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Article

A new representation of extended Mittage-Leffler function and its properties

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Abstract: In this article, our main purpose is to establish a new extension of Mittag-Leffler function by using the known extended beta function $\mathbf{B}_{\omega}(a,b;p)$ introduced in [1]. It led to a novel extension of the applicability of Mittag-Leffler function that introduced them as distributions defined for a specific set of functions. We also, investigate some of its important properties, namely recursion relation, Mellin transform and differential formulas.

Keywords: Mittag-Leffler function; extended Mittag-Leffler function; Extended beta function; Fox Wright function

MSC: 33B20, 33C20, 33C45, 33C60, 33B15, 33C05

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0. Introduction

Special functions are particular mathematical functions which are vital tools in higher calculus, and generally play a key role in almost all branches of applied and pure mathematics. Many well-known special functions are helpful to solve boundary value problems. In addition, solutions of many differential equations come out in terms of special functions. The zoo of special function contains Gamma function, Beta function, Hypergeometric function, Bessel's function, Mittag-Leffler function and many more. Special functions have been one of the most popular area of mathematics. In 17^{th} century, beta function was investigated by Legendre and Euler. Beta function is commonly used in probability theory, physics, engineering and other areas of mathematics.

Due to various implementations of generalized hypergeometric and Mittag-Leffler functions, many researchers have made their contribution to extend it in various forms. Recently, many authors [2–11] introduced several extensions of the well-known special functions due to their vital importance in mathematical and functional analysis.

We start with the Gosta Mittag-Leffler function $\mathbf{E}_{v}(z)$ [12] represented by the following series as:

$$\mathbf{E}_{v}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(vk+1)},\ (z,v\in\mathbb{C};\mathbf{Re}(v)>0). \tag{1}$$

A more general form of Mittag-Leffler function given by Wiman [13] has the following form

$$\mathbf{E}_{v,\tau}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(vk+\tau)},\ (z,v,\tau\in\mathbb{C};\mathbf{Re}(v)>0,\mathbf{Re}(\tau)>0). \tag{2}$$

Throughout this article, we will denote by $\mathbb{C}, \mathbf{N}, \mathbf{Z}_0^-$, and $\mathbb{R}^+$ the set of Complex numbers, natural numbers, non-positive integers and positive real numbers respectively. Prabhakar [14] established the generalization of $\mathbf{E}_{v,\tau}(z)$ in the following form

$$\mathbf{E}_{v,\tau}^{\lambda}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)_k z^k}{\Gamma(vk+\tau)k!}, \ (z, v, \tau, \lambda \in \mathbb{C}; \mathbf{Re}(v) > 0, \mathbf{Re}(\tau) > 0), \tag{3}$$

where $(\lambda)_k$ represents the Pochhammer symbol [15] defined as follows

$$\begin{aligned} (\lambda)_k &= \begin{cases} 1; & k = 0, \lambda \in \mathbb{C} - \{0\}, \\ \lambda(\lambda+1)...(\lambda+k-1); & k \in \mathbf{N}, \lambda \in \mathbb{C} \end{cases} \\ &= \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} \quad \lambda \in \mathbb{C} - \mathbf{Z}_0^-. \end{aligned}$$

Generalized Mittag-Leffler function $\mathbf{E}_{v,\tau}^{\lambda,s}(z)$ introduced by Prajapati and Shukla [16] has the following form

$$\mathbf{E}_{v,\tau}^{\lambda,s}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)_{sk} z^k}{\Gamma(vk+\tau)k!}, \ (z, v, \tau, \lambda \in \mathbb{C}; \mathbf{Re}(v) > 0, \mathbf{Re}(\tau) > 0, \mathbf{Re}(\lambda) > 0; s \in (0,1) \cup \mathbf{N}), \tag{4}$$

where $(\lambda)_{sk} = \frac{\Gamma(\lambda+sk)}{\Gamma(\lambda)}$ represents the generalized Pochhammer symbol. Ozarslan and Yilmaz [17] extended the Mittag-Leffler function $\mathbf{E}_{v,\tau}^{\lambda,s}(z)$ in the following way

$$\mathbf{E}_{v,\tau}^{\lambda,s}(z;p) = \sum_{k=0}^{\infty} \frac{\mathbf{B}_p(\lambda+k, s-\lambda)}{\mathbf{B}(\lambda, s-\lambda)} \frac{(s)_k}{\Gamma(vk+\tau)} \frac{z^k}{k!}, \tag{5}$$

where $(z, v, \tau, \lambda \in \mathbb{C}; \mathbf{Re}(v) > 0, \mathbf{Re}(\tau) > 0, \mathbf{Re}(\lambda) > 0; \mathbf{Re}(p) > 0, p = 0; \mathbf{Re}(s) > \mathbf{Re}(\lambda) > 0)$ and

$$\mathbf{B}_p(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1} \exp\left(-\frac{p}{u(1-u)}\right) du, \ (\mathbf{Re}(p) > 0, \mathbf{Re}(a) > 0, \mathbf{Re}(b) > 0). \tag{6}$$

is the extended Euler's beta function [18] and $\mathbf{B}_0(a,b) = \mathbf{B}(a,b)$ where $\mathbf{B}(a,b)$ is the classical beta function [15]. An interesting extension of extended beta function $\mathbf{B}_p(a,b)$ introduced by Parmar et al. [1] is of the following form

$$\mathbf{B}_\omega(a,b;p) = \sqrt{\frac{2p}{\pi}} \int_0^1 u^{a-\frac{3}{2}}(1-u)^{b-\frac{3}{2}} \mathbf{K}_{\omega+\frac{1}{2}}\left(\frac{p}{u(1-u)}\right) du, \ (\omega \in \mathbb{C}, \mathbf{Re}(p) > 0, \mathbf{Re}(a) > 0, \mathbf{Re}(b) > 0), \tag{7}$$

where $\mathbf{K}_{\omega+\frac{1}{2}}$ is the modified Bessel's function. Motivated essentially by the demonstrated potential for applications of these generalized Mittag-Leffler functions in many diverse areas of mathematical, physical, engineering and statistical sciences, we introduce here another interesting extension of the extended Mittag-Leffler function as follows

$$\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) = \sum_{k=0}^{\infty} \frac{\mathbf{B}_\omega(\lambda+k, s-\lambda;p)}{\mathbf{B}(\lambda, s-\lambda)} \frac{(s)_k}{\Gamma(vk+\tau)} \frac{z^k}{k!}, \tag{8}$$

where $(z, v, \tau, \lambda, \omega \in \mathbb{C}; \mathbf{Re}(v) > 0, \mathbf{Re}(\tau) > 0, \mathbf{Re}(\lambda) > 0; \mathbf{Re}(p) > 0; \mathbf{Re}(s) > \mathbf{Re}(\lambda) > 0)$. For our ambition, we remember the Fox- Wright function ${}_r\Psi_s$ [19]

$${}_r\mathbf{\Psi}_s(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1+\tau_1 k)...\Gamma(\alpha_r+\tau_r k)}{\Gamma(\beta_1+\mu_1 k)...\Gamma(\beta_s+\mu_s k)} \frac{z^k}{k!}. \tag{9}$$

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where τ_i and $\mu_m \in \mathbb{R}^+$ such that

$$1 + \sum_{m=1}^{s} \mu_m - \sum_{i=1}^{r} \tau_i > 0.$$

Further, making use of the extended Mittag-Leffler function (8). For each of these new extensions we obtain various integral representations, properties and Mellin transforms, together with differentiation, transformation, summation, generating function and asymptotic formulas.

1. Integral Formulae

In this section, we obtain several integral representations of Mittag-Leffler function ($\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p)$) and consider certain special cases.

Theorem 1. *Let* $s, v, \tau, \lambda, \in \mathbb{C}$ *with* $\mathbf{Re}(s) > \mathbf{Re}(\lambda) > 0, \mathbf{Re}(v) > 0, \mathbf{Re}(\tau) > 0,$ *and let* $\mathbf{Re}(p) > 0$. *Then*

$$\begin{aligned} \mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) &= \frac{1}{\mathbf{B}(\lambda, s-\lambda)} \sqrt{\frac{2p}{\pi}} \times \\ & \int_0^1 \left(u^{\lambda-\frac{3}{2}} (1-u)^{s-\lambda-\frac{3}{2}} \mathbf{K}_{\omega+\frac{1}{2}} \left(\frac{p}{u(1-u)} \right) \mathbf{E}_{v,\tau}^{s}(uz) \right) du. \end{aligned} \tag{10}$$

Proof. By using equation (7) in (8), we get

$$\begin{aligned} \mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) &= \sum_{k=0}^{\infty} \left[\sqrt{\frac{2p}{\pi}} \int_0^1 u^{\lambda+k-\frac{3}{2}} (1-u)^{s-\lambda-\frac{3}{2}} \mathbf{K}_{\omega+\frac{1}{2}} \left(\frac{p}{u(1-u)} \right) du \right. \\ & \left. \frac{(s)_k}{\mathbf{B}(\lambda, s-\lambda)\Gamma(vk+\tau)} \frac{z^k}{k!} \right]. \end{aligned}$$

After interchanging the order of integration and summation in the above equation which is justified under the supposition of theorem (1) ,we obtain

$$\begin{aligned} \mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) &= \frac{1}{\mathbf{B}(\lambda, s-\lambda)} \sqrt{\frac{2p}{\pi}} \int_0^1 \left(u^{\lambda-\frac{3}{2}} (1-u)^{s-\lambda-\frac{3}{2}} \mathbf{K}_{\omega+\frac{1}{2}} \left(\frac{p}{u(1-u)} \right) \right. \\ & \left. \sum_{k=0}^{\infty} \frac{(s)_k}{\Gamma(vk+\tau)} \frac{(zu)^k}{k!} \right) du. \end{aligned}$$

With the help of equation (3) in above equation, we attain the desired consequence.
Now, we obtain two different integral formulae of the extended MLF (10) as corollaries. □

Corollary 1. *Let* $s, v, \tau, \lambda, \in \mathbb{C}$ *with* $\mathbf{Re}(s) > \mathbf{Re}(\breve{}) > \mathbf{0}, \mathbf{Re}(\text{Æ}) > \mathbf{0}, \mathbf{Re}(\text{ø}) > 0$ *and let* $\mathbf{Re}(p) > 0$. *Then*

$$\begin{aligned} \mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) &= \frac{1}{\mathbf{B}(\lambda, s-\lambda)} \sqrt{\frac{2p}{\pi}} \int_0^{\infty} \left[y^{\lambda-\frac{3}{2}} (1+y)^{1-s} \mathbf{K}_{\omega+\frac{1}{2}} \left(\frac{p(1+y)^2}{y} \right) \right. \\ & \left. \mathbf{E}_{v,\tau}^{s} \left(\frac{yz}{1+y} \right) \right] dy. \end{aligned} \tag{11}$$

Proof. If we set $u = \frac{y}{1+y}$, in theorem (1),we acquire the consequence we wish to prove. □

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Corollary 2. *Let* $s, \upsilon, \tau, \lambda, \in \mathbb{C}$ *with* $\mathbf{Re}(s) > \mathbf{Re}(\breve{\ }) > \mathbf{0}, \mathbf{Re}(\text{Æ}) > \mathbf{0}, \mathbf{Re}(\text{ø}) > 0$ *and let* $\mathbf{Re}(p) > 0$. *Then*

$$\begin{aligned} \mathbf{E}_{\upsilon,\tau}^{\lambda,s;\omega}(z;p) &= \frac{2}{\mathbf{B}(\lambda, s-\lambda)}\sqrt{\frac{2p}{\pi}}\int_0^{\frac{\pi}{2}}\left[(\sin\psi)^{2(\lambda-1)}(\cos\psi)^{2(s-\lambda-1)}\right. \\ & \left.\mathbf{K}_{\omega+\frac{1}{2}}\left(\frac{p}{\sin^2\psi\cos^2\psi}\right)\mathbf{E}_{\upsilon,\tau}^{s}\left(z\sin^2\psi\right)\right]dy. \end{aligned} \tag{12}$$

Proof. If we set $u = \sin^2\psi$, in theorem (1), we acquire the consequence we want to prove. □

2. Mellin Transform Formula

Theorem 2. *Let* $s, \upsilon, \tau, \lambda, \in \mathbb{C}$ *with* $\mathbf{Re}(s) > \mathbf{Re}(\breve{\ }) > \mathbf{0}, \mathbf{Re}(\text{Æ}) > \mathbf{0}, \mathbf{Re}(\text{ø}) > 0$ *and let* $\mathbf{Re}(p) > 0$. *Then the Mellin ransform formula for the extended MLF (**??**) is*

$$\begin{aligned} \mathbb{M}\left(\mathbf{E}_{\upsilon,\tau}^{\lambda,s;\omega}(z;p);r\right) &= \frac{2^{r-1}}{\sqrt{\pi}}\frac{\Gamma(s+r-\lambda)}{\Gamma(\lambda)\Gamma(s-\lambda)}\Gamma\left(\frac{r-\omega}{2}\right)\Gamma\left(\frac{r+\omega+1}{2}\right)\times \\ & {}_2\Psi_2\left[\begin{array}{ll} (s,1),(\lambda+r,1); & \\ & |z \\ (\tau,\upsilon),(2r+s,1); & \end{array}\right]. \end{aligned} \tag{13}$$

Proof. By definition, Mellin transform of the MLF is

$$\mathbb{M}\left(\mathbf{E}_{\upsilon,\tau}^{\lambda,s;\omega}(z;p);r\right) = \int_0^\infty p^{r-1}\mathbf{E}_{\upsilon,\tau}^{\lambda,s;\omega}(z;p)dp.$$

From equation (10), we have

$$\begin{aligned} \mathbb{M}\left(\mathbf{E}_{\upsilon,\tau}^{\lambda,s;\omega}(z;p);r\right) &= \int_0^\infty\left[p^{r-1}\frac{1}{\mathbf{B}(\lambda,s-\lambda)}\sqrt{\frac{2p}{\pi}}\int_0^1 u^{\lambda-\frac{3}{2}}(1-u)^{s-\lambda-\frac{3}{2}}\right. \\ & \left.\mathbf{K}_{\omega+\frac{1}{2}}\left(\frac{p}{u(1-u)}\right)\mathbf{E}_{\upsilon,\tau}^{s}(uz)du\right]dp. \end{aligned} \tag{14}$$

By interchanging the order of integration in equation (14) that is verified by the assumptions of this theorem, we get

$$\begin{aligned} \mathbb{M}\left(\mathbf{E}_{\upsilon,\tau}^{\lambda,s;\omega}(z;p);r\right) &= \frac{1}{\mathbf{B}(\lambda,s-\lambda)}\sqrt{\frac{2}{\pi}}\int_0^1\left[u^{\lambda-\frac{3}{2}}(1-u)^{s-\lambda-\frac{3}{2}}\mathbf{E}_{\upsilon,\tau}^{s}(uz)\right] \\ & \int_0^\infty p^{r-\frac{1}{2}}\mathbf{K}_{\omega+\frac{1}{2}}\left(\frac{p}{u(1-u)}\right)dpdu. \end{aligned} \tag{15}$$

By putting $y = \frac{p}{u(1-u)}$ in the internal integral in equation (15), we have

$$\begin{aligned} \mathbb{M}\left(\mathbf{E}_{\upsilon,\tau}^{\lambda,s;\omega}(z;p);r\right) &= \frac{1}{\mathbf{B}(\lambda,s-\lambda)}\sqrt{\frac{2}{\pi}}\int_0^1 u^{\lambda+r-1}(1-u)^{s+r-\lambda-1}\mathbf{E}_{\upsilon,\tau}^{s}(uz) \\ & \left[\int_0^\infty y^{r-\frac{1}{2}}\mathbf{K}_{\omega+\frac{1}{2}}(y)dy\right]du. \end{aligned} \tag{16}$$

By using the consequence (Olver *et al.* (2010))

$$\int_0^\infty y^{r-\frac{1}{2}}\mathbf{K}_{\omega+\frac{1}{2}}(y)dy = 2^{r-\frac{3}{2}}\Gamma(\frac{r-\omega}{2})\Gamma(\frac{r+\omega+1}{2}),$$

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and then applying definition (9) in equation (16), we have

$$\mathbb{M}\left(\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p);r\right) = \frac{1}{\mathbf{B}(\lambda,s-\lambda)}\frac{2^{r-1}}{\sqrt{\pi}}\Gamma\left(\frac{r-\omega}{2}\right)\Gamma\left(\frac{r+\omega+1}{2}\right) \int_0^1 u^{\lambda+r-1}(1-u)^{s+r-\lambda-1}\sum_{k=0}^{\infty}\frac{(s)_k}{\Gamma(vk+\tau)}\frac{(uz)^k}{k!}du. \tag{17}$$

By interchanging the order of summation and integration and then applying the definition (2) in equation (17), we obtain

$$\mathbb{M}\left(\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p);r\right) = \frac{2^{r-1}}{\sqrt{\pi}}\frac{\Gamma(s+r-\lambda)}{\Gamma(\lambda)\Gamma(s-\lambda)}\Gamma\left(\frac{r-\omega}{2}\right)\Gamma\left(\frac{r+\omega+1}{2}\right) \sum_{k=0}^{\infty}\frac{\Gamma(s+k)\Gamma(\lambda+r+k)}{\Gamma(\tau+vk)\Gamma(2r+s+k)}\frac{z^k}{k!}. \tag{18}$$

By the implementation of equation (9) in equation (18), we obtain the required consequence. □

Corollary 3. *Let $s,v,\tau,\lambda,\in\mathbb{C}$ with* $\mathbf{Re}(s) > \mathbf{Re}(˘) > \mathbf{0}, \mathbf{Re}(Æ) > \mathbf{0}, \mathbf{Re}(ø) > 0$ *and let* $\mathbf{Re}(p) > 0$. *Then the following result holds true*

$$\int_0^{\infty}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p)dp = \frac{1}{\sqrt{\pi}}\frac{\Gamma(s+1-\lambda)}{\Gamma(\lambda)\Gamma(s-\lambda)}\Gamma\left(\frac{1-\omega}{2}\right)\Gamma\left(\frac{\omega+2}{2}\right) {}_2\Psi_2\left[\begin{array}{c}(s,1),(\lambda+1,1);\\ \qquad |z\\ (\tau,v),(2+s,1);\end{array}\right]. \tag{19}$$

Proof. If we take $r=1$ in equation (13), we obtain the required result. □

3. Derivative Formulae

Theorem 3. *Let $s,v,\tau,\lambda,\in\mathbb{C}$ with* $\mathbf{Re}(s) > \mathbf{Re}(˘) > \mathbf{0}, \mathbf{Re}(Æ) > \mathbf{0}, \mathbf{Re}(ø) > 0$ *and let* $\mathbf{Re}(p) > 0$. *Then the derivative formula for the extended MLF (8)is*

$$\frac{d^k}{dz^k}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) = (\lambda)_k\mathbf{E}_{v,kv+\tau}^{\lambda+k,s+k;\omega}(z;p), \quad (k\in\mathbb{N}\cup\{0\}). \tag{20}$$

Proof. From equation (8), we have

$$\begin{aligned}\frac{d}{dz}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) &= \sum_{k=1}^{\infty}\left(\frac{\mathbf{B}_{\omega}(\lambda+k,s-\lambda;p)}{\mathbf{B}(\lambda,s-\lambda)}\frac{(s)_k}{\Gamma(vk+\tau)}\frac{z^{k-1}}{(k-1)!}\right)\\ &= s\sum_{k=0}^{\infty}\left(\frac{\mathbf{B}_{\omega}(\lambda+1+k,s-\lambda;p)}{\mathbf{B}(\lambda,s-\lambda)}\frac{(s+1)_k}{\Gamma(vk+v+\tau)}\frac{z^k}{k!}\right).\end{aligned}$$

Using (5), we obtain

$$\frac{d}{dz}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) = \lambda\sum_{k=0}^{\infty}\left(\frac{\mathbf{B}_{\omega}(\lambda+1+k,s-\lambda;p)}{\mathbf{B}(\lambda+1,s-\lambda)}\frac{(s+1)_k}{\Gamma(vk+v+\tau)}\frac{z^k}{k!}\right).$$

In terms of (8), we get

$$\frac{d}{dz}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) = \lambda\mathbf{E}_{v,v+\tau}^{\lambda+1,s+1;\omega}(z;p).$$

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Similarly,we have

$$\frac{d^2}{dz^2}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) = (\lambda)_2\mathbf{E}_{v,2v+\tau}^{\lambda+2,s+2;\omega}(z;p).$$

By taking nth order derivative of the MLF (9), we obtain the desired result. □

Theorem 4. *Let s, v, τ, λ, and $\eta \in \mathbb{C}$ with $\mathbf{Re}(s) > \mathbf{Re}(\lambda) > 0, \mathbf{Re}(v) > 0, \mathbf{Re}(\tau) > 0$ and let $\mathbf{Re}(p) > 0$. Then*

$$\frac{d^k}{dz^k}\left(z^{\tau-1}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(\eta z^v;p)\right) = z^{\tau-k-1}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(\eta z^v;p), \quad (k \in \mathbb{N}\cup\{0\}). \tag{21}$$

Proof. From equation (8), we get

$$\begin{aligned}\frac{d}{dz}\left[z^{\tau-1}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(\eta z^v;p)\right] &= \frac{d}{dz}\sum_{k=0}^{\infty}\left[\frac{\mathbf{B}_\omega(\lambda+k,s-\lambda;p)}{\mathbf{B}(\lambda,s-\lambda)}\frac{(s)_k}{\Gamma(vk+\tau)}(vk+\tau-1)\right.\\ &\quad\left.\frac{\eta^k z^{vk+\tau-1}}{k!}\right]\\ &= z^{\tau-2}\sum_{k=0}^{\infty}\frac{\mathbf{B}_\omega(\lambda+k,s-\lambda;p)}{\mathbf{B}(\lambda,s-\lambda)}\frac{(s)_k}{\Gamma(vk+\tau-1)}\frac{(\eta z^v)^k}{k!}.\end{aligned}$$

By using equation (8),above equation can be written as

$$\frac{d}{dz}\left[z^{\tau-1}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(\eta z^v;p)\right] = z^{\tau-2}\mathbf{E}_{v,\tau-1}^{\lambda,s;\omega}(\eta z^v;p).$$

Similarly, we have

$$\frac{d^2}{dz^2}\left[z^{\tau-1}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(\eta z^v;p)\right] = z^{\tau-3}\mathbf{E}_{v,\tau-2}^{\lambda,s;\omega}(\eta z^v;p).$$

After differentiating equation (8) k times, we get the desired result. □

4. Recurrence Relation

Theorem 5. *Let $s, v, \tau, \lambda, \in \mathbb{C}$ with $\mathbf{Re}(s) > \mathbf{Re}(\lambda) > 0, \mathbf{Re}(v) > 0, \mathbf{Re}(\tau) > 0$ and let $\mathbf{Re}(p) > 0$, then*

$$\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) = \tau\mathbf{E}_{v,\tau+1}^{\lambda,s;\omega}(z;p) + vsz\mathbf{E}_{v,v+\tau+1}^{\lambda+1,s+1;\omega}(z;p). \tag{22}$$

Proof. By using (8) in the integrand of (10), we get

$$\begin{aligned}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) &= \frac{1}{\mathbf{B}(\lambda,s-\lambda)}\sqrt{\frac{2p}{\pi}}\int_0^1\left[u^{\lambda-\frac{3}{2}}(1-u)^{s-\lambda-\frac{3}{2}}\mathbf{K}_{\omega+\frac{1}{2}}\left(\frac{p}{u(1-u)}\right)\right.\\ &\quad\left.\tau\mathbf{E}_{v,\tau+1}^{s}(uz)\right]du + \frac{1}{\mathbf{B}(\lambda,s-\lambda)}\sqrt{\frac{2p}{\pi}}\int_0^1\left[u^{\lambda-\frac{3}{2}}(1-u)^{s-\lambda-\frac{3}{2}}\right.\\ &\quad\left.\mathbf{K}_{\omega+\frac{1}{2}}\left(\frac{p}{u(1-u)}\right)vz\frac{d}{dz}\mathbf{E}_{v,\tau+1}^{s}(uz)\right]du.\end{aligned} \tag{23}$$

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Using equation (21) for $k = 1$ in equation (23), we get

$$\begin{aligned}\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) &= \tau\mathbf{E}_{v,\tau+1}^{\lambda,s;\omega}(z;p) + vsz\frac{1}{\mathbf{B}(\lambda,s-\lambda)}\sqrt{\frac{2p}{\pi}} \\ &\int_0^1 \Big[u^{(\lambda+1)-\frac{3}{2}}(1-u)^{(s+1)-(\lambda+1)-\frac{3}{2}} \\ &\mathbf{K}_{\omega+\frac{1}{2}}\left(\frac{p}{u(1-u)}\right)\mathbf{E}_{v,v+\tau+1}^{s+1}(uz)\Big]\,du.\end{aligned}$$

Using equation (10) in above equation, we obtain

$$\mathbf{E}_{v,\tau}^{\lambda,s;\omega}(z;p) = \tau\mathbf{E}_{v,\tau+1}^{\lambda,s;\omega}(z;p) + vsz\mathbf{E}_{v,v+\tau+1}^{\lambda+1,s+1;\omega}(z;p).$$

□

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