Deflection angle of photon from magnetized black hole and effect of nonlinear electrodynamics

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In this paper, we proposed a new model of non-linear electrodynamics with parameter. Firstly, we study the weak limit approximation and by using the Gauss Bonnet theorem we obtain the deflection angle of photon from magnetized black hole and effect of non-linear electrodynamics. In doing so, we find the corresponding optical metric after that we calculate the Gaussian curvature which is used in Gauss Bonnet theorem. Then we show the deflection angle in the leading order terms. We also analyzed that our results reduces into Maxwell's electrodynamics and RN solution with the reduction of parameters. Moreover, we also investigate the graphical behavior of deflection angle w.r.t correction parameter, black hole charge and impact parameter.

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I. INTRODUCTION

In 1916, Einstein cleverly predicted the existence of gravitational waves and gravitational lensing as part of the theory of general relativity [1]. In 2015, these waves were first detected by LIGO [2]. This shows that the theoretical predictions are well fitted with experimental observations. Hence, this new detection of gravitational waves marks not only a culmination of a decades-long search, but also the beginning of a new way to look at the universe. After detection of gravitational waves by LIGO, there is renewed interest in the topic of gravitational lensing of gravitational waves [3]. Gravitational lensing first proposed by Soldner in 1801 in context of Newtonian theory [4]. Many useful results for cosmology have come out of using this property of matter and light. Then Using data taken during a solar eclipse in 1919, Eddington measured a value close to that of the GR prediction [5, 6]. Then gravitational lensing has been worked in the literature in various space-times using different methods [7–15].

Moreover, over the years, there have been many studies linking gravitational lensing with the Gauss-Bonnet theorem (GBT). First, Gibbons and Werner (GW) elegantly showed that the possibile way of calculation to deflection angle using the GBT for asymptotically flat static black holes [16]:

$$\alpha = -\int\int_{S_\infty} \mathcal{K} \mathrm{d}\sigma,$$

Here $\mathcal{K}$ and $\mathrm{d}\sigma$ are the Gaussian curvature and surface element of optical metric.

Afterwards, Werner extended this method for stationary black holes [17]. Next, Ishihara et al. [18] showed that it is possible to find deflection angle for the finite-distances (large impact parameter) because the GW only found the deflection angle using the optical Fermat geometry of the black hole's spacetime in weak field limits and for the observers at asymptotically flat region. Recently, Crisnejo and Gallo have studied the deflection of light in a plasma medium [19]. Since then, there is a continuously growing interest to the weak gravitational lensing via the method of GW and GBT of black holes, wormholes or cosmic strings [20–40].

The main aim of this paper is to investigate the effect of the nonlinear electrodynamical (magnetized) charge on the deflection angle where we use the GBT in which the deflection of light become a global effect [41]. Because we only focus the nonsingular region outside of a photon rays. It is expected that the horizon of a black hole is detected

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using the Event Horizon Telescope (EHT) in this year so that the properties of the black hole which are encoded in the event horizon might be appears.

Gravitational singularities are mainly considered within general relativity, where density apparently becomes infinite at the center of a black hole, and within astrophysics and cosmology as the earliest state of the universe during the Big Bang. In General Relativity, spacetime singularities raise a number of problems, both mathematical and physical [42–48]. Using the nonlinear electrodynamics it is possible to remove these singularities by constructing a regular black hole solution [49–53]. Recently, Kruglov proposed a new model of nonlinear electrodynamics with two parameters β and γ where the the specific range of magnetic field, the causality and unitarity principles are satisfied [41]. Moreover, it is shown that there is no singularity of the electric field strength at the origin for the point-like particles and it has a magnetic charge.

This paper is organized as follows. In section 2, we briefly review the solution of magnetized black hole and then we calculate its optical geometry and the optical curvature. In section 3, deflection angle of photon using the GBT is studied in the case of magnetized black hole. In section 4, we analysis the deflection angle in details using the graphical analytsis. And paper concludes in section 4 with a discussion regarding the results obtained from the present work.

II. WEAK GRAVITATIONAL LENSING AND MAGNETIZED BLACK HOLE

In this section, our NLED's model as [41]

$$J = \int d^4x\sqrt{-g}\left(\frac{1}{2k^2R} + \mathcal{L}\right). \tag{1}$$

Where $k^2 = 8\pi G \equiv M_{pl}^{-2}$, M_{pl} is for reduced planck mass, G is Newtonian constant and R is Ricci scalar. From above equation we derived the Einstein equation as

$$R_{ab} - \frac{1}{2}g_{ab}R = -k^2T_{ab}. \tag{2}$$

Now, we derive the equation of motion for electromagnetic fields by varying (2.1)

$$\partial_a(\sqrt{-g}(F^{ab}\mathcal{L}_G + \tilde{F}^{ab}\mathcal{L}_G)) = 0. \tag{3}$$

Now, we analyze the static magnetic black hole solution by using the Einstein field equation and equation of motion for electromagnetic field with above equations. In the case of pure magnetic field, Bronnikov shown[49] that when spherical symmetry holds, the invariant is $\mathcal{F} = q^2/(2r^2)$, here q is a magnetic charge. In this case we find the static and spherical symmetric space-time, the line element is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4}$$

By assuming both the source and observer are in the equatorial plane likewise trajectory of the null photon is also on the same plane with$(\theta = \frac{\pi}{2})$, we obtain the optical metric as follows

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\varphi^2. \tag{5}$$

For moving photon in the equatorial plane, $k_2 = 0$ and null geodesics, $ds^2 = 0$, we get

$$dt^2 = \frac{1}{f(r)^2}dr^2 + \frac{r^2}{f(r)}d\varphi^2. \tag{6}$$

Subsequently, we make the transformation into new coordinate u, the metric function $\zeta(u)$ as

$$du = \frac{dr}{f(r)}, \zeta = \frac{r}{\sqrt{f(r)}}. \tag{7}$$

Then the optical metric tensor $\bar{g}_{ab}$ as follows

$$dt^2 = \bar{g}_{ab}dx^a dx^b = du^2 + \zeta^2 d\varphi^2. \tag{8}$$

It is to be noted that $(a,b) \to (r,\varphi)$ and determinant is $det\bar{g}_{ab} = \frac{r^2}{f(r)^3}$. Now by using $Eq.(2.8)$, the non-zero Christopher symbols are $\Gamma^r_{rr} = -\frac{f'(r)}{f(r)}$, $\Gamma^r_{\varphi\varphi} = \frac{r(rf'(r)-2f(r))}{2}$, $\Gamma^{\varphi}_{r\varphi} = \frac{-rf'(r)+2f(r)}{2rf(r)} = \Gamma^{\varphi}_{\varphi r}$. Hence, we can find the Gaussian optical curvature $\mathcal{K}$ [16] as follows

$$\mathcal{K} = -\frac{R_{r\varphi r\varphi}}{det\bar{g}_{r\varphi}} = -\frac{1}{\zeta}\frac{d^2\zeta}{du^2}. \tag{9}$$

Now, we rewritten Gaussian optical curvature in terms of Schwarzschild radial coordinate r[54]. We have

$$\mathcal{K} = -\frac{1}{\zeta}\left[\frac{dr}{du}\frac{d}{dr}(\frac{dr}{du})\frac{d\zeta}{dr} + (\frac{dr}{du})^2\frac{d^2\zeta}{dr^2}\right]. \tag{10}$$

By applying the Eq.(2.10) into our metric (2.5) we obtain the Gaussian optical curvature of photon from magnetized black hole, which yields

$$\mathcal{K} = -\frac{\sqrt{f(r)}}{r}\left[\frac{r(f'(r)^2 - 2f(r)f''(r))}{4\sqrt{f(r)}}\right], \tag{11}$$

where the function $f(r)$ is [41]

$$f(r) = 1 - \frac{2Gm}{r} + \frac{Gq^2}{r^2} - \frac{\beta G q^4}{5r^6} + \mathcal{O}(r^{-10}). \tag{12}$$

After substituting the value of $f(r)$, we get the value of optical curvature in terms of r as

$$\begin{aligned}\mathcal{K} &= \frac{G[-G(-5q^2r^4 + 5mr^5 + 3q^4\beta)^2]}{25r^{14}} + \frac{G(-15q^2r^4 + 10mr^5 + 21q^4\beta)}{25r^{14}} \\ &\times \frac{G(-5r^6 + G(-5q^2r^4 + 10mr^5 + q^4\beta))}{25r^{14}},\end{aligned} \tag{13}$$

which asymptotically behaves as

$$\mathcal{K} \simeq -\frac{2mG}{r^3} + \frac{3q^2G}{r^4} - \frac{21\beta q^4}{5r^8}. \tag{14}$$

III. DEFLECTION ANGLE OF PHOTON AND GAUSS-BONNET THEOREM

Now, we use the Gauss-Bonnet theorem to derive the deflection angle of photon for magnetized black hole. We apply the Gauss-Bonnet theorem to the region $\mathcal{M}_R$, stated as [16]

$$\int_{\mathcal{M}_R}\mathcal{K}dS + \oint_{\partial\mathcal{M}_R}kdt + \sum_j \theta_j = 2\pi\mathcal{X}(\mathcal{M}_R). \tag{15}$$

Here $\mathcal{K}$ is for Gaussian curvature and the geodesic curvature is k, given as $k = \bar{g}(\nabla_{\dot{\alpha}}\dot{\alpha}, \ddot{\alpha})$ in such a way that $\bar{g}(\dot{\alpha},\dot{\alpha}) = 1$, where θ_j is the representation for exterior angle at the j^{th} vertex and $\ddot{\alpha}$ is unit acceleration vector. The jump angles become $\pi/2$ as $R \to \infty$ and we get $\theta_O + \theta_S \to \pi$. The Euler characteristic is $\mathcal{X}(\mathcal{M}_R) = 1$, as $\mathcal{M}_R$ is non singular. Therefore we get

$$\int_{\mathcal{M}_R}\int \mathcal{K}dS + \oint_{\partial\mathcal{M}_R}kdt + \theta_j = 2\pi\mathcal{X}(\mathcal{M}_R). \tag{16}$$

Where $\theta_j = \pi$ shows the total jump angle and $\alpha_{\bar{g}}$ is a geodesic; since the Euler characteristic number $\mathcal{X}$ is 1. As $R \to \infty$ the remaining part yield that $k(D_R) = \mid \nabla_{\dot{D}_R}\dot{D}_R \mid$. The radial component of the geodesic curvature is given by

$$(\nabla_{\dot{D}_R}\dot{D}_R)^r = \dot{D}^{\varphi}_R\partial_{\varphi}\dot{D}^r_R + \Gamma^r_{\varphi\varphi}(\dot{D}^{\varphi}_R)^2. \tag{17}$$

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At very large R, $D_R := r(\varphi) = R = const$. Therefore, the first term of equation (3.3) vanish and $(\dot{D}_R^{\varphi})^2 = \frac{1}{\zeta}$. Recalling $\Gamma^r_{\varphi\varphi} = \frac{r(rf'(r)-2f(r))}{2}$, we get

$$(\nabla_{\dot{D}_R^r}\dot{D}_R^r)^r \rightarrow \frac{-1}{R}, \tag{18}$$

and it shows that the geodesic curvature is independent of topological defects, $k(D_R) \rightarrow R^{-1}$. But from the optical metric (2.8), we can say that $dt = Rd\varphi$. Therefore we found that

$$k(D_R)dt = \frac{1}{R}Rd\varphi. \tag{19}$$

Taking into account the above results, we obtain

$$\int \mathcal{M}_R \int \mathcal{K}ds + \oint kdt =^{R\rightarrow\infty} \int_{S_\infty} \int \mathcal{K}dS + \int_0^{\pi+\Theta} d\varphi. \tag{20}$$

In the weak deflection limit, by assuming that at the zeroth order the light ray is given by $r(t) = b/\sin\varphi$. Thus by using (2.14) and (3.6), the deflection angle becomes [16]

$$\Theta = -\int_0^{\pi}\int_{b/\sin\varphi}^{\infty} \mathcal{K}\sqrt{det\bar{g}}dud\varphi, \tag{21}$$

where,

$$\sqrt{det\bar{g}}du = rdr(1 - \frac{6mG}{r} + \frac{3Gq^2}{r^2} - \frac{3q^4G\beta}{5r^6}). \tag{22}$$

After substituting the leading order term of Gaussian curvature (2.14) into equation(3.7), the deflection angle upto second order term as follows:

$$\Theta \approx \frac{4mG}{b} - \frac{3\pi Gq^2}{4b^2} + \frac{7\pi\beta q^4}{32b^6} + \frac{32m\beta Gq^2}{25b^7}. \tag{23}$$

IV. GRAPHICAL ANALYSIS

This section is devoted to discuss the graphically behavior of deflection angle Θ. We also demonstrate the physical significance of these graphs to analyze the impact of correction parameter β, BH charge q, and impact parameter b on deflection angle by examining the stability and instability of BH.

A. Deflection angle with Impact parameter b

This subsection gives the analysis of deflection angle Θ with impact parameter b for different values of correction parameter β and BH charge q, for fixed $m = 1, G = 1$ and $\pi = 3.14$.

- **Figure 1** shows the behavior of Θ with b for varying q and for fixed value for correction parameter.
  1. In figure (i), we examined that the deflection angle exponentially decreases for small variation of q.
  2. In figure (ii), we analyzed that the deflection angle gradually decreasing and then eventually goes to infinity for large variation of q which is the unstable state of magnetized BH.
     Therefore, we conclude that for small values of q the magnetized BH is stable but as q increases it shows the unstable behavior of magnetized BH.

  In **Figure 2**, indicates the behavior of deflection angle with impact parameter by varying correction parameter β.

  1. In figure (i), we noticed that the deflection angle decreasing constantly for small values of β but in
  2. In figure (ii), as β increases the deflection angle gradually decreasing and then goes to infinity.

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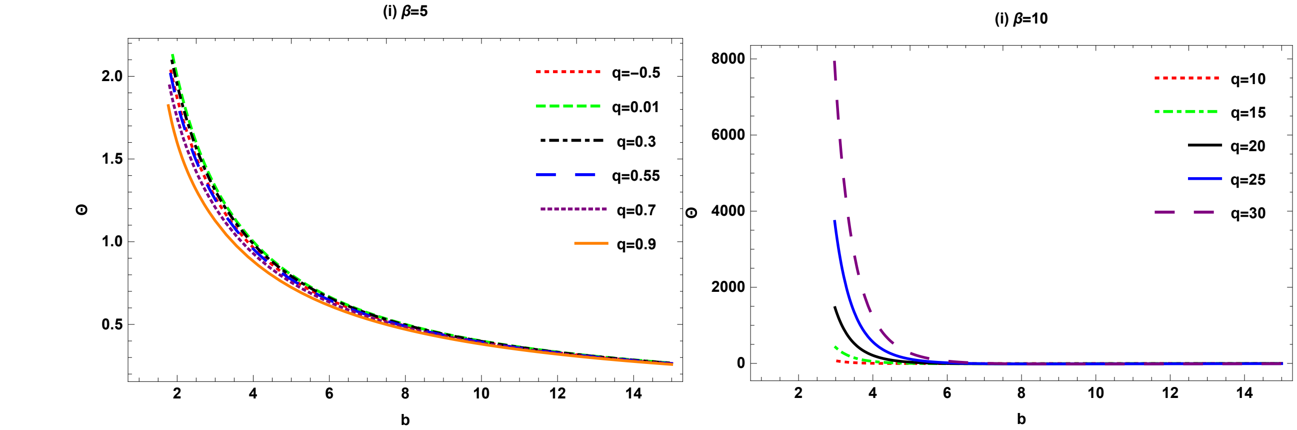

Figure 1: Relation between Θ and impact parameter b.

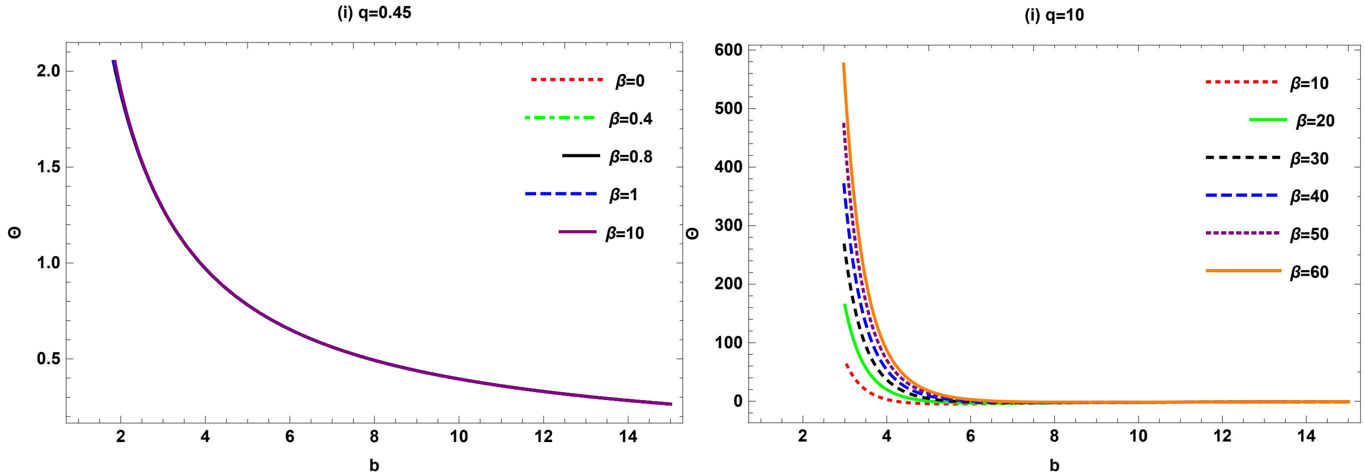

Figure 2: Relation between Θ and impact parameter b.

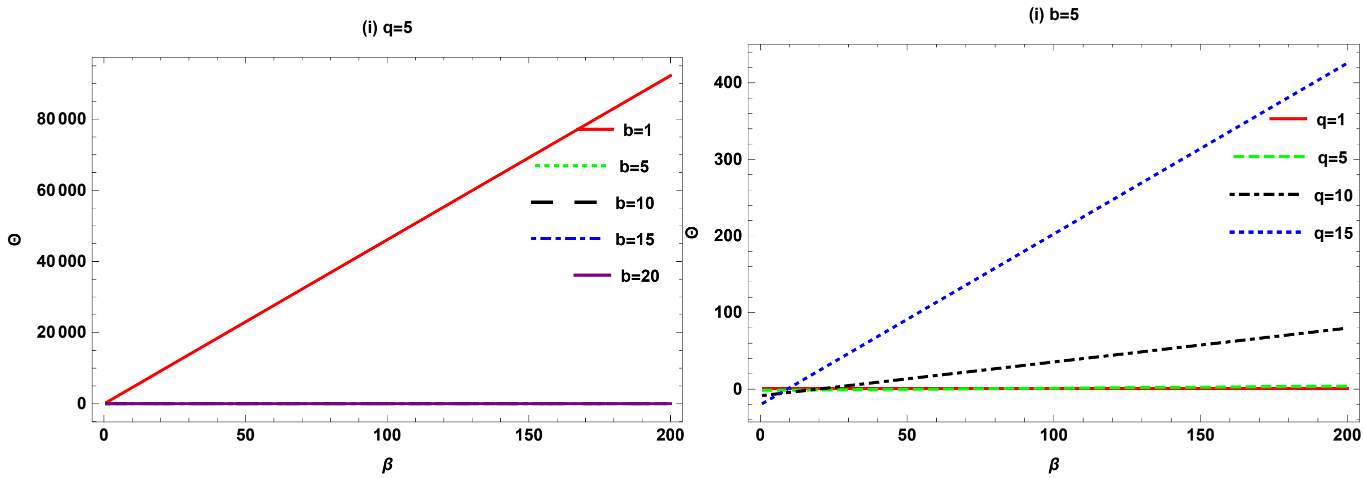

Figure 3: Relation between Θ and correction parameter β.

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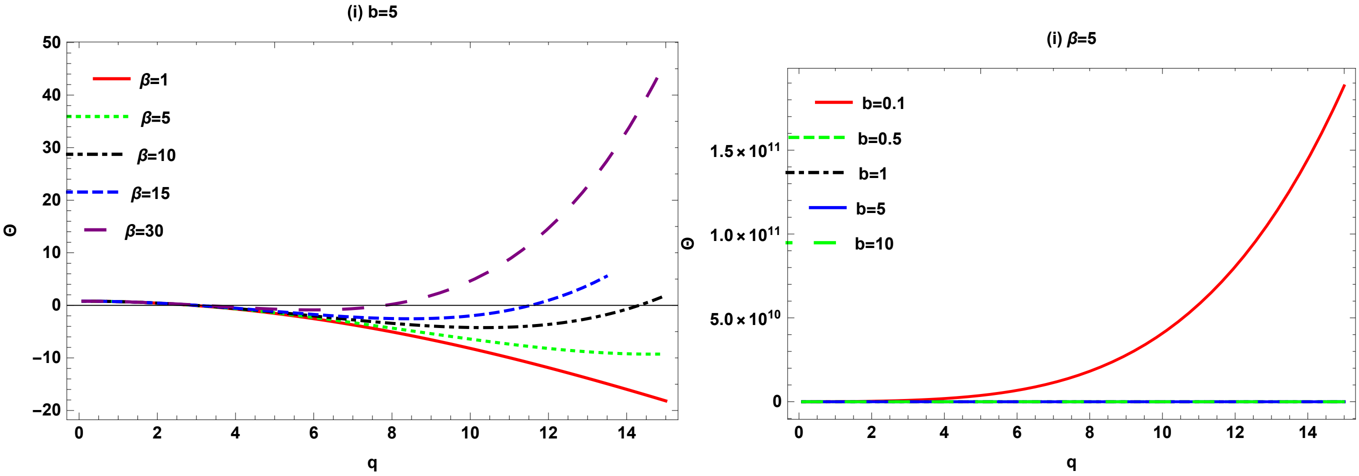

Figure 4: Relation between Θ and BH charge q.

- **Figure 3** shows the behavior of Θ with correction parameter β.
  1. In figure (i), shows the behavior of Θ with β, for varying b and fixed q. This shows that when $b < 0$ it gives the uniform negative behavior but for $b > 0$ it behave as a positive slope. We also conclude that for $4 \leq b < 15$ the behavior is negative slope but for $b \geq 15$ it gives positive slope.
  2. In figure (ii), indicate the behavior of Θ with β, for the variation of q and fixed b. We examined that the deflection angle is positively increasing with increase of q but for $5 \leq q \leq 10$ the behavior is negative and then it becomes positive.

In **Figure 4**, represent the behavior of deflection angle with BH charge q.

1. In figure (i), represents the behavior of Θ with q, by varying β and fixed b. We analyzed that the deflection angle initially decreases but as β increases the the deflection angle firstly decreases and then increases. We also observe that for $\beta > 20$ the behavior is increasing.
2. In figure (ii), shows the behavior of Θ with q, by varying b and fixed β. We observed that the deflection angle is increasing for smaller values of b but as b increases the deflection angle become negatively decreasing.

V. CONCLUSION

In this paper, we have proposed a new model of NLED with parameter β. Then, we analyzed the magnetized black hole and obtained the regular black hole solution. After that we calculate the optical Gaussian curvature for magnetized black hole. Then by using Gauss-Bonnet theorem, we calculate the weak gravitational lensing. We obtain the following angle of deflection for magnetized black hole

$$\Theta \approx \frac{4mG}{b} - \frac{3\pi G q^2}{4b^2} + \frac{7\pi\beta q^4}{32b^6} + \frac{32m\beta G q^2}{25b^7}. \tag{24}$$

We conclude that for $f(r)$ if $r \to \infty$ the space-time becomes flat, if $\beta = 0$ the NLED model converted into Maxwell's electrodynamics and the solution becomes RN solution. We have analyzed the behavior of deflection angle w.r.t impact parameter b, correction parameter β and BH charge q.
The results obtained from the analysis of deflection angle given in the paper are summarized as follows:
Deflection angle with respect to impact parameter:

- In our analysis we have discussed the behavior of deflection angle, for this we choose different values of BH charge q and fixed correction parameter. For smaller values the deflection angle gradually decreasing but for large values of q the deflection angle gradually decreases and then goes to infinity.
- While for fixed q and different values of β, the deflection angle constantly decreases for small range of β and gradually decreases for greater values.

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- Thus, for smaller values BH indicates the stability and for large values shows the instability of BH.

Deflection angle with respect to correction parameter:

- The behavior of Θ w.r.t β, for fixed q and varying b, positive behavior can be observed only for $b > 0$ except $4 \leq b < 15$.
- The behavior of Θ w.r.t β, for fixed b and varying q, the behavior is positively increasing for $q > 0$ except $5 \leq q \leq 10$.

Deflection angle with respect to black hole charge:

- The behavior of Θ w.r.t q, for fixed b and varying β, the deflection angle initially decreases but with the increase of β firstly decreases and then increases. For $\beta > 20$ the deflection angle positively increases.
- The behavior of Θ w.r.t q, for fixed β and choose different values of b, the deflection angle positively increases for small b and negatively decreases for greater b.

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