis for this space, can be written as the outer product of the X^μ eigenvectors with the Dirac notation

$$|y^0> |y^1>|y^2> \dots|y^{n-1}> = | y^0, y^1, y^2, \dots y^{n-1} > = |y> \tag{2}$$

where the eigenvalues y^μ label the associated eigenvectors $|y>$ of the X^μ operators.
We use the notation

$$X^\mu|y> = y^\mu|y> \text{ where eigenvalues } y^\mu \text{ of the operators } X^\mu \text{ are real numbers.} \tag{3}$$

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These independent real variables y^μ can be thought of as the coordinates of an n-dimensional space R_n since each set of values defines a point in R_n. Let the eigenvectors be normalized to be orthonormal with the Dirac function scalar product

$$<y_a|y_b> = \delta(y^0{}_a - y^0{}_b)\, \delta(y^1{}_a - y^1{}_b) \ldots \delta(y^{n-1}{}_a - y^{n-1}{}_b). \qquad (4)$$

Let the decomposition of unity

$$1 = \int_{dy} |y><y| \qquad (5)$$

project the entire space onto the basis vectors $|y>$ where $<y|$, using Dirac notation, is the dual vector to $|y>$. A general vector in the representation (Hilbert) space of this Lie algebra can then be written as

$$|\Psi> = \int_{dy} |y><y|\Psi> \; = \int_{dy} \Psi(y)\, |y>, \qquad (6)$$

where the function $\Psi(y)$ gives the "components" of the abstract vector $|\Psi>$ on the basis vectors $|y>$. Thus

$$<\Psi|\Psi> = 1 = \int_{dy} <\Psi|y><y|\Psi> = \int_{dy} \Psi^*(y)\Psi(y). \qquad (7)$$

Now consider another set of n linear operators, X'^μ, which are independent analytic functions, $X'^\mu(X^\mu)$, of the X^μ operators also forming an Abelian Lie algebra on the same representation space for this algebra where it follows that

$$[X'^\mu, X'^\nu] = 0. \qquad (8)$$

Let the X'^μ have eigenvectors $|y'>$ and eigenvalues y'^μ given by

$$X'^\mu|y'> = y'^\mu|y'> \text{ where } y'^\mu \text{ are real numbers}. \qquad (9)$$

The same orthonormality and decomposition of unity also obtain for the $|y'>$ vectors which are also a complete basis for the space. Then we can let the $X'^\mu(X^\nu)$ act to the left on the dual vector $<y'|$ and also act to the right on the vector $|y>$ as

$$<y'|\, X'^\mu(X^\nu)|y> = <y'|\, X'^\mu(X^\nu)|y> \text{ to give} \qquad (10)$$

$$y'^\mu<y'|y> = y'^\mu(y)<y'|y>. \qquad (11)$$

Thus the eigenvalues $y'^\mu = y'^\mu(y)$ give the transformation from the y coordinates to the y' coordinates if the Jacobian does not vanish i.e. $|\partial y'^\mu/\partial y^\nu| \neq 0$ which we require to be the case. Thus the operators $X'^\mu(X^\mu)$ define a coordinate transformation in R_n between the eigenvalues (coordinates) y and the eigenvalues y' (transformed coordinates) R_n'. Then the set of n real variables y^μ and the alternative variables y'^μ both can be interpreted as specifying the coordinates of points in this n-dimensional real space R_n with coordinate transformations given by the functions

$$y'^\mu = y'^\mu(y). \qquad (12)$$

It now follows that

$$dy'^\mu = (\partial y'^\mu/\partial y^\nu)\, dy^\nu \qquad (13)$$

and any set of n functions $V^\mu(y)$ that transform as the coordinates,

$$V'^\mu(y') = (\partial y'^\mu/\partial y^\nu)\, V^\nu(y) \text{ is to be called a contravariant vector.} \qquad (14)$$

We use the summation convention for repeated identical indices. The derivatives $\partial/\partial y^\nu$ transform as

$$\partial/\partial y'^\mu = (\partial y^\nu/\partial y'^\mu)\, \partial/\partial y^\nu \qquad (15)$$

and any such vector $V_\mu(y)$ which transforms in this manner as

$$V'_\mu(y') = (\partial y^\nu/\partial y'^\mu)\, V_\nu(y) \text{ is defined as a covariant vector.} \qquad (16)$$

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Upper indices are defined as contravariant indices while lower indices are covariant indices. Functions with multiple upper and lower indices that transform as the contravariant and covariant indices just shown are defined as tensors of the rank of the associated indices.

One would like to have transformations that translate one in the R_n space of the X operators (and thus their eigenvalues y). We now define a new additional set of n operators, D^μ, that by definition are to translate an infinitesimal distance, ds, respectively in each corresponding directions y^μ by using the group generated by the elements D^μ of the algebra via the exponential map with transformations:

$$G(ds, \eta) = \exp(ds\,(-i/\hbar)\eta_\mu D^\mu)\ . \tag{17}$$

In this transformation η_μ is to be a unit vector in the y space, ħ is a real constant, and ds is defined to be the distance moved in the direction η_μ as defined below. Then

$$X'^\lambda = G\, X^\lambda\, G^{-1}. \tag{18}$$

By taking ds to be infinitesimal, then to one gets

$$\begin{aligned} X'^\lambda = X^\lambda(s+ds) &= \exp(\, ds\,(-i/\hbar)\,\eta_\mu D^\mu)\, X^\lambda(s)\, \exp(-\, ds\,(-i/\hbar)\,\eta_\nu D^\nu) \\ &= (1 + ds\,(-i/\hbar)\,\eta_\mu D^\mu)\, X^\lambda(s)\, (1 - ds(-i/\hbar)\,\eta_\nu D^\nu) \\ &= X^\lambda(\,s\,) + ds\,(-i/\hbar)\,\eta_\mu\,[\, D^\mu, X^\lambda] + \text{higher order in ds.}, \end{aligned} \tag{19}$$

Thus the commutator $[D^\mu, X^\lambda]$ defines the way in which the transformations commute (interact) with each other in executing the transformation in keeping with the theory of Lie algebras and Lie groups. If the space is flat then there is no dependence of the commutator upon location, and thus there is no interference among the D^μ. Then $[D^\mu, X^\lambda]$ can be normalized to $I\,\delta_\pm^{\mu\lambda}$ (since D^μ is defined to translate X^μ) thus

$$[D^\mu, X^\lambda] = I\,\delta_\pm^{\mu\lambda} \tag{20}$$

where $\delta_\pm$ is the diagonal n x n matrix with ±1 on the diagonal with off-diagonal terms zero. This is the customary Heisenberg Lie algebra with structure constants $\delta_\pm^{\mu\lambda}$ and with $[D^\mu, D^\lambda] = 0$ for $\mu \neq \lambda$. The operator I commutes with all elements, by definition has a single eigenvalue iħ, and is needed to close the basis of the Lie algebra which is now of dimension 2n+1. Thus in the position representation

$$dy^\lambda(\,s\,) = ds\,(-i/\hbar)\,\eta_\mu\,(i\hbar)\,\delta_\pm^{\mu\lambda} = ds\,\eta^\lambda + \text{higher order terms in ds.} \tag{21}$$

We now wish to allow for curvature in the space R_n of the X eigenvalues. Thus the [D, X] commutator is now allowed to be dependent upon the X operators and can vary from point to point in the space. We define the functions $g^{\mu\nu}(X)$ as generalized structure constants:

$$[D^\mu, X^\nu] = I\, g^{\mu\nu}(X) \quad \text{(with the requirement that } |g| \neq 0\text{)} \tag{22}$$

where I has the single eigenvalue iħ with the commutators

$$[D^\mu, I\,] = 0 = [X^\mu, I\,], \text{ and } [X^\mu, X^\nu] = 0\ . \tag{23}$$

These (generalized structure constant) functions can also be written as

$$g^{\mu\nu}(X) = (-i/\hbar)\,[D^\mu, X^\nu] \tag{24}$$

where $g^{\mu\nu}(X)$ are assumed to be analytic with $g_{\mu\nu}(X)$ defined by

$$g_{\mu\alpha}(X)\, g^{\alpha\nu}(X) = \delta_\mu{}^\nu. \tag{25}$$

Then using (19) one gets

$$y^\mu(s+ds) - y^\mu(s) = dy^\mu = ds\,\eta_\lambda\, g^{\mu\lambda}(y) \quad = ds\,\eta^\mu. \tag{26}$$

Then

$$g_{\mu\nu}(y)\, dy^\mu\, dy^\nu = ds^2\, g_{\mu\nu}(y)\,\eta^\mu\,\eta^\nu = ds^2 \text{ since } \eta^\mu \text{ is a unit vector on this metric} \tag{27}$$

Thus it follows that

$$ds^2 = g_{\mu\nu}(y)\, dy^\mu\, dy^\nu \quad \text{showing that } g_{\mu\nu}(y) \text{ is the metric for the space.} \tag{28}$$

One notes that $g_{\mu\nu}(X)$ can have an antisymmetric component as well as a symmetric. But only the symmetric portion of $g_{\mu\nu}(X)$ contributes to the metric for the space since it is contracted with the symmetric form $dX^\mu\, dX^\nu$. The antisymmetric component of $g_{\mu\nu}(X)$ can however support a torsion (twisting) for the transformation although not contributing to the distance function ds. We thus obtain a 2n+1 dimensional "generalized Lie algebra" (GLA) with D^μ, X^ν, and I as the basis elements of the algebra. One notes that the commutator $[D^\mu, D^\nu]$ has not yet been defined. D^μ can be represented on the basis vectors of the Hilbert representation space where X^ν is diagonal as

$$\langle y|[D^\mu , X^\nu] = \langle y|\, i\hbar\, g^{\mu\nu}(X) \quad \text{as} \tag{29}$$

$$\langle y|\, D^\mu = (i\hbar g^{\mu\beta}(y)\partial/y^\beta + A^\mu(y))\langle y| = (i\hbar\, \partial^\mu + A^\mu(y)) \text{ where } \partial^\mu = g^{\mu\nu}(y)\, (\partial/\partial y^\nu) \tag{30}$$

as the representation of D^μ on the space of eigenvectors $\langle y|$ and where $A^\mu(y)$ is an arbitrary collection of vector functions of y. Note that this arbitrary vector function $A^\mu(y)$ can include other terms such as $i\hbar\, g^{\mu\beta}(y)\, \partial\Lambda(y)/y^\beta$. So one could write

$$D^\mu = D^\mu + A^\mu(X) \tag{31}$$

in the commutators with X as this would not alter the commutation rules of D with X. This is the most general representation of the commutation rules with the operators available using the scalar, vector and second rank tensor representations. Both the vector function $A^\mu(y)$ and a scalar function $\Lambda(y)$ could consist of multiple higher order tensor components including $g^{\mu\nu}(X)$, arbitrary scalar functions, arbitrary contravariant vector function $A^\mu(X)$ and derivatives of such objects because any contravariant vector function of the X^μ will commute with the X in the defining commutator of D and X.

The $A^\mu(y)$ can also support a Yang Mills gauge transformation group, acting simultaneously on the representation space $|\Psi\rangle$, and on the $A^\mu(y)$ vector functions. In that case the $A^\mu(y)$ will have the commutation rules of that algebra with additional indices supporting Yang Mills gauge transformations. If that gauge algebra were to be extended to include $g^{\mu\nu}(X)$ then the commutators are more complex.

Since $[D^\mu, X^\nu] = I\, g^{\mu\nu}(X)$, this is a generalization of the normal definition of a Lie algebra since $g^{\mu\nu}(X)$ is now a function of the position operators, X which, in the position representation $|y\rangle$, become the eigenvalues which determine the position in the n dimensional space. Consequently, this "generalized Lie Algebra" has "structure constants", $g^{\mu\nu}(y)$, which vary from point to point in the space. From now on we assume the general case where $g^{\alpha\beta} = g^{\alpha\beta}(y)$ is to be understood in the position representation.

In the position representation one now has

$$\langle y|\, D^\mu|\Psi\rangle = (i\hbar\, g^{\mu\upsilon}(y)\, (\partial/\partial y^\nu) + A^\mu (y)))\, \Psi(y) = (\, i\hbar\, \partial^\mu + A^\mu (y))\, \Psi(y) \tag{32}$$

$$\text{where } \Psi(y) = \langle y|\Psi\rangle \quad \text{and} \tag{33}$$

$$\partial^\mu = g^{\mu\upsilon}(y)\, (\partial/\partial y^\nu) \tag{34}$$

and $A^\mu(y)$ is a yet undetermined vector function of X^ν. In the position representation, one can write

$$g^{\mu\nu}\, (\partial /\partial y^\nu)\, \psi(y) \langle y| = \partial^\mu\, \psi (y) \langle y| = \langle y|\, (-i/ \hbar)\, [D^\mu, \psi(X)] \tag{35}$$

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for any function ψ (X) allowing one to convert differential operators into commutators with D^μ. It follows that $[D^\mu, [D^\nu, X^\lambda]] \neq 0$ so that this Heisenberg algebra is no longer nilpotent. But instead one gets

$$<y|\, [D^\mu, [D^\nu, X^\lambda]] = (i\hbar)^2\, g^{\mu\alpha}\, (\partial g^{\upsilon\lambda}/\partial y^\alpha)\, <y| \text{ since} \tag{36}$$

$$[A^\mu, g^{\alpha\beta}] = 0 \tag{37}$$

as they both are only functions of X. We have not specified the commutators $[D^\mu, D^\nu]$ yet as they are no longer zero but which in the position representation give

$$<y|\, [D^\mu, D^\nu] = [(i\hbar\, g^{\mu\alpha}(y)(\partial/\partial y^\alpha) + A^\mu(y)), (i\hbar\, g^{\upsilon\beta}(y)\, (\partial/\partial y^\beta) + A^\nu(y))]\, <y| \tag{38}$$

$$= (-\hbar^2\, (\, g^{\mu\alpha}(y)\, (\partial g^{\upsilon\beta}(y)\, /\partial y^\alpha)\, (\partial/\partial y^\beta) - g^{\nu\beta}(y)\, (\partial g^{\mu\alpha}(y)\, /\partial y^\beta)\, (\partial/\partial y^\alpha) + g^{\mu\alpha}(y)\, g^{\upsilon\beta}(y)$$
$$(\partial/\partial y^\alpha)\, (\partial/\partial y^\beta) - g^{\nu\alpha}(y)\, g^{\mu\beta}(y)\, (\partial/\partial y^\alpha)\, (\partial/\partial y^\beta)\,) + [D^\mu, A^\nu] + [A^\mu, A^\nu\,])\, <y|. \tag{39}$$

The third and fourth terms cancel and the last term vanishes allowing one to re-express the D commutator as

$$<y|[D^\mu, D^\nu] = (-\hbar^2 (g^{\mu\alpha}(y)(\partial g^{\upsilon\beta}(y)/\partial y^\alpha) - g^{\nu\alpha}(y)(\partial g^{\mu\beta}(y)/\partial y^\alpha))(\partial/\partial y^\beta) + [D^\mu, A^\nu])\, <y| \tag{40}$$

One can write $(\partial/\partial y^\beta) = -(i/\hbar)\, D_\beta$ to get (41)

$$[D^\mu, D^\nu]\, <y| = (\, i\hbar\, B^{\mu\nu\beta}\, D_\beta + [D^\mu, A^\nu\,]\,)\, <y| \tag{42}$$

$$= (\, i\hbar\, B^{\mu\nu}{}_\beta\, D^\beta + [D^\mu, A^\nu])\, <y| \tag{43}$$

But since this is true on all states $<y|$, it follows that

$$[\, D^\mu, D^\nu] = i\hbar\, B^{\mu\nu}{}_\gamma\, D^\gamma + [D^\mu, A^\nu] \text{ where we define} \tag{44}$$

$$B^{\mu\nu}{}_\gamma = (g^{\mu\alpha}(y)\, (\partial g^{\upsilon\beta}(y)\, /\partial y^\alpha) - g^{\nu\alpha}(y)\, (\partial g^{\mu\beta}(y)\, /\partial y^\alpha))\, g_{\beta\gamma}(y) \tag{45}$$

and where these "structure constants" depend upon the both the metric and its derivatives. The term $[A^\mu, A^\nu]$ is zero unless A^μ contains additional operators such as with a Yang Mills gauge transformation. One also notes in the following, that since $[A^\mu, g^{\nu\alpha}(X)] = 0$, the A terms will no longer be present.

3. Results

The Christoffel symbols are given by

$$\Gamma_{\gamma\alpha\beta} = (½)\, (\partial_\beta, g_{\gamma\alpha} + \partial_\alpha, g_{\gamma\beta} - \partial_\gamma, g_{\alpha\beta}) \tag{46}$$

and can be written in the position diagonal representation, in terms of the commutators of D with the metric as

$$\Gamma_{\gamma\alpha\beta} = (½)\, (-i/\hbar)\, ([D_\beta, g_{\gamma\alpha}] + [D_\alpha, g_{\gamma\beta}] - [\, D_\gamma, g_{\alpha\beta}]\,). \tag{47}$$

Then using

$$g_{\alpha\beta}\, (X) = (-i/\, \hbar)\, [D_\alpha, X_\beta] \text{ one obtains} \tag{48}$$

$$\Gamma_{\gamma\alpha\beta} = (-½)\, (1/\hbar^2)\, (\, [D_\beta, [D_\gamma, X_\alpha]] + [D_\alpha, [D_\gamma, X_\beta]] - [\, D_\gamma, [D_\alpha, X_\beta]\,]\,). \tag{49}$$

The Riemann tensor then becomes

$$R_{\lambda\alpha\beta\gamma} = (-i/\hbar)\, ([D_\beta, \Gamma_{\lambda\alpha\gamma}\,] - [D_\gamma, \Gamma_{\lambda\alpha\beta}\,]\,) + (\Gamma_{\lambda\beta\sigma}\Gamma^\sigma{}_{\alpha\gamma} - \Gamma_{\lambda\gamma\sigma}\Gamma^\sigma{}_{\alpha\beta}\,) \tag{50}$$

where $\Gamma_{\gamma\alpha\beta}$ is to be inserted for the Christoffel symbols using (49) giving only commutators. One then defines the Ricci tensor using (50) for the Riemann tensor as

$$R_{\alpha\beta} = g^{\mu\nu}\, R_{\alpha\mu\beta\nu} = (-i/\, \hbar)\, [D^\mu, X^\nu]\, R_{\alpha\mu\beta\nu} \text{ and also defines} \tag{51}$$

$$R = g^{\alpha\beta}\, R_{\alpha\beta} = (-i/\, \hbar)\, [D^\alpha, X^\beta]\, R_{\alpha\beta}. \tag{52}$$

where the D is not to act on the Riemann or Ricci tensor.

It is well known that the ordinary derivative of a scalar function, $V_\mu = \partial\Lambda/\partial y^\mu$, in Riemann geometry will transform under arbitrary coordinate transformations as a covariant

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vector. But such a derivative of a vector function of the coordinates will not transform as a tensor. The covariant derivative with respect y^{ν} of a contravariant vector A^{μ} is

$$A^{\mu}{}_{,\nu} = \partial A^{\mu} / \partial y^{\nu} + A^{\sigma} \Gamma^{\mu}{}_{\sigma\nu} \quad (53)$$

and the covariant derivative of a covariant vector A_{μ} is given by

$$A_{\mu,\nu} = \partial A_{\mu} / \partial y^{\nu} - A_{\sigma} \Gamma^{\sigma}{}_{\mu\nu} \quad (54)$$

where both $A^{\mu}{}_{,\nu}$ and $A_{\mu,\nu}$ transform as tensors with respect to the metric $g^{\alpha\beta}$.

One recalls for Riemannian geometry that there is a Christoffel symbol on the right hand side for each index of the tensor being differentiated. In our algebraic framework one can write the covariant differentiation of a contravariant vector A^{μ} as:

$$A^{\mu}{}_{,\nu} = i\,[D_{\nu},A^{\mu}] + (-\tfrac{1}{2})A^{\sigma}\,([D_{\nu},[D^{\mu},X_{\sigma}]] + [D_{\sigma},[D^{\mu},X_{\nu}]] - [D^{\mu},[D_{\sigma},X_{\nu}]]) \quad (55)$$

assuming that A is at most a function of the X operators. Thus we are able to write both the regular derivative (first term) and complete it with the index contraction with the Christoffel symbol (second term). It is important to distinguish this covariant differentiation from the regular differentiation that occurs as a representation of the operator D^{μ} in the position representation. It follows that we can write the covariant derivative of any tensor in the same way but with a contraction of the Christoffel symbol with each of the tensor indices as is well known in Riemannian geometry.

Finally, the generalization of the Fourier transform follows from $<y|D^{\mu}|k> = <y|D^{\mu}|k>$ where the D^{μ} acts first to the left on the bra vector and then to the right on the ket vector which is to be an eigenstate of D^{μ} with eigenvalue k^{μ} giving the differential equation:

$$(i\hbar\, g^{\mu\upsilon}(y)\, (\partial/\partial y^{\nu}) + A^{\mu}(y)\,) <y|k> \;= (k^{\mu} + A^{\mu}(y)) <y|k>. \quad (56)$$

When there is no vector field A^{μ} present and when $g^{\mu\upsilon}$ is constant (no y dependence & Minkowski metric), then this can be solved (with normalization for a four dimensional space-time) with:

$$<y|k> \;= (2\,\pi\,)^{-2} \exp\,(g_{\mu\nu}\, y^{\mu} k^{\nu}\,). \quad (57)$$

But in the general case with $g^{\mu\upsilon}(y)$ as a function of y this is no longer a solution and in the general case one cannot solve this equation except formally. In fact, since the D^{μ} do not commute among themselves, one does not generally have a complete set of simultaneous D^{μ} eigenvectors. However, one can consider very small regions of space where the metric is effectively a constant and giving the traditional Fourier transform. Then the general solution would be approximately the smoothing of these local traditional solutions into a global solution maintaining functional and derivative continuity.

4. Discussion of Applications to General Relativity

In general relativity the Einstein equations

$$R_{\alpha\beta} - \tfrac{1}{2}\, g_{\alpha\beta}\, R + g_{\alpha\beta}\Lambda = (8\,\pi\, G/c^{4})\, T_{\alpha\beta} \quad \text{become} \quad (58)$$

$$R_{\alpha\beta} + ((i/\hbar)\,[D_{\alpha}, X_{\beta}])\,(\tfrac{1}{2} R - \Lambda) = (8\,\pi\, G/c^{4})\, T_{\alpha\beta} \quad (59)$$

where $R_{\alpha\beta}$ and R are now given in terms of commutators as shown above in (51) and (52) while $T_{\alpha\beta}$ is the energy-momentum tensor. <u>Thus all terms on the LHS consist only of commutators of operators, and (59) is an exact reproduction of the Einstein equations in GR expressed totally as GLA commutators</u> thus framing GR as a generalized Lie algebra making it more mathematically compatible with quantum theory and standard model. In a strong gravitational field near a star, such as a non-rotating white dwarf, one can treat the metric as

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constant using the Schwarzschild solution over a very small region such as atomic dimensions. The radial direction can be taken as the y^1 direction as the distance to the center of the star, with

$$g_{00} = (1 - r_s/y^1) \quad \text{and } g_{11} = -1/(1 - r_s/y^1) \qquad (60)$$

$$\text{where } r_s = 2GM/c^2 \text{ with } g_{22} = g_{33} = -1 \qquad (61)$$

and where G is the gravitational constant, M is the mass of the star, c is the speed of light, and y^1 is the distance to the center of the star.

One would now use traditional quantum field theory with the standard model intact, with all fields quantized as creation and annihilation operators as representations of the Poincare algebra. Then the gravitational field $g_{\alpha\beta}$ now included in D_β, would be quantized as the spin 2, ($b_0 = 0$, $b_1 = 3$ symmetric tensor Poincare representation) massless field determined by equation (59) with other spin and helicity components gauged away. Then the RHS would be expressed in terms of the symmetrized operator $T_{\alpha\beta} = \gamma_\alpha D_\beta$ operator where D_β not only contains the vector fields of the standard model but now also contains the gravitational tensor field $g_{\alpha\beta}$ in parallel with the mediating forces of the vector fields.

The $T_{\alpha\beta}$ would be taken in the traditional way between the spin ½ quark and lepton fields to give the energy-momentum tensor in the Lagrangian as the source of the gravitational field with the standard additional Lagrangian terms for the vector fields. The Λ term would approximate dark energy. It is known that dark matter has gravitational interactions and could possibly be expressed as a non-zero mass (spin 2 $g_{\alpha\beta}$ representation) particle if it turns out not to have weak interactions. This might be reasonable since the other vector particles have both massive and massless representations which might also exist for the tensor (spin 2) field. This approach reduces to exactly the current QT and SM if gravitation is negligible and reduces to exactly the current Einstein GR theory if quantum effects are negligible. When both theories contribute, the theory is far more complex.

A specific prediction of this approach is that with the Schwarzschild metric, one gets an altered uncertainty principle

$$\Delta X^1 \, \Delta P^1 \geq (\hbar/2)(1/(1\text{-}r_s/r)) \text{ and} \qquad (62)$$

$$\Delta X^0 \, \Delta P^0 \geq (\hbar/2)\,(1\text{-}r_s/r) \qquad (63)$$

where $r_s = 2GM/c^2$ and where r = the distance to the center of the spherical mass. This is because the generalized Lie algebra effectively alters the value of Planks constant as a result of the curvature of space-time. This would in turn alter the creation rate of virtual pairs in the vacuum in a gravitational field, certainly around a black hole and near singular conditions. It could also have other implications which we are now investigating. What is maintained is a more general form of the Heisenberg uncertainty principle obtained by multiplying (62) and (63) together to obtain

$$\Delta X^0 \, \Delta P^0 \, \Delta X^1 \, \Delta P^1 \geq (\hbar/2)^2 \qquad (64)$$

while the other two uncertainty relations remain the same. Because the metric is quantized, it follows that distance and angle in space-time are now "granular" or "quantized". The Lorentz algebra is now defined by

$$L^{\mu\nu} = X^\mu D^\nu - X^\nu D^\mu \qquad (65)$$

determining their generalized commutation rules.

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5. Conclusions

It is not necessary to reexpress the numerous theorems that already exist in Riemannian geometry because the essential foundation is established above. If the metric $g^{\mu\nu}(X)$ is a well behaved function of the operators X^{μ} then the same results again will be obtained. One notes that the commutators $[D^{\mu}, D^{\nu}]$ are not arbitrary and are fixed by the metric and their commutators with the X^{μ}. Likewise while the commutators among the rotation generators in this space $L^{\mu\nu} = X^{\mu}D^{\nu} - X^{\nu}D^{\mu}$ and other commutators are complex in structure, they are still determined from derivatives of the metric and can be used to generate other groups of transformations such as rotations and Lorentz transformations thus generalizing this extended Poincare algebra. Naturally, the truly different aspect is that the metric function is defined in the enveloping algebra of the underlying algebra and the algebra does not have the same kind of closure that one normally has for a Lie algebra. If the metric functions are sufficiently smooth, then in a sufficiently small neighborhood of a gravitational field, one gets a standard Heisenberg Lie algebra with constant (but different) numerical values for the structure constants as with the Schwarzschild or Kerr metric. Even among the $[D^{\mu}, D^{\nu}]$ commutators, the derivatives of the metric result in fixed values in that small neighborhood as well as for the rotation group. The system is reminiscent of the group contraction concepts introduced by E. Inonu and E. P. Wigner and subsequent work where the structure constants are dependent upon other parameters as referenced above. Since the D^{μ} operators generate infinitesimal translations in the Riemann space defined by the metric of the [D, X] commutator, then it follows that this approach gives the framework of all groups of motions in all Riemann spaces via the exponential map. The linking of two domains of mathematics such as Lie algebras & groups with Riemannian geometry, may allow each to inform the other. This is especially true when one of the domains is generalized as we have done here with the structure constants of the basic Heisenberg Lie algebra. One can now ask if the framework of Lie algebras and groups tells us something new about allowable metrics of the associated Riemann spaces. Likewise does the generalization of Lie algebras give one new tools and challenges.

From the physics point of view, there are extensive implications because the metric (and thus the commutation rules) is determined by the distribution of matter and energy as expressed in the energy momentum tensor operators with Einstein's equations. The basic generalized Heisenberg algebra equation introduced here, $[D^{\mu}, X^{\lambda}] = I\, g^{\mu\nu}(X)$, could tell us something specific about the fundamental nature of the universe, namely that the interference among four-momentum and four-position (space time) observations is given by the Einstein metric along with all other resulting commutation relations. As the primary equations of motion in quantum theory are built upon the D^{μ} operators with the SM, it follows that observable effects will follow this assumption which offers an alternate framework for beginning to unify general relativity with quantum theory. With this framework one can now extend the Poincare algebra from its Heisenberg algebra component. [12, 20, 21]. It is also of interest to observe that the representation of the D^{μ} operator, $(i\hbar\, g^{\mu\upsilon}(y)\, (\partial/\partial y^{\nu}) + A^{\mu}(y))$, contains arbitrary vector fields $A^{\mu}(y)$ in a natural manner that are necessary for the SM to support Yang Mills gauge transformations. It is also of interest to note that the functions

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$g^{\mu\upsilon}(y)$ can contain an antisymmetric component related to torsion although this component does not contribute to the metric for distance [22, 23, 24]. This framework has several freedoms as it can allow for an antisymmetric component to $g_{\alpha\beta}$ which, as discussed above, does not contribute to the metric distance but does allow more freedom in the Γ connection as explored by Einstein and Cartan. And finally, this framework can be extended to higher dimensions as with string theory as there is no restriction of the space-time to four dimensions.

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