Article

Distinguishing familiar random variables through the use of risk measures

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Abstract: The use of risk measures such as the Value at Risk (VaR) or Tail Conditional Expectation (TCE) is required by the Basel Committee on Banking Supervision in determining a bank's risk profile. However, both measures can be shown to have shortcomings in the information that they provide to regulators and investors. In this paper we present an introduction to risk measure calculations before demonstrating the weaknesses of these measures. Through the exploration of specific cases we show how familiar yet differing risk profiles have identical values for combinations of these measures. From this evidence we recommend that a sequence of several risk measures should be used to give a more accurate representation of the risk contained on banking balance sheet.

Keywords: Risk measure; Value at Risk; Tail Conditional Expectation; Expected Shortfall; Bank capital; Basel Accords

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In a world of crumbling visions of economic security, most individuals are all too familiar with the negative side of investments–losses, which are sometimes enormous. With most retirement investment now i markets, risk has become an even more pressing reality than it has been in decades past. As such, the study of measuring and managing risk is now given more attention than in previous times. In this paper, we will focus on risk measures as tools to evaluate risk and better understand financial profiles. We will look at how risk measures have traditionally been used and then move to an information-based mindset that incorporates the significant elements of the risk measure. If we can better understand and quantify risk then we can more effectively prepare for and secure ourselves against devastating losses.

As we progress, we will first introduce the two risk measures being used, the Value at Risk and the Tail Conditional Expectation. Next we will explore the information-based approach that characterizes our perspective. Then we will undertake several calculations using probability distributions before concluding that several risk measures are necessary to truly understand the risk of a financial portfolio.

1. Some definitions

1.1. Risk and Risk Measures

The situation of risk requires both uncertainty and exposure. If a company already knows that a loan will default, there is no uncertainty and thus no risk. And if the bank decides not to loan to a business that is considered likely to default, there is also no risk for that bank as the bank has no exposure to the possibility of loss. Thus, Merriam-Webster (16) defines risk as the possibility of loss or injury.

So now one goal could be to have a single risk measure that will account for all the risk that a bank or securities firm might encounter. Some have objected to the risk measure being a single number, but there is some support for this idea. Investing is always a binary decision- either one invests or one chooses not to invest. Thus the argument is that, given a single number, one should have enough information to decide whether to invest or not. There have been some general agreements about the kinds of properties that such a risk measure ought to possess. These agreements must be acknowledged as being biases of what we consider important, but they have provided the definition that is now universal.

So, we will define a risk measure in the universally accepted fashion of Artzner et al. (1). Let X be a random variable. Then ρ is a risk measure if it satisfies the following properties:

- (Monotonicity) if $X \geq 0$, then $\rho(X) \leq 0$
- (Positive Homogeneity) $\rho(\beta X) = \beta\rho(X)\ \forall\ \beta \geq 0$
- (Translation Invariance) $\rho(X+a) = \rho(X) - a\ \forall\ a \in$.

The idea behind this definition is that a positive number implies that one is at risk for losing capital and should have that positive number of a cash balance on hand to offset this potential loss. A negative number would say that the company has enough capital to take on more risk or to return some of its cash to other operations or to its shareholders. The monotonicity property states that an investment that always has positive payoff gives the company the ability to take on more risk. Positive homogeneity implies that multiplying your investment by β times gives you a risk of a loss that is β times larger. Translation invariance implies that a company holding a in cash lowers its measure of risk by a.

1.2. The Value at Risk

The most well-known measure seems to be the Value at Risk (henceforth referred to as the "VaR"). In order to define VaR, we must first recall some basic concepts. Let X be a random variable and $\alpha \in [0,1]$.

- q is called an **α-quantile** if $Pr[X < q] \leq 1-\alpha \leq Pr[X \leq q]$,
- the largest **α-quantile** is $q_\alpha(X) = \inf\{x|Pr[X \leq x] > 1-\alpha\}$, and
- the smallest **α-quantile** is $q_\alpha^- = \inf\{x|Pr[X \leq x] \geq 1-\alpha\}$.

It is clear that $q_\alpha \geq q_\alpha^-$ and that q is an α-quantile if and only if $q_\alpha^- \leq q \leq q_\alpha$.

Given a position X and a number $\alpha \in [0,1]$, we define the α-Value at Risk, α-VaR or $VaR_\alpha(X)$, by $VaR_\alpha(X) = -q_\alpha(X)$. The α-VaR can be seen as the amount of cash that a firm needs in order to make the probability of that firm going bankrupt to be equal to α. This leads to the following property of VaR: $VaR_\alpha(X + VaR_\alpha(X)) = 0$. This states that one may offset the risk of an investment by having an amount of cash on hand equal to the Value at Risk inherent in holding the asset.

The Value at Risk has historically been the most important of the risk measures, as it has been the one used by the Basel Committee on Banking Supervision in order to determine capital requirements for banks (2).

1.3. The Tail Conditional Expectation

We now introduce the notion of tail conditional expectation. Given a base probability measure P on a probability space Ω, and a level α, the tail conditional expectation (TCE) is the measure of risk defined by

$$TCE_\alpha(X) = -E_P[X|X \leq -VaR_\alpha(X)] \tag{1}$$

For continuous random variables this concept is variously called the expected shortfall, expected tail loss, conditional value at risk, mean excess loss, loss given default, or mean shortfall by several authors in the literature, Rockafellar and Uryasev, (17). In mathematical terms, the TCEs are measures on probability spaces.

The TCE is often considered preferable to the VaR because it respects diversification. Since the paper of Markowitz [15] in 1952, diversification has been valued in investment. In fact Artzner et al. call any measure that fails to favor diversification an incoherent measure in (1). Accordingly, the TCE is a coherent measure.
Finally, we can remark that for each risk X one has the equality

$$VaR_\alpha(X) = \inf\{\rho(X)|\rho \ \text{ coherent}, \ \ \rho \geq VaR_\alpha(X)\}.$$

Thus, knowing that more restrictive measures are available to them, the question is why regulators would use the Value at Risk, noting that no known organized exchanges use VaR as the basis of risk measurement for margin requirements. ADEH [1] immediately answer their own question, with a quote from Stulz (18), "Regulators like Value at Risk because they can regulate it." An expansion of this discussion can be found in Guégan and Tarrant (11).

1.4. Calculation

To calculate the α-VaR we seek that value V which, when the probability density function (pdf) P(x) is integrated from negative infinity to $-V$, provides the value $1-\alpha$. That is, if $\int_{-\infty}^{-V} P(x)dx = (1-\alpha)$, then the α-VaR is V.

Similarly, the calculation of some α-TCE is reached by integrating $xP(x)$ from negative infinity to the negative α-VaR and then dividing that value by $(1-\alpha)$. This simply calculates a weighted average of the distribution's tail. Symbolically then,

$$\frac{1}{(1-\alpha)} \int_{-\infty}^{-(\alpha\text{-VaR})} xP(x)dx = \alpha\text{-TCE}$$

2. Example of Risk Measures as Information

Let us recall the information provided by the definition of the VaR: there is a 100(1-α)% chance that a return will be less than or equal to the α-VaR. While this is important information to have about a probability distribution, Artzner et al. (1) have shown that a single VaR does not provide significant insight into the risk portfolio of investments. This can be demonstrated in the following example, in which we only show the lowest 5% of the returns distribution:

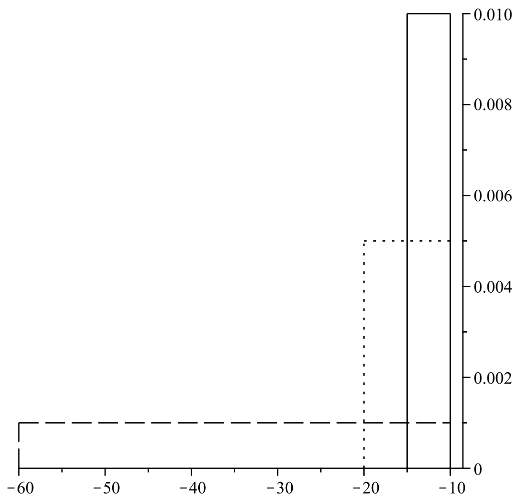

Figure 1. Three uniform distributions with equal 95%-VaRs

Consider the 95%-VaR and several uniform distributions. Notice that in the figure above, all distributions have an area of 5%. In other words,

$$\int_{-\infty}^{-10} P(x)dx = 0.05$$

Also, the right-most boundary is -10 for each pdf, making the 95%-VaR= 10 for each distribution. This calculation does not take into account the differences of the distributions, nor does it even make note of differences. Which is a "better" distribution? Which is least risky? Risk measures should at least be able to provide some insight into such questions. If we did not have direct access to these distributions, we would be making blind decisions. The α-VaR alone gives an investor or regulator no information about what actually happens inside of the worst $(1-\alpha)$ of returns; it provides no knowledge of the tail end itself, only where the tail begins. As Simon Johnson of MIT has put it, "VaR misses everything that matters when it matters."

The practical application of this idea to the traditional use of a VaR is as follows. Three banks with the probability distributions above would be advised by the VaR to keep the same amount of cash on hand to legally hedge their risks. As we can see, there is a significant difference between these distributions, a difference that is not indicated by the VaR.

Notice in Figure 2 that the same idea applies for three uniform distributions with identical values for the 95%-TCE. The last 5% of returns for each distribution is shown graphically and, for each distribution, the 95%-TCE = 10. Again, the use of the TCE as a risk measure would give the three banks identical advice under quite different circumstances.

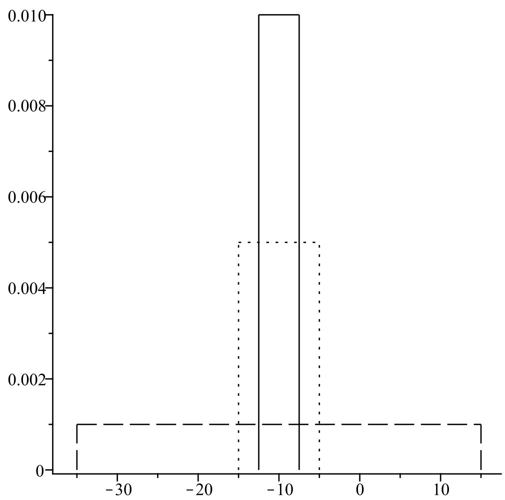

Figure 2. Three uniform distributions with equal 95%-TCEs

Hence acquiring more information about a risk distribution is important; one measure does not provide sufficient information. When one more clearly sees the true nature of a probability distribution, one can make a better decision about the amount of cash needed to secure a portfolio. Although our preceding examples focus on uniform distributions and the 95% level, we will show more realistic examples in the coming sections.

3. Standard Normal versus Uniform

Comparing different uniform distributions is surely contrived, so we move to comparing uniform distributions with the standard normal distribution. Now since we have seen how easily one risk measure may be "fooled," let us explore the possibility of using several risk measures in conjunction. This idea is recommended and further explained in Guegan and Tarrant (2012). We will first explore the possibilities of two risk measures together. The combinations of two risk measures that we will use are the 95%-VaR and the 99%-VaR, the 95%-VaR and 95%-TCE, and the 95%-TCE and 99%-TCE.

3.1. Two Risk Measures

3.1.1. 95%-VaR and 99%-VaR

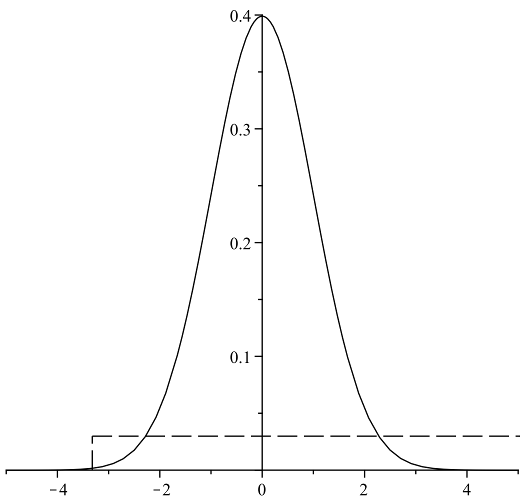

Figure 3. Standard normal and uniform distributions

Notice in Figure 3 that there are two significantly different distributions that are graphed on the same axes. In Figure 4, we have magnified the relevant sections of the distributions under consideration for our example.

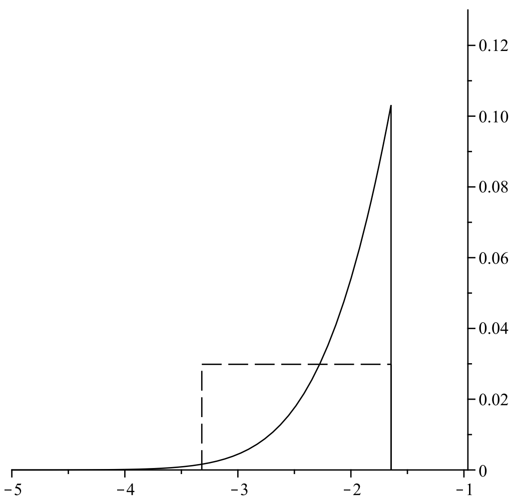

Figure 4. A magnification of the last 5%

Recall that the 95%-VaR is that V for which $\int_{-\infty}^{-V} P(x)dx = 0.05$. The PDFs of the two graphs shown are $\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$, the standard normal distribution $(N(0,1))$, and $0.029863 \cdot 1_X$, where $X = [-3.31915, 30.16672]$, a uniform distribution. Upon calculating, we find that the 95%-VaR of both distributions is 1.64485. Similarly, the 99%-VaR of each is 2.32635. So the two VaRs that we have selected make no differentiation between these two very different risk distributions. However, the risk distributions are shown to be different first by simple inspection and then also by the two TCEs. The 95%-TCE and 99%-TCE for the standard normal distribution are 2.06271 and 2.66522, respectively, while the same statistics for this uniform distribution are 2.48200 and 2.82275, respectively. At least the use of a third measure makes some sort of delineation between the distributions, enabling a more educated decision about investment.

We collect these results in the following table and, henceforth, we shall present these statistics in a similar tabular format so as to reduce potential confusion.

	standard normal	uniform
95%-VaR	1.64485	1.64485
95%-TCE	2.06271	2.48200
99%-VaR	2.32635	2.32635
99%-TCE	2.66522	2.82275

Table 1: Comparison of TCEs and VaRs of N(0,1) & $0.029863 \cdot 1_X$

3.1.2. 95%-VaR and 95%-TCE

As in the example above, Figure 5 presents the two distributions that we are dealing with and Figure 6 is the enlargement of the last 5% of both distributions.

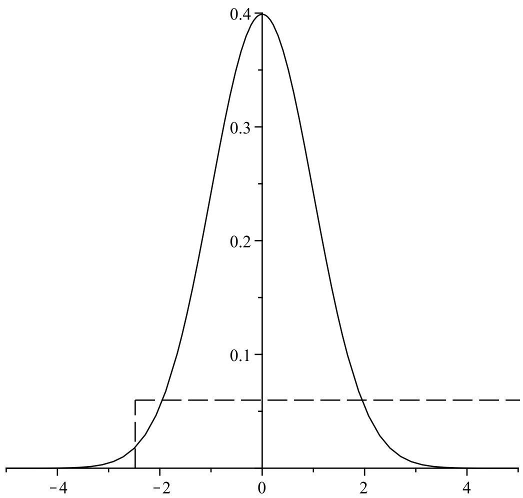

Figure 5. Standard normal and uniform distributions

While the standard normal has remained the same, our selection of the uniform distribution here, $0.059828 \cdot 1_Y$, where $Y = [-2.48059, 14.23405]$, has produced the same 95%-VaR and 95%-TCE as the standard normal. This can be seen more explicitly in the table below. Here, a combination of the 95%-VaR and the 95%-TCE shows no differences between the two distributions, though either the 99% VaR or the 99% TCE would distinguish between the two.

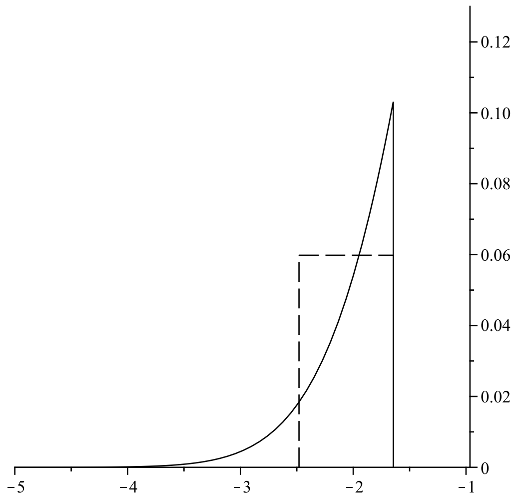

Figure 6. A magnification of the last 5%

	standard normal	uniform
95%-VaR	1.64485	1.64485
95%-TCE	2.06271	2.06271
99%-VaR	2.32635	2.31344
99%-TCE	2.66522	2.39701

Table 2: Comparison of TCEs and VaRs of N(0,1) & $0.059828 \cdot 1_Y$

3.1.3. 95%-TCE and 99%-TCE

Once again, the images below represent, respectively, a new uniform distribution, $0.033195 \cdot 1_Z$, where $Z = [-2.81585, 27.30916]$, compared to the standard normal and the magnification of the last 5% of each distribution.

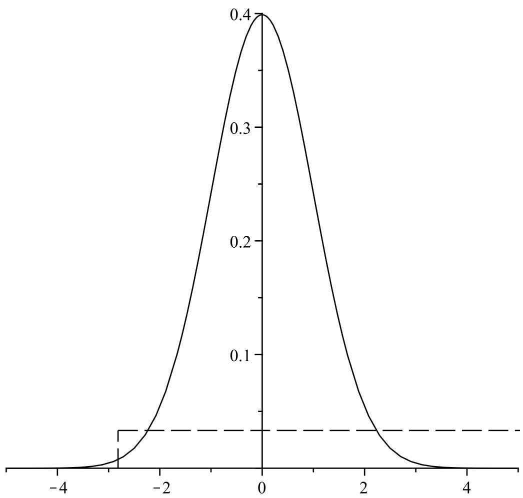

Figure 7. standard normal and uniform distributions

The most obvious graphical difference between these distributions and the previous selections is that the standard normal and the uniform do not begin their last 5% at the same point (Figure 8). This is a simple illustration that the 95%-VaRs are different. Similarly, the 99%-VaR differentiates the distributions from one another. However, the two

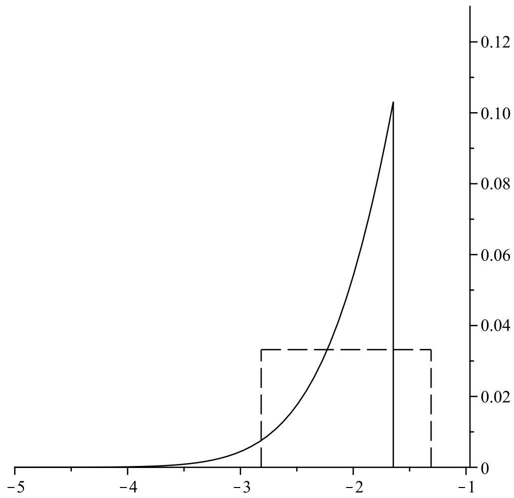

Figure 8. A magnification of the last 5%

TCEs are identical and thus yield no differentiation without further information.

	standard normal	uniform
95%-VaR	1.64485	1.30960
95%-TCE	2.06271	2.06271
99%-VaR	2.32635	2.51460
99%-TCE	2.66522	2.66522

Table 3: Comparison of TCEs and VaRs of N(0,1) & $0.033195 \cdot 1_Z$

This is the final combination of two risk measures, and has been included to exhaust possible combinations. The reader will now understand that each set of two risk measures is insufficient for differentiating among all possible standard normal and uniform pairs. Before moving on, we must note that we have compared the uniform distribution to a very specific instance of normal Gaussian distributions (with variables $\mu = 0$ and $\sigma = 1$). This is for simplicity's sake, because the standard normal is used quite often and thus it is familiar. Clearly, it is reasonable to see that this idea applies to the entire family of normal Gaussian distributions: that any combination of two risk measures can fail to differentiate between some normal Gaussian and some uniform distribution.

3.2. Three Risk Measures

The possibility of finding three risk measures that are equal for two separate probability distributions is somewhat more complex than the earlier results with two measures. In order to test the possibility, we first develop ratios that make experimentation possible. At this point in the paper, it will be beneficial for the reader to understand the approach to these ratios. Furthermore, these ratios make the results with two risk measures much more simple in retrospect.

The nature of the uniform distribution, geometrically, gives it certain simple properties which must always be respected. For instance, a rectangle's area is equal to its length multiplied by its width. In the same way, the 95%-VaR subtracted from the Maximum Loss (length) multiplied by the value (or height) of the uniform distribution must be equal to the integral over $[-\infty, -(95\%\text{-VaR})]$.

In figure 9, the worst 5% of returns of an arbitrary uniform distribution is shown. The two rightmost (dashed) lines represent 95% risk measures (the TCE and VaR, respectively, from left to right) and the two leftmost (dotted) lines

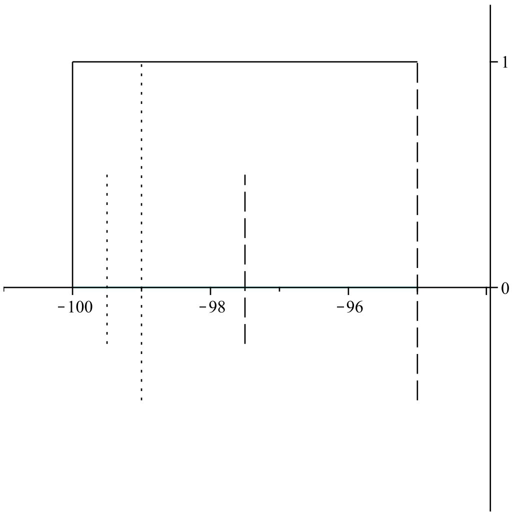

Figure 9. uniform Ratios

represent 99% risk measures (also the TCE and VaR, respectively, from left to right). The TCEs are shown as half-lines extended beyond the x-axis while the VaRs are full-length and also extended beyond the x-axis.

We have defined two variables as follows: $x = 95\%\text{-TCE} - 95\%\text{-VaR}$ (the length between the dashed lines in Fig. 9) and $y = 99\%\text{-TCE} - 99\%\text{-VaR}$ (one fifth of that length). Using these variables, we arrive at relatively simple ratios such as $95\%\text{-TCE} - 99\%\text{-TCE} + x = y$, $\frac{1}{5}$ ($95\%\text{-TCE} + x - 95\%\text{-VaR}$) $= 2y$, and $\frac{1}{4}$ ($99\%\text{-VaR} - 95\%\text{-VaR}$) $= 2y$. These may be combined to form the ratios used in our results:

$$10[(99\%\text{-TCE}) - (95\%\text{-TCE})] = 8[(95\%\text{-TCE}) - (95\%\text{-VaR})]$$

$$16(95\%\text{-TCE}) - 6(95\%\text{-VaR}) = 10(99\%\text{-VaR})$$

$$9[(95\%\text{-VaR}) - (90\%\text{-VaR})] = 5[(99\%\text{-VaR}) - (90\%\text{-VaR})]$$

Note that the three ratios above must be satisfied by any uniform distribution. Also, each ratio uses a different combination of three risk measures. From this we determine that any distribution which might have three risk measure calculations in common with a uniform distribution must satisfy the corresponding ratio.

The simple example here is the standard normal, whose risk measure values are as follows. Notice that we have included in this table the value for the 90%-VaR, which is necessary as a third VaR value.

	standard normal
90%-VaR	1.28155
95%-VaR	1.64485
95%-TCE	2.06271
99%-VaR	2.32635
99%-TCE	2.66522

Table 4: TCEs and VaRs for N(0,1)

If we substitute the values of the standard normal into the ratios for the uniform distribution, we will then be able to determine whether three risk measures can differentiate between a standard normal distribution and any uniform distribution by checking to see if the ratios for the uniform are respected or violated.

3.2.1. 95% / 99%-VaR, 95%-TCE

So, we find as we substitute values,

$$1.6(95\%\text{-TCE}) - 0.6(95\%\text{-VaR}) = 1.6(2.06271) - 0.6(1.64485)$$
$$= 3.30035 - 0.98691 = 2.31344 \neq 2.32635 = (99\%\text{-VaR})$$

3.2.2. 95%-VaR, 95% / 99%-TCE

The ratio with values substituted,

$$10[(99\%\text{-TCE}) - (95\%\text{-TCE})] = 10[(2.66522) - (2.06271)]$$
$$= 10[0.60250] = 6.02500 \neq 3.34296 = 8[0.41787]$$
$$= 8[(2.06271) - (1.64485)] = 8[(95\%\text{-TCE}) - (95\%\text{-VaR})]$$

3.2.3. 90% / 95% / 99%-VaR

The possibility of three VaRs, while not discussed previously, is shown briefly here as an illustration that this combination will not work either. We have the ratio, with values substituted,

$$9[(95\%\text{-VaR}) - (90\%\text{-VaR})] = 9[1.64485 - 1.28155]$$
$$= 9(0.36330) = 3.26970 \neq 5.22400 = 5(1.0448)$$
$$= 5[2.32635 - 1.28155] = 5[(99\%\text{-VaR}) - (90\%\text{-VaR})]$$

3.2.4. 90% / 95% / 99%-TCE

It is important here to note that the TCE is a much more comprehensive (or coherent) measure than the VaR. In fact, the VaR is used in the calculation of the TCE. In the particular case of the uniform distribution, the TCE will effectively act like a VaR (i.e. the 95%-TCE is the 97.5%-VaR) but in distributions such as the standard normal, the TCE will contain much more information than the VaR. As a result of the TCE behaving like a VaR for uniform distributions, the simple result that 3 VaRs fail to determine between two probability distributions shows that 3 TCEs will do the same for the specific case where a uniform distribution is involved.

By the previous results we see that there is no uniform distribution in the entire family of uniform distributions that will ever achieve the same values over three risk measures as the standard normal distribution.

Essentially, we have proved the following theorem:

Theorem Let N be a random variable with standard normal distribution, N(0,1). Also, let ρ, ϕ, and ψ be selections of VaR or TCE risk measures. Then there exists no random variable U with uniform distribution such that $\rho(N) = \rho(U)$, $\varphi(N) = \varphi(U)$, and $\psi(N) = \psi(U)$.

4. Pareto versus Uniform

Now we turn to the Pareto probability random variable since Pareto distributions often arise in economic and financial applications.

First, it is important to recall that the pdf and cdf of the family of Pareto distributions are as follows, with shape parameter $\delta, (0 < \delta < 100)$:

$$\text{PDF: g(x)} = \frac{\delta\beta^{\delta}}{x^{\delta+1}} \text{for } x \geq \beta$$

$$\text{CDF: G(t)} = \int_{\beta}^{t} \frac{\delta\beta^{\delta}}{x^{\delta+1}} \text{dX for } t \geq \beta$$

Both of these will become useful in the calculations to come. Due to the definition of the Pareto function, we shall here consider the pdf that is the negative Pareto distribution.

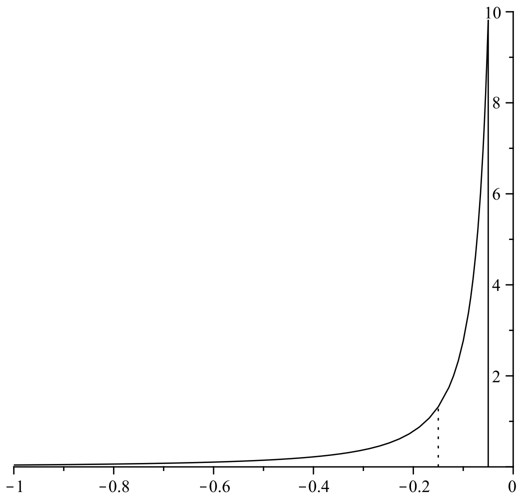

Figure 10. An instance of the negative Pareto distribution on $-1 \leq X \leq -0.05 = -\beta$

Recall that the α-VaR of some Pareto distribution $f(x)$ is V if and only if

$$\int_{-\infty}^{-A} f(x)dx = (1-\alpha)$$

This is because

$$\int_{-\infty}^{-\beta} f(x)dx = 1$$

Similarly, the α-TCE of a Pareto distribution $f(x)$ is T if and only if V is the α-VaR and

$$-\frac{1}{(1-\alpha)}\int_{-\infty}^{-A} xf(x)dx = B$$

Now we present ratios that are particular to the Pareto distribution. The first of these ratios is derived directly from the calculation of the 95%-VaR in the Appendix. For these calculations, $V_{95\%}$ is the 95%-VaR of the Pareto distribution (and $V_{99\%}$ is the 99%-VaR of the Pareto distribution, etc.). This first ratio is:

$$\beta = (0.05)^{\frac{1}{\delta}} V_{95\%}$$

The next ratio that we have, also derived in the Appendix, relates a Pareto distribution's 95%-VaR with its 99%-VaR. This ratio is:

$$V_{99\%} = 5^{\frac{1}{\delta}} V_{95\%}$$

Similarly, we have:

$$V_{90\%} = 2^{-\frac{1}{\delta}} V_{95\%}$$

These are the most important and most often used ratios; their derivations can be found in the Appendix.

4.1. 95% / 99%-VaR , 95%-TCE

In the process of testing the combination of these risk measures, we approach the Pareto in quite a different manner. Rather than choosing a single instance of the family of Normal distributions as we did before, we will here work with the entire family of Pareto distributions in general.

Recall that the earlier ratio of these three risk measures in the uniform is:

$$16(95\%\text{-TCE}) - 6(95\%\text{-VaR}) = 10(99\%\text{-VaR})$$

Here we insert the calculation of these values and find

$$16(20\int_{V_{95\%}}^{\infty} \frac{\delta\beta^{\delta}}{x^{\delta}} dx) - 6V_{95\%} = 10V_{99\%}$$

Which simplifies to

$$10 \cdot [1 - 5^{\frac{1}{\delta}}] + \frac{16}{\delta - 1} = 0$$

This is solved by $\delta = 35.14694$, which implies that two distributions can have identical values for all three of these measures. In this one isolated case a fourth measure would be necessary to delineate between the two distributions. To see an example of this idea applied to specific distributions, see the Appendix.

4.2. 95%-VaR , 95% / 99%-TCE

Similarly, recall that the ratio for these three measures in a uniform distribution is:

$$10[(99\%\text{-TCE}) - (95\%\text{-TCE})] = 8[(95\%\text{-TCE}) - (95\%\text{-VaR})]$$

As above, we insert the general calculations of these measures to find

$$10[(100\int_{V_{99\%}}^{\infty} \frac{\delta\beta^{\delta}}{x^{\delta}} dx) - (20\int_{V_{95\%}}^{\infty} \frac{\delta\beta^{\delta}}{x^{\delta}} dx)] = 8[(20\int_{V_{95\%}}^{\infty} \frac{\delta\beta^{\delta}}{x^{\delta}} dx) - (V_{95\%})]$$

This simplifies to

$$45\int_{V_{95\%}}^{\infty} \frac{\delta\beta^{\delta}}{x^{\delta}} dx - 125\int_{V_{99\%}}^{\infty} \frac{\delta\beta^{\delta}}{x^{\delta}} dx = V_{95\%}$$

And finally to

$$\frac{\delta}{-\delta+1}(-45+125\cdot 5^{\frac{1}{\delta}-1})=20$$

On the available values of δ, $0<\delta\leq 100$, this equality does not have a solution. This confirms the idea that these three measures together will differentiate any uniform distribution from any Pareto distribution. More on these simplifications and the proof of this portion of the result can be found in the Appendix.

4.3. 90% / 95% / 99%-VaR

The necessary ratio that we apply here is

$$9[(95\%\text{-VaR})-(90\%\text{-VaR})]=5[(99\%\text{-VaR})-(90\%\text{-VaR})]$$

Substituting for these calculations in the family of Pareto distributions, we find

$$9[V_{95\%}-V_{90\%}]=5[V_{99\%}-V_{90\%}]$$

This simplifies finally to

$$9-4\cdot(2^{-\frac{1}{\delta}})=5\cdot(5^{\frac{1}{\delta}})$$

This has no solution on $0<\delta\leq 100$. Hence we come to a similar result as before. These details can also be found in the Appendix.

4.4. 90% / 95% / 99%-TCE

As was done before, it is important to recognize that in the uniform distribution the $(1-\alpha)$-TCE is the $(1-\frac{\alpha}{2})$-VaR. This is true only for the uniform distribution. As a result, the earlier proof of 3 VaRs is sufficient to prove that 3 TCEs will differentiate among any Pareto and any uniform distributions.

Combining the results above, we arrive at the theorem below.

Theorem Let P be any random variable with Pareto distribution, and ρ, ϕ, ψ, and ξ selections of VaR or TCE risk measures. Then there is no random variable U with uniform distribution such that $\rho(P)=\rho(U)$, $\phi(P)=\phi(U)$, $\psi(P)=\psi(U)$, and $\xi(P)=\xi(U)$. (*Note, in some cases four measures will indeed be necessary*)

5. Standard Normal versus Pareto

Now we will see if combinations of risk measures can differentiate between normal and fatter tailed Pareto distributions. Here we return to a method analogous to that of section 3.2. We will be comparing a single instance of the normal distribution, the standard normal (N(0,1)), to the entire family of Pareto distributions.

So we once again start with the standard normal distribution. All of its VaRs are set and are easily found by looking at standard charts. We can also easily find all of its TCEs with simple definite integral calculations. Equating the desired VaRs and/or TCEs with the general Pareto distribution will allow us to solve for the parameters in the Pareto. Since there are two parameters in the Pareto, we can choose any two VaRs and/or TCEs to set the parameters and then see if a third measure will also align with that of the standard normal. That is our approach here.

5.1. 95% / 99%-VaR

Consider the standard normal distribution and the Pareto distribution with parameters $\delta = 4.64281$ and $\beta = 0.86278$. In this case, the 95%-VaR for both distributions is 1.64885, while the 99%-VaR for each is 2.32635. However, as we return to tabular format below, we can see that any third measure will differentiate between the two distributions.

	standard normal	pareto
90%-VaR	1.28155	1.41674
90%-TCE	1.75498	1.80565
95%-VaR	1.64485	1.64485
95%-TCE	2.06271	2.06069
99%-VaR	2.32635	2.32635
99%-TCE	2.66522	2.91448

Table 5: Comparison of TCEs and VaRs of N(0,1) & Pareto(0.86278, 4.64281)

5.2. 95% / 99%-TCE

Now we take the standard normal and the Pareto with $\delta = 5.43736$ and $\beta = 0.98509$. Here the 95%-TCE for each is 2.06271 and the 99%-TCE for each is 2.66522. Again, any third measure will differentiate between the distributions.

	standard normal	pareto
90%-VaR	1.28155	1.50450
90%-TCE	1.75498	1.84355
95%-VaR	1.64485	1.70905
95%-TCE	2.06271	2.06271
99%-VaR	2.32635	2.29776
99%-TCE	2.66522	2.66522

Table 6: Comparison of TCEs and VaRs of N(0,1) & Pareto(0.98509, 5.43735)

5.3. 95% VaR / 95% TCE

Here, our Pareto distribution is given by $\delta = 1.04153$ and $\beta = 0.092677$. As with the earlier sections, we again see that a combination of two measures fails to differentiate between this Pareto and the standard normal; as such, a third measure is necessary.

	standard normal	pareto
90%-VaR	1.28155	1.42937
90%-TCE	1.75498	1.79249
95%-VaR	1.64485	1.64485
95%-TCE	2.06271	2.06271
99%-VaR	2.32635	2.27889
99%-TCE	2.66522	2.85782

Table 7: Comparison of TCEs and VaRs of N(0,1) & Pareto(0.092676, 1.04153)

After considering the above results, we conclude that at least three risk measures be used to differentiate a standard normal distribution from any Pareto distribution. This result is similar to previous sections and implies the following theorem.

Theorem Let N be a random variable with standard normal distribution, N(0,1), and let ρ, ϕ, and ψ be selections of VaR or TCE risk measures. Then there is no random variable P with Pareto distribution such that $\rho(N) = \rho(P)$, $\phi(N) = \phi(P)$, and $\psi(N) = \psi(P)$.

6. Conclusion and future directions

We have shown that using more risk measures improves information about loss distributions, which should help investment decisions. When it is not possible to have the distribution itself, a quantity of risk measures is a good option for investors and regulators to more clearly understand the risk presented by an investment portfolio. In fact it is interesting to consider whether regulation could even be rendered unnecessary if investors had access to such information.

Since one and two risk measures are shown as possessing too little information about a risk level, and in light of our one anomaly, we recommend that at least four risk measures be required in investment practices to get a more accurate idea of how much risk is truly present.

Further study in this area is certainly feasible, since there are a great many more complex distributions to which this idea might be applied. Also, there is potential to explore the pros and cons of various risk measures. Testing of this idea with live data should prove most insightful for practical purposes. The beginnings of such work appears in a paper of Tarrant, (19).

As evidenced by the world's current economic turmoil, risk is an area which cannot be forgotten. Perhaps as the methods of studying and quantifying risk improve, more knowledgeable decisions concerning investment will help to prohibit the catastrophes that have been experienced in the recent past.

Appendix

Derivation of Pareto ratios

Note that $\int_{V_{95\%}}^{\infty} \frac{\delta\beta^{\delta}}{x^{\delta+1}}dx = 0.05$. So $-\frac{\beta^{\delta}}{x^{\delta}}|_{V_{95\%}}^{\infty} = 0.05$. Then $(\frac{\beta}{V_{95\%}})^{\delta} = 0.05$ or $\frac{\beta}{V_{95\%}} = (0.05)^{\frac{1}{\delta}}$. Finally, our ratio, which holds for all Pareto distributions, is

$$\beta = (0.05)^{\frac{1}{\delta}} V_{95\%}$$

Similarly, for a 99%-VaR $V_{99\%}$,

$$\beta = (0.01)^{\frac{1}{\delta}} V_{99\%}$$

And, generally, for an α-VaR V_{α} ,

$$\beta = (1-\alpha)^{\frac{1}{\delta}} V_{\alpha}$$

From above, $\beta = (0.05)^{\frac{1}{\delta}} V_{95\%}$ and $\beta = (0.01)^{\frac{1}{\delta}} V_{99\%}$. So $(0.05)^{\frac{1}{\delta}} V_{95\%} = (0.01)^{\frac{1}{\delta}} V_{99\%}$.

And finally, we have

$$V_{99\%} = 5^{\frac{1}{\delta}} V_{95\%}$$

Furthermore, the similar $\beta = (0.1)^{\frac{1}{\delta}} V_{90\%}$ and $\beta = (0.05)^{\frac{1}{\delta}} V_{95\%}$ provides the result

$$V_{90\%} = 2^{-\frac{1}{\delta}} V_{95\%}$$

Computation of 95% / 99%-VaR , 95%-TCE with Pareto and uniform

Recall that the ratio from above, with values substituted, comes to:

$$16(20\int_{V_{95\%}}^{\infty} \frac{\delta\beta^{\delta}}{x^{\delta}} dx) - 6V_{95\%} = 10V_{99\%}$$

Using the ratios that we have previously defined and basic integration, we find:

$$16(20\int_{V_{95\%}}^{\infty} \frac{\delta\beta^{\delta}}{x^{\delta}} dx) - 6V_{95\%} = \frac{320\cdot\delta\cdot\beta^{\delta}}{\delta-1}\cdot[\frac{1}{(V_{95\%})^{\delta-1}}] - 6V_{95\%} =$$

$$\frac{320\cdot\delta\cdot[(0.05)^{\frac{1}{\delta}}V_{95\%}]^{\delta}}{\delta-1}\cdot[\frac{1}{(V_{95\%})^{\delta-1}}] - 6V_{95\%} = \frac{320\cdot\delta}{\delta-1}\cdot[(0.05)(V_{95\%})^{\delta}]\cdot[\frac{1}{(V_{95\%})^{\delta-1}}] - 6V_{95\%} =$$

$$\frac{16\cdot\delta}{\delta-1}\cdot[\frac{1}{(V_{95\%})^{-1}}] - 6V_{95\%} = \frac{16\cdot\delta}{\delta-1}\cdot[V_{95\%}] - 6V_{95\%} = 10V_{99\%} = 10(5^{\frac{1}{\delta}}V_{95\%})$$

Then this simplifies to:

$$\frac{16\cdot\delta}{\delta-1} - 6 = 16(1+\frac{1}{\delta-1}) - 6 = 16 + \frac{16}{\delta-1} - 6 = 10 + \frac{16}{\delta-1} = 10(5^{\frac{1}{\delta}})$$

And finally,

$$10\cdot[1-5^{\frac{1}{\delta}}] + \frac{16}{\delta-1} = 0$$

This function has a unique solution at $\delta = 35.14694$ on $0 < \delta < 100$.

As we check this value, we obtain: $\beta = (0.05)^{\frac{1}{\delta}} V_{95\%} = (0.05)^{\frac{1}{35.146942}} V_{95\%} \approx 0.918297 \cdot V_{95\%}$.
For the standard normal the 95%-VaR $= 1.64485$, and thus we find

$$\beta = 0.91830 \cdot V_{95\%} = 0.91830 \cdot 1.64485 = 1.51046$$

Then,

$$95\%\text{-TCE } = 20\cdot\int_{V_{95\%}}^{\infty} \frac{\delta\beta^{\delta}}{x^{\delta}} dx = 20\cdot\int_{1.64485}^{\infty} \frac{35.14694\cdot(1.51046)^{35.14694}}{x^{35.14694}} dx = 1.69302$$

Finally, the 99%-VaR and 99%-TCE are:

$$V_{99\%} = 5^{\frac{1}{\delta}} V_{95\%} = 5^{\frac{1}{35.14694}} \cdot 1.64485 = 1.72192$$

$$99\%\text{-TCE } = 100\cdot\int_{V_{99\%}}^{\infty} \frac{\delta\beta^{\delta}}{x^{\delta}} dx = 100\cdot\int_{1.72192}^{\infty} \frac{35.14694\cdot(1.51046)^{35.14694}}{x^{35.14694}} dx = 1.77235$$

By analyzing the relationship between the 95%-VaR and 99%-Var, we arrive at a uniform distribution $0.51900 \cdot 1_A$, where $A = [-1.74119, 0.18560]$. The risk measure values for these two distributions are compared below:

	Pareto	uniform
95%-VaR	1.64485	1.64485
95%-TCE	1.69302	1.69302
99%-VaR	1.72192	1.72192
99%-TCE	1.77235	1.73156

Table 5: Comparison of TCEs and VaRs of Pareto($\delta = 35.14694$) & $0.51900 \cdot 1_A$

Note that the values for the 95%-VaR, 95%-TCE, and 99%-VaR are all equal. This one instance serves as an illustration and confirmation of our result in a particular case.

Computing 95%-VaR , 95% / 99%-TCE for uniform and standard normal

Here we begin with the necessary ratio and substituted values:

$$10[(100\int_{V_{99\%}}^{\infty}\frac{\delta\beta^{\delta}}{x^{\delta}}dx)-(20\int_{V_{95\%}}^{\infty}\frac{\delta\beta^{\delta}}{x^{\delta}}dx)]=8[(20\int_{V_{95\%}}^{\infty}\frac{\delta\beta^{\delta}}{x^{\delta}}dx)-(V_{95\%})]$$

Simplifying terms gives us

$$45\int_{V_{95\%}}^{\infty}\frac{\delta\beta^{\delta}}{X^{\delta}}dX-125\int_{V_{99\%}}^{\infty}\frac{\delta\beta^{\delta}}{X^{\delta}}dX=V_{95\%}$$

Then we integrate and substitute above ratios to find

$$45\int_{V_{95\%}}^{\infty}\frac{\delta\beta^{\delta}}{x^{\delta}}dx-125\int_{V_{99\%}}^{\infty}\frac{\delta\beta^{\delta}}{x^{\delta}}dx=\frac{45\cdot\delta\cdot\beta^{\delta}}{(V_{95\%})^{\delta-1}\cdot(\delta-1)}-\frac{125\cdot\delta\cdot\beta^{\delta}}{(V_{99\%})^{\delta-1}\cdot(\delta-1)}=$$

$$\frac{\delta}{\delta-1}\cdot[\frac{45\cdot\beta^{\delta}}{(V_{95\%})^{\delta-1}}-\frac{125\cdot\beta^{\delta}}{(V_{99\%})^{\delta-1}}]=\frac{\delta}{\delta-1}\cdot[\frac{45\cdot((0.05)^{\frac{1}{\delta}}V_{95\%})^{\delta}}{(V_{95\%})^{\delta-1}}-\frac{125\cdot((0.05)^{\frac{1}{\delta}}V_{95\%})^{\delta}}{(5^{\frac{1}{\delta}}V_{95\%})^{\delta-1}}]=$$

$$\frac{\delta}{\delta-1}\cdot[\frac{45\cdot(0.05)\cdot(V_{95\%})^{\delta}}{(V_{95\%})^{\delta-1}}-\frac{125\cdot(0.05)\cdot(V_{95\%})^{\delta}}{(5^{1-\frac{1}{\delta}})\cdot(V_{95\%})^{\delta-1}}]=$$

$$\frac{0.05\cdot\delta}{\delta-1}\cdot[45\cdot(V_{95\%})-125\cdot(V_{95\%})\cdot(5^{\frac{1}{\delta}-1})]=\frac{0.05\cdot\delta\cdot(V_{95\%})}{\delta-1}\cdot[45-125\cdot(5^{\frac{1}{\delta}-1})]=V_{95\%}$$

Or, finally,

$$\frac{\delta}{\delta-1}\cdot[45-125\cdot(5^{\frac{1}{\delta}-1})]=20$$

The only solution to this function is $\delta=-1$, which is not on the available values of δ, $(0<\delta<100)$.

6.1. Computing 90% / 95% / 99%-VaR for standard normal and uniform

Beginning with the necessary ratio,

$$9[V_{95\%}-V_{90\%}]=5[V_{99\%}-V_{90\%}]$$

We find

$$9\cdot V_{95\%}-9\cdot V_{90\%}=9[V_{95\%}-V_{90\%}]=5[V_{99\%}-V_{90\%}]=5\cdot V_{99\%}-5\cdot V_{90\%}=5\cdot(5^{\frac{1}{\delta}}V_{95\%})-5\cdot V_{90\%}$$

By simplifying like terms, we arrive at

$$(9-5^{\frac{1}{\delta}+1})\cdot V_{95\%} = 4\cdot V_{90\%} = 4\cdot 2^{-\frac{1}{\delta}}V_{95\%}$$

And finally

$$9-5^{\frac{1}{\delta}+1} = 4\cdot 2^{-\frac{1}{\delta}} = 2^{2-\frac{1}{\delta}}$$

As before, the only possible solution to this equation is $\delta = -1$, which is not on the domain of possible values of δ, $(0 < \delta < 100)$.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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