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# Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Compactness

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Abstract. Several specific types of generalized compactness of generalized topological spaces have been defined, investigated and related to compactness in ordinary topological spaces from time to time in the literature of topological spaces. Our recent research in the field of a new class of generalized compactness in a generalized topological space is reported herein as a starting point for more generalized classes.

Key words and phrases. *Generalized topological space, generalized compactness, generalized countable compactness, generalized sequential compactness, generalized local connectedness*

## 1. Introduction

Among the most fundamental topological properties (briefly, $\mathcal{T}$-properties relative to ordinary topology, and $\mathcal{T}_{\mathfrak{g}}$-properties relative to generalized topology), the $\mathcal{T}$-properties[1], termed $\mathfrak{T}$-compactness and $\mathfrak{g}$-$\mathfrak{T}$-compactness in $\mathcal{T}$-spaces (ordinary and generalized compactness in ordinary topological spaces) and the $\mathcal{T}_{\mathfrak{g}}$-properties termed $\mathfrak{T}_{\mathfrak{g}}$-compactness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathcal{T}_{\mathfrak{g}}$-spaces (ordinary and generalized compactness in generalized topological spaces) are verily the most important invariant properties [3, 4, 5, 7, 15, 16, 17, 20, 21, 22, 23, 25, 28, 29, 30, 31, 32, 33, 34, 35, 36]. In actual truth, $\mathfrak{T}$-compactness is an absolute property of a $\mathfrak{T}$-set [2, 13, 38, 33], and $\mathfrak{g}$-$\mathfrak{T}$-compactness, $\mathfrak{T}_{\mathfrak{g}}$-compactness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness, respectively, are absolute properties of a $\mathfrak{g}$-$\mathfrak{T}$-set, a $\mathfrak{T}_{\mathfrak{g}}$-set, and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set [18, 25, 29]. Typical examples of $\mathfrak{g}$-$\mathfrak{T}$-compactness in $\mathcal{T}$-spaces are $\alpha$, $\beta$, $\gamma$-compactness [10, 19, 26]; examples of $\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathcal{T}_{\mathfrak{g}}$-spaces are semi-$^{*}\alpha$, s, gb-compactness [7, 14, 29], whereas examples of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathcal{T}_{\mathfrak{g}}$-spaces are $\mathrm{bT}^{\mu}$, $\mu$-rgb, $\pi$p-compactness [5, 22, 37], among others.

In the literature of $\mathcal{T}_{\mathfrak{g}}$-spaces, the study of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets by various authors has produced some new classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathcal{T}_{\mathfrak{g}}$-spaces, similar in descriptions to $\mathfrak{g}$-$\mathfrak{T}$-compactness in $\mathcal{T}$-spaces [20, 21, 22, 25, 28, 30, 34, 35, 36]. In the paper

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[1]Notes to the reader: The structures $\mathfrak{T} = (\Omega, \mathcal{T})$ and $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, respectively, are topological spaces (briefly, $\mathcal{T}$-space and $\mathcal{T}_{\mathfrak{g}}$-space) with ordinary topology $\mathcal{T}$ and generalized topology $\mathcal{T}_{\mathfrak{g}}$ (briefly, topology and $\mathfrak{g}$-topology). Subsets of $\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{T}$, $\mathfrak{T}_{\mathfrak{g}}$-sets; subsets of $\mathcal{T}$, $\mathcal{T}_{\mathfrak{g}}$, respectively, are called $\mathcal{T}$, $\mathcal{T}_{\mathfrak{g}}$-open sets, and their complements are called $\mathcal{T}$, $\mathcal{T}_{\mathfrak{g}}$-closed sets. Generalizations of $\mathfrak{T}$-sets, $\mathcal{T}$-open and $\mathcal{T}$-closed sets in $\mathfrak{T}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathcal{T}$-open and $\mathfrak{g}$-$\mathcal{T}$-closed sets; generalizations of $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets in $\mathfrak{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed sets. Compactness in $\mathfrak{T}$ with $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-open coverings are called $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-compactness, respectively; compactness in $\mathfrak{T}_{\mathfrak{g}}$ with $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open coverings are called $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness, respectively.

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of [21], the authors introduced, gave characterizations and studied the properties of a new kind of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness, called soft $\mu$-compactness, in $\mathcal{T}_{\mathfrak{g}}$-spaces. [25] introduced two types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness, called $\mu$-semi-compactness and $\mu$-semi-Lindelöfness in $\mathcal{T}_{\mathfrak{g}}$-spaces and, studied and gave characterizations of some of their properties. [34] introduced five types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in $\mathcal{T}_{\mathfrak{g}}$-spaces, called g-compactness, g-semi compactness, g-precompactness, g-$\alpha$-compactness, and g-$\beta$-compactness, and gave the characterizations of each of such g-compactness like properties and established the relationships among them. [36] gave the definition of the notion of $\mathrm{D}_{\mu}$-compactness in $\mathcal{T}_{\mathfrak{g}}$-spaces and studied its properties. The authors also investigated $\mathrm{D}_{\mu}$-compactness in $\mathcal{T}_{\mathfrak{g}}$-subspaces and products of $\mathcal{T}_{\mathfrak{g}}$-spaces, just to name a few.

In regards to the above references, it results that, from every new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set defined in a $\mathcal{T}_{\mathfrak{g}}$-space, there can be defined a new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-connectedness in the $\mathcal{T}_{\mathfrak{g}}$-space. Having defined a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets and studied from it some $\mathcal{T}_{\mathfrak{g}}$-properties in a $\mathcal{T}_{\mathfrak{g}}$-space (see our works on *theories of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-sets*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-maps*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-separation axioms*, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-connectedness*), it seems, therefore, worth considering to introduce a new type of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in the $\mathcal{T}_{\mathfrak{g}}$-space and discuss its $\mathcal{T}_{\mathfrak{g}}$-properties accordingly. In this paper, we attempt to make a contribution to such a development by introducing a new theory, called *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Compactness*, in which it is presented a new generalized version of $\mathfrak{T}_{\mathfrak{g}}$-connectedness in terms of the notions of $\mathfrak{g}$-$\mathfrak{T}$-sets, discussing the basic properties and giving its characterizations in this direction.

The paper is organised as follows: In Sect. 2, preliminary notions are described in Sect. 2.1 and the main results of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in a $\mathcal{T}_{\mathfrak{g}}$-space are reported in Sect. 3. In Sect. 4, the establishment of the relationships among various types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness are discussed in Sect. 4.1. To support the work, a nice application of the concept of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in a $\mathcal{T}_{\mathfrak{g}}$-space is presented in Sect. 4.2. Finally, Sect. 4.3 provides concluding remarks and future directions of the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in a $\mathcal{T}_{\mathfrak{g}}$-space.

## 2. Theory

2.1. Preliminaries. Though many of the notations and definitions had already been neatly discussed in complementary papers (see papers on theories of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-maps, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation axioms) however, we thought it necessary to recall some basic definitions and notations of most basic concepts presented in those papers.

The set $\mathfrak{U}$ denotes the universe of discourse, fixed within the framework of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness and containing as elements all sets ($\Lambda$-sets: $\Lambda \in \{\Omega, \Sigma\}$; $\mathcal{T}_{\Lambda}$, $\mathfrak{g}$-$\mathcal{T}_{\Lambda}$, $\mathfrak{T}_{\Lambda}$, $\mathfrak{g}$-$\mathfrak{T}_{\Lambda}$-sets; $\mathcal{T}_{\mathfrak{g},\Lambda}$, $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g},\Lambda}$, $\mathfrak{T}_{\mathfrak{g},\Lambda}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-sets, to name a few) considered in this theory, and $I_n^0 \stackrel{\text{def}}{=} \{\nu \in \mathbb{N}^0 : \nu \leq n\}$; index sets $I_{\infty}^0$, $I_n^*$, $I_{\infty}^*$ are defined in an analogous way. Let $\Lambda \in \{\Omega, \Sigma\} \subset \mathfrak{U}$ be a given set and let $\mathcal{P}(\Lambda) \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g},\nu} \subseteq \Lambda : \nu \in I_{\infty}^*\}$ be the collection of all subsets $\mathcal{O}_{\mathfrak{g},1}$, $\mathcal{O}_{\mathfrak{g},2}$, ..., of $\Lambda$. Then every one-valued map of the type $\mathcal{T}_{\mathfrak{g},\Lambda} : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$ satisfying $\mathcal{T}_{\mathfrak{g},\Lambda}(\emptyset) = \emptyset$, $\mathcal{T}_{\mathfrak{g},\Lambda}(\mathcal{O}_{\mathfrak{g}}) \subseteq \mathcal{O}_{\mathfrak{g}}$, and $\mathcal{T}_{\mathfrak{g},\Lambda}\left(\bigcup_{\nu\in I_{\infty}^*} \mathcal{O}_{\mathfrak{g},\nu}\right) = \bigcup_{\nu\in I_{\infty}^*} \mathcal{T}_{\mathfrak{g},\Lambda}(\mathcal{O}_{\mathfrak{g},\nu})$ is called a $\mathfrak{g}$-topology on $\Lambda$, and the structure $\mathfrak{T}_{\mathfrak{g},\Lambda} \stackrel{\text{def}}{=} (\Lambda, \mathcal{T}_{\mathfrak{g},\Lambda})$ is called a $\mathcal{T}_{\mathfrak{g},\Lambda}$-space, on which no separation axioms are assumed unless otherwise mentioned [12, 11, 27]. On the other hand, if $\mathcal{P}(\Gamma) \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g},\nu} \subset \Gamma : \nu \in I_{\infty}^*\}$ denotes the family of all subsets $\mathcal{O}_{\mathfrak{g},1}$, $\mathcal{O}_{\mathfrak{g},2}$, ..., of any subset $\Gamma \subseteq \Lambda$ of $\Lambda$, then every one-valued restriction map of the type $\mathcal{T}_{\mathfrak{g},\Gamma} : \mathcal{P}(\Gamma) \longmapsto \mathcal{T}_{\mathfrak{g},\Gamma} \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g}} \cap \Gamma : \ \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g},\Lambda}\}$ defines a relative $\mathfrak{g}$-topology on $\Gamma$, and the structure $\mathfrak{T}_{\mathfrak{g},\Gamma} \stackrel{\text{def}}{=} (\Gamma, \mathcal{T}_{\mathfrak{g},\Gamma})$ is called a $\mathcal{T}_{\mathfrak{g}}$-subspace. The operator $\mathrm{cl}_{\mathfrak{g},\Lambda} : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$ carrying each $\mathfrak{T}_{\mathfrak{g},\Lambda}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$ into its closure $\mathrm{cl}_{\mathfrak{g},\Lambda}(\mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g},\Lambda} - \mathrm{int}_{\mathfrak{g},\Lambda}(\mathfrak{T}_{\mathfrak{g},\Lambda} \setminus \mathcal{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$ is called a $\mathfrak{g}$-closure operator and the operator $\mathrm{int}_{\mathfrak{g},\Lambda} : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$ carrying each $\mathfrak{T}_{\mathfrak{g},\Lambda}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$ into its interior $\mathrm{int}_{\mathfrak{g},\Lambda}(\mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g},\Lambda} - \mathrm{cl}_{\mathfrak{g},\Lambda}(\mathfrak{T}_{\mathfrak{g},\Lambda} \setminus \mathcal{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$ is called a $\mathfrak{g}$-interior operator; for clarity, we will use $\mathrm{cl}_{\mathfrak{g}}(\cdot)$, $\mathrm{int}_{\mathfrak{g}}(\cdot)$, respectively, instead of $\mathrm{cl}_{\mathfrak{g},\Lambda}(\cdot)$, $\mathrm{int}_{\mathfrak{g},\Lambda}(\cdot)$.

Let $\mathfrak{T}_{\mathfrak{g},\Lambda}$ be a $\mathcal{T}_{\mathfrak{g},\Lambda}$-space, let $\complement_{\Lambda} : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$ denotes the absolute complement with respect to the underlying set $\Lambda \subset \mathfrak{U}$, and let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$ be any $\mathfrak{T}_{\mathfrak{g},\Lambda}$-set. The classes

$$\begin{aligned} \mathcal{T}_{\mathfrak{g},\Lambda} &\stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : \ \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g},\Lambda}\}, \\ \neg\mathcal{T}_{\mathfrak{g},\Lambda} &\stackrel{\text{def}}{=} \{\mathcal{K}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : \ \complement_{\Lambda}(\mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g},\Lambda}\}, \end{aligned} \tag{2.1}$$

respectively, denote the classes of all $\mathcal{T}_{\mathfrak{g},\Lambda}$-open and $\mathcal{T}_{\mathfrak{g},\Lambda}$-closed sets relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g},\Lambda}$, and the classes

$$\begin{aligned} \mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g},\Lambda}}[\mathcal{S}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g},\Lambda} : \ \mathcal{O}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}\}, \\ \mathrm{C}^{\mathrm{sup}}_{\neg\mathcal{T}_{\mathfrak{g},\Lambda}}[\mathcal{S}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathcal{K}_{\mathfrak{g}} \in \neg\mathcal{T}_{\mathfrak{g},\Lambda} : \ \mathcal{K}_{\mathfrak{g}} \supseteq \mathcal{S}_{\mathfrak{g}}\}, \end{aligned} \tag{2.2}$$

respectively, denote the classes of $\mathcal{T}_{\mathfrak{g},\Lambda}$-open subsets and $\mathcal{T}_{\mathfrak{g},\Lambda}$-closed supersets (complements of the $\mathcal{T}_{\mathfrak{g},\Lambda}$-open subsets) of the $\mathfrak{T}_{\mathfrak{g},\Lambda}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$ relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g},\Lambda}$. To this end, the $\mathfrak{g}$-closure and the $\mathfrak{g}$-interior of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g},\Lambda}$-space [6] define themselves as

$$\mathrm{int}_{\mathfrak{g},\Lambda}(\mathcal{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcup_{\mathcal{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g},\Lambda}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{O}_{\mathfrak{g}}, \quad \mathrm{cl}_{\mathfrak{g},\Lambda}(\mathcal{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcap_{\mathcal{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathcal{T}_{\mathfrak{g},\Lambda}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{K}_{\mathfrak{g}}. \tag{2.3}$$

In this work, by $\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\cdot)$, $\mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot)$, and $\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot)$, respectively, are meant $\mathrm{cl}_{\mathfrak{g}}(\mathrm{int}_{\mathfrak{g}}(\cdot))$, $\mathrm{int}_{\mathfrak{g}}(\mathrm{cl}_{\mathfrak{g}}(\cdot))$, and $\mathrm{cl}_{\mathfrak{g}}(\mathrm{int}_{\mathfrak{g}}(\mathrm{cl}_{\mathfrak{g}}(\cdot)))$; other composition operators are defined in a similar way. Furthermore, the backslash $\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}$ refers to the set-theoretic difference $\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}}$. The mapping $\mathrm{op}_{\mathfrak{g}} : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$ is called a $\mathfrak{g}$-operation on $\mathcal{P}(\Lambda)$ if the following statements hold:

$$\begin{aligned} &\forall \mathcal{S}_{\mathfrak{g}} \in \mathcal{P}(\Lambda) \setminus \{\emptyset\}, \ \exists (\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g},\Lambda} \setminus \{\emptyset\} \times \neg\mathcal{T}_{\mathfrak{g},\Lambda} \setminus \{\emptyset\} : \\ &\left(\mathrm{op}_{\mathfrak{g}}(\emptyset) = \emptyset\right) \vee \left(\neg\,\mathrm{op}_{\mathfrak{g}}(\emptyset) = \emptyset\right), \ \left(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\right) \vee \left(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\right), \end{aligned} \tag{2.4}$$

where $\neg\,\mathrm{op}_{\mathfrak{g}} : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$ is called the "complementary $\mathfrak{g}$-operation" on $\mathcal{P}(\Lambda)$ and, for all $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathcal{S}_{\mathfrak{g}}$, $\mathcal{S}_{\mathfrak{g},\nu}$, $\mathcal{S}_{\mathfrak{g},\mu} \in \mathcal{P}(\Lambda) \setminus \{\emptyset\}$, the following axioms are satisfied:

- Ax. I. $\left(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\right) \vee \left(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\right)$,
- Ax. II. $\left(\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\right) \vee \left(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}} \circ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\right)$,
- Ax. III. $\left(\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathcal{S}_{\mathfrak{g},\mu} \ \rightarrow \ \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\mu})\right) \vee \left(\mathcal{S}_{\mathfrak{g},\mu} \subseteq \mathcal{S}_{\mathfrak{g},\nu} \ \leftarrow \ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\mu}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu})\right)$,
- Ax. IV. $\left(\mathrm{op}_{\mathfrak{g}}\left(\bigcup_{\sigma=\nu,\mu} \mathcal{S}_{\mathfrak{g},\sigma}\right) \subseteq \bigcup_{\sigma=\nu,\mu} \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma})\right) \vee \left(\neg\,\mathrm{op}_{\mathfrak{g}}\left(\bigcup_{\sigma=\nu,\mu} \mathcal{S}_{\mathfrak{g},\sigma}\right) \supseteq \bigcup_{\sigma=\nu,\mu} \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\sigma})\right)$,

for some $\mathcal{T}_{\mathfrak{g},\Lambda}$-open sets $\mathcal{O}_{\mathfrak{g}}$, $\mathcal{O}_{\mathfrak{g},\nu}$, $\mathcal{O}_{\mathfrak{g},\mu} \in \mathcal{T}_{\mathfrak{g},\Lambda} \setminus \{\emptyset\}$ and $\mathcal{T}_{\mathfrak{g},\Lambda}$-closed sets $\mathcal{K}_{\mathfrak{g}}$, $\mathcal{K}_{\mathfrak{g},\nu}$, $\mathcal{K}_{\mathfrak{g},\mu} \in \neg\mathcal{T}_{\mathfrak{g},\Lambda}$ [8, 24]. The class $\mathcal{L}_{\mathfrak{g}}[\Omega] = \mathcal{L}^{\omega}_{\mathfrak{g}}[\Lambda] \times \mathcal{L}^{\kappa}_{\mathfrak{g}}[\Omega]$, where

$$\mathcal{L}_{\mathfrak{g}}[\Lambda] \stackrel{\text{def}}{=} \left\{\mathbf{op}_{\mathfrak{g},\nu\mu}(\cdot) = \left(\mathrm{op}_{\mathfrak{g},\nu}(\cdot), \neg\,\mathrm{op}_{\mathfrak{g},\mu}(\cdot)\right) : \ (\nu,\mu) \in I^0_3 \times I^0_3\right\} \tag{2.5}$$

in the $\mathcal{T}_{\mathfrak{g},\Lambda}$-space $\mathfrak{T}_{\mathfrak{g},\Lambda}$, stands for the class of all possible $\mathfrak{g}$-operators and their complementary $\mathfrak{g}$-operators in the $\mathcal{T}_{\mathfrak{g},\Lambda}$-space $\mathfrak{T}_{\mathfrak{g},\Lambda}$. Its elements are defined as:

$$\begin{aligned} \mathrm{op}_{\mathfrak{g}}(\cdot) &\in \mathcal{L}^{\omega}_{\mathfrak{g}}[\Lambda] \stackrel{\text{def}}{=} \left\{\mathrm{op}_{\mathfrak{g},0}(\cdot),\ \mathrm{op}_{\mathfrak{g},1}(\cdot),\ \mathrm{op}_{\mathfrak{g},2}(\cdot),\ \mathrm{op}_{\mathfrak{g},3}(\cdot)\right\} \\ &= \left\{\mathrm{int}_{\mathfrak{g}}(\cdot),\ \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}(\cdot),\ \mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\cdot),\ \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\cdot)\right\}; \\ \neg\,\mathrm{op}_{\mathfrak{g}}(\cdot) &\in \mathcal{L}^{\kappa}_{\mathfrak{g}}[\Lambda] \stackrel{\text{def}}{=} \left\{\neg\,\mathrm{op}_{\mathfrak{g},0}(\cdot),\ \neg\,\mathrm{op}_{\mathfrak{g},1}(\cdot),\ \neg\,\mathrm{op}_{\mathfrak{g},2}(\cdot),\ \neg\,\mathrm{op}_{\mathfrak{g},3}(\cdot)\right\} \\ &= \left\{\mathrm{cl}_{\mathfrak{g}}(\cdot),\ \mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\cdot),\ \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}(\cdot),\ \mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}(\cdot)\right\}. \end{aligned} \tag{2.6}$$

A $\mathfrak{T}_{\mathfrak{g},\Lambda}$-set $\mathcal{S}_{\mathfrak{g},\Lambda} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g},\Lambda}$-space is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-set if and only if there exist a pair $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g},\Lambda} \times \neg\mathcal{T}_{\mathfrak{g},\Lambda}$ of $\mathcal{T}_{\mathfrak{g},\Lambda}$-open and $\mathcal{T}_{\mathfrak{g},\Lambda}$-closed sets, and a $\mathfrak{g}$-operator $\mathbf{op}_{\mathfrak{g}}(\cdot) \in \mathcal{L}_{\mathfrak{g}}[\Lambda]$ such that the following statement holds:

$$(\exists\xi)\left[(\xi \in \mathcal{S}_{\mathfrak{g}}) \wedge \left((\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})) \vee (\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}}))\right)\right]. \tag{2.7}$$

The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$ is said to be of category $\nu$ if and only if it belongs to the following class of $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-sets:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g},\Lambda}] \stackrel{\text{def}}{=} \{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} :\ & (\exists \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)) \\ & \left[(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}})) \vee (\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g}}))\right]\}. \end{aligned} \tag{2.8}$$

It is called a $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-open set if it satisfies the first property in $\mathfrak{g}$-$\nu$-$\mathrm{S}[\mathfrak{T}_{\mathfrak{g},\Lambda}]$ and a $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-closed set if it satisfies the second property in $\mathfrak{g}$-$\nu$-$\mathrm{S}[\mathfrak{T}_{\mathfrak{g},\Lambda}]$. The classes of $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-open and $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g},\Lambda}$-closed sets, respectively, are defined by

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g},\Lambda}] &\stackrel{\text{def}}{=} \left\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} :\ (\exists \mathcal{O}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot))\left[\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}})\right]\right\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g},\Lambda}] &\stackrel{\text{def}}{=} \left\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} :\ (\exists \mathcal{K}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot))\left[\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g}})\right]\right\}. \end{aligned} \tag{2.9}$$

From these classes, the following relation holds:

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g},\Lambda}] &= \bigcup\nolimits_{\nu \in I^0_3} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g},\Lambda}] \\ &= \bigcup\nolimits_{\nu \in I^0_3} \left(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]\right) \\ &= \left(\bigcup\nolimits_{\nu \in I^0_3} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g},\Lambda}]\right) \cup \left(\bigcup\nolimits_{\nu \in I^0_3} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g},\Lambda}]\right) \\ &= \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g},\Lambda}] \cup \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g},\Lambda}]. \end{aligned} \tag{2.10}$$

By omitting the subscript $\mathfrak{g}$ in almost all symbols of the above definitions, we obtain very similar definitions but in a $\mathcal{T}_{\Lambda}$-space.

A $\mathfrak{T}_{\Lambda}$-set $\mathcal{S} \subset \mathfrak{T}_{\Lambda}$ in a $\mathcal{T}_{\Lambda}$-space is called a $\mathfrak{g}$-$\mathfrak{T}_{\Lambda}$-set if and only if there exists a pair $(\mathcal{O}, \mathcal{K}) \in \mathcal{T}_{\Lambda} \times \neg\mathcal{T}_{\Lambda}$ of $\mathcal{T}_{\Lambda}$-open and $\mathcal{T}_{\Lambda}$-closed sets, and an operator $\mathbf{op}(\cdot) \in \mathcal{L}[\Lambda]$ such that the following statement holds:

$$(\exists\xi)\left[(\xi \in \mathcal{S}) \wedge \left((\mathcal{S} \subseteq \mathrm{op}(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg\,\mathrm{op}(\mathcal{K}))\right)\right]. \tag{2.11}$$

The $\mathfrak{g}$-$\mathfrak{T}_\Lambda$-set $\mathcal{S} \subset \mathfrak{T}_\Lambda$ is said to be of category $\nu$ if and only if it belongs to the following class of $\mathfrak{g}$-$\nu$-$\mathfrak{T}_\Lambda$-sets:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\big[\mathfrak{T}_\Lambda\big] \overset{\text{def}}{=} \big\{\mathcal{S} \subset \mathfrak{T}_\Lambda :\ (\exists \mathcal{O}, \mathcal{K}, \mathbf{op}_\nu(\cdot)) \\ \big[(\mathcal{S} \subseteq \mathrm{op}_\nu(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg\, \mathrm{op}_\nu(\mathcal{K}))\big]\big\}. \end{aligned} \tag{2.12}$$

It is called a $\mathfrak{g}$-$\nu$-$\mathfrak{T}_\Lambda$-open set if it satisfies the first property in $\mathfrak{g}$-$\nu$-$\mathrm{S}\big[\mathfrak{T}_\Lambda\big]$ and a $\mathfrak{g}$-$\nu$-$\mathfrak{T}_\Lambda$-closed set if it satisfies the second property in $\mathfrak{g}$-$\nu$-$\mathrm{S}\big[\mathfrak{T}_\Lambda\big]$. The classes of $\mathfrak{g}$-$\nu$-$\mathfrak{T}_\Lambda$-open and $\mathfrak{g}$-$\nu$-$\mathfrak{T}_\Lambda$-closed sets, respectively, are defined by

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_\Lambda\big] &\overset{\text{def}}{=} \big\{\mathcal{S} \subset \mathfrak{T}_\Lambda :\ (\exists \mathcal{O}, \mathbf{op}_\nu(\cdot))\ \big[\mathcal{S} \subseteq \mathrm{op}_\nu(\mathcal{O})\big]\big\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_\Lambda\big] &\overset{\text{def}}{=} \big\{\mathcal{S} \subset \mathfrak{T}_\Lambda :\ (\exists \mathcal{K}, \mathbf{op}_\nu(\cdot))\ \big[\mathcal{S} \supseteq \neg\, \mathrm{op}_\nu(\mathcal{K})\big]\big\}. \end{aligned} \tag{2.13}$$

As in the previous definitions, from these classes, the following relation holds:

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{S}\big[\mathfrak{T}_\Lambda\big] &= \textstyle\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}\big[\mathfrak{T}_\Lambda\big] \\ &= \textstyle\bigcup_{\nu \in I_3^0} \big(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_\Lambda\big] \cup \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_\Lambda\big]\big) \\ &= \big(\textstyle\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_\Lambda\big]\big) \cup \big(\textstyle\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_\Lambda\big]\big) \\ &= \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_\Lambda\big] \cup \mathfrak{g}\text{-}\mathrm{K}\big[\mathfrak{T}_\Lambda\big]. \end{aligned} \tag{2.14}$$

The classes $\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right]$ and $\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right]$ denote the families of $\mathfrak{T}_{\mathfrak{g},\Lambda}$-open and $\mathfrak{T}_{\mathfrak{g},\Lambda}$-closed sets, respectively, in $\mathfrak{T}_{\mathfrak{g},\Lambda}$, with $\mathrm{S}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right] = \mathrm{O}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right] \cup \mathrm{K}\left[\mathfrak{T}_{\mathfrak{g},\Lambda}\right]$; the classes $\mathrm{O}\left[\mathfrak{T}_\Lambda\right]$ and $\mathrm{K}\left[\mathfrak{T}_\Lambda\right]$ denote the families of $\mathfrak{T}$-open and $\mathfrak{T}_\Lambda$-closed sets, respectively, in $\mathfrak{T}_\Lambda$, with $\mathrm{S}\left[\mathfrak{T}_\Lambda\right] = \mathrm{O}\left[\mathfrak{T}_\Lambda\right] \cup \mathrm{K}\left[\mathfrak{T}_\Lambda\right]$. (Whenever we feel that the subscript $\Lambda \in \{\Omega, \Sigma\}$ is understood from the context, it will be omitted for clarity.) We are now in a position to present a carefully chosen set of terms used in the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-maps between $\mathcal{T}_{\mathfrak{g}}$-spaces.

A $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-map and a $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map, respectively, are mappings in the usual sense between $\mathcal{T}$-spaces and $\mathcal{T}_{\mathfrak{g}}$-spaces.

Definition 2.1 ($(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$, $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-Maps). Let $\mathfrak{T}_\Omega = (\Omega, \mathcal{T}_\Omega)$ and $\mathfrak{T}_\Sigma = (\Sigma, \mathcal{T}_\Sigma)$ be $\mathcal{T}$-spaces and, let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$ be $\mathcal{T}_{\mathfrak{g}}$-spaces. Then, a map:

- I. $\pi : \mathfrak{T}_\Omega \to \mathfrak{T}_\Sigma$ is called a $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-map from $\mathfrak{T}_\Omega$ into $\mathfrak{T}_\Sigma$.
- II. $\pi : \mathfrak{T}_{\mathfrak{g},\Omega} \to \mathfrak{T}_{\mathfrak{g},\Sigma}$ is called a $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map from $\mathfrak{T}_{\mathfrak{g},\Omega}$ into $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

A $\mathfrak{g}$-$(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-map is a generalization of a $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-map and, hence, is a distinguished mapping between $\mathcal{T}$-spaces which does not exhibit mapping properties in the usual sense but does exhibit mapping properties in the generalized sense.

Definition 2.2 ($\mathfrak{g}$-$\nu$-$(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-Map). Let $\mathfrak{T}_\Omega = (\Omega, \mathcal{T}_\Omega)$ and $\mathfrak{T}_\Sigma = (\Sigma, \mathcal{T}_\Sigma)$ be $\mathcal{T}$-spaces, and let $\mathbf{op}(\cdot) \in \mathcal{L}\big[\Sigma\big]$. Then, a map $\pi_{\mathfrak{g}} : \mathfrak{T}_\Omega \to \mathfrak{T}_\Sigma$ is called a $\mathfrak{g}$-$(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-map if and only if, for every pair $(\mathcal{O}_\omega, \mathcal{K}_\omega) \in \mathcal{T}_\Omega \times \neg\mathcal{T}_\Omega$ of $\mathcal{T}_\Omega$-open and $\mathcal{T}_\Omega$-closed sets in $\mathfrak{T}_\Omega$ there corresponds a pair $(\mathcal{O}_\sigma, \mathcal{K}_\sigma) \in \mathcal{T}_\Sigma \times \neg\mathcal{T}_\Sigma$ of $\mathcal{T}_\Sigma$-open and $\mathcal{T}_\Sigma$-closed sets in $\mathfrak{T}_\Sigma$ such that the following statement holds:

$$\big[\pi_{\mathfrak{g}}(\mathcal{O}_\omega) \subseteq \mathrm{op}(\mathcal{O}_\sigma)\big] \vee \big[\pi_{\mathfrak{g}}(\mathcal{K}_\omega) \supseteq \neg\, \mathrm{op}(\mathcal{K}_\sigma)\big]. \tag{2.15}$$

A $\mathfrak{g}$- $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-map is said to be of category $\nu$ if and only if it belongs to the following class of $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-maps:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-M}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right] &\stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} : \ (\forall \mathcal{O}_\omega, \mathcal{K}_\omega)\left(\exists \mathcal{O}_\sigma, \mathcal{K}_\sigma, \mathbf{op}_\nu\left(\cdot\right)\right) \\ &\quad \left[\left(\pi_{\mathfrak{g}}\left(\mathcal{O}_\omega\right) \subseteq \mathrm{op}_\nu\left(\mathcal{O}_\sigma\right)\right) \vee \left(\pi_{\mathfrak{g}}\left(\mathcal{K}_\omega\right) \supseteq \neg\, \mathrm{op}_\nu\left(\mathcal{K}_\sigma\right)\right)\right]\big\}. \end{aligned} \tag{2.16}$$

It is called a $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-open map if it satisfies the first property in $\mathfrak{g}$-$\nu$-M$\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right]$ and a $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-closed map if it satisfies the second property in $\mathfrak{g}$-$\nu$-M$\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right]$. The classes of $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-open and $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-closed maps, respectively, are defined by

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-M}_{\text{O}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right] &\stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} : (\forall \mathcal{O}_\omega)\left(\exists \mathcal{O}_\sigma, \mathbf{op}_\nu\left(\cdot\right)\right)\left[\pi_{\mathfrak{g}}\left(\mathcal{O}_\omega\right) \subseteq \mathrm{op}_\nu\left(\mathcal{O}_\sigma\right)\right]\big\}, \\ \mathfrak{g}\text{-}\nu\text{-M}_{\text{K}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right] &\stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} : (\forall \mathcal{K}_\omega)\left(\exists \mathcal{K}_\sigma, \mathbf{op}_\nu\left(\cdot\right)\right)\left[\pi_{\mathfrak{g}}\left(\mathcal{K}_\omega\right) \supseteq \mathrm{op}_\nu\left(\mathcal{K}_\sigma\right)\right]\big\}. \end{aligned} \tag{2.17}$$

From the class $\mathfrak{g}$-$\nu$-M$\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right]$, consisting of the classes $\mathfrak{g}$-$\nu$-M$_{\text{O}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right]$ and $\mathfrak{g}$-$\nu$-M$_{\text{K}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right]$, respectively, of $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-open and $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-closed maps, where $\nu \in I_3^0$, there results in the following definition.

Definition 2.3. Let $\mathfrak{T}_\Omega = (\Omega, \mathcal{T}_\Omega)$ and $\mathfrak{T}_\Sigma = (\Sigma, \mathcal{T}_\Sigma)$ be $\mathcal{T}$-spaces. If, for each $\nu \in I_3^0$, $\mathfrak{g}$-$\nu$-M$_{\text{O}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right]$ and $\mathfrak{g}$-$\nu$-M$_{\text{K}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right]$, respectively, denote the classes of $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-open and $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_\Omega, \mathfrak{T}_\Sigma)$-closed maps, then

$$\begin{aligned} \mathfrak{g}\text{-M}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right] &= \textstyle\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-M}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right] \\ &= \textstyle\bigcup_{\nu \in I_3^0} \left(\mathfrak{g}\text{-}\nu\text{-M}_{\text{O}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right] \cup \mathfrak{g}\text{-}\nu\text{-M}_{\text{K}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right]\right) \\ &= \left(\textstyle\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-M}_{\text{O}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right]\right) \cup \left(\textstyle\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-M}_{\text{K}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right]\right) \\ &= \mathfrak{g}\text{-M}_{\text{O}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right] \cup \mathfrak{g}\text{-M}_{\text{K}}\left[\mathfrak{T}_\Omega; \mathfrak{T}_\Sigma\right]. \end{aligned} \tag{2.18}$$

As above, the $\mathfrak{g}$- $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map is a generalization of the $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map and, thus, is a distinguished mapping between $\mathcal{T}_{\mathfrak{g}}$-spaces which does not exhibit mapping properties in the usual sense but does exhibit mapping properties in the generalized sense.

Definition 2.4 ($\mathfrak{g}$-$\nu$- $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-Map). Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$ be $\mathcal{T}_{\mathfrak{g}}$-spaces, and let $\mathbf{op}_{\mathfrak{g}}\left(\cdot\right) \in \mathcal{L}_{\mathfrak{g}}\left[\Sigma\right]$. Then, a map $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \to \mathfrak{T}_{\mathfrak{g},\Sigma}$ is called a $\mathfrak{g}$- $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map if and only if, for every pair $(\mathcal{O}_{\mathfrak{g},\omega}, \mathcal{K}_{\mathfrak{g},\omega}) \in \mathcal{T}_{\mathfrak{g},\Omega} \times \neg\mathcal{T}_{\mathfrak{g},\Omega}$ of $\mathcal{T}_{\mathfrak{g},\Omega}$-open and $\mathcal{T}_{\mathfrak{g},\Omega}$-closed sets in $\mathfrak{T}_{\mathfrak{g},\Omega}$ there corresponds a pair $(\mathcal{O}_{\mathfrak{g},\sigma}, \mathcal{K}_{\mathfrak{g},\sigma}) \in \mathcal{T}_{\mathfrak{g},\Sigma} \times \neg\mathcal{T}_{\mathfrak{g},\Sigma}$ of $\mathcal{T}_{\mathfrak{g},\Sigma}$-open and $\mathcal{T}_\Sigma$-closed sets in $\mathfrak{T}_{\mathfrak{g},\Sigma}$ such that the following statement holds:

$$\left[\pi_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\omega}\right) \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\sigma}\right)\right] \vee \left[\pi_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\omega}\right) \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\sigma}\right)\right]. \tag{2.19}$$

A $\mathfrak{g}$- $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-map is said to be of category $\nu$ if and only if it belongs to the following class of $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-maps:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-M}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right] &\stackrel{\text{def}}{=} \big\{\pi_{\mathfrak{g}} : \ (\forall \mathcal{O}_{\mathfrak{g},\omega}, \mathcal{K}_{\mathfrak{g},\omega})\left(\exists \mathcal{O}_{\mathfrak{g},\sigma}, \mathcal{K}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}\left(\cdot\right)\right) \\ &\quad \left[\left(\pi_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\omega}\right) \subseteq \mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g},\sigma}\right)\right) \vee \left(\pi_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\omega}\right) \supseteq \neg\, \mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{K}_{\mathfrak{g},\sigma}\right)\right)\right]\big\}. \end{aligned} \tag{2.20}$$

In the above description, it is called a $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-open map if it satisfies the first property in $\mathfrak{g}$-$\nu$-M$\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ and a $\mathfrak{g}$-$\nu$- $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-closed map if it satisfies the second property in $\mathfrak{g}$-$\nu$-M$\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. The classes of $\mathfrak{g}$-$\nu$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-open maps and $\mathfrak{g}$-$\nu$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-closed maps, respectively, are defined by

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{M}_{\mathrm{O}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] &\stackrel{\mathrm{def}}{=} \big\{\pi_{\mathfrak{g}}:\left(\forall\mathcal{O}_{\mathfrak{g},\omega}\right)\left(\exists\mathcal{O}_{\mathfrak{g},\sigma},\mathbf{op}_{\mathfrak{g},\nu}\left(\cdot\right)\right)\\ &\qquad\left[\pi_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\omega}\right)\subseteq\mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g},\sigma}\right)\right]\big\},\\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{M}_{\mathrm{K}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] &\stackrel{\mathrm{def}}{=} \big\{\pi_{\mathfrak{g}}:\left(\forall\mathcal{K}_{\omega}\right)\left(\exists\mathcal{K}_{\mathfrak{g},\sigma},\mathbf{op}_{\mathfrak{g},\nu}\left(\cdot\right)\right)\\ &\qquad\left[\pi_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)\supseteq\mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{K}_{\mathfrak{g},\sigma}\right)\right]\big\}.\end{aligned}\tag{2.21}$$

From the class $\mathfrak{g}$-$\nu$-M$\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, consisting of the classes $\mathfrak{g}$-$\nu$-$\mathrm{M}_{\mathrm{O}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ and $\mathfrak{g}$-$\nu$-$\mathrm{M}_{\mathrm{K}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ of $\mathfrak{g}$-$\nu$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-open and $\mathfrak{g}$-$\nu$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-closed maps, where $\nu\in I_3^0$, respectively, there results in the following definition.

Definition 2.5. Let $\mathfrak{T}_{\mathfrak{g},\Omega}=\left(\Omega,\mathcal{T}_{\mathfrak{g},\Omega}\right)$ and $\mathfrak{T}_{\mathfrak{g},\Sigma}=\left(\Sigma,\mathcal{T}_{\mathfrak{g},\Sigma}\right)$ be $\mathcal{T}_{\mathfrak{g}}$-spaces. If, for each $\nu\in I_3^0$, $\mathfrak{g}$-$\nu$-$\mathrm{M}_{\mathrm{O}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ and $\mathfrak{g}$-$\nu$-$\mathrm{M}_{\mathrm{K}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, respectively, denote the classes of $\mathfrak{g}$-$\nu$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-open and $\mathfrak{g}$-$\nu$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-closed maps, then

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{M}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] &= \textstyle\bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{M}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\\ &= \textstyle\bigcup_{\nu\in I_3^0}\left(\mathfrak{g}\text{-}\nu\text{-}\mathrm{M}_{\mathrm{O}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\cup\mathfrak{g}\text{-}\nu\text{-}\mathrm{M}_{\mathrm{K}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\right)\\ &= \left(\textstyle\bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{M}_{\mathrm{O}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\right)\cup\left(\textstyle\bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{M}_{\mathrm{K}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\right)\\ &= \mathfrak{g}\text{-}\mathrm{M}_{\mathrm{O}}\left[\mathfrak{T}_{\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]\cup\mathfrak{g}\text{-}\mathrm{M}_{\mathrm{K}}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right].\end{aligned}\tag{2.22}$$

Definition 2.6 ($\mathfrak{g}$-$\nu$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-Continuous). Let $\mathfrak{T}_{\mathfrak{g},\Omega}=\left(\Omega,\mathcal{T}_{\mathfrak{g},\Omega}\right)$ and $\mathfrak{T}_{\mathfrak{g},\Sigma}=\left(\Sigma,\mathcal{T}_{\mathfrak{g},\Sigma}\right)$ be $\mathcal{T}_{\mathfrak{g}}$-spaces, and let $\mathbf{op}_{\mathfrak{g}}\left(\cdot\right)\in\mathcal{L}_{\mathfrak{g}}\big[\Omega\big]$. Then, a map $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\to\mathfrak{T}_{\mathfrak{g},\Sigma}$ is said to be $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-continuous if and only if, for every pair $\left(\mathcal{O}_{\mathfrak{g},\sigma},\mathcal{K}_{\mathfrak{g},\sigma}\right)\in\mathcal{T}_{\mathfrak{g},\Sigma}\times\neg\mathcal{T}_{\mathfrak{g},\Sigma}$ of $\mathcal{T}_{\mathfrak{g},\Sigma}$-open and $\mathcal{T}_{\Sigma}$-closed sets in $\mathfrak{T}_{\mathfrak{g},\Sigma}$ there corresponds a pair $\left(\mathcal{O}_{\mathfrak{g},\omega},\mathcal{K}_{\mathfrak{g},\omega}\right)\in\mathcal{T}_{\mathfrak{g},\Omega}\times\neg\mathcal{T}_{\mathfrak{g},\Omega}$ of $\mathcal{T}_{\mathfrak{g},\Omega}$-open and $\mathcal{T}_{\mathfrak{g},\Omega}$-closed sets in $\mathfrak{T}_{\mathfrak{g},\Omega}$ such that the following statement holds:

$$\left[\pi_{\mathfrak{g}}^{-1}\left(\mathcal{O}_{\mathfrak{g},\sigma}\right)\subseteq\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\omega}\right)\right]\vee\left[\pi_{\mathfrak{g}}^{-1}\left(\mathcal{K}_{\mathfrak{g},\sigma}\right)\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)\right].\tag{2.23}$$

A $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-continuous map is said to be of category $\nu$ if and only if it belongs to the following class of $\mathfrak{g}$-$\nu$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-continuous maps:

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right] &\stackrel{\mathrm{def}}{=} \big\{\pi_{\mathfrak{g}}:\ \left(\forall\mathcal{O}_{\mathfrak{g},\sigma},\mathcal{K}_{\mathfrak{g},\sigma}\right)\left(\exists\mathcal{O}_{\mathfrak{g},\omega},\mathcal{K}_{\mathfrak{g},\omega},\mathbf{op}_{\mathfrak{g},\nu}\left(\cdot\right)\right)\\ &\left[\left(\pi_{\mathfrak{g}}^{-1}\left(\mathcal{O}_{\mathfrak{g},\sigma}\right)\subseteq\mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g},\omega}\right)\right)\vee\left(\pi_{\mathfrak{g}}^{-1}\left(\mathcal{K}_{\mathfrak{g},\sigma}\right)\supseteq\neg\,\mathrm{op}_{\mathfrak{g},\nu}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)\right)\right]\big\}.\end{aligned}\tag{2.24}$$

Definition 2.7. Let $\mathfrak{T}_{\mathfrak{g},\Omega}=\left(\Omega,\mathcal{T}_{\mathfrak{g},\Omega}\right)$ and $\mathfrak{T}_{\mathfrak{g},\Sigma}=\left(\Sigma,\mathcal{T}_{\mathfrak{g},\Sigma}\right)$ be $\mathcal{T}_{\mathfrak{g}}$-spaces. If, for each $\nu\in I_3^0$, $\mathfrak{g}$-$\nu$-C$\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ denotes the class of $\mathfrak{g}$-$\nu$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-continuous maps, then

$$\mathfrak{g}\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right]=\textstyle\bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}\right].\tag{2.25}$$

Definition 2.8 ($\mathfrak{g}$-$\nu$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-Irresolute). Let $\mathfrak{T}_{\mathfrak{g},\Omega}=\left(\Omega,\mathcal{T}_{\mathfrak{g},\Omega}\right)$ and $\mathfrak{T}_{\mathfrak{g},\Sigma}=\left(\Sigma,\mathcal{T}_{\mathfrak{g},\Sigma}\right)$ be $\mathcal{T}_{\mathfrak{g}}$-spaces, and let $\mathbf{op}_{\mathfrak{g}}\left(\cdot\right)\in\mathcal{L}_{\mathfrak{g}}\big[\Omega\big]$. Then, a map $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\to\mathfrak{T}_{\mathfrak{g},\Sigma}$ is said to be $\mathfrak{g}$-$\left(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma}\right)$-irresolute if and only if, for every pair $\left(\mathcal{O}_{\mathfrak{g},\sigma},\mathcal{K}_{\mathfrak{g},\sigma}\right)\in\mathcal{T}_{\mathfrak{g},\Sigma}\times\neg\mathcal{T}_{\mathfrak{g},\Sigma}$ of $\mathcal{T}_{\mathfrak{g},\Sigma}$-open and $\mathcal{T}_{\Sigma}$-closed sets in $\mathfrak{T}_{\mathfrak{g},\Sigma}$ there corresponds a pair $\left(\mathcal{O}_{\mathfrak{g},\omega},\mathcal{K}_{\mathfrak{g},\omega}\right)\in\mathcal{T}_{\mathfrak{g},\Omega}\times\neg\mathcal{T}_{\mathfrak{g},\Omega}$ of $\mathcal{T}_{\mathfrak{g},\Omega}$-open and $\mathcal{T}_{\mathfrak{g},\Omega}$-closed sets in $\mathfrak{T}_{\mathfrak{g},\Omega}$ such that the

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following statement holds:

$$\left[\pi_{\mathfrak{g}}^{-1}\left(\operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\sigma}\right)\right) \subseteq \operatorname{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\omega}\right)\right] \vee \left[\pi_{\mathfrak{g}}^{-1}\left(\neg \operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\sigma}\right)\right) \supseteq \neg \operatorname{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)\right]. \tag{2.26}$$

A $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute map is said to be of category $\nu$ if and only if it belongs to the following class of $\mathfrak{g}$-$\nu$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute maps:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right] \stackrel{\text{def}}{=} \Big\{\pi_{\mathfrak{g}} :\ & \left(\forall \mathcal{O}_{\mathfrak{g},\sigma}, \mathcal{K}_{\mathfrak{g},\sigma}\right)\left(\exists \mathcal{O}_{\mathfrak{g},\omega}, \mathcal{K}_{\mathfrak{g},\omega}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)\right) \\ & \left[\left(\pi_{\mathfrak{g}}^{-1}\left(\operatorname{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g},\sigma}\right)\right) \subseteq \operatorname{op}_{\mathfrak{g},\nu}\left(\mathcal{O}_{\mathfrak{g},\omega}\right)\right) \vee \left(\pi_{\mathfrak{g}}^{-1}\left(\neg \operatorname{op}_{\mathfrak{g},\nu}\left(\mathcal{K}_{\mathfrak{g},\sigma}\right)\right) \supseteq \right.\right. \\ & \left.\left. \neg \operatorname{op}_{\mathfrak{g},\nu}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)\right)\right]\Big\}. \end{aligned} \tag{2.27}$$

Definition 2.9. Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$ be $\mathcal{T}_{\mathfrak{g}}$-spaces. If, for each $\nu \in I_3^0$, $\mathfrak{g}$-$\nu$-$\text{I}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ denotes the class of $\mathfrak{g}$-$\nu$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute maps, then

$$\mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right] = \bigcup\nolimits_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right]. \tag{2.28}$$

In regards to the above descriptions, by a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open set and a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed set are meant a $\mathcal{T}_{\mathfrak{g}}$-open set $\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}$ and a $\mathcal{T}_{\mathfrak{g}}$-closed set $\mathcal{K}_{\mathfrak{g}} \in \neg\mathcal{T}_{\mathfrak{g}}$ satisfying $\mathcal{O}_{\mathfrak{g}} \subseteq \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$ and $\mathcal{K}_{\mathfrak{g}} \supseteq \neg \operatorname{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$, respectively. Likewise, by a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open set of category $\nu$ and a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed set of category $\nu$ are meant a $\mathcal{T}_{\mathfrak{g}}$-open set $\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}$ and a $\mathcal{T}_{\mathfrak{g}}$-closed set $\mathcal{K}_{\mathfrak{g}} \in \neg\mathcal{T}_{\mathfrak{g}}$ satisfying $\mathcal{O}_{\mathfrak{g}} \subseteq \operatorname{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}})$ and $\mathcal{K}_{\mathfrak{g}} \supseteq \neg \operatorname{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g}})$, respectively; $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-sets of category $\nu$ will be called $\mathfrak{g}$-$\nu$-$\mathcal{T}_{\mathfrak{g}}$-sets.

Given the $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathcal{R}_{\mathfrak{g}}$, $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, $\mathcal{R}_{\mathfrak{g}}$ is said to be *equivalent* to $\mathcal{S}_{\mathfrak{g}}$, written $\mathcal{R}_{\mathfrak{g}} \sim \mathcal{S}_{\mathfrak{g}}$ if and only if, there exists a $\mathfrak{T}_{\mathfrak{g}}$-map $\pi_{\mathfrak{g}} : \mathcal{R}_{\mathfrak{g}} \to \mathcal{S}_{\mathfrak{g}}$ which is bijective. A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is *finite* if and only if $\mathcal{S}_{\mathfrak{g}} = \emptyset$ or $\mathcal{S}_{\mathfrak{g}} \sim I_{\mu}^{*}$ for some $\mu \in I_{\infty}^{*}$; otherwise, the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}}$ is said to be *infinite*. A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is *denumerable* and satisfies the condition $\operatorname{card}(\mathcal{R}_{\mathfrak{g}}) = \aleph_0$ (*aleph-null*) if and only if $\mathcal{S}_{\mathfrak{g}} \sim I_{\infty}^{*}$. The $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{R}_{\mathfrak{g}}$ is called *countable* if and only if it is *finite* or *denumerable*.

By adding a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation axiom of type H, called $\mathfrak{g}$-$\text{T}_{\mathfrak{g},\text{H}}$-*axiom*, to the axioms for a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ to obtain a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\text{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\text{H})} \stackrel{\text{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\text{H})}\right)$ is meant that, for every disjoint pair $(\xi, \zeta) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ of points in $\mathfrak{T}_{\mathfrak{g}}$, there exists a disjoint pair $(\mathcal{O}_{\mathfrak{g},\xi}, \mathcal{O}_{\mathfrak{g},\zeta}) \in \mathcal{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}}$ of $\mathcal{T}_{\mathfrak{g}}$-open sets such that $\xi \in \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$ and $\zeta \in \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\zeta})$. The definition follows:

Definition 2.10 ($\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\text{H})}$-Space). A $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ endowed with a $\mathfrak{g}$-$\text{T}_{\mathfrak{g},\text{H}}$-axiom is called a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\text{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\text{H})} \stackrel{\text{def}}{=} \left(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\text{H})}\right)$.

Definition 2.11 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Sets Sequence). Let $\mathfrak{g}$-$\nu$-$\text{S}\left[\mathfrak{T}_{\mathfrak{g}}\right] \subseteq \mathfrak{T}_{\mathfrak{g}}$ be the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of category $\nu$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$. The symbol $\left\langle \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right\rangle_{\alpha \in I_{\sigma}^{*}}$ denotes a sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of category $\nu$ in $\mathfrak{T}_{\mathfrak{g}}$ that has been indexed by $I_{\sigma}^{*} \subseteq I_{\infty}^{*}$, inheriting its order from $I_{\sigma}^{*}$, and the corresponding index mapping $\phi : \alpha \mapsto \mathcal{S}_{\mathfrak{g},\alpha}$ denotes the $\alpha^{\text{th}}$ term of the sequence.

Throughout, the relation $\left\langle \mathcal{R}_{\mathfrak{g},\alpha}\right\rangle_{\alpha \in I_{\infty}^{*}} \prec \left\langle \mathcal{S}_{\mathfrak{g},\alpha}\right\rangle_{\alpha \in I_{\infty}^{*}}$ means that the one preceding "$\prec$" is a subsequence of the other following "$\prec$." Suppose a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is related to a sequence $\left\langle \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right\rangle_{\alpha \in I_{\sigma}^{*}}$ by the relation $\mathcal{R}_{\mathfrak{g}} \subseteq \bigcup_{\alpha \in I_{\sigma}^{*}} \mathcal{S}_{\mathfrak{g},\alpha}$, then $\mathcal{R}_{\mathfrak{g}}$ is said to be *covered* by a sequence $\left\langle \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right]\right\rangle_{\alpha \in I_{\sigma}^{*}}$ whose *cardinality is at most* $\sigma \in I_{\infty}^{*}$. The definition follows:

DEFINITION 2.12 ($\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-Covering). Let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Then, for every $\nu \in I_3^0$:

- I. $\mathcal{S}_{\mathfrak{g}}$ is said to be "covered" by a sequence $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_\sigma^*}$ of $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-open sets whose cardinality is at most $\sigma \in I_\infty^*$ if and only if $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{\alpha \in I_\sigma^*} \mathcal{U}_{\mathfrak{g},\alpha}$.
- II. $\mathcal{S}_{\mathfrak{g}}$ is said to be "covered" by a sequence $\langle \mathcal{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_\sigma^*}$ of $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets whose cardinality is at most $\sigma \in I_\infty^*$ if and only if $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{\alpha \in I_\sigma^*} \mathcal{V}_{\mathfrak{g},\alpha}$.

Since $\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, and $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, the sequences $\langle \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_\sigma^*}$, $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_\sigma^*}$, and $\langle \mathcal{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_\sigma^*}$, respectively, are simply said to be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*covering*, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*open covering*, and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*closed covering* of $\mathcal{S}_{\mathfrak{g}}$ whose cardinality is at most $\sigma \in I_\infty^*$.

DEFINITION 2.13 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Subcovering). Let $\langle \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_\sigma^*}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and let $\vartheta : I_\sigma^* \to I_{\vartheta(\sigma)}^* \subseteq I_\sigma^*$ be an index mapping. Then the map

$$\vartheta : \langle \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_\sigma^*} \longrightarrow \langle \mathcal{S}_{\mathfrak{g},\vartheta(\alpha)} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{(\alpha,\vartheta(\alpha)) \in I_\sigma^* \times I_{\vartheta(\sigma)}^*} \tag{2.29}$$

is said to realise a " $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-subcovering" $\langle \mathcal{S}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_\sigma^* \times I_{\vartheta(\sigma)}^*}$ of $\mathcal{S}_{\mathfrak{g}}$ from the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering $\langle \mathcal{S}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_\sigma^*}$ if and only if $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_\sigma^* \times I_{\vartheta(\sigma)}^*} \mathcal{S}_{\mathfrak{g},\vartheta(\alpha)}$.

Thus, $\langle \mathcal{S}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_\sigma^* \times I_{\vartheta(\sigma)}^*} \prec \langle \mathcal{S}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_\sigma^*}$ is equivalent to this definition, meaning that, for every $\vartheta(\alpha) \in I_{\vartheta(\sigma)}^* \subseteq I_\sigma^*$, there exists $\alpha \in I_\sigma^* \subseteq I_\infty^*$ such that $\mathcal{S}_{\mathfrak{g},\vartheta(\alpha)} = \mathcal{S}_{\mathfrak{g},\alpha}$. It is plain that, for every $\sigma \in I_\infty^*$, $\vartheta(\sigma) = \mathrm{card}\big(I_{\vartheta(\sigma)}^*\big) \leq \mathrm{card}(I_\sigma^*) = \sigma$.

DEFINITION 2.14 ($\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-Compact Set). A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ is said to be $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-compact if and only if, for every $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_\sigma^*}$,

$$\exists \langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_\sigma^* \times I_{\vartheta(\sigma)}^*} : \quad \mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_\sigma^* \times I_{\vartheta(\sigma)}^*} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}, \tag{2.30}$$

where $\vartheta(\sigma) = \mathrm{card}\big(I_{\vartheta(\sigma)}^*\big) \leq \mathrm{card}(I_\sigma^*) = \sigma$. The class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets of category $\nu \in I_3^0$ is:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-A}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\mathrm{def}}{=} \Big\{ \mathcal{S}_{\mathfrak{g}} : \big[\forall \langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_\sigma^*}\big] \big[\exists \langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_\sigma^* \times I_{\vartheta(\sigma)}^*}\big] \\ \Big( \mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_\sigma^* \times I_{\vartheta(\sigma)}^*} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \Big) \Big\}. \end{aligned} \tag{2.31}$$

Thus, by a $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-compact set is meant a type of set $\mathfrak{T}_{\mathfrak{g}}$-set every $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of which has a finite $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering [25, 34, 35]. Further, it is clear from the context that, $\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$; its elements, then, are simply called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets. Stated differently, the above definition says that, given any sequence $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets of $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ such that every point $\xi \in \mathcal{S}_{\mathfrak{g}}$ belongs to at least one $\mathcal{U}_{\mathfrak{g},\alpha}$, $\alpha \in I_{\sigma}^{*}$, it is possible to select from $\langle \mathcal{U}_{\mathfrak{g},\alpha}\rangle_{\alpha \in I_{\sigma}^{*}}$ a finite number of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets $\mathcal{U}_{\mathfrak{g},\vartheta(1)}, \mathcal{U}_{\mathfrak{g},\vartheta(2)}, \ldots, \mathcal{U}_{\mathfrak{g},\vartheta(\sigma)}$ whose union covers all of $\mathcal{S}_{\mathfrak{g}}$.

Remark 2.15. Since $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness of a $\mathfrak{T}_{\mathfrak{g}}$-set is defined in terms of *relatively* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open sets*.

The concept of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-refinement is now given in the following definition.

Definition 2.16 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Refinement). A $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering $\langle \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$ of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-refinement" of another $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering $\langle \mathcal{R}_{\mathfrak{g},\beta} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\beta \in I_{\mu}^{*}}$ of the same $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}}$ if and only if:

$$(\forall \alpha \in I_{\sigma}^{*})(\exists \beta \in I_{\mu}^{*})[\mathcal{S}_{\mathfrak{g},\alpha} \subseteq \mathcal{R}_{\mathfrak{g},\beta}]. \tag{2.32}$$

In the event that $\mathcal{S}_{\mathfrak{g}} = \Omega$, $\langle \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-covering of $\mathfrak{T}_{\mathfrak{g}}$ if $\Omega = \bigcup_{\alpha \in I_{\sigma}^{*}} \mathcal{S}_{\mathfrak{g},\alpha}$. Accordingly, $\langle \mathcal{S}_{\mathfrak{g},\vartheta(\alpha)} \in \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}}$ is a $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-subcovering of $\mathfrak{T}_{\mathfrak{g}}$ if the relation $\Omega = \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \mathcal{S}_{\mathfrak{g},\vartheta(\alpha)}$ holds, where $\vartheta(\sigma) = \operatorname{card}(I_{\vartheta(\sigma)}^{*}) < \operatorname{card}(I_{\sigma}^{*}) < \infty$. The definition follows.

Definition 2.17 ($\mathfrak{g}$-$\nu$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-Space). A $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is called a $\mathfrak{g}$-$\nu$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ if and only if each $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathfrak{T}_{\mathfrak{g}}$ has a finite $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering.

In the sequel, by $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{CA}]} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{CA}]})$, $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{SA}]} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{SA}]})$, and $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{LA}]} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{LA}]})$, respectively, are meant *countably*, *sequentially*, and *locally* $\mathfrak{g}$-$\nu$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-spaces, as are easily understood. Finally, by a $\mathfrak{g}$-$\nu$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{E}]}$-space $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{E}]})$ is meant $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} = \bigvee_{\nu \in I_{3}^{0}} \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} = (\Omega, \bigvee_{\nu \in I_{3}^{0}} \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{E}]}) = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{E}]})$, where $\mathrm{E} \in \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$.

The main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness are presented in the following sections.

## 3. Main Results

In a $\mathcal{T}_{\mathfrak{g}}$-space, any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-subcovering is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-refinement as proved in the following theorem.

Theorem 3.1 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Refinement). *In a* $\mathcal{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, *any* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-subcovering* $\langle \mathcal{S}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}}$ *derived from a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-covering* $\langle \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$ *is a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-refinement.*

Proof. Let $\langle \mathcal{S}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}}$ be any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-subcovering derived from a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering $\langle \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha \in I_{\sigma}^{*}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$. Then, it results that $\langle \mathcal{S}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathcal{S}_{\mathfrak{g},\alpha}\rangle_{\alpha \in I_{\sigma}^{*}}$. Thus,

$$(\forall \vartheta(\alpha) \in I_{\vartheta(\sigma)}^{*})(\exists \alpha \in I_{\sigma}^{*})[\mathcal{S}_{\mathfrak{g},\vartheta(\alpha)} \subseteq \mathcal{S}_{\mathfrak{g},\alpha}].$$

Therefore, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-subcovering $\langle \mathcal{S}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ derived from the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-covering $\langle \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$ is therefore a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-refinement. This completes the proof of the theorem. Q.E.D.

A necessary and sufficient condition for a $T_{\mathfrak{g}}$-set of a $\mathcal{T}_{\mathfrak{g}}$-space to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact may well be stated as thus.

THEOREM 3.2. *Let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$. Then, $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ if and only if, for each $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathcal{S}_{\mathfrak{g}}$, there is a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ of $\mathcal{S}_{\mathfrak{g}}$:*

$$\begin{aligned}\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}] \;\Leftrightarrow\; & \big(\forall \langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}\big) \big(\exists \langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \\ & \prec \langle \mathcal{U}_{\mathfrak{g},\alpha} \rangle_{\alpha\in I_{\sigma}^{*}}\big) \Big[\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\Big].\end{aligned}$$

(3.1)

PROOF. *Necessity.* Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{T}_{\mathfrak{g}}$, and let $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathcal{S}_{\mathfrak{g}}$. Then, $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{\alpha\in I_{\sigma}^{*}} \mathcal{U}_{\mathfrak{g},\alpha}$ and, consequently, $\mathcal{S}_{\mathfrak{g}} = \bigcup_{\alpha\in I_{\sigma}^{*}} (\mathcal{U}_{\mathfrak{g},\alpha} \cap \mathcal{S}_{\mathfrak{g}})$. Therefore, $\langle \mathcal{U}_{\mathfrak{g},\alpha} \cap \mathcal{S}_{\mathfrak{g}} \rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathcal{S}_{\mathfrak{g}}$ by relatively $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets $\mathcal{U}_{\mathfrak{g},1} \cap \mathcal{S}_{\mathfrak{g}}$, $\mathcal{U}_{\mathfrak{g},2} \cap \mathcal{S}_{\mathfrak{g}}$, $\ldots$, $\mathcal{U}_{\mathfrak{g},\sigma} \cap \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. Since $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$, there is a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ of $\mathcal{S}_{\mathfrak{g}}$ such that $\mathcal{S}_{\mathfrak{g}} = \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} (\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathcal{S}_{\mathfrak{g}})$. Thus, it follows that $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathcal{S}_{\mathfrak{g}}$.

*Sufficiency.* Conversely, suppose that, for every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathcal{S}_{\mathfrak{g}}$, $\langle \mathcal{U}_{\mathfrak{g},\alpha} \rangle_{\alpha\in I_{\sigma}^{*}}$ has a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of the type $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ of $\mathcal{S}_{\mathfrak{g}}$. It must be shown that, given a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \hat{\mathcal{U}}_{\mathfrak{g},\beta} \rangle_{\beta\in I_{\mu}^{*}}$ of $\mathcal{S}_{\mathfrak{g}}$ by relatively $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets $\hat{\mathcal{U}}_{\mathfrak{g},1}$, $\hat{\mathcal{U}}_{\mathfrak{g},2}$, $\ldots$, $\hat{\mathcal{U}}_{\mathfrak{g},\mu} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, there is a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\beta)} \rangle_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}}$ of $\mathcal{S}_{\mathfrak{g}}$ such that $\mathcal{S}_{\mathfrak{g}} = \bigcup_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}} \hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\beta)}$. For every $\beta \in I_{\mu}^{*}$, since $\hat{\mathcal{U}}_{\mathfrak{g},\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ is a relatively $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set in $\mathcal{S}_{\mathfrak{g}}$, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathcal{U}_{\mathfrak{g},\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\hat{\mathcal{U}}_{\mathfrak{g},\beta} = \mathcal{U}_{\mathfrak{g},\beta} \cap \mathcal{S}_{\mathfrak{g}}$. But $\mathcal{S}_{\mathfrak{g}} = \bigcup_{\beta\in I_{\mu}^{*}} \hat{\mathcal{U}}_{\mathfrak{g},\beta} = \bigcup_{\beta\in I_{\mu}^{*}} (\mathcal{U}_{\mathfrak{g},\beta} \cap \mathcal{S}_{\mathfrak{g}}) \subseteq \bigcup_{\beta\in I_{\mu}^{*}} \mathcal{U}_{\mathfrak{g},\beta}$ and, consequently, $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{\beta\in I_{\mu}^{*}} \mathcal{U}_{\mathfrak{g},\beta}$, implying that $\langle \mathcal{U}_{\mathfrak{g},\beta} \rangle_{\beta\in I_{\mu}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathcal{S}_{\mathfrak{g}}$ by $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets $\mathcal{U}_{\mathfrak{g},1}$, $\mathcal{U}_{\mathfrak{g},2}$, $\ldots$, $\mathcal{U}_{\mathfrak{g},\mu} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. By hypothesis, there exists a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\beta)} \rangle_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}}$ of $\mathcal{S}_{\mathfrak{g}}$ such that $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\beta)}$. Thus,

$$\begin{aligned}\mathcal{S}_{\mathfrak{g}} = \Bigg(\bigcup_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\beta)}\Bigg) \cap \mathcal{S}_{\mathfrak{g}} &= \bigcup_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}} (\mathcal{U}_{\mathfrak{g},\vartheta(\beta)} \cap \mathcal{S}_{\mathfrak{g}}) \\ &= \bigcup_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}} \hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\beta)}.\end{aligned}$$

Hence, it results that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle\hat{\mathcal{U}}_{\mathfrak{g},\beta} \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\beta\in I_{\mu}^{*}}$ of $\mathcal{S}_{\mathfrak{g}}$ by relatively $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets $\hat{\mathcal{U}}_{\mathfrak{g},1}, \hat{\mathcal{U}}_{\mathfrak{g},2}, \ldots, \hat{\mathcal{U}}_{\mathfrak{g},\sigma} \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ has a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle\hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\beta)}\rangle_{(\beta,\vartheta(\beta))\in I_{\mu}^{*}\times I_{\vartheta(\mu)}^{*}}$ of $\mathcal{S}_{\mathfrak{g}}$. Q.E.D.

The following theorem states that, any finite union of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g}}$.

THEOREM 3.3. *If* $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \ldots, \mathcal{S}_{\mathfrak{g},\mu} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ *be* $\mu \geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact sets in a* $\mathcal{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$*, then* $\bigcup_{\alpha\in I_{\mu}^{*}} \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ *in* $\mathfrak{T}_{\mathfrak{g}}$*:*

$$\bigwedge_{\alpha\in I_{\mu}^{*}}\big(\mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big) \;\Rightarrow\; \bigcup_{\alpha\in I_{\mu}^{*}} \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g}}\big]. \tag{3.2}$$

PROOF. Let $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \ldots, \mathcal{S}_{\mathfrak{g},\mu} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ be $\mu \geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets in $\mathfrak{T}_{\mathfrak{g}}$. Then, for every $\alpha \in I_{\mu}^{*}$, there exists $\langle\mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\beta)}\rangle_{(\vartheta(\alpha),\vartheta(\alpha,\beta))\in I_{\sigma}^{*}\times I_{\beta(\sigma)}^{*}} \prec \langle\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\rangle_{\vartheta(\alpha)\in I_{\sigma}^{*}}$, where $I_{\beta(\sigma)}^{*} \subseteq I_{\sigma}^{*}$, such that $\mathcal{S}_{\mathfrak{g},\alpha} \subseteq \bigcup_{(\vartheta(\alpha),\vartheta(\alpha,\beta))\in I_{\sigma}^{*}\times I_{\beta(\sigma)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\beta)}$ holds. Consequently,

$$\begin{aligned}\textstyle\bigcup_{\alpha\in I_{\mu}^{*}}\mathcal{S}_{\mathfrak{g},\alpha} \;&\subseteq\; \bigcup_{\alpha\in I_{\mu}^{*}}\Bigg(\bigcup_{(\vartheta(\alpha),\vartheta(\alpha,\beta))\in I_{\sigma}^{*}\times I_{\beta(\sigma)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\beta)}\Bigg)\\ &\subseteq\; \bigcup_{(\alpha,\vartheta(\alpha),\vartheta(\alpha,\beta))\in I_{\mu}^{*}\times I_{\sigma}^{*}\times I_{\beta(\sigma)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\beta)}.\end{aligned}$$

Hence, it follows that, $\bigcup_{\alpha\in I_{\mu}^{*}} \mathcal{S}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ in $\mathfrak{T}_{\mathfrak{g}}$. The proof of the theorem is complete. Q.E.D.

An arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathcal{T}_{\mathfrak{g}}$-space which is equivalent to the index set $I_{\mu}^{*}$ for some $\mu < \infty$ is necessarily $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact and thus, the theorem follows.

THEOREM 3.4. *If* $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be any finite* $\mathfrak{T}_{\mathfrak{g}}$*-set of a* $\mathcal{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$*, then* $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}\left[\mathfrak{T}_{\mathfrak{g}}\right]$*:*

$$\big(\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}\big) \wedge \big(\mathrm{card}\left(\mathcal{S}_{\mathfrak{g}}\right) < \infty\big) \;\Rightarrow\; \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}\left[\mathfrak{T}_{\mathfrak{g}}\right]. \tag{3.3}$$

PROOF. Let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any finite $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$. Then, there exist $\langle\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \prec \langle\mathcal{O}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ such that the relation $\bigcup_{\xi\in\mathcal{S}_{\mathfrak{g}}}\{\xi\} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}$ holds. Since $\mathcal{O}_{\mathfrak{g},\alpha} \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\alpha}\right)$ for every $\alpha \in I_{\sigma}^{*}$ and, $\bigcup_{\xi\in\mathcal{S}_{\mathfrak{g}}}\{\xi\} = \mathcal{S}_{\mathfrak{g}}$, it results that,

$$\begin{aligned}\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathcal{O}_{\mathfrak{g},\vartheta(\alpha)} \;&\subseteq\; \mathrm{op}_{\mathfrak{g}}\Bigg(\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\Bigg)\\ &\subseteq\; \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\big)\\ &\subseteq\; \bigcup_{\alpha\in I_{\sigma}^{*}} \mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\alpha}\big),\end{aligned}$$

Therefore, $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\big)$. But, for every $(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}$, $\mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\big) \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. Consequently, for every $(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}$,

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there exists $\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} = \mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\big)$. Thus, $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}$ and hence, $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$. This completes the proof of the theorem. Q.E.D.

Since finite $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathcal{T}_{\mathfrak{g}}$-space are always $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact, an immediate consequence of the above theorem is the following corollary.

Corollary 3.5. *Let $\mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set of a discrete $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$. Then, $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ if and only if it is a finite $\mathfrak{T}_{\mathfrak{g}}$-set.*

An immediate consequence of the above theorem is the following proposition.

Proposition 3.6. *If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is a finite strong $\mathcal{T}_{\mathfrak{g}}$-space, then it is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big)$:*

$$(3.4)\qquad \big(\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})\big) \wedge \big(\mathrm{card}\,(\Omega) < \infty\big) \;\Rightarrow\; \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big).$$

Proof. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a finite strong $\mathcal{T}_{\mathfrak{g}}$-space with $\Omega = \{\xi_{\alpha} :\ \alpha \in I_{\mu}^{*}\}$ and $\mu < \infty$. Since $\mathfrak{T}_{\mathfrak{g}}$ is a finite strong $\mathcal{T}_{\mathfrak{g}}$-space, if $\langle \mathcal{O}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-open covering of $\Omega$, then, for every $\alpha \in I_{\mu}^{*}$, there exists a $\vartheta(\alpha) \in I_{\sigma}^{*}$ such that $\xi_{\alpha} \in \mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}$. Thus, $\Omega = \bigcup_{\alpha\in I_{\mu}^{*}} \{\xi_{\alpha}\} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\mu}^{*}\times I_{\sigma}^{*}} \mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}$ and consequently, $\langle \mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\mu}^{*}\times I_{\sigma}^{*}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-open subcovering of $\Omega$. But, for every $(\alpha, \vartheta(\alpha)) \in I_{\mu}^{*} \times I_{\sigma}^{*}$, $\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)} \subseteq \mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\big) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, for each $(\alpha, \vartheta(\alpha)) \in I_{\mu}^{*} \times I_{\sigma}^{*}$, there corresponds a $\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} = \mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\big)$. Thus, $\Omega \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\mu}^{*}\times I_{\sigma}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}$. Hence, $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big)$. The proof of the proposition is complete. Q.E.D.

To prove that a $\mathfrak{T}_{\mathfrak{g}}$-set is not $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact, one only has to exhibit one $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of the $\mathfrak{T}_{\mathfrak{g}}$-set with no finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering. The proposition follows.

Proposition 3.7. *If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathcal{T}_{\mathfrak{g}}$-space generated by unit $\mathfrak{T}_{\mathfrak{g}}$-sets of $\Omega$, then any infinite $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is not $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact.*

Proof. Let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any infinite $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ generated by unit $\mathfrak{T}_{\mathfrak{g}}$-sets of $\Omega$. Then, since $\{\xi\} \in \mathfrak{T}_{\mathfrak{g}}$ and $\{\xi\} \subseteq \mathrm{op}_{\mathfrak{g}}\big(\{\xi\}\big)$ hold for every $\{\xi\} \subset \mathcal{S}_{\mathfrak{g}}$, it follows that, for every $\xi \in \mathcal{S}_{\mathfrak{g}}$, $\{\xi\} \subseteq \mathrm{op}_{\mathfrak{g}}\big(\{\xi\}\big)$. Consequently, $\mathcal{S}_{\mathfrak{g}} = \bigcup_{\xi\in\mathcal{S}_{\mathfrak{g}}} \{\xi\} \subseteq \bigcup_{\xi\in\mathcal{S}_{\mathfrak{g}}} \mathrm{op}_{\mathfrak{g}}\big(\{\xi\}\big)$. Clearly, $\mathrm{op}_{\mathfrak{g}}\big(\{\xi\}\big) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\xi \in \mathcal{S}_{\mathfrak{g}}$ and therefore, there exists, for each $\xi \in \mathcal{S}_{\mathfrak{g}}$, a $\mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\mathcal{U}_{\mathfrak{g},\xi} = \mathrm{op}_{\mathfrak{g}}\big(\{\xi\}\big)$. Hence, $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{\xi\in\mathcal{S}_{\mathfrak{g}}} \mathcal{U}_{\mathfrak{g},\xi}$, implying that $\langle \mathcal{U}_{\mathfrak{g},\xi}\rangle_{\xi\in\mathcal{S}_{\mathfrak{g}}}$ is an infinite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathcal{S}_{\mathfrak{g}}$. Consequently, there exists no finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\xi)}\rangle_{(\xi,\vartheta(\xi))\in\mathcal{S}_{\mathfrak{g}}\times I_{\sigma}^{*}} \prec \langle \mathcal{U}_{\mathfrak{g},\xi}\rangle_{\xi\in\mathcal{S}_{\mathfrak{g}}}$ of $\mathcal{S}_{\mathfrak{g}}$ such that $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\xi,\vartheta(\xi))\in\mathcal{S}_{\mathfrak{g}}\times I_{\sigma}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\xi)}$. Hence, $\mathcal{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. This completes the proof of the theorem. Q.E.D.

From the above two propositions, the corollary follows.

Corollary 3.8. *If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathcal{T}_{\mathfrak{g}}$-space generated by unit $\mathfrak{T}_{\mathfrak{g}}$-sets of $\Omega$ and $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, then $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g}}\big]$ if and only if it is a finite $\mathfrak{T}_{\mathfrak{g}}$-set in $\mathfrak{T}_{\mathfrak{g}}$.*

A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is called a $\mathfrak{T}_{\mathfrak{g}}$-open set if $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{\mathcal{O}_{\mathfrak{g}}\in \mathrm{C}_{\mathcal{T}_{\mathfrak{g}}}^{\mathrm{sub}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{O}_{\mathfrak{g}}$. But, for every $\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}$, $\mathcal{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$ and, consequently, $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{\mathcal{O}_{\mathfrak{g}}\in \mathrm{C}_{\mathcal{T}_{\mathfrak{g}}}^{\mathrm{sub}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{O}_{\mathfrak{g}} \subseteq$

$\bigcup_{\mathcal{O}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]} \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$, meaning that, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness is implied by $\mathfrak{T}_{\mathfrak{g}}$-openness. Accordingly, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies $\mathfrak{T}_{\mathfrak{g}}$-compactness. The theorem follows.

THEOREM 3.9. *Let $\mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$. If $\mathcal{S}_{\mathfrak{g}}$ be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact, then it is also $\mathfrak{T}_{\mathfrak{g}}$-compact:*

$$\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{A}\,[\mathfrak{T}_{\mathfrak{g}}] \;\Rightarrow\; \mathcal{S}_{\mathfrak{g}} \in \mathrm{A}\,[\mathfrak{T}_{\mathfrak{g}}]. \tag{3.5}$$

PROOF. Let $\mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ and suppose $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{A}\,[\mathfrak{T}_{\mathfrak{g}}]$. Since $\mathcal{S}_{\mathfrak{g}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\rangle_{\alpha\in I^{*}_{\sigma}}$ of $\mathcal{S}_{\mathfrak{g}}$ which has a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I^{*}_{\sigma}\times I^{*}_{\vartheta(\sigma)}}$ such that $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I^{*}_{\sigma}\times I^{*}_{\vartheta(\sigma)}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}$. The assertion that, $\mathcal{U}_{\mathfrak{g},\vartheta(\xi)} \in \mathfrak{g}\text{-}\mathrm{A}\,[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\alpha, \vartheta(\alpha)) \in I^{*}_{\sigma} \times I^{*}_{\vartheta(\sigma)}$ implies the existence of $\mathcal{O}_{\mathfrak{g},\vartheta(\xi)} \in \mathcal{T}_{\mathfrak{g}}$ such that, $\mathcal{U}_{\mathfrak{g},\vartheta(\xi)} \subseteq \mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\vartheta(\xi)}\big)$ for every $(\alpha, \vartheta(\alpha)) \in I^{*}_{\sigma} \times I^{*}_{\vartheta(\sigma)}$. Consequently,

$$\begin{aligned}
\mathcal{S}_{\mathfrak{g}} &= \bigcup_{(\alpha,\vartheta(\alpha))\in I^{*}_{\sigma}\times I^{*}_{\vartheta(\sigma)}} \big(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathcal{S}_{\mathfrak{g}}\big)\\
&\subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I^{*}_{\sigma}\times I^{*}_{\vartheta(\sigma)}} \big(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathrm{op}_{\mathfrak{g}}\big(\mathcal{O}_{\mathfrak{g},\vartheta(\xi)}\big)\big)\\
&\subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I^{*}_{\sigma}\times I^{*}_{\vartheta(\sigma)}} \mathcal{O}_{\mathfrak{g},\vartheta(\xi)},
\end{aligned}$$

thereby implying, $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I^{*}_{\sigma}\times I^{*}_{\vartheta(\sigma)}} \mathcal{O}_{\mathfrak{g},\vartheta(\xi)}$. Hence, $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{A}\,[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathcal{S}_{\mathfrak{g}} \in \mathrm{A}\,[\mathfrak{T}_{\mathfrak{g}}]$. The proof of the theorem is complete. Q.E.D.

A situation in which a $\mathfrak{T}_{\mathfrak{g}}$-set fails to be $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact is contained in the following proposition.

PROPOSITION 3.10. *If $\mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ be any infinite $\mathfrak{T}_{\mathfrak{g}}$-set of a discrete $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, then $\mathcal{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-}\mathrm{A}\,[\mathfrak{T}_{\mathfrak{g}}]$.*

PROOF. Let $\mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set of a discrete $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$. Then, $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{A}\,[\mathfrak{T}_{\mathfrak{g}}]$ if and only if it is a finite $\mathfrak{T}_{\mathfrak{g}}$-set. Since $\mathfrak{T}_{\mathfrak{g}}$ is a discrete $\mathcal{T}_{\mathfrak{g}}$-space, consider the class $\big\{\{\xi\} :\ \xi \in \mathcal{S}_{\mathfrak{g}}\big\}$ of unit $\mathfrak{T}_{\mathfrak{g}}$-sets of $\mathcal{S}_{\mathfrak{g}}$. Clearly, the relation $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{\xi\in\mathcal{S}_{\mathfrak{g}}} \{\xi\} \subseteq \bigcup_{\xi\in\mathcal{S}_{\mathfrak{g}}} \mathrm{op}_{\mathfrak{g}}\big(\{\xi\}\big)$ holds and, for every $\xi \in \mathcal{S}_{\mathfrak{g}}$, $\mathrm{op}_{\mathfrak{g}}\big(\{\xi\}\big) \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. Accordingly, for every $\xi \in \mathcal{S}_{\mathfrak{g}}$, set $\mathrm{op}_{\mathfrak{g}}\big(\{\xi\}\big) = \mathcal{U}_{\mathfrak{g},\xi}$. Then, $\langle \mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\rangle_{\xi\in\mathcal{S}_{\mathfrak{g}}}$ is an infinite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathcal{S}_{\mathfrak{g}}$. Consequently, $\langle \mathcal{U}_{\mathfrak{g},\xi}\rangle_{\xi\in\mathcal{S}_{\mathfrak{g}}}$ contains no finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\xi)}\rangle_{(\xi,\vartheta(\xi))\in\mathcal{S}_{\mathfrak{g}}\times I^{*}_{\sigma}} \prec \langle \mathcal{U}_{\mathfrak{g},\xi}\rangle_{\xi\in\mathcal{S}_{\mathfrak{g}}}$ of $\mathcal{S}_{\mathfrak{g}}$ such that $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\xi,\vartheta(\xi))\in\mathcal{S}_{\mathfrak{g}}\times I^{*}_{\sigma}} \mathcal{U}_{\mathfrak{g},\vartheta(\xi)}$. Hence, $\mathcal{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-}\mathrm{A}\big[\mathfrak{T}_{\mathfrak{g}}\big]$. The proof of the theorem is complete. Q.E.D.

The above proposition motivates us to postulate the following corollary.

COROLLARY 3.11. *Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ to be a $\mathcal{T}_{\mathfrak{g}}$-space. If $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}^{[\mathrm{A}]}_{\mathfrak{g}}$-space $\mathfrak{g}$-$\mathfrak{T}^{[\mathrm{A}]}_{\mathfrak{g}} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}^{[\mathrm{A}]}_{\mathfrak{g}}\big)$, then it is also a $\mathcal{T}^{[\mathrm{A}]}_{\mathfrak{g}}$-space $\mathfrak{T}^{[\mathrm{A}]}_{\mathfrak{g}} = \big(\Omega, \mathcal{T}^{[\mathrm{A}]}_{\mathfrak{g}}\big)$.*

In terms of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness may be characterized in the following way.

THEOREM 3.12. *A necessary and sufficient conditions for a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ to be a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ is that, whenever a sequence $\langle \mathcal{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets is such that $\bigcap_{\alpha \in I_{\sigma}^{*}} \mathcal{V}_{\mathfrak{g},\alpha} = \emptyset$, then there exists $\langle \mathcal{V}_{\mathfrak{g},\beta(\alpha)} \rangle_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \prec \langle \mathcal{V}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$ such that the relation $\bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{V}_{\mathfrak{g},\alpha} = \emptyset$ holds.*

PROOF. *Necessity.* Suppose $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ and a sequence $\langle \mathcal{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets is given such that $\bigcap_{\alpha \in I_{\sigma}^{*}} \mathcal{V}_{\mathfrak{g},\alpha} = \emptyset$. Then, $\bigcup_{\alpha \in I_{\sigma}^{*}} \mathcal{U}_{\mathfrak{g},\alpha} = \bigcup_{\alpha \in I_{\sigma}^{*}} \complement(\mathcal{V}_{\mathfrak{g},\alpha}) = \complement(\bigcap_{\alpha \in I_{\sigma}^{*}} \mathcal{V}_{\mathfrak{g},\alpha}) = \Omega$, so that $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. Thus, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\beta(\alpha)} \rangle_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \prec \langle \mathcal{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$ and, thus, $\bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{V}_{\mathfrak{g},\beta(\alpha)} = \complement(\bigcup_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{U}_{\mathfrak{g},\beta(\alpha)}) = \emptyset$.

*Sufficiency.* Conversely, suppose that, for every $\langle \mathcal{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that $\bigcap_{\alpha \in I_{\sigma}^{*}} \mathcal{V}_{\mathfrak{g},\alpha} = \emptyset$, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering given by $\langle \mathcal{V}_{\mathfrak{g},\beta(\alpha)} \rangle_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \prec \langle \mathcal{V}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$ such that $\bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{V}_{\mathfrak{g},\alpha}$. Further, let $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\mu}^{*}}$ stand for a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. Then $\langle \complement(\mathcal{U}_{\mathfrak{g},\alpha}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\mu}^{*}}$ is a sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that $\bigcap_{\alpha \in I_{\mu}^{*}} \complement(\mathcal{U}_{\mathfrak{g},\alpha}) = \emptyset$. Thus $\bigcap_{(\alpha,\beta(\alpha)) \in I_{\mu}^{*} \times I_{n}^{*}} \complement(\mathcal{U}_{\mathfrak{g},\beta(\alpha)}) = \emptyset$ and $\langle \mathcal{U}_{\mathfrak{g},\beta(\alpha)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{(\alpha,\beta(\alpha)) \in I_{\mu}^{*} \times I_{n}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of $\mathfrak{T}_{\mathfrak{g}}$. Q.E.D.

If $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathcal{T}_{\mathfrak{g},\Gamma})$ be a $\mathcal{T}_{\mathfrak{g}}$-space such that $(\Gamma, \mathcal{T}_{\mathfrak{g},\Gamma}) \subseteq (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ and $(\Gamma, \mathcal{T}_{\mathfrak{g},\Gamma}) \subseteq (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$, where $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$ are two $\mathcal{T}_{\mathfrak{g}}$-spaces satisfying $(\Omega, \mathcal{T}_{\mathfrak{g},\Omega}) \neq (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$, then $\mathcal{T}_{\mathfrak{g},\Gamma} : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is the same whether $\mathfrak{T}_{\mathfrak{g},\Gamma} \subseteq \mathfrak{T}_{\mathfrak{g},\Omega}$ or $\mathfrak{T}_{\mathfrak{g},\Gamma} \subseteq \mathfrak{T}_{\mathfrak{g},\Sigma}$ and, hence, the assertion that, $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathcal{T}_{\mathfrak{g},\Gamma})$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},\Gamma}^{[\mathrm{A}]} = (\Gamma, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\Gamma}^{[\mathrm{A}]})$ depends only on the elements forming the structure $(\Gamma, \mathcal{T}_{\mathfrak{g},\Gamma})$. Therefore, the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness of a $\mathcal{T}_{\mathfrak{g}}$-subspace $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathcal{T}_{\mathfrak{g},\Gamma})$ of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ may be related to $\mathcal{T}_{\mathfrak{g},\Omega} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ by virtue of the following theorem.

THEOREM 3.13. *Let $\Gamma \subset \Omega$ be a $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$. Then, the following statements are equivalent:*

- I. $\Gamma \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ *with respect to the absolute $\mathfrak{g}$-topology* $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}_{\mathfrak{g}}(\Omega) \to \mathcal{P}_{\mathfrak{g}}(\Omega)$.
- II. $\Gamma \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ *with respect to the relative $\mathfrak{g}$-topology* $\mathcal{T}_{\mathfrak{g},\Gamma} : \mathcal{P}_{\mathfrak{g}}(\Gamma) \mapsto \mathcal{T}_{\mathfrak{g},\Gamma} \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g}} \cap \Gamma : \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}\}$.

PROOF. I. $\to$ II. Let $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\Gamma$ with respect to the relative $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g},\Gamma} : \mathcal{P}_{\mathfrak{g}}(\Gamma) \mapsto \mathcal{T}_{\mathfrak{g},\Gamma}$. The relative $\mathfrak{g}$-topology being $\mathcal{T}_{\mathfrak{g},\Gamma} : \mathcal{P}_{\mathfrak{g}}(\Gamma) \mapsto \mathcal{T}_{\mathfrak{g},\Gamma} \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g}} \cap \Gamma : \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}\}$, it consequently follows that, for every $\alpha \in I_{\sigma}^{*}$, there exists $\hat{\mathcal{O}}_{\mathfrak{g},\alpha} \in \mathcal{T}_{\mathfrak{g}}$ such that $\mathcal{U}_{\mathfrak{g},\alpha} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\alpha}) = \mathrm{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\alpha} \cap \Gamma) \subseteq \mathrm{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\alpha})$. For every $\alpha \in I_{\sigma}^{*}$, set $\hat{\mathcal{U}}_{\mathfrak{g},\alpha} = \mathrm{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\alpha} \cap \Gamma)$. Thus, $\Gamma \subseteq \bigcup_{\alpha \in I_{\sigma}^{*}} \hat{\mathcal{U}}_{\mathfrak{g},\alpha}$ and therefore, $\langle \hat{\mathcal{U}}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\Gamma$ with respect to the absolute $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}_{\mathfrak{g}}(\Omega) \to \mathcal{P}_{\mathfrak{g}}(\Omega)$. By virtue of I., $\Gamma \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ with respect to $\mathcal{T}_{\mathfrak{g}}$ and consequently, a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \prec \langle \hat{\mathcal{U}}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$

exists where, for every $(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}$, $\hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\alpha)}=\mathrm{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\vartheta(\alpha)}\cap\Gamma)$. But then

$$\begin{aligned}\Gamma\subseteq\Gamma\cap\Bigg(\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\hat{\mathcal{O}}_{\mathfrak{g},\vartheta(\alpha)}\Bigg) &= \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}(\hat{\mathcal{O}}_{\mathfrak{g},\vartheta(\alpha)}\cap\Gamma)\\ &= \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\alpha)}.\end{aligned}$$

Thus, it follows that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle\mathcal{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ contains a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle\hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ of $\Gamma$ with respect to the relative $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g},\Gamma}:\mathcal{P}_{\mathfrak{g}}(\Gamma)\mapsto\mathcal{T}_{\mathfrak{g},\Gamma}$. Hence, $(\Gamma,\mathcal{T}_{\mathfrak{g},\Gamma})$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space. This proves that I. implies II.

I. $\leftarrow$ II. Let $\langle\hat{\mathcal{U}}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\Gamma$ with respect to the absolute $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}:\mathcal{P}_{\mathfrak{g}}(\Omega)\to\mathcal{P}_{\mathfrak{g}}(\Omega)$. For every $\alpha\in I_{\sigma}^{*}$, there exists, then, $\hat{\mathcal{O}}_{\mathfrak{g},\alpha}\in\mathcal{T}_{\mathfrak{g}}$ such that $\hat{\mathcal{U}}_{\mathfrak{g},\alpha}=\mathrm{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\alpha})$. For every $\alpha\in I_{\sigma}^{*}$, set $\mathcal{O}_{\mathfrak{g},\alpha}=\hat{\mathcal{O}}_{\mathfrak{g},\alpha}\cap\Gamma$. Consequently, $\Gamma\subseteq\bigcup_{\alpha\in I_{\sigma}^{*}}\hat{\mathcal{U}}_{\mathfrak{g},\alpha}$ implies

$$\begin{aligned}\Gamma\subseteq\Gamma\cap\Bigg(\bigcup_{\alpha\in I_{\sigma}^{*}}\hat{\mathcal{U}}_{\mathfrak{g},\alpha}\Bigg) &= \bigcup_{\alpha\in I_{\sigma}^{*}}(\Gamma\cap\hat{\mathcal{U}}_{\mathfrak{g},\alpha})\\ &= \bigcup_{\alpha\in I_{\sigma}^{*}}(\Gamma\cap\mathrm{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\alpha}))\\ &= \bigcup_{\alpha\in I_{\sigma}^{*}}\mathrm{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\alpha}\cap\Gamma)=\bigcup_{\alpha\in I_{\sigma}^{*}}\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\alpha})\end{aligned}$$

and from which it results that, $\Gamma\subseteq\bigcup_{\alpha\in I_{\sigma}^{*}}\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\alpha})$. Since $\mathcal{O}_{\mathfrak{g},\alpha}\in\mathcal{T}_{\mathfrak{g},\Gamma}$ and $\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\alpha})\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha\in I_{\sigma}^{*}$, set $\mathcal{U}_{\mathfrak{g},\alpha}=\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\alpha})$. Then, $\langle\mathcal{U}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\Gamma$ with respect to the relative $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g},\Gamma}:\mathcal{P}_{\mathfrak{g}}(\Gamma)\mapsto\mathcal{T}_{\mathfrak{g},\Gamma}$. But, by hypothesis, $\Gamma\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ with respect to the relative $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g},\Gamma}:\mathcal{P}_{\mathfrak{g}}(\Gamma)\mapsto\mathcal{T}_{\mathfrak{g},\Gamma}$ and, therefore, a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\prec\langle\mathcal{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ exists. Accordingly,

$$\begin{aligned}\Gamma &\subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}=\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)})\\ &= \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathrm{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\vartheta(\alpha)}\cap\Gamma)=\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}(\Gamma\cap\mathrm{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\vartheta(\alpha)}))\\ &= \Gamma\cap\Bigg(\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathrm{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\vartheta(\alpha)})\Bigg)\subseteq\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathrm{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\vartheta(\alpha)})\\ &= \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\alpha)}.\end{aligned}$$

Thus, it results that, $\langle\hat{\mathcal{U}}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ is reducible to a a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle\hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ with respect to the absolute $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}:\mathcal{P}_{\mathfrak{g}}(\Omega)\to$

$\mathcal{P}_{\mathfrak{g}}(\Omega)$. Hence, $\Gamma \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ with respect to the absolute $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}_{\mathfrak{g}}(\Omega) \to \mathcal{P}_{\mathfrak{g}}(\Omega)$. Thus proves that I. is implied by II. Q.E.D.

Of the concept of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-compactness, an equivalent statement is contained in the following theorem.

THEOREM 3.14. *Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathcal{T}_{\mathfrak{g}}$-space. Then, the following statements are equivalent:*

- I. *$\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[A]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[A]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[A]})$.*
- II. *For every sequence $\langle \mathcal{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of $\mathfrak{T}_{\mathfrak{g}}$, $\bigcap_{\alpha \in I_{\sigma}^{*}} \mathcal{V}_{\mathfrak{g},\alpha} = \emptyset$ implies $\langle \mathcal{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ contains a finite subsequence $\langle \mathcal{V}_{\mathfrak{g},\beta(\alpha)} \rangle_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets with $\bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{V}_{\mathfrak{g},\beta(\alpha)} = \emptyset$.*

PROOF. I. $\rightarrow$ II. Suppose $\bigcap_{\alpha \in I_{\sigma}^{*}} \mathcal{V}_{\mathfrak{g},\alpha} = \emptyset$. Then, by virtue of De Morgan's Law, it follows that $\Omega = \complement(\emptyset) = \complement\left(\bigcap_{\alpha \in I_{\sigma}^{*}} \mathcal{V}_{\mathfrak{g},\alpha}\right) = \bigcup_{\alpha \in I_{\sigma}^{*}} \complement(\mathcal{V}_{\mathfrak{g},\alpha}) = \bigcup_{\alpha \in I_{\sigma}^{*}} \mathcal{U}_{\mathfrak{g},\alpha}$. Therefore, $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. But since $\mathfrak{T}_{\mathfrak{g}}$ is, by hypothesis, a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[A]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[A]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[A]})$, there exists a finite subsequence $\langle \mathcal{U}_{\mathfrak{g},\beta(\alpha)} \rangle_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets such that $\Omega = \bigcup_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{U}_{\mathfrak{g},\beta(\alpha)}$. Thus, by De Morgan's Law, it follows that $\emptyset = \complement(\Omega) = \complement\left(\bigcup_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{U}_{\mathfrak{g},\beta(\alpha)}\right) = \bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \complement(\mathcal{U}_{\mathfrak{g},\beta(\alpha)}) = \bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{V}_{\mathfrak{g},\beta(\alpha)}$. This proves that I. implies II.

I. $\leftarrow$ II. Let $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. Then, $\Omega = \bigcup_{\alpha \in I_{\sigma}^{*}} \mathcal{U}_{\mathfrak{g},\alpha}$. Moreover, by De Morgan's Law, $\emptyset = \complement(\Omega) = \complement\left(\bigcup_{\alpha \in I_{\sigma}^{*}} \mathcal{U}_{\mathfrak{g},\alpha}\right) = \bigcap_{\alpha \in I_{\sigma}^{*}} \complement(\mathcal{U}_{\mathfrak{g},\alpha}) = \bigcap_{\alpha \in I_{\sigma}^{*}} \mathcal{V}_{\mathfrak{g},\alpha}$. Thus, $\langle \mathcal{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ is a sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets and, by above, has an empty intersection. By hypothesis, it follows, then, that there exists a finite subsequence $\langle \mathcal{V}_{\mathfrak{g},\beta(\alpha)} \rangle_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets such that $\bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{V}_{\mathfrak{g},\beta(\alpha)} = \emptyset$. Thus, by virtue of De Morgan's Law, it results that $\Omega = \complement(\emptyset) = \complement\left(\bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{V}_{\mathfrak{g},\beta(\alpha)}\right) = \bigcup_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \complement(\mathcal{V}_{\mathfrak{g},\beta(\alpha)}) = \bigcup_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{U}_{\mathfrak{g},\beta(\alpha)}$. Accordingly, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[A]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[A]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[A]})$ and, hence, I. is implied by II. Q.E.D.

It might be conjectured that a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[A]}$-space is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact. In actual fact this conjecture holds, as proved in the following proposition.

PROPOSITION 3.15. *If $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[A]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[A]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[A]})$, then $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$:*

$$\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \;\Rightarrow\; \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}] \tag{3.6}$$

PROOF. Let it be assumed that $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set of a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[A]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[A]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[A]})$. Then, $\complement(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$; that is, $\Omega \setminus \mathcal{S}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set in $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[A]}$. Let $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}^{[A]}] \rangle_{\alpha \in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathcal{S}_{\mathfrak{g}}$ in $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[A]}$ and, for every $\alpha \in I_{\sigma}^{*}$, set $\hat{\mathcal{U}}_{\mathfrak{g},\alpha} = \mathcal{U}_{\mathfrak{g},\alpha} \cup \complement(\mathcal{S}_{\mathfrak{g}})$. Then, $\langle \hat{\mathcal{U}}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\Omega$. But since $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[A]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[A]})$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[A]}$-space, there exists a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathcal{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$ such that $\Omega \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\alpha)}$, where $\hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\alpha)} = \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \cup \complement(\mathcal{S}_{\mathfrak{g}})$ for every $(\alpha, \vartheta(\alpha)) \in$

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$I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}$. Therefore, $\langle\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ is a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of $\mathcal{S}_{\mathfrak{g}}$. Hence, $\mathcal{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-A}\left[\mathfrak{T}_{\mathfrak{g}}\right]$. The proof of the proposition is complete. Q.E.D.

Another way of stating the notion of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-compactness is by the aid of the notion of *finite intersection property.*

DEFINITION 3.16. A sequence $\langle\mathcal{S}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-S}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets is said to have the "finite intersection property" if and only if every finite subsequence of the type $\langle\mathcal{S}_{\mathfrak{g},\beta(\alpha)}\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}$ has a non-empty intersection:

$$\forall\langle\mathcal{S}_{\mathfrak{g},\beta(\alpha)}\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\prec\langle\mathcal{S}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}:\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathcal{S}_{\mathfrak{g},\beta(\alpha)}\neq\emptyset. \tag{3.7}$$

Granted the above definition, we can now state the notion of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-compactness in terms of the families of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed sets of a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space.

THEOREM 3.17. *A $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$ if and only if every sequence $\langle\mathcal{V}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets which has the finite intersection property has a non-empty intersection.*

PROOF. *Necessity.* Let the $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$ be a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$, and let $\langle\mathcal{V}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ be a sequence of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ such that $\bigcup_{\alpha\in I_{\sigma}^{*}}\mathcal{V}_{\mathfrak{g},\alpha}=\emptyset$. For every $\alpha\in I_{\sigma}^{*}$, set $\mathcal{U}_{\mathfrak{g},\alpha}=\complement\left(\mathcal{V}_{\mathfrak{g},\alpha}\right)$, and consider the sequence $\langle\mathcal{U}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets. Since $\bigcup_{\alpha\in I_{\sigma}^{*}}\mathcal{U}_{\mathfrak{g},\alpha}=\bigcup_{\alpha\in I_{\sigma}^{*}}\complement\left(\mathcal{V}_{\mathfrak{g},\alpha}\right)=\complement\left(\bigcap_{\alpha\in I_{\sigma}^{*}}\mathcal{V}_{\mathfrak{g},\alpha}\right)=\Omega$, it follows that $\langle\mathcal{U}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$. But $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$ and, therefore, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle\mathcal{U}_{\mathfrak{g},\beta(\alpha)}\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\prec\langle\mathcal{U}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ such that

$$\begin{aligned}\Omega=\bigcup_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathcal{U}_{\mathfrak{g},\beta(\alpha)}&=\bigcup_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\complement\left(\mathcal{V}_{\mathfrak{g},\beta(\alpha)}\right)\\&=\complement\Bigg(\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathcal{V}_{\mathfrak{g},\beta(\alpha)}\Bigg).\end{aligned}$$

This implies that $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathcal{V}_{\mathfrak{g},\beta(\alpha)}=\emptyset$. Hence, it follows that, if a sequence $\langle\mathcal{V}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ has the finite intersection property, then $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathcal{V}_{\mathfrak{g},\beta(\alpha)}\neq\emptyset$.

*Sufficiency.* Conversely, suppose that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space in which every sequence $\langle\mathcal{V}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets which has the finite intersection property has a non-empty intersection. Then, for every subsequence $\langle\mathcal{V}_{\mathfrak{g},\beta(\alpha)}\rangle_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\prec\langle\mathcal{V}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, the relation $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathcal{V}_{\mathfrak{g},\beta(\alpha)}\neq\emptyset$ holds. Consequently, $\bigcap_{\alpha\in I_{\sigma}^{*}}\mathcal{V}_{\mathfrak{g},\alpha}\neq\emptyset$. In other words, $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathcal{V}_{\mathfrak{g},\beta(\alpha)}\neq\emptyset$ for every $I_{n}^{*}\subseteq I_{\sigma}^{*}$ implies $\bigcap_{\alpha\in I_{\sigma}^{*}}\mathcal{V}_{\mathfrak{g},\alpha}\neq\emptyset$. But this is the contrapositive statement of $\bigcap_{\alpha\in I_{\sigma}^{*}}\mathcal{V}_{\mathfrak{g},\alpha}=\emptyset$ implies that there exists $I_{n}^{*}\subseteq I_{\sigma}^{*}$ such that $\bigcap_{(\alpha,\beta(\alpha))\in I_{\sigma}^{*}\times I_{n}^{*}}\mathcal{V}_{\mathfrak{g},\beta(\alpha)}=\emptyset$. It results that, every sequence $\langle\mathcal{V}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-K}\left[\mathfrak{T}_{\mathfrak{g}}\right]\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of $\mathfrak{T}_{\mathfrak{g}}$, $\bigcap_{\alpha\in I_{\sigma}^{*}}\mathcal{V}_{\mathfrak{g},\alpha}=\emptyset$ implies

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$\langle \mathcal{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ contains a finite subsequence $\langle \mathcal{V}_{\mathfrak{g},\beta(\alpha)} \rangle_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}}$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets with $\bigcap_{(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{n}^{*}} \mathcal{V}_{\mathfrak{g},\beta(\alpha)} = \emptyset$. Hence, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]})$. Q.E.D.

An interesting remark may well be given at this stage.

REMARK 3.18. In particular, if the $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})})$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ and the elements of $\langle \mathcal{V}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ forms a descending sequence $\mathcal{V}_{\mathfrak{g},1} \supset \mathcal{V}_{\mathfrak{g},2} \supset \cdots \supset \mathcal{V}_{\mathfrak{g},\alpha} \supset \cdots$ of non-empty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, then $\bigcap_{\alpha \in I_{\sigma}^{*}} \mathcal{V}_{\mathfrak{g},\alpha} \neq \emptyset$. Such property in its own right is weaker than $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness. In fact, it indicates the sense in which $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness asserts that the $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ has enough points, namely, at least enough points to yield one point in each such intersection of a descending sequence $\mathcal{V}_{\mathfrak{g},1} \supset \mathcal{V}_{\mathfrak{g},2} \supset \cdots \supset \mathcal{V}_{\mathfrak{g},\alpha} \supset \cdots$ of non-empty $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets.

The notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness may be characterized in terms of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open neighborhood in the following manner.

THEOREM 3.19. *A necessary and sufficient conditions for a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ to be a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]})$ is that, whenever for each $\xi \in \mathfrak{T}_{\mathfrak{g}}$ a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open neighborhood of $\xi$ is given, there is a finite collection $\mathcal{C}_{\xi} = \{\xi_{\eta} : \eta \in I_{n}^{*}\}$ of points $\xi_1, \xi_2, \ldots, \xi_n \in \mathfrak{T}_{\mathfrak{g}}$ such that $\Omega = \bigcup_{\xi \in \mathcal{C}_{\xi}} \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$.*

PROOF. *Necessity.* Suppose $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]})$. Let there be given for each $\xi \in \mathfrak{T}_{\mathfrak{g}}$ a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open neighborhood of $\xi$. For each $\xi \in \mathfrak{T}_{\mathfrak{g}}$, there is a $\mathfrak{T}_{\mathfrak{g}}$-open set $\mathcal{U}_{\mathfrak{g},\xi} \subset \mathfrak{T}_{\mathfrak{g}}$ satisfying $\xi \in \mathcal{U}_{\mathfrak{g},\xi} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$. Thus, for every $\xi \in \mathfrak{T}_{\mathfrak{g}}$, $\mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and, consequently, $\langle \mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\xi \in \mathfrak{T}_{\mathfrak{g}}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. Since $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]})$, there is a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\xi_{\mu}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\mu \in I_{n}^{*}}$. But, for every $\mu \in I_{n}^{*}$, $\xi_{\mu} \in \mathcal{U}_{\mathfrak{g},\xi_{\mu}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi_{\mu}})$, whence $\Omega = \bigcup_{\mu \in I_{n}^{*}} \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi_{\mu}}) = \bigcup_{\xi \in \mathcal{C}_{\xi}} \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$.

*Sufficiency.* Conversely, suppose that whenever, for each $\xi \in \mathfrak{T}_{\mathfrak{g}}$ a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open neighborhood of $\xi$ is given, there is a finite collection $\mathcal{C}_{\xi} = \{\xi_{\eta} : \eta \in I_{n}^{*}\}$ of points $\xi_1, \xi_2, \ldots, \xi_n \in \mathfrak{T}_{\mathfrak{g}}$ such that $\Omega = \bigcup_{\xi \in \mathcal{C}_{\xi}} \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$. Let $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathfrak{T}_{\mathfrak{g}}$. Then for each $\xi \in \mathfrak{T}_{\mathfrak{g}}$, there exists an $\alpha = \alpha(\xi)$ such that $\xi \in \mathcal{U}_{\mathfrak{g},\alpha(\xi)}$, and hence, $\mathcal{U}_{\mathfrak{g},\alpha(\xi)} = \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$ for every $(\xi, \alpha(\xi)) \in \mathfrak{T}_{\mathfrak{g}} \times I_{n}^{*}$. By hypothesis, there is, then, a finite collection $\mathcal{C}_{\xi} = \{\xi_{\eta} : \eta \in I_{n}^{*}\}$ of points $\xi_1, \xi_2, \ldots, \xi_n \in \mathfrak{T}_{\mathfrak{g}}$ such that $\Omega = \bigcup_{\xi \in \mathcal{C}_{\xi}} \mathcal{U}_{\mathfrak{g},\alpha(\xi)}$ and thus, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]})$. Q.E.D.

In the following statements, the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness is related to the $\mathfrak{g}$-$\mathrm{T}_{\mathfrak{g},\alpha}$-axioms of their $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\alpha)}$-spaces.

LEMMA 3.20. *If $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})})$ and suppose $\xi \notin \mathcal{S}_{\mathfrak{g}}$, then there exists $(\mathcal{U}_{\mathfrak{g},\alpha}, \mathcal{U}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $(\{\xi\}, \mathcal{S}_{\mathfrak{g}}) \subseteq (\mathcal{U}_{\mathfrak{g},\alpha}, \mathcal{U}_{\mathfrak{g},\beta})$ and $\bigcap_{\mu=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\mu} = \emptyset$.*

PROOF. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})})$ and suppose $\xi \notin \mathcal{S}_{\mathfrak{g}}$. Since $\xi \notin \mathcal{S}_{\mathfrak{g}}$, it results that $\zeta \in \mathcal{S}_{\mathfrak{g}}$ implies

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$\xi \notin \{\zeta\}$. But by hypothesis, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})})$ and therefore, there exists $(\mathcal{U}_{\mathfrak{g},\zeta}, \hat{\mathcal{U}}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $(\xi, \zeta) \in \mathcal{U}_{\mathfrak{g},\zeta} \times \hat{\mathcal{U}}_{\mathfrak{g},\zeta}$ and $\mathcal{U}_{\mathfrak{g},\zeta} \cap \hat{\mathcal{U}}_{\mathfrak{g},\zeta} = \emptyset$. Hence, it follows that $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{\zeta \in \mathcal{S}_{\mathfrak{g}}} \hat{\mathcal{U}}_{\mathfrak{g},\zeta}$, meaning that $\langle \hat{\mathcal{U}}_{\mathfrak{g},\zeta} \rangle_{\zeta \in \mathcal{S}_{\mathfrak{g}}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathcal{S}_{\mathfrak{g}}$. But $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, there exists $\langle \hat{\mathcal{U}}_{\mathfrak{g},\zeta(\mu)} \rangle_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathcal{S}_{\mathfrak{g}}} \prec \langle \hat{\mathcal{U}}_{\mathfrak{g},\zeta} \rangle_{\zeta \in \mathcal{S}_{\mathfrak{g}}}$ such that $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathcal{S}_{\mathfrak{g}}} \hat{\mathcal{U}}_{\mathfrak{g},\zeta(\mu)}$. Now let

$$\mathcal{U}_{\mathfrak{g},\alpha} = \bigcap_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathcal{S}_{\mathfrak{g}}} \mathcal{U}_{\mathfrak{g},\zeta(\mu)}, \quad \mathcal{U}_{\mathfrak{g},\beta} = \bigcup_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathcal{S}_{\mathfrak{g}}} \hat{\mathcal{U}}_{\mathfrak{g},\zeta(\mu)}.$$

It is evidently that, $(\mathcal{U}_{\mathfrak{g},\alpha}, \mathcal{U}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, since $(\mathcal{U}_{\mathfrak{g},\zeta(\mu)}, \hat{\mathcal{U}}_{\mathfrak{g},\zeta(\mu)}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\mu, \zeta(\mu)) \in I_{\sigma}^{*} \times \mathcal{S}_{\mathfrak{g}}$. Furthermore, $(\{\xi\}, \mathcal{S}_{\mathfrak{g}}) \subseteq (\mathcal{U}_{\mathfrak{g},\alpha}, \mathcal{U}_{\mathfrak{g},\beta})$, since $\xi \in \mathcal{U}_{\mathfrak{g},\zeta(\mu)}$ for every $(\mu, \zeta(\mu)) \in I_{\sigma}^{*} \times \mathcal{S}_{\mathfrak{g}}$. Lastly, let it be claimed that $\bigcap_{\mu=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\mu} = \emptyset$. Then, $\mathcal{U}_{\mathfrak{g},\zeta(\mu)} \cap \hat{\mathcal{U}}_{\mathfrak{g},\zeta(\mu)} = \emptyset$ for every $(\mu, \zeta(\mu)) \in I_{\sigma}^{*} \times \mathcal{S}_{\mathfrak{g}}$ which, in turn, implies that $\mathcal{U}_{\mathfrak{g},\alpha} \cap \hat{\mathcal{U}}_{\mathfrak{g},\zeta(\mu)} = \emptyset$ for every $(\mu, \zeta(\mu)) \in I_{\sigma}^{*} \times \mathcal{S}_{\mathfrak{g}}$. Hence,

$$\begin{aligned} \bigcap_{\mu=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\mu} = \mathcal{U}_{\mathfrak{g},\alpha} \cap \Bigg( \bigcup_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathcal{S}_{\mathfrak{g}}} \hat{\mathcal{U}}_{\mathfrak{g},\zeta(\mu)} \Bigg) &= \bigcup_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathcal{S}_{\mathfrak{g}}} (\mathcal{U}_{\mathfrak{g},\alpha} \cap \hat{\mathcal{U}}_{\mathfrak{g},\zeta(\mu)}) \\ &= \bigcup_{(\mu,\zeta(\mu)) \in I_{\sigma}^{*} \times \mathcal{S}_{\mathfrak{g}}} \emptyset = \emptyset. \end{aligned}$$

This completes the proof of the lemma. Q.E.D.

THEOREM 3.21. *Let* $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compact set of a* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-*space* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})})$. *If* $\xi \notin \mathcal{S}_{\mathfrak{g}}$, *then there exists a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*open set* $\mathcal{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ *such that* $\xi \in \mathcal{U}_{\mathfrak{g}} \subseteq \complement(\mathcal{S}_{\mathfrak{g}})$.

PROOF. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})})$ and suppose $\xi \notin \mathcal{S}_{\mathfrak{g}}$. Since $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})})$, there exists then $(\mathcal{U}_{\mathfrak{g}}, \hat{\mathcal{U}}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $(\{\xi\}, \mathcal{S}_{\mathfrak{g}}) \subseteq (\mathcal{U}_{\mathfrak{g}}, \hat{\mathcal{U}}_{\mathfrak{g}})$ and $\mathcal{U}_{\mathfrak{g}} \cap \hat{\mathcal{U}}_{\mathfrak{g}} = \emptyset$. Hence, $\mathcal{U}_{\mathfrak{g}} \cap \mathcal{S}_{\mathfrak{g}} = \emptyset$ and consequently, $\xi \in \mathcal{U}_{\mathfrak{g}} \subseteq \complement(\mathcal{S}_{\mathfrak{g}})$. This proves the theorem. Q.E.D.

PROPOSITION 3.22. *If* $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compact set of a* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-*space* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})})$, *then* $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ *in* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$.

PROOF. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})})$. It must be proved that $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ which is equivalent to prove that $\complement(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$. Let $\xi \in \complement(\mathcal{S}_{\mathfrak{g}})$; that is, $\xi \notin \mathcal{S}_{\mathfrak{g}}$. Since $\xi \notin \mathcal{S}_{\mathfrak{g}}$ there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\xi \in \mathcal{U}_{\mathfrak{g},\xi} \subseteq \complement(\mathcal{S}_{\mathfrak{g}})$. Consequently, $\complement(\mathcal{S}_{\mathfrak{g}}) = \bigcup_{\xi \in \complement(\mathcal{S}_{\mathfrak{g}})} \mathcal{U}_{\mathfrak{g},\xi}$. Therefore, $\complement(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, since $\mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\xi \in \complement(\mathcal{S}_{\mathfrak{g}})$. Hence, $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ in $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$. This proves the proposition. Q.E.D.

LEMMA 3.23. *If* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ *be a* $\mathcal{T}_{\mathfrak{g}}$-*space whose* $\mathfrak{g}$-*topology* $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ *is cofinite on* $\Omega$, *then* $\mathfrak{T}_{\mathfrak{g}}$ *is a* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-*space* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]})$.

PROOF. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathcal{T}_{\mathfrak{g}}$-space whose $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ is cofinite on $\Omega$ and suppose $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open covering of

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$\Omega$. Then, $\complement(\mathcal{U}_{\mathfrak{g},\alpha}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ for any chosen $\alpha \in I_{\sigma}^{*}$. Furthermore, since $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ is cofinite on $\Omega$, $\mathcal{U}_{\mathfrak{g},\alpha}$, it follows that, for every $\alpha \in I_{\sigma}^{*}$, $\complement(\mathcal{U}_{\mathfrak{g},\alpha})$ is a finite $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed set. Set $\complement(\mathcal{U}_{\mathfrak{g},\alpha}) = \left\{\xi_{\beta(\alpha)} : (\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}\right\}$. Since $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open covering of $\Omega$, for every $(\alpha,\beta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}$, $\xi_{\beta(\alpha)} \in \complement(\mathcal{U}_{\mathfrak{g},\alpha})$ implies the existence of $\mathcal{U}_{\mathfrak{g},\gamma(\alpha)}$, where $\langle \mathcal{U}_{\mathfrak{g},\gamma(\alpha)}\rangle_{(\alpha,\gamma(\alpha))\in I_{\sigma}^{*}\times I_{\gamma(\sigma)}^{*}} \prec \langle \mathcal{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$, satisfying $\xi_{\beta(\alpha)} \in \mathcal{U}_{\mathfrak{g},\gamma(\alpha)}$. Hence, $\complement(\mathcal{U}_{\mathfrak{g},\alpha}) \subseteq \bigcup_{(\alpha,\gamma(\alpha))\in I_{\sigma}^{*}\times I_{\gamma(\sigma)}^{*}} \mathcal{U}_{\mathfrak{g},\gamma(\alpha)}$ and therefore,

$$\Omega = \mathcal{U}_{\mathfrak{g},\alpha} \cup \complement(\mathcal{U}_{\mathfrak{g},\alpha}) = \mathcal{U}_{\mathfrak{g},\alpha} \cup \Bigg(\bigcup_{(\alpha,\gamma(\alpha))\in I_{\sigma}^{*}\times I_{\gamma(\sigma)}^{*}} \mathcal{U}_{\mathfrak{g},\gamma(\alpha)}\Bigg).$$

Thus, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \left(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$. This completes the proof of the lemma. Q.E.D.

THEOREM 3.24. *If* $(\mathcal{R}_{\mathfrak{g}},\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ *be a pair of disjoint* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-*compact sets of a* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-*space* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \left(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$, *then there exists a pair* $(\mathcal{U}_{\mathfrak{g},\alpha},\mathcal{U}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ *of disjoint* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-*open sets such that* $(\mathcal{R}_{\mathfrak{g}},\mathcal{S}_{\mathfrak{g}}) \subseteq (\mathcal{U}_{\mathfrak{g},\alpha},\mathcal{U}_{\mathfrak{g},\beta})$.

PROOF. Let $(\mathcal{R}_{\mathfrak{g}},\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a pair of disjoint $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-compact sets of a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})} = \left(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$ and suppose $\xi \in \mathcal{R}_{\mathfrak{g}}$. Then, since $\mathcal{R}_{\mathfrak{g}} \cap \mathcal{S}_{\mathfrak{g}} = \emptyset$, it results that $\xi \notin \mathcal{S}_{\mathfrak{g}}$. But by hypothesis, $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ and consequently, there exists $\left(\mathcal{U}_{\mathfrak{g},\xi},\hat{\mathcal{U}}_{\mathfrak{g},\xi}\right) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $(\{\xi\},\mathcal{S}_{\mathfrak{g}}) \subseteq \left(\mathcal{U}_{\mathfrak{g},\xi},\hat{\mathcal{U}}_{\mathfrak{g},\xi}\right)$ and $\mathcal{U}_{\mathfrak{g},\xi} \cap \hat{\mathcal{U}}_{\mathfrak{g},\xi} = \emptyset$. Since $\xi \in \mathcal{U}_{\mathfrak{g},\xi}$, it follow that $\langle \mathcal{U}_{\mathfrak{g},\xi}\rangle_{\xi\in\mathcal{R}_{\mathfrak{g}}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathcal{R}_{\mathfrak{g}}$. Since $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering

$$\langle \mathcal{U}_{\mathfrak{g},\upsilon(\xi)}\rangle_{(\xi,\upsilon(\xi))\in\hat{\mathcal{R}}_{\mathfrak{g}}\times\mathcal{R}_{\mathfrak{g}}} \prec \langle \mathcal{U}_{\mathfrak{g},\xi}\rangle_{\xi\in\mathcal{R}_{\mathfrak{g}}},$$

where $\hat{\mathcal{R}}_{\mathfrak{g}} \subseteq \mathcal{R}_{\mathfrak{g}}$ is finite, can be selected so that $\mathcal{R}_{\mathfrak{g}} \subseteq \bigcup_{(\xi,\upsilon(\xi))\in\hat{\mathcal{R}}_{\mathfrak{g}}\times\mathcal{R}_{\mathfrak{g}}} \mathcal{U}_{\mathfrak{g},\upsilon(\xi)}$. Furthermore, $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcap_{(\zeta,\vartheta(\zeta))\in\hat{\mathcal{S}}_{\mathfrak{g}}\times\mathcal{S}_{\mathfrak{g}}} \hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\xi)}$, where $\hat{\mathcal{S}}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}$ is finite, since $\mathcal{S}_{\mathfrak{g}} \subseteq \hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\xi)}$ for every $(\zeta,\vartheta(\zeta)) \in \hat{\mathcal{S}}_{\mathfrak{g}} \times \mathcal{S}_{\mathfrak{g}}$. Now let

$$\mathcal{U}_{\mathfrak{g},\alpha} = \bigcup_{(\xi,\upsilon(\xi))\in\hat{\mathcal{R}}_{\mathfrak{g}}\times\mathcal{R}_{\mathfrak{g}}} \mathcal{U}_{\mathfrak{g},\upsilon(\xi)}, \quad \mathcal{U}_{\mathfrak{g},\beta} = \bigcap_{(\zeta,\vartheta(\zeta))\in\hat{\mathcal{S}}_{\mathfrak{g}}\times\mathcal{S}_{\mathfrak{g}}} \hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\zeta)}.$$

Observe that $(\mathcal{R}_{\mathfrak{g}},\mathcal{S}_{\mathfrak{g}}) \subseteq (\mathcal{U}_{\mathfrak{g},\alpha},\mathcal{U}_{\mathfrak{g},\beta})$. Moreover, $(\mathcal{U}_{\mathfrak{g},\alpha},\mathcal{U}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, since $\mathcal{U}_{\mathfrak{g},\upsilon(\xi)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\xi,\upsilon(\xi)) \in \hat{\mathcal{R}}_{\mathfrak{g}} \times \mathcal{R}_{\mathfrak{g}}$ and, $\hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\zeta)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\zeta,\vartheta(\zeta)) \in \hat{\mathcal{S}}_{\mathfrak{g}} \times \mathcal{S}_{\mathfrak{g}}$. The proof of the theorem is complete when the statement $\mathcal{U}_{\mathfrak{g},\alpha} \cap \mathcal{U}_{\mathfrak{g},\beta} = \emptyset$ is proved. First observe that, for every $(\xi,\zeta,\upsilon(\xi),\vartheta(\zeta)) \in \hat{\mathcal{R}}_{\mathfrak{g}} \times \hat{\mathcal{S}}_{\mathfrak{g}} \times \mathcal{R}_{\mathfrak{g}} \times \mathcal{S}_{\mathfrak{g}}$, the relation $\mathcal{U}_{\mathfrak{g},\upsilon(\xi)} \cap \hat{\mathcal{U}}_{\mathfrak{g},\vartheta(\zeta)} = \emptyset$ implies $\mathcal{U}_{\mathfrak{g},\upsilon(\xi)} \cap \mathcal{U}_{\mathfrak{g},\beta} = \emptyset$. Consequently,

$$\begin{aligned}\bigcap_{\mu=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\mu} = \Bigg(\bigcup_{(\xi,\upsilon(\xi))\in\hat{\mathcal{R}}_{\mathfrak{g}}\times\mathcal{R}_{\mathfrak{g}}} \mathcal{U}_{\mathfrak{g},\upsilon(\xi)}\Bigg) \cap \mathcal{U}_{\mathfrak{g},\beta} &= \bigcup_{(\xi,\upsilon(\xi))\in\hat{\mathcal{R}}_{\mathfrak{g}}\times\mathcal{R}_{\mathfrak{g}}} \left(\mathcal{U}_{\mathfrak{g},\upsilon(\xi)} \cap \mathcal{U}_{\mathfrak{g},\beta}\right)\\ &= \bigcup_{(\xi,\upsilon(\xi))\in\hat{\mathcal{R}}_{\mathfrak{g}}\times\mathcal{R}_{\mathfrak{g}}} \emptyset = \emptyset.\end{aligned}$$

This proves the theorem. Q.E.D.

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We next show that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness is an absolute property that is preserved under $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-continuous maps.

THEOREM 3.25. *Let* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ *and* $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$ *be* $\mathcal{T}_{\mathfrak{g}}$*-spaces. If* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *is a* $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-continuous map and,* $\mathcal{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ *in* $\mathfrak{T}_{\mathfrak{g},\Omega}$*, then* $\operatorname{im}\big(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$ *in* $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

PROOF. Let $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$ be given $\mathcal{T}_{\mathfrak{g}}$-spaces, $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, $\mathcal{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ in $\mathfrak{T}_{\mathfrak{g},\Omega}$ and, suppose $\langle \mathcal{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\operatorname{im}\big(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}\big)$ in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. Then,

$$\mathcal{S}_{\mathfrak{g},\omega} \subseteq \pi_{\mathfrak{g}}^{-1} \circ \pi_{\mathfrak{g}} \left(\mathcal{S}_{\mathfrak{g},\omega}\right) \subseteq \pi_{\mathfrak{g}}^{-1}\Big(\bigcup_{\alpha\in I_{\sigma}^{*}} \mathcal{U}_{\mathfrak{g},\alpha}\Big) \subseteq \bigcup_{\alpha\in I_{\sigma}^{*}} \pi_{\mathfrak{g}}^{-1}\left(\mathcal{U}_{\mathfrak{g},\alpha}\right).$$

Thus, $\langle \pi_{\mathfrak{g}}^{-1}\left(\mathcal{U}_{\mathfrak{g},\alpha}\right)\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathcal{S}_{\mathfrak{g},\omega}$ in $\mathfrak{T}_{\mathfrak{g},\Sigma}$, because $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ and, for every $\alpha \in I_{\sigma}^{*}$, $\mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Omega}\right]$ implies $\pi_{\mathfrak{g}}^{-1}\left(\mathcal{U}_{\mathfrak{g},\alpha}\right) \in \mathfrak{g}\text{-O}\left[\mathfrak{T}_{\mathfrak{g},\Sigma}\right]$. But, the relation $\mathcal{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g},\Omega}\big]$ holds and, consequently, there exists $\langle \pi_{\mathfrak{g}}^{-1}\left(\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\right)\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \prec \langle \pi_{\mathfrak{g}}^{-1}\left(\mathcal{U}_{\mathfrak{g},\alpha}\right)\rangle_{\alpha\in I_{\sigma}^{*}}$ such that the relation $\mathcal{S}_{\mathfrak{g},\omega} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \pi_{\mathfrak{g}}^{-1}\left(\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\right)$ holds. Accordingly,

$$\pi_{\mathfrak{g}}\left(\mathcal{S}_{\mathfrak{g},\omega}\right) \subseteq \pi_{\mathfrak{g}} \circ \pi_{\mathfrak{g}}^{-1}\Bigg(\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\Bigg) = \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}.$$

Thus, $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of $\operatorname{im}\big(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}\big)$ and hence, $\operatorname{im}\big(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$ in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. The proof of the theorem is complete. Q.E.D.

We now show that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness is an absolute property that is also preserved under $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-irresolute maps.

THEOREM 3.26. *Let* $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ *be a* $\mathfrak{T}_{\mathfrak{g}}$*-set and let* $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ *be a* $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-irresolute map, where* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ *and* $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$ *are* $\mathcal{T}_{\mathfrak{g}}$*-spaces. If* $\mathcal{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\Omega}\big]$*, then* $\operatorname{im}\big(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$.

PROOF. Let $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set and let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right]$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-irresolute map, where $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$ are $\mathcal{T}_{\mathfrak{g}}$-spaces. Suppose $\mathcal{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\Omega}\big]$, let $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}\big[\mathfrak{T}_{\Sigma}\big]\rangle_{\alpha\in I_{\sigma}^{*}}$ be any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\pi_{\mathfrak{g}}\left(\mathcal{S}_{\mathfrak{g},\omega}\right) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$. Then, since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}\right]$, it follows, evidently, that the relation $\mathcal{S}_{\mathfrak{g},\omega} \bigcup_{\alpha\in I_{\sigma}^{*}} \pi_{\mathfrak{g}}^{-1}\left(\mathcal{U}_{\mathfrak{g},\alpha}\right)$ holds. On the other hand, since $\mathcal{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\Omega}\big]$, it results that, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathcal{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_{\sigma}^{*}}$ exists such that the relation $\mathcal{S}_{\mathfrak{g},\omega} \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \pi_{\mathfrak{g}}^{-1}\big(\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\big)$ holds. Consequently, it follows, then, that $\pi_{\mathfrak{g}}\left(\mathcal{S}_{\mathfrak{g},\omega}\right) \subseteq \bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}$ and, hence, $\operatorname{im}\big(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}\big) \in \mathfrak{g}\text{-A}\big[\mathfrak{T}_{\mathfrak{g},\Sigma}\big]$. The proof of the theorem is complete. Q.E.D.

LEMMA 3.27. *Let* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big)$ *be a* $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space. If* $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$*, then* $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$ *in* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$.

PROOF. Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \big(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big)$ be a $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space and suppose $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$. Suppose $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]\rangle_{\alpha\in I_{\sigma}^{*}}$ be a $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-open covering of $\mathcal{S}_{\mathfrak{g}}$, then $\Omega =$

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$\left(\bigcup_{\alpha\in I_{\sigma}^{*}}\mathcal{U}_{\mathfrak{g},\alpha}\right)\cup\complement\left(\mathcal{S}_{\mathfrak{g}}\right)=\bigcup_{\alpha\in I_{\sigma}^{*}}\left(\mathcal{U}_{\mathfrak{g},\alpha}\cup\complement\left(\mathcal{S}_{\mathfrak{g}}\right)\right)$, meaning that $\left\langle\mathcal{U}_{\mathfrak{g},\alpha}\cup\complement\left(\mathcal{S}_{\mathfrak{g}}\right)\right\rangle_{\alpha\in I_{\sigma}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-open covering of $\mathcal{S}_{\mathfrak{g}}$ because, $\mathcal{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-K}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$ implies $\complement\left(\mathcal{S}_{\mathfrak{g}}\right)\in\mathfrak{g}\text{-O}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$. On the other hand, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$ is, by hypothesis, a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space. Thus, there exists $\left\langle\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\right\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\prec\left\langle\mathcal{U}_{\mathfrak{g},\alpha}\right\rangle_{\alpha\in I_{\sigma}^{*}}$ such that $\Omega=\left(\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\right)\cup\complement\left(\mathcal{S}_{\mathfrak{g}}\right)$. But $\mathcal{S}_{\mathfrak{g}}\cap\complement\left(\mathcal{S}_{\mathfrak{g}}\right)=\emptyset$ and, hence, $\mathcal{S}_{\mathfrak{g}}\subseteq\bigcup_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}$. This shows that any $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-open covering $\left\langle\mathcal{U}_{\mathfrak{g},\alpha}\cup\complement\left(\mathcal{S}_{\mathfrak{g}}\right)\right\rangle_{\alpha\in I_{\sigma}^{*}}$ of $\mathcal{S}_{\mathfrak{g}}$ contains a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-open subcovering $\left\langle\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\right\rangle_{(\alpha,\vartheta(\alpha))\in I_{\sigma}^{*}\times I_{\vartheta(\sigma)}^{*}}$ and hence, $\mathcal{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-A}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$ in $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$. The proof of the lemma is complete. Q.E.D.

THEOREM 3.28. *Let* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\right)$ *be a* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space and let* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}=\left(\Sigma,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\right)$ *be a* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$*-space. If the* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-map* $\pi_{\mathfrak{g}}:\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\rightarrow\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}$ *is a one-one* $\mathfrak{g}$-$\left(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]},\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\right)$*-continuous map, then* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\cong\pi_{\mathfrak{g}}\left(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\right)$.

PROOF. Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\right)$ be a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space and let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}=\left(\Sigma,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\right)$ be a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space, and suppose $\pi_{\mathfrak{g}}:\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\rightarrow\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}$ is a one-one $\mathfrak{g}$-$\left(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]},\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\right)$-continuous map. Clearly, $\pi_{\mathfrak{g}}:\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\rightarrow\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}$ is onto, and since it is, by hypothesis a one-one $\mathfrak{g}$-$\left(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]},\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\right)$-continuous map, it follows that $\pi_{\mathfrak{g}}^{-1}:\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\rightarrow\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}$ exists. It must be shown that $\pi_{\mathfrak{g}}^{-1}\in\mathfrak{g}\text{-C}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})};\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\big]$. It must be shown that $\pi_{\mathfrak{g}}^{-1}\in\mathfrak{g}\text{-C}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})};\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\big]$. Recall that $\pi_{\mathfrak{g}}^{-1}:\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\rightarrow\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}$ is $\mathfrak{g}$-$\left(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})},\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\right)$-continuous if and only if, for every $\mathcal{K}_{\mathfrak{g},\omega}\in\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}$, $\left(\pi_{\mathfrak{g}}^{-1}\right)^{-1}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)=\pi_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)\in\mathfrak{g}\text{-K}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{(\mathrm{H})}\big]$ and $\pi_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)\subseteq\mathrm{im}\left(\pi_{\mathfrak{g}|\Sigma}\right)$. Clearly, $\mathcal{K}_{\mathfrak{g},\omega}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)$, so $\mathcal{K}_{\mathfrak{g},\omega}\in\mathfrak{g}\text{-K}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$. But, $\mathcal{K}_{\mathfrak{g},\omega}\in\mathfrak{g}\text{-K}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$ implies $\mathcal{K}_{\mathfrak{g},\omega}\in\mathfrak{g}\text{-A}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\big]$ in $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$. Furthermore, since $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-C}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]};\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\big]$, it follows that $\pi_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)\in\mathfrak{g}\text{-A}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{(\mathrm{H})}\big]$ and $\pi_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)\subseteq\mathrm{im}\left(\pi_{\mathfrak{g}|\Sigma}\right)$. But, $\pi_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\omega}\right)\in\mathfrak{g}\text{-A}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{(\mathrm{H})}\big]$ implies $\mathcal{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-K}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}\big]$. Accordingly, $\pi_{\mathfrak{g}}^{-1}\in\mathfrak{g}\text{-C}\big[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})};\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\big]$ and hence, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\cong\pi_{\mathfrak{g}}\left(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\right)$. The proof of the theorem is complete. Q.E.D.

A $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space coincides with a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space upon satisfaction of some condition as given in the following proposition.

PROPOSITION 3.29. *Let* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$ *be a* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space and let* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$ *be a* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$*-space. If* $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\supseteq\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},}^{(\mathrm{H})}$*, then* $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},}^{(\mathrm{H})}$.

PROOF. Let $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$ be a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}=\left(\Omega,\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}\right)$, a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space, and suppose $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\supseteq\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},}^{(\mathrm{H})}$. Further, consider the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-map $\pi_{\mathfrak{g}}:\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}\rightarrow\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\mathrm{H})}$ defined by $\pi_{\mathfrak{g}}\left(\xi\right)=\xi$. Since $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\supseteq\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},}^{(\mathrm{H})}$, for every $\mathcal{O}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},}^{(\mathrm{H})}$, there exist $\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\in\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$ such that $\pi_{\mathfrak{g}}^{-1}\left(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\right)=\mathcal{O}_{\mathfrak{g},\alpha}\subseteq\mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\alpha}\right)$. Consequently, $\pi_{\mathfrak{g}}:\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\rightarrow\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}$ is a one-one and onto $\mathfrak{g}$-$\left(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]},\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}\right)$-continuous map from a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}$ to a $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{(\mathrm{H})}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g},\Sigma}^{(\mathrm{H})}$ and therefore, $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\cong\pi_{\mathfrak{g}}\left(\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{[\mathrm{A}]}\right)$. Hence, $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}=\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},}^{(\mathrm{H})}$. The proof of the proposition is complete. Q.E.D.

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Below is defined a new concept called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point, by means of which further characterisation on $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness can be establised.

DEFINITION 3.30 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Accumulation Point). A point $\xi \in \mathfrak{T}_{\mathfrak{g}}$ of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point" (or, "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-limit point," "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-cluster point," "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived point") of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of $\mathfrak{T}_{\mathfrak{g}}$ if and only if every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}\,[\mathfrak{T}_{\mathfrak{g}}]$ containing $\xi$ (whether $\xi \in \mathcal{S}_{\mathfrak{g}}$ or $\xi \notin \mathcal{S}_{\mathfrak{g}}$) contains at least a point $\zeta \in \mathcal{S}_{\mathfrak{g}} \setminus \{\xi\}$:

$$\xi \in \mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}\,[\mathfrak{T}_{\mathfrak{g}}] \;\Rightarrow\; \mathcal{S}_{\mathfrak{g}} \cap \left(\mathcal{U}_{\mathfrak{g},\xi} \setminus \{\xi\}\right) \neq \emptyset. \tag{3.8}$$

The set of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation points, denoted by $\operatorname{der}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$, is called the "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-derived set of $\mathcal{S}_{\mathfrak{g}}$."

Making use of the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point, we further introduce another concept called *countable* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compactness*, possessed by all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets.

DEFINITION 3.31. A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is said to be "countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact" if and only if every infinite $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathcal{R}_{\mathfrak{g}} \subset \mathcal{S}_{\mathfrak{g}}$ of $\mathcal{S}_{\mathfrak{g}}$ has at least one $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point $\xi \in \mathcal{S}_{\mathfrak{g}}$.

The statement relating the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness and countable $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness is contained in the following theorem.

THEOREM 3.32. *If $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}\,[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, then it is also countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g}}$.*

PROOF. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}\,[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ and, suppose $\mathcal{R}_{\mathfrak{g}} \subset \mathcal{S}_{\mathfrak{g}}$ be any infinite $\mathfrak{T}_{\mathfrak{g}}$-subset of $\mathcal{S}_{\mathfrak{g}}$. Equivalently proved, it must be shown that, the assumption that $\mathcal{R}_{\mathfrak{g}}$ has no $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point $\xi \in \mathcal{S}_{\mathfrak{g}}$ leads to a contradiction. Since $\mathcal{R}_{\mathfrak{g}} \subset \mathcal{S}_{\mathfrak{g}}$ is, by assumption, an infinite $\mathfrak{T}_{\mathfrak{g}}$-subset of $\mathcal{S}_{\mathfrak{g}}$ with no $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point $\xi \in \mathcal{S}_{\mathfrak{g}}$, it follows, then, that, for every $\xi \in \mathcal{S}_{\mathfrak{g}}$, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}\,[\mathfrak{T}_{\mathfrak{g}}]$ which contains at most one point $\zeta \in \mathcal{R}_{\mathfrak{g}}$. It may be remarked, in passing, that $\langle \mathcal{U}_{\mathfrak{g},\xi} \rangle_{\xi \in \mathcal{S}_{\mathfrak{g}}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}\,[\mathfrak{T}_{\mathfrak{g}}]$ for, $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{\xi \in \mathcal{S}_{\mathfrak{g}}} \mathcal{U}_{\mathfrak{g},\xi}$. Consequently, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\xi)} \rangle_{(\xi,\vartheta(\xi)) \in \mathcal{S}_{\mathfrak{g}} \times \hat{\mathcal{S}}_{\mathfrak{g}}} \prec \langle \mathcal{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$, where $\hat{\mathcal{S}}_{\mathfrak{g}} \subset \mathcal{S}_{\mathfrak{g}}$, such $\mathcal{R}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\xi,\vartheta(\xi)) \in \mathcal{S}_{\mathfrak{g}} \times \hat{\mathcal{S}}_{\mathfrak{g}}} \mathcal{U}_{\mathfrak{g},\vartheta(\xi)}$. But, for every $(\xi, \vartheta(\xi)) \in \mathcal{S}_{\mathfrak{g}} \times \hat{\mathcal{S}}_{\mathfrak{g}}$, $\mathcal{U}_{\mathfrak{g},\vartheta(\xi)}$ contains at most one point $\zeta \in \mathcal{R}_{\mathfrak{g}}$. Therefore, the infinite $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathcal{R}_{\mathfrak{g}}$ of $\mathcal{S}_{\mathfrak{g}}$, satisfying $\mathcal{R}_{\mathfrak{g}} \subseteq \bigcup_{(\xi,\vartheta(\xi)) \in \mathcal{S}_{\mathfrak{g}} \times \hat{\mathcal{S}}_{\mathfrak{g}}} \mathcal{U}_{\mathfrak{g},\vartheta(\xi)}$, can contain at most $\eta = \operatorname{card}(\hat{\mathcal{S}}_{\mathfrak{g}}) < \infty$ points. Accordingly, it follows that every infinite $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathcal{R}_{\mathfrak{g}} \subset \mathcal{S}_{\mathfrak{g}}$ of $\mathcal{S}_{\mathfrak{g}}$ contains a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point $\xi \in \mathcal{S}_{\mathfrak{g}}$. Hence, $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}\,[\mathfrak{T}_{\mathfrak{g}}]$ is also countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g}}$. This completes the proof of the theorem. Q.E.D.

An immediate consequence of the above theorem is the following corollary.

COROLLARY 3.33. *Every $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ having the property that every countable $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}\,[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathfrak{T}_{\mathfrak{g}}$ contains a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\sigma}^{*} \times I_{\vartheta(\sigma)}^{*}} \prec \langle \mathcal{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_{\sigma}^{*}}$ of $\mathfrak{T}_{\mathfrak{g}}$ is a countably $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-space $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = \left(\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}\right)$.*

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DEFINITION 3.34. A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is "sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact" if and only if every sequence $\langle \xi_{\alpha} \in \mathcal{S}_{\mathfrak{g}} \rangle_{\alpha \in I_{\infty}^{*}}$ in $\mathcal{S}_{\mathfrak{g}}$ contains a subsequence $\langle \xi_{\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\infty}^{*} \times I_{\infty}^{*}} \prec \langle \xi_{\alpha} \rangle_{\alpha \in I_{\infty}^{*}}$ which converges to a point $\xi \in \mathcal{S}_{\mathfrak{g}}$

THEOREM 3.35. *Let* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *be a* $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-continuous map, where* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ *and* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ *are* $\mathcal{T}_{\mathfrak{g}}$*-spaces. If* $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ *be a sequentially* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Omega}$*, then* $\operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ *is also a sequentially* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

PROOF. Let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, where $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ are $\mathcal{T}_{\mathfrak{g}}$-spaces, and suppose $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be a sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set in $\mathfrak{T}_{\mathfrak{g},\Omega}$. If $\langle \zeta_{\alpha} \in \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\omega}) \rangle_{\alpha \in I_{\infty}^{*}}$ be a sequence in $\operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$, then there exists a sequence $\langle \xi_{\alpha} \in \mathcal{S}_{\mathfrak{g}} \rangle_{\alpha \in I_{\infty}^{*}}$ in $\mathcal{S}_{\mathfrak{g}}$ such $\pi_{\mathfrak{g}}(\xi_{\alpha}) = \zeta_{\alpha}$ that for every $\alpha \in I_{\infty}^{*}$. But, by hypothesis, $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ is sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Therefore, there exists a subsequence $\langle \xi_{\alpha} \rangle_{\alpha \in I_{\infty}^{*}} \prec \langle \xi_{\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\infty}^{*} \times I_{\infty}^{*}}$ which converges to a point $\xi \in \mathcal{S}_{\mathfrak{g}}$. On the other hand, $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ and, therefore, $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ is sequentially $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous. Consequently, $\langle \pi_{\mathfrak{g}}(\xi_{\vartheta(\alpha)}) \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\infty}^{*} \times I_{\infty}^{*}} = \langle \zeta_{\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\infty}^{*} \times I_{\infty}^{*}}$ converges to $\pi_{\mathfrak{g}}(\xi) \in \operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}})$. Hence, it follows that $\operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ is sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. Q.E.D.

PROPOSITION 3.36. *Let* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *be a* $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-continuous map, where* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ *and* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ *are* $\mathcal{T}_{\mathfrak{g}}$*-spaces. If* $\mathcal{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ *be a* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Omega}$*, then* $\operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) \in \text{A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ *is also* $\mathfrak{T}_{\mathfrak{g}}$*-compact in* $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

PROOF. Let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, where $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ are $\mathcal{T}_{\mathfrak{g}}$-spaces, and suppose $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \text{O}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \rangle_{\alpha \in I_{\eta}^{*}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathcal{S}_{\mathfrak{g},\sigma} = \operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$. Then, since the relation $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ holds, it results that $\langle \pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\alpha}) \rangle_{\alpha \in I_{\eta}^{*}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\mathcal{S}_{\mathfrak{g},\omega} = \pi_{\mathfrak{g}}^{-1}(\mathcal{S}_{\mathfrak{g},\sigma})$, because $\text{O}[\mathfrak{T}_{\mathfrak{g},\Omega}] \subseteq \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]$. Since $\mathcal{S}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Omega}]$, a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}) \rangle_{(\alpha,\vartheta(\alpha)) \in I_{\eta}^{*} \times I_{\vartheta(\eta)}^{*}} \prec \langle \pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\alpha}) \rangle_{\alpha \in I_{\eta}^{*}}$ exists, and such that, $\mathcal{S}_{\mathfrak{g},\omega} \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\eta}^{*} \times I_{\vartheta(\eta)}^{*}} \pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)})$. Since $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, it follows, consequently, that $\pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\omega}) \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I_{\eta}^{*} \times I_{\vartheta(\eta)}^{*}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}$. Therefore, $\langle \pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\alpha}) \in \text{O}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \rangle_{\alpha \in I_{\eta}^{*}}$ is a finite $\mathfrak{T}_{\mathfrak{g}}$-open subcovering of $\mathcal{S}_{\mathfrak{g},\sigma} \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$. Hence, $\operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) \in \text{A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ is also $\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. The proof of the proposition is complete. Q.E.D.

THEOREM 3.37. *Let* $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ *be a* $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$*-continuous map, where* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ *and* $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ *are* $\mathcal{T}_{\mathfrak{g}}$*-spaces. If* $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ *be a countably* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Omega}$*, then* $\operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ *is also a countably* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-compact set in* $\mathfrak{T}_{\mathfrak{g},\Sigma}$.

PROOF. Let $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$, where $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ are $\mathcal{T}_{\mathfrak{g}}$-spaces, and suppose $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ be a countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set in $\mathfrak{T}_{\mathfrak{g},\Omega}$. To prove that $\operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ is countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Sigma}$, let $\mathcal{S}_{\mathfrak{g},\sigma} \subseteq \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\omega})$ be an infinite $\mathfrak{T}_{\mathfrak{g}}$-subset of $\operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}})$. Then, a denumerable $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathcal{R}_{\mathfrak{g},\sigma} = \{\zeta_{\alpha} : \alpha \in I_{\infty}^{*}\} \subset \mathcal{S}_{\mathfrak{g},\sigma}$ exists. Since $\mathcal{R}_{\mathfrak{g},\sigma} \subset \mathcal{S}_{\mathfrak{g},\sigma} \subseteq \operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) = \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\omega})$,

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there exists a denumerable $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathcal{R}_{\mathfrak{g},\omega} = \{\xi_\alpha : \alpha \in I_\infty^*\} \subset \mathcal{S}_{\mathfrak{g},\omega}$, with $\pi_{\mathfrak{g}}(\xi_\alpha) = \zeta_\alpha$ for every $\alpha \in I_\infty^*$. But, by hypothesis, $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$ is countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Omega}$, so $\mathcal{R}_{\mathfrak{g},\omega}$ contains a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point $\xi \in \mathcal{S}_{\mathfrak{g},\omega}$. Thus, $\xi \in \mathcal{R}_{\mathfrak{g},\omega} \cup \mathrm{der}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\omega}) \subseteq \mathcal{R}_{\mathfrak{g},\omega}$ and $\pi_{\mathfrak{g}}(\xi) \in \mathrm{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) = \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\omega})$; evidently, $\mathrm{der}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\omega}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ and, therefore, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed set $\mathcal{V}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ exists such that, $\mathrm{der}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\omega}) = \mathcal{V}_{\mathfrak{g},\omega}$. But, by hypothesis, $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$. Consequently, $\pi_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\omega} \cup \mathrm{der}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\omega})) \subseteq \pi_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\omega}) \cup \mathrm{der}_{\mathfrak{g}}(\pi_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\omega})) = \mathcal{R}_{\mathfrak{g},\sigma} \cup \mathrm{der}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\sigma})$. But, $\xi \in \mathcal{R}_{\mathfrak{g},\omega} \cup \mathrm{der}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\omega})$ and, therefore, $\pi_{\mathfrak{g}}(\xi) \in \mathcal{R}_{\mathfrak{g},\sigma} \cup \mathrm{der}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\sigma})$. Now, $\pi_{\mathfrak{g}}(\xi) \in \mathcal{R}_{\mathfrak{g},\sigma} \cup \mathrm{der}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\sigma})$, so let it be claimed that $\pi_{\mathfrak{g}}(\xi)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\mathcal{R}_{\mathfrak{g},\sigma}$. There are, then, two cases, namely, $\xi \notin \mathcal{R}_{\mathfrak{g},\omega}$ and $\xi \in \mathcal{R}_{\mathfrak{g},\omega}$.

I. *Case* $\xi \notin \mathcal{R}_{\mathfrak{g},\omega}$. If $\xi \notin \mathcal{R}_{\mathfrak{g},\omega}$, then $\pi_{\mathfrak{g}}(\xi) \notin (\mathcal{R}_{\mathfrak{g},\omega}) = \mathcal{R}_{\mathfrak{g},\sigma}$. But, $\pi_{\mathfrak{g}}(\xi) \in \mathcal{R}_{\mathfrak{g},\sigma} \cup \mathrm{der}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\sigma})$ and, consequently, $\pi_{\mathfrak{g}}(\xi)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\mathcal{R}_{\mathfrak{g},\sigma}$.

II. *Case* $\xi \in \mathcal{R}_{\mathfrak{g},\omega}$. If $\xi \in \mathcal{R}_{\mathfrak{g},\omega}$, choose a $\mu \in I_\infty^*$ such that $\xi = \xi_\mu$. Then, $\xi \notin \hat{\mathcal{R}}_{\mathfrak{g},\omega} = \{\xi_\alpha : \alpha \in I_\infty^* \setminus \{\mu\}\}$ and, every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ containing $\xi$ contains at least a point $\hat{\xi} \in \hat{\mathcal{R}}_{\mathfrak{g},\omega} = \{\xi_\alpha : \alpha \in I_\infty^* \setminus \{\mu\}\}$ and, therefore, $\xi$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\hat{\mathcal{R}}_{\mathfrak{g},\omega}$. But, $\pi_{\mathfrak{g}}(\hat{\mathcal{R}}_{\mathfrak{g},\omega}) = \{\zeta_\alpha : \alpha \in I_\infty^* \setminus \{\mu\}\}$ since, by hypothesis, $\pi_{\mathfrak{g}}(\xi_\alpha) = \zeta_\alpha$ for every $\alpha \in I_\infty^*$. Thus, $\pi_{\mathfrak{g}}(\xi)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\pi_{\mathfrak{g}}(\hat{\mathcal{R}}_{\mathfrak{g},\omega})$ where, $\pi_{\mathfrak{g}}(\hat{\mathcal{R}}_{\mathfrak{g},\omega}) \subseteq \mathcal{R}_{\mathfrak{g},\sigma}$. Moreover, since $\pi_{\mathfrak{g}}(\hat{\mathcal{R}}_{\mathfrak{g},\omega} \cup \mathrm{der}_{\mathfrak{g}}(\hat{\mathcal{R}}_{\mathfrak{g},\omega})) \subseteq \pi_{\mathfrak{g}}(\hat{\mathcal{R}}_{\mathfrak{g},\omega}) \cup \mathrm{der}_{\mathfrak{g}}(\pi_{\mathfrak{g}}(\hat{\mathcal{R}}_{\mathfrak{g},\omega})) = \hat{\mathcal{R}}_{\mathfrak{g},\sigma} \cup \mathrm{der}_{\mathfrak{g}}(\hat{\mathcal{R}}_{\mathfrak{g},\sigma})$, it follows that, $\pi_{\mathfrak{g}}(\xi)$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\hat{\mathcal{R}}_{\mathfrak{g},\sigma}$. Since $\mathcal{R}_{\mathfrak{g},\sigma} \subset \mathcal{S}_{\mathfrak{g},\sigma} \subseteq \mathrm{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) = \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\omega})$, $\pi_{\mathfrak{g}}(\xi)$ is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point of $\mathcal{S}_{\mathfrak{g},\sigma}$ and, $\pi_{\mathfrak{g}}(\xi) \in \mathrm{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) = \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\omega})$. Therefore, every infinite $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathcal{S}_{\mathfrak{g},\sigma} \subseteq \mathrm{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}})$ of $\pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\omega})$ contains a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point in $\pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\omega})$ and, hence, $\mathrm{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}}) \subset \mathfrak{T}_{\mathfrak{g},\Sigma}$ is also a countably $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. The proof of the theorem is complete. Q.E.D.

PROPOSITION 3.38. *If $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, then every countable $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_\sigma^*}$ of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set $\mathcal{S}_{\mathfrak{g}}$ is reducible to a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of the type $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_\sigma^* \times I_{\vartheta(\sigma)}^*} \prec \langle \mathcal{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_\sigma^*}$ of $\mathcal{S}_{\mathfrak{g}}$.*

PROOF. Let it be assumed that $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is a sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact infinite set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$. Furthermore, assume that there exists a countable $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I_\sigma^*}$ of $\mathcal{S}_{\mathfrak{g}}$ with no finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_\sigma^* \times I_{\vartheta(\sigma)}^*} \prec \langle \mathcal{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I_\sigma^*}$ of $\mathcal{S}_{\mathfrak{g}}$. Finally, introduce the sequence $\langle \xi_\alpha \in \mathcal{S}_{\mathfrak{g}} \rangle_{\alpha \in I_\infty^*}$ and define its elements in the following manner. Let $\vartheta(1) \in I_{\vartheta(\sigma)}^* \subset I_\sigma^*$ be the smallest integer in $I_{\vartheta(\sigma)}^*$ such that $\mathcal{S}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\vartheta(1)} \neq \emptyset$; choose $\xi_1 \in \mathcal{S}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\vartheta(1)}$. Let $\vartheta(2) \in I_{\vartheta(\sigma)}^* \subset I_\sigma^*$ be the least integer larger than $\vartheta(1)$ in $I_{\vartheta(\sigma)}^*$ such that $\mathcal{S}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\vartheta(2)} \neq \emptyset$; choose $\xi_2 \in (\mathcal{S}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\vartheta(2)}) \setminus (\mathcal{S}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\vartheta(1)})$. Note that, such a point $\xi_2$ always exists, for otherwise $\mathcal{U}_{\mathfrak{g},\vartheta(1)}$ covers $\mathcal{S}_{\mathfrak{g}}$. Continuing in this way, the properties of $\langle \xi_\alpha \rangle_{\alpha \in I_\infty^*}$, for every $\alpha \in I_\infty^* \setminus \{1\}$, are

$$\xi_\alpha \in \mathcal{S}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}, \quad \xi_\alpha \notin \bigcup_{\nu \in I_{\alpha-1}^*} (\mathcal{S}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\vartheta(\nu)}), \quad \vartheta(\alpha) > \vartheta(\alpha - 1).$$

Let it be claimed that the sequence $\langle \xi_\alpha \rangle_{\alpha \in I_\infty^*}$ has no convergent subsequence of the type $\langle \xi_{\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I_\infty^* \times I_\infty^*} \prec \langle \xi_\alpha \rangle_{\alpha \in I_\infty^*}$ in $\mathcal{S}_{\mathfrak{g}}$. Suppose $\xi \in \mathcal{S}_{\mathfrak{g}}$, then there exists a $\mu \in I_{\vartheta(\sigma)}^*$ such that $\xi \in \mathcal{U}_{\mathfrak{g},\vartheta(\mu)}$. Now, $\mathcal{S}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\vartheta(\mu)} \neq \emptyset$ since, $\xi \in \mathcal{S}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\vartheta(\mu)}$.

Thus, there exists $\nu \in I^{*}_{\vartheta(\sigma)}$ such that, $\mathcal{U}_{\mathfrak{g},\vartheta(\nu)} = \mathcal{U}_{\mathfrak{g},\vartheta(\mu)}$. But, by the properties of the sequence $\langle \xi_{\alpha} \rangle_{\alpha \in I^{*}_{\infty}}$, $\alpha > \vartheta(\nu)$ implies $\xi_{\alpha} \notin \mathcal{U}_{\mathfrak{g},\vartheta(\mu)}$. Accordingly, since $\xi \in \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ no subsequence $\langle \xi_{\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I^{*}_{\infty} \times I^{*}_{\infty}} \prec \langle \xi_{\alpha} \rangle_{\alpha \in I^{*}_{\infty}}$ of $\langle \xi_{\alpha} \rangle_{\alpha \in I^{*}_{\infty}}$ converges to $\xi \in \mathcal{S}_{\mathfrak{g}}$. But, $\xi$ was arbitrary and, hence, $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is not sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g}}$. The proof of the proposition is complete. Q.E.D.

Definition 3.39 ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Neighborhood). Let $\xi \in \mathfrak{T}_{\mathfrak{g}}$ be a point in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$. A $\mathfrak{T}_{\mathfrak{g}}$-subset $\mathcal{N}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ of $\mathfrak{T}_{\mathfrak{g}}$ is a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhood of $\xi$" if and only if $\mathcal{N}_{\mathfrak{g}}$ is a $\mathfrak{T}_{\mathfrak{g}}$-superset of a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ containing $\xi$:

$$(3.9) \qquad (\xi, \mathcal{N}_{\mathfrak{g}}, \mathcal{U}_{\mathfrak{g},\xi}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] : \quad \xi \in \mathcal{U}_{\mathfrak{g},\xi} \subseteq \mathcal{N}_{\mathfrak{g}}.$$

The class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhoods of $\xi \in \mathfrak{T}_{\mathfrak{g}}$, defined as

$$(3.10) \qquad \mathfrak{g}\text{-N}[\xi] \stackrel{\text{def}}{=} \{\mathcal{N}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : (\exists \mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]) [\xi \in \mathcal{U}_{\mathfrak{g},\xi} \subseteq \mathcal{N}_{\mathfrak{g}}]\},$$

is called the "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhood system of $\xi$."

There exist $\mathcal{T}_{\mathfrak{g}}$-spaces which are not $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact, but have instead a *local* version of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness, and such local $\mathcal{T}_{\mathfrak{g}}$-property, called *local* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*compactness*, is formalized in the following definition.

Definition 3.40 (Locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Compact). A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ is said to be "locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact" if and only if, given any $(\xi, \mathcal{N}_{\mathfrak{g},\xi}) \in \mathcal{S}_{\mathfrak{g}} \times \mathfrak{g}\text{-N}[\xi]$, there is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhood $\hat{\mathcal{N}}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-N}[\xi]$ of $\xi$ such that $\hat{\mathcal{N}}_{\mathfrak{g},\xi} \subset \mathcal{N}_{\mathfrak{g},\xi}$ and $\hat{\mathcal{N}}_{\mathfrak{g},\xi} \cup \operatorname{der}_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}) \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$.

The localisation of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness is the requirement that *small* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*open sets* have the desired $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness property even though the $\mathcal{T}_{\mathfrak{g}}$-space as a whole may not. The following theorem shows that we are dealing with a valid generalization of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness.

Theorem 3.41. *If $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, then it is also locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g}}$.*

Proof. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$. Since $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$, for every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle \mathcal{U}_{\mathfrak{g},\alpha} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \rangle_{\alpha \in I^{*}_{\sigma}}$, there exists a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \rangle_{(\alpha,\vartheta(\alpha)) \in I^{*}_{\sigma} \times I^{*}_{\vartheta(\sigma)}} \prec \langle \mathcal{U}_{\mathfrak{g},\alpha} \rangle_{\alpha \in I^{*}_{\sigma}}$ such that $\mathcal{S}_{\mathfrak{g}} \subseteq \bigcup_{(\alpha,\vartheta(\alpha)) \in I^{*}_{\sigma} \times I^{*}_{\vartheta(\sigma)}} \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}$. It is clear that, for every $\xi \in \mathcal{S}_{\mathfrak{g}}$, there exists $\mathcal{U}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ such that $\mathcal{S}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\xi} = \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathcal{U}_{\mathfrak{g},\xi}$ for some $(\alpha, \vartheta(\alpha)) \in I^{*}_{\sigma} \times I^{*}_{\vartheta(\sigma)}$. For every $(\alpha, \xi, \vartheta(\alpha), \upsilon(\alpha,\xi)) \in I^{*}_{\sigma} \times \mathcal{S}_{\mathfrak{g}} \times I^{*}_{\vartheta(\sigma)} \times I^{*}_{\upsilon(\sigma)}$, set $\mathcal{U}_{\mathfrak{g},\upsilon(\alpha,\xi)} = \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathcal{U}_{\mathfrak{g},\xi}$. Then, since $(\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}, \mathcal{U}_{\mathfrak{g},\xi}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\alpha, \xi, \vartheta(\alpha)) \in I^{*}_{\sigma} \times \mathcal{S}_{\mathfrak{g}} \times I^{*}_{\vartheta(\sigma)}$, there exists, for every $(\alpha, \xi, \vartheta(\alpha)) \in I^{*}_{\sigma} \times \mathcal{S}_{\mathfrak{g}} \times I^{*}_{\vartheta(\sigma)}$, a pair $(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}, \mathcal{O}_{\mathfrak{g},\xi}) \in \mathcal{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}}$ of $\mathcal{T}_{\mathfrak{g}}$-open sets such that, $(\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}, \mathcal{U}_{\mathfrak{g},\xi}) \subseteq (\operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}), \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi}))$. Consequently,

$$\begin{aligned} \mathcal{U}_{\mathfrak{g},\upsilon(\alpha,\xi)} = \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathcal{U}_{\mathfrak{g},\xi} &\subseteq \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}) \cap \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi}) \\ &\subseteq \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathcal{O}_{\mathfrak{g},\xi}) = \operatorname{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\upsilon(\alpha,\xi)}), \end{aligned}$$

where $\mathcal{U}_{\mathfrak{g},\upsilon(\alpha,\xi)} = \mathcal{U}_{\mathfrak{g},\vartheta(\alpha)} \cap \mathcal{U}_{\mathfrak{g},\xi}$ for every $(\alpha, \xi, \vartheta(\alpha), \upsilon(\alpha,\xi)) \in I^{*}_{\sigma} \times \mathcal{S}_{\mathfrak{g}} \times I^{*}_{\vartheta(\sigma)} \times I^{*}_{\upsilon(\sigma)}$. Therefore, $\mathcal{U}_{\mathfrak{g},\upsilon(\alpha,\xi)} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $(\alpha, \xi, \vartheta(\alpha), \upsilon(\alpha,\xi)) \in I^{*}_{\sigma} \times \mathcal{S}_{\mathfrak{g}} \times I^{*}_{\vartheta(\sigma)} \times$

$I^{*}_{\upsilon(\sigma)}$. But since $\xi\in\mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\xi)}\subseteq\mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\xi)}\cup\operatorname{der}_{\mathfrak{g}}(\mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\xi)})$ and $\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\supset\mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\xi)}\cup\operatorname{der}_{\mathfrak{g}}(\mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\xi)})\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$, it results that,

$$\xi\in\mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\xi)}\subseteq\mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\xi)}\cup\operatorname{der}_{\mathfrak{g}}(\mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\xi)})\subset\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}.$$

Thus, given any $(\xi,\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)})\in\mathcal{S}_{\mathfrak{g}}\times\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$, there is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open neighborhood $\mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\xi)}\in\mathfrak{g}\text{-N}[\xi]$ of $\xi$ such that $\mathcal{U}_{\mathfrak{g},\vartheta(\alpha,\xi)}\subset\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}$ and $\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\cup\operatorname{der}_{\mathfrak{g}}(\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)})\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$. Hence, $\mathcal{S}_{\mathfrak{g}}\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g}}]$ implies that it is also locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g}}$. The proof of the theorem is complete. Q.E.D.

By virtue of the above theorem, it follows that a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set of a $\mathcal{T}_{\mathfrak{g}}$-space has a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering necessarily. This is embodied in the following corollary.

Corollary 3.42. *Every $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathcal{T}_{\mathfrak{g}})$ having the property that every local $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering $\langle\mathcal{U}_{\mathfrak{g},\alpha}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\rangle_{\alpha\in I^{*}_{\sigma}}$ of $\mathfrak{T}_{\mathfrak{g}}$ contains a finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle\mathcal{U}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in I^{*}_{\sigma}\times I^{*}_{\vartheta(\sigma)}}\prec\langle\mathcal{U}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I^{*}_{\sigma}}$ of $\mathfrak{T}_{\mathfrak{g}}$ is a locally $\mathfrak{g}$-$\mathcal{T}^{[\mathrm{A}]}_{\mathfrak{g}}$-space $\mathfrak{g}$-$\mathfrak{T}^{[\mathrm{A}]}_{\mathfrak{g}}=(\Omega,\mathfrak{g}\text{-}\mathcal{T}^{[\mathrm{A}]}_{\mathfrak{g}})$.*

For $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous maps, the following theorem, which shows that local $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness is a $\mathcal{T}_{\mathfrak{g}}$-invariant, presents itself.

Theorem 3.43. *Let $\pi_{\mathfrak{g}}:\mathfrak{T}_{\mathfrak{g},\Omega}\to\mathfrak{T}_{\mathfrak{g},\Sigma}$ be a $\mathfrak{g}$-$(\mathfrak{T}_{\mathfrak{g},\Omega},\mathfrak{T}_{\mathfrak{g},\Sigma})$-continuous map, where $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathcal{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathcal{T}_{\mathfrak{g},\Omega})$ are $\mathcal{T}_{\mathfrak{g}}$-spaces. If $\mathcal{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be a locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set in $\mathfrak{T}_{\mathfrak{g},\Omega}$, then $\operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}})\subset\mathfrak{T}_{\mathfrak{g},\Sigma}$ is also a locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact set in $\mathfrak{T}_{\mathfrak{g},\Sigma}$.*

Proof. Let $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$, where $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathcal{T}_{\mathfrak{g},\Omega})$ and $\mathfrak{T}_{\mathfrak{g},\Omega}=(\Omega,\mathcal{T}_{\mathfrak{g},\Omega})$ are $\mathcal{T}_{\mathfrak{g}}$-spaces, and suppose $\mathcal{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ be locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Omega}$. Since $\mathcal{S}_{\mathfrak{g},\omega}$ is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact, for any given $(\xi,\mathcal{N}_{\mathfrak{g},\xi})\in\mathcal{S}_{\mathfrak{g},\omega}\times\mathfrak{g}\text{-N}[\xi]$, there is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhood $\hat{\mathcal{N}}_{\mathfrak{g},\xi}\in\mathfrak{g}\text{-N}[\xi]$ of $\xi$ such that $\hat{\mathcal{N}}_{\mathfrak{g},\xi}\subset\mathcal{N}_{\mathfrak{g},\xi}$ and $\hat{\mathcal{N}}_{\mathfrak{g},\xi}\cup\operatorname{der}_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi})\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Omega}]$. Consequently, $\xi\in\hat{\mathcal{N}}_{\mathfrak{g},\xi}\subseteq\hat{\mathcal{N}}_{\mathfrak{g},\xi}\cup\operatorname{der}_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi})\subset\mathcal{N}_{\mathfrak{g},\xi}$ and thus, $\pi_{\mathfrak{g}}(\xi)\in\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi})\subseteq\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}\cup\operatorname{der}_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}))\subset\pi_{\mathfrak{g}}(\mathcal{N}_{\mathfrak{g},\xi})$. But, $\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}\cup\operatorname{der}_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}))\subseteq\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi})\cup\pi_{\mathfrak{g}}(\operatorname{der}_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}))$ because, by hypothesis, $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$. Therefore,

$$\begin{aligned}\pi_{\mathfrak{g}}(\xi)\in\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}) &\subseteq \pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}\cup\operatorname{der}_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}))\\ &\subseteq \pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi})\cup\operatorname{der}_{\mathfrak{g}}(\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}))\subset\pi_{\mathfrak{g}}(\mathcal{N}_{\mathfrak{g},\xi}).\end{aligned}$$

Since $\hat{\mathcal{N}}_{\mathfrak{g},\xi}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhood in $\mathfrak{T}_{\mathfrak{g},\Omega}$ containing $\xi\in\mathcal{S}_{\mathfrak{g},\omega}\subset\mathfrak{T}_{\mathfrak{g},\Omega}$, $\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi})\subset\mathfrak{T}_{\mathfrak{g},\Sigma}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhood in $\mathfrak{T}_{\mathfrak{g},\Sigma}$ containing $\pi_{\mathfrak{g}}(\xi)\in\pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\omega})\in\mathfrak{T}_{\mathfrak{g},\Sigma}$. Now $\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi})\cup\operatorname{der}_{\mathfrak{g}}(\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}))\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ by virtue of the statements $\hat{\mathcal{N}}_{\mathfrak{g},\xi}\cup\operatorname{der}_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi})\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ and $\pi_{\mathfrak{g}}\in\mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega};\mathfrak{T}_{\mathfrak{g},\Sigma}]$. In other words, for any given $(\pi_{\mathfrak{g}}(\xi),\pi_{\mathfrak{g}}(\mathcal{N}_{\mathfrak{g},\zeta}))=(\zeta,\mathcal{N}_{\mathfrak{g},\zeta})\in\mathcal{S}_{\mathfrak{g},\sigma}\times\mathfrak{g}\text{-N}[\zeta]$, there is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-neighborhood $\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi})=\hat{\mathcal{N}}_{\mathfrak{g},\zeta}\in\mathfrak{g}\text{-N}[\zeta]$ of $\pi_{\mathfrak{g}}(\xi)=\zeta$ such that $\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi})=\hat{\mathcal{N}}_{\mathfrak{g},\zeta}\subseteq\mathcal{N}_{\mathfrak{g},\zeta}=\pi_{\mathfrak{g}}(\mathcal{N}_{\mathfrak{g},\xi})$ and $\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi})\cup\operatorname{der}_{\mathfrak{g}}(\pi_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\xi}))=\hat{\mathcal{N}}_{\mathfrak{g},\zeta}\cup\operatorname{der}_{\mathfrak{g}}(\hat{\mathcal{N}}_{\mathfrak{g},\zeta})\in\mathfrak{g}\text{-A}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$. Therefore, $\mathcal{S}_{\mathfrak{g},\sigma}\subset\mathfrak{T}_{\mathfrak{g},\Sigma}$ is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. But, $\mathcal{S}_{\mathfrak{g},\sigma}=\pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\omega})=\operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}})$. Hence, $\operatorname{im}(\pi_{\mathfrak{g}|\mathcal{S}_{\mathfrak{g},\omega}})\subset\mathfrak{T}_{\mathfrak{g},\Sigma}$ is locally $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact in $\mathfrak{T}_{\mathfrak{g},\Sigma}$. The proof of the theorem is complete. Q.E.D.

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The categorical classifications of $\mathfrak{T}$-compactness and $\mathfrak{g}$-$\mathfrak{T}$-compactness in the $\mathcal{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$ and, $\mathfrak{T}_{\mathfrak{g}}$-compactness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are discussed and diagrammed on this basis in the next sections.

## 4. Discussion

4.1. Categorical Classifications. Having adopted a categorical approach in the classifications of the $\mathcal{T}_{\mathfrak{g}}$-property called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the dual purposes of the this section are firstly, to establish the various relationships amongst the elements of the sequences $\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{E}]})\rangle_{\nu\in I_3^0}$ and $\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}^{[\mathrm{E}]} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}^{[\mathrm{E}]})\rangle_{\nu\in I_3^0}$ of $\mathfrak{g}$-$\nu$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{E}]}$-spaces and $\mathfrak{g}$-$\nu$-$\mathcal{T}^{[\mathrm{E}]}$-spaces, respectively, where $\mathrm{E} \in \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$, and secondly, to illustrate them through diagrams.

Let $\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}$ be any $\mathcal{T}_{\mathfrak{g}}$-open set in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ and, for every $\nu \in I_3^0$, let there exist a $\mu \in I_3^0$ such that the relation $\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},\mu}(\mathcal{O}_{\mathfrak{g}})$ holds. Then, since $\mathcal{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}})$ for every $\nu \in I_3^0$, it follows that $\mathfrak{T}_{\mathfrak{g}}$-openness implies $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-openness and the latter, in turn, implies $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-openness. But since the statement that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies $\mathfrak{T}_{\mathfrak{g}}$-compactness is a consequence of the statement that $\mathfrak{T}_{\mathfrak{g}}$-openness implies $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness, it evidently follows that, $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-compactness and the latter, in turn, implies $\mathfrak{T}_{\mathfrak{g}}$-compactness. On the other hand, for every $\mathfrak{T}_{\mathfrak{g}}$-open set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the relation $\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ holds (see *Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Sets*). Consequently,

$$\mathrm{op}_{\mathfrak{g},0}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},1}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},3}(\mathcal{S}_{\mathfrak{g}}) \supseteq \mathrm{op}_{\mathfrak{g},2}(\mathcal{S}_{\mathfrak{g}}) \quad \forall \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}.$$

Therefore, from the statement that, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness of category 0 implies $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness of category 1, it results that, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness of category 1 implies $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness of category 0; from the statement that, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness of category 1 implies $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness of category 3, it results that, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness of category 3 implies $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness of category 1; from the statement that, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness of category 2 implies $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness of category 3, it results that, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness of category 3 implies $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness of category 1. Thus, if $\mathcal{U}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set then, with respect to $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-openness, the following system of implications holds:

$$\begin{array}{ccc} \mathcal{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-0-O}[\mathfrak{T}_{\mathfrak{g}}] & \Longrightarrow & \mathcal{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-1-O}[\mathfrak{T}_{\mathfrak{g}}] \\ & & \Downarrow \\ \mathcal{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-2-O}[\mathfrak{T}_{\mathfrak{g}}] & \Longrightarrow & \mathcal{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-3-O}[\mathfrak{T}_{\mathfrak{g}}]; \end{array}$$

Such system with respect to $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness, in turn, implies the following system of implications:

$$\begin{array}{ccc} \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-0-O}[\mathfrak{T}_{\mathfrak{g}}] & \Longleftarrow & \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-1-O}[\mathfrak{T}_{\mathfrak{g}}] \\ & & \Uparrow \\ \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-2-O}[\mathfrak{T}_{\mathfrak{g}}] & \Longleftarrow & \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-3-O}[\mathfrak{T}_{\mathfrak{g}}]. \end{array}$$

For visualization, a so-called *categorical compactness diagram*, expressing the various relationships amongst the classes of $\mathfrak{g}$-$\mathfrak{T}$-compact and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets, is presented in Fig. 1. According to the previous section, it is plain that, $\mathfrak{T}_{\mathfrak{g}}$-compactness in the ordinary sense implies both countable $\mathfrak{T}_{\mathfrak{g}}$-compactness in the

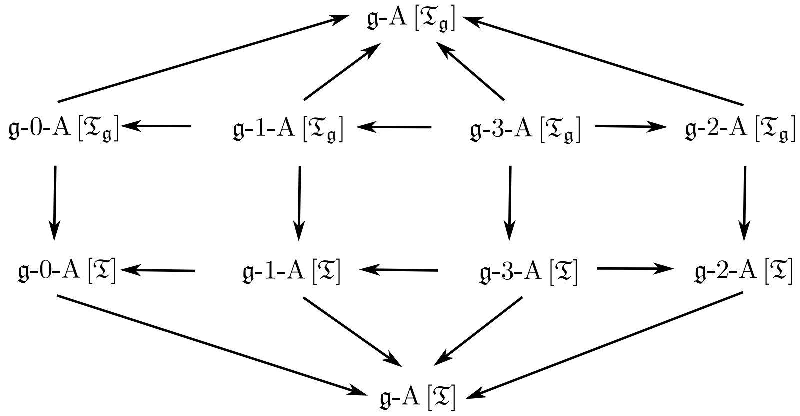


FIGURE 1. Relationships: classes of $\mathfrak{g}$-$\mathfrak{T}$-compact and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact sets.

ordinary sense and local countable $\mathfrak{T}_{\mathfrak{g}}$-compactness in the ordinary sense; sequential $\mathfrak{T}_{\mathfrak{g}}$-compactness in the ordinary sense implies countable $\mathfrak{T}_{\mathfrak{g}}$-compactness in the ordinary sense. Moreover, the following implications also hold: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\mathrm{LA}} \leftarrow \mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\mathrm{A}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\mathrm{CA}} \leftarrow \mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\mathrm{A}}$, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\mathrm{CA}} \leftarrow \mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{\mathrm{SA}}$. Since the relation $\mathfrak{T}^{[\mathrm{E}]} \leftarrow \mathfrak{g}$-$\mathfrak{T}^{[\mathrm{E}]}$ holds for every $\mathrm{E} \in \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$, taking this last statement together with those preceding it into account, another compactness diagram is obtained. The diagram presented in FIG. 2 illustrates the various relationships amongst the elements of $\langle \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} \rangle_{\mathrm{E}\in\Lambda}$ and $\langle \mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} \rangle_{\mathrm{E}\in\Lambda}$, where $\Lambda = \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$. It is interesting to present a third compactness diagram illustrating both the implications and the categorical classifications of the elements of $\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} \rangle_{\mathrm{E}\in\Lambda}$, where $\nu \in I_3^0$ and, obviously, $\Lambda = \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$.

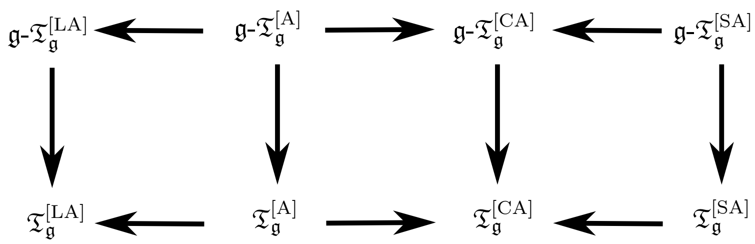


FIGURE 2. Relationships: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact spaces and $\mathfrak{T}_{\mathfrak{g}}$-compact spaces.

For each $\nu \in I_3^0$, it is immediate that these implications hold: $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{LA}]} \leftarrow \mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$, $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{CA}]} \leftarrow \mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]}$, and $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{CA}]} \leftarrow \mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{SA}]}$. Furthermore, for each $\mathrm{E} \in \Lambda = \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$, it is also plain that these implications hold: $\mathfrak{g}$-0-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} \leftarrow \mathfrak{g}$-1-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]}$, $\mathfrak{g}$-1-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} \leftarrow \mathfrak{g}$-3-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]}$, and $\mathfrak{g}$-2-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]} \leftarrow \mathfrak{g}$-3-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]}$. When all these implications are taken into consideration, the resulting compactness diagram so obtained is that presented in FIG. 3. It is reasonably correct to call them $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]}$*-spaces of type* E *and of category* $\nu$, where $(\nu, \mathrm{E}) \in I_3^0 \times \{\mathrm{A}, \mathrm{CA}, \mathrm{SA}, \mathrm{LA}\}$. As in the papers of [16], [18], and [34], among others, the manner we have positioned the arrows is solely to stress that, in general, none of the implications in FIGS 1, 2 and 3 is reversible.

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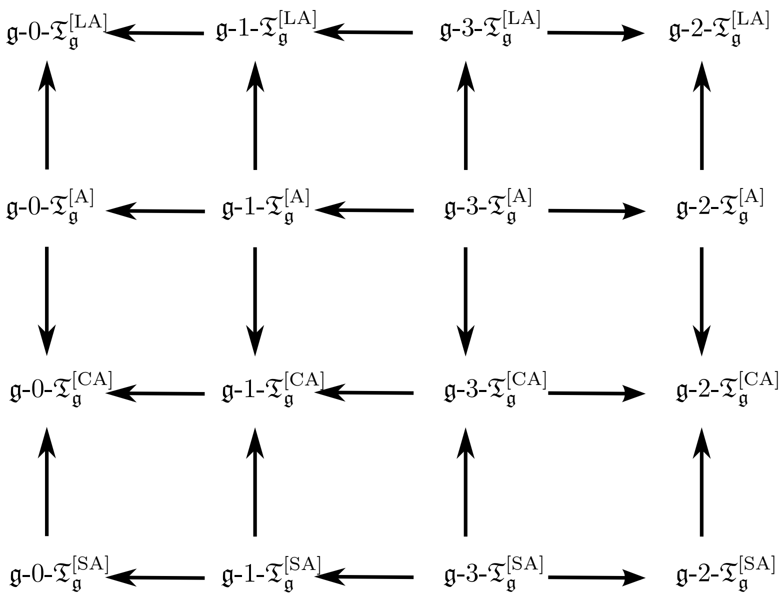


FIGURE 3. Relationships: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact spaces.

In order to exemplify the notion of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{E}]}$-spaces of type E and of category $\nu$, where $(\nu,\mathrm{E})\in I_3^0\times\{\mathrm{A},\mathrm{CA},\mathrm{SA},\mathrm{LA}\}$, a nice application is presented in the following section.

4.2. A NICE APPLICATION. Focusing on basic concepts from the standpoint of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness, we shall now present a nice application comprising of some interesting cases.

Let $\mathcal{T}_{\mathfrak{g}}:\mathcal{P}(\Omega)\longrightarrow\mathcal{P}(\Omega)$ be the $\mathfrak{g}$-topology on $\Omega=(0,1)\subset\mathbb{R}$ (set of real numbers) generated by $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets belonging to:

$$\mathcal{T}_{\mathfrak{g}} \stackrel{\text{def}}{=} \left\{\mathcal{O}_{\mathfrak{g},\mu}:\ (\forall\mu\in I_\infty^*\setminus I_2^*)\left(\left[\mathcal{O}_{\mathfrak{g},\mu}=\emptyset\right]\vee\left[\mathcal{O}_{\mathfrak{g},\mu}=\left(\frac{1}{\mu},1-\frac{1}{\mu}\right)\right]\right)\right\};$$

$$\neg\mathcal{T}_{\mathfrak{g}} \stackrel{\text{def}}{=} \left\{\mathcal{K}_{\mathfrak{g},\mu}:\ (\forall\mu\in I_\infty^*\setminus I_2^*)\left(\left[\mathcal{K}_{\mathfrak{g},\mu}=\Omega\right]\vee\left[\mathcal{K}_{\mathfrak{g},\mu}=\complement\left(\frac{1}{\mu},1-\frac{1}{\mu}\right)\right]\right)\right\},$$

respectively. Clearly, the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}:\mathcal{P}(\Omega)\rightarrow\mathcal{P}(\Omega)$ satisfies the relations $\mathcal{T}_{\mathfrak{g}}(\emptyset)=\emptyset$, $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\mu})\subseteq\left(\frac{1}{\mu},1-\frac{1}{\mu}\right)=\mathcal{O}_{\mathfrak{g},\mu}$ and, moreover, $\mathcal{T}_{\mathfrak{g}}\left(\bigcap_{\mu\in I_\sigma^*\setminus I_2^*}\mathcal{O}_{\mathfrak{g},\mu}\right)=\bigcap_{\mu\in I_\sigma^*\setminus I_2^*}\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\mu})$ and $\mathcal{T}_{\mathfrak{g}}\left(\bigcup_{\mu\in I_\infty^*\setminus I_2^*}\mathcal{O}_{\mathfrak{g},\mu}\right)=\bigcup_{\mu\in I_\infty^*\setminus I_2^*}\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\mu})$ are also satisfied, since $\bigcap_{\mu\in I_\sigma^*\setminus I_2^*}\mathcal{O}_{\mathfrak{g},\mu}=\mathcal{O}_{\mathfrak{g},3}\in\mathcal{T}_{\mathfrak{g}}$ and $\bigcup_{\mu\in I_\infty^*\setminus I_2^*}\mathcal{O}_{\mathfrak{g},\mu}=\Omega\in\mathcal{T}_{\mathfrak{g}}$, respectively. Thus, $\mathfrak{T}_{\mathfrak{g}}=(\mathcal{T}_{\mathfrak{g}},\Omega)$ is a $\mathcal{T}_{\mathfrak{g}}$-space and, since $\mathfrak{T}_{\mathfrak{g}}=(\mathcal{T}_{\mathfrak{g}},\Omega)=(\mathcal{T},\Omega)=\mathfrak{T}$, it is also a $\mathcal{T}$-space. Observe that $\langle\mathcal{O}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_\infty^*\setminus I_2^*}$ is a $\mathfrak{T}_{\mathfrak{g}}$-open covering of $\Omega$, since $\mathcal{O}_{\mathfrak{g},\alpha}\in\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha\in I_\infty^*\setminus I_2^*$ and moreover, it is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\Omega$, since $\mathcal{O}_{\mathfrak{g},\alpha}\subseteq\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\alpha})\in\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha\in I_\infty^*\setminus I_2^*$. On the other hand, for each $\sigma>3$, the relation $\frac{1}{\sigma}\in\bigcup_{\mu\in I_\sigma^*\setminus I_2^*}\mathcal{O}_{\mathfrak{g},\mu}=\left(\frac{1}{\sigma},1-\frac{1}{\sigma}\right)$. Hence, from every $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering $\langle\mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha))\in J_\infty^*\times J_{\vartheta(\infty)}^*}\prec\langle\mathcal{O}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I_\infty^*\setminus I_2^*}$, where $J_\infty^*=I_\infty^*\setminus I_2^*$ and

$J^*_{\vartheta(\infty)} = I^*_{\vartheta(\infty)} \setminus I^*_2$, the union $\bigcup_{(\alpha,\vartheta(\alpha))\in J^*_\infty \times J^*_{\vartheta(\infty)}} \mathcal{O}_{\mathfrak{g},\vartheta(\alpha)}$ must fail to contain some point of $\Omega$ and, hence, there exist no finite $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subcovering of $\langle \mathcal{O}_{\mathfrak{g},\alpha}\rangle_{\alpha\in I^*_\infty\setminus I^*_2}$. This proves that $\mathfrak{T}_{\mathfrak{g}} = (\mathcal{T}_{\mathfrak{g}}, \Omega)$, where $\Omega = (0,1)$, is not a $\mathfrak{T}^{[\mathrm{A}]}_{\mathfrak{g}}$-space. Since $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies $\mathfrak{T}_{\mathfrak{g}}$-compactness, it follows, consequently, that it is also not a $\mathfrak{g}$-$\mathfrak{T}^{[\mathrm{A}]}_{\mathfrak{g}}$-space. Finally, from this case, it results that, *not every $\mathfrak{T}_{\mathfrak{g}}$-set of a $\mathfrak{g}$-$\mathfrak{T}^{[\mathrm{A}]}_{\mathfrak{g}}$-space is itself $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact.*

Let $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ be the $\mathfrak{g}$-topology on $\Omega = \mathbb{N}$ (set of positive integers) generated by $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets belonging to:

$$\begin{aligned}\mathcal{T}_{\mathfrak{g}} &\overset{\text{def}}{=} \big\{\mathcal{O}_{\mathfrak{g},(2\mu-1,2\mu)} :\ (\forall \mu\in I^*_\infty)\big([\mathcal{O}_{\mathfrak{g},(2\mu-1,2\mu)} = \emptyset] \\ &\qquad \vee[\mathcal{O}_{\mathfrak{g},(2\mu-1,2\mu)} = \{2\mu-1,2\mu\}]\big)\big\};\\ \neg\mathcal{T}_{\mathfrak{g}} &\overset{\text{def}}{=} \big\{\mathcal{K}_{\mathfrak{g},(2\mu-1,2\mu)} :\ (\forall \mu\in I^*_\infty)\big([\mathcal{K}_{\mathfrak{g},(2\mu-1,2\mu)} = \mathbb{N}] \\ &\qquad \vee[\mathcal{K}_{\mathfrak{g},(2\mu-1,2\mu)} = \complement(\{2\mu-1,2\mu\})]\big)\big\},\end{aligned}$$

respectively. As in the above case, it results that $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ satisfies the relations $\mathcal{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},(2\mu-1,2\mu)}) \subseteq \{2\mu-1,2\mu\} = \mathcal{O}_{\mathfrak{g},(2\mu-1,2\mu)}$ and, the relation $\mathcal{T}_{\mathfrak{g}}\big(\bigcap_{\mu\in I^*_\sigma}\mathcal{O}_{\mathfrak{g},(2\mu-1,2\mu)}\big) = \bigcap_{\mu\in I^*_\sigma}\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},(2\mu-1,2\mu)})$ as well as the relation $\mathcal{T}_{\mathfrak{g}}\big(\bigcup_{\mu\in I^*_\infty}\mathcal{O}_{\mathfrak{g},(2\mu-1,2\mu)}\big) = \bigcup_{\mu\in I^*_\infty}\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},(2\mu-1,2\mu)})$, since the two relations $\bigcap_{\mu\in I^*_\sigma}\mathcal{O}_{\mathfrak{g},(2\mu-1,2\mu)} = \emptyset \in \mathcal{T}_{\mathfrak{g}}$ and $\bigcup_{\mu\in I^*_\infty}\mathcal{O}_{\mathfrak{g},(2\mu-1,2\mu)} = \Omega \in \mathcal{T}_{\mathfrak{g}}$, respectively, hold. Therefore, $\mathfrak{T}_{\mathfrak{g}} = (\mathcal{T}_{\mathfrak{g}}, \Omega)$ is a $\mathcal{T}_{\mathfrak{g}}$-space and, moreover, since the relation $\mathfrak{T}_{\mathfrak{g}} = (\mathcal{T}_{\mathfrak{g}}, \Omega) = (\mathcal{T}, \Omega) = \mathfrak{T}$ holds, it is also a $\mathcal{T}$-space. Notice that $\langle \mathcal{O}_{\mathfrak{g},(2\alpha-1,2\alpha)}\rangle_{\alpha\in I^*_\infty}$ is a $\mathfrak{T}_{\mathfrak{g}}$-open covering of $\Omega$, since $\mathcal{O}_{\mathfrak{g},(2\alpha-1,2\alpha)} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha\in I^*_\infty$ and furthermore, it is also a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open covering of $\Omega$, since $\mathcal{O}_{\mathfrak{g},(2\alpha-1,2\alpha)} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},(2\alpha-1,2\alpha)}) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha\in I^*_\infty$. However, $\mathfrak{T}_{\mathfrak{g}} = (\mathcal{T}_{\mathfrak{g}}, \Omega)$, where $\Omega = \mathbb{N}$, is not a $\mathfrak{T}^{[\mathrm{A}]}_{\mathfrak{g}}$-space because $\langle \mathcal{O}_{\mathfrak{g},(2\alpha-1,2\alpha)}\rangle_{\alpha\in I^*_\infty}$ is a $\mathfrak{T}_{\mathfrak{g}}$-open covering of $\Omega$ with no finite $\mathfrak{T}_{\mathfrak{g}}$-open subcovering.

As stated above, since $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness implies $\mathfrak{T}_{\mathfrak{g}}$-compactness, it follows, obviously, that it is also not a $\mathfrak{g}$-$\mathfrak{T}^{[\mathrm{A}]}_{\mathfrak{g}}$-space. On the other hand, the $\mathfrak{T}_{\mathfrak{g}} = (\mathcal{T}_{\mathfrak{g}}, \Omega)$, where $\Omega = \mathbb{N}$, is also not a sequentially $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compact $\mathcal{T}_{\mathfrak{g}}$-space for the simple reason that sequence $\langle \xi_\alpha = \alpha \in \Omega\rangle_{\alpha\in I^*_\infty}$ in $\mathfrak{T}_{\mathfrak{g}}$ contains no subsequence of the type $\langle \xi_{\vartheta(\alpha)}\rangle_{(\alpha,\vartheta(\alpha)\in\Omega)\in I^*_\infty\times I^*_\infty} \prec \langle \xi_\alpha\rangle_{\alpha\in I^*_\infty}$ which converges to a point $\xi\in\Omega$. Hence, $\mathfrak{T}_{\mathfrak{g}}$ is not a $\mathfrak{g}$-$\mathfrak{T}^{[\mathrm{SA}]}_{\mathfrak{g}}$-space which, then, implies that it is also not a $\mathfrak{T}^{[\mathrm{SA}]}_{\mathfrak{g}}$-space.

Let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a non-empty $\mathfrak{T}_{\mathfrak{g}}$-set in $\mathfrak{T}_{\mathfrak{g}}$. Then, it is no error to express it as $\mathcal{S}_{\mathfrak{g}} = \mathcal{S}^{\mathrm{even}}_{\mathfrak{g}} \cup \mathcal{S}^{\mathrm{odd}}_{\mathfrak{g}}$, where $\mathcal{S}^{\mathrm{even}}_{\mathfrak{g}} = \{\mu :\ (\forall\alpha\in I^*_\infty)[\mu = 2\alpha]\}$ and $\mathcal{S}^{\mathrm{odd}}_{\mathfrak{g}} = \{\mu :\ (\forall\alpha\in I^*_\infty)[\mu = 2\alpha-1]\}$. Since $\mathcal{S}_{\mathfrak{g}} \neq \emptyset$, consider an arbitrary point $\xi\in\mathcal{S}_{\mathfrak{g}}$. If $\xi\in\mathcal{S}^{\mathrm{even}}_{\mathfrak{g}}$ then, for every $\mathfrak{T}_{\mathfrak{g}}$-open set $\mathcal{U}_{\mathfrak{g},\xi}\in\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ containing $\xi$, $\mathcal{S}^{\mathrm{even}}_{\mathfrak{g}}\cap(\mathcal{U}_{\mathfrak{g},\xi}\setminus\{\xi\}) = \emptyset$ and $\mathcal{S}^{\mathrm{odd}}_{\mathfrak{g}}\cap(\mathcal{U}_{\mathfrak{g},\xi}\setminus\{\xi\}) \neq \emptyset$. But, if $\xi\in\mathcal{S}^{\mathrm{odd}}_{\mathfrak{g}}$ then, for every $\mathfrak{T}_{\mathfrak{g}}$-open set $\mathcal{U}_{\mathfrak{g},\xi}\in\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ containing $\xi$, $\mathcal{S}^{\mathrm{even}}_{\mathfrak{g}}\cap(\mathcal{U}_{\mathfrak{g},\xi}\setminus\{\xi\}) \neq \emptyset$ and $\mathcal{S}^{\mathrm{odd}}_{\mathfrak{g}}\cap(\mathcal{U}_{\mathfrak{g},\xi}\setminus\{\xi\}) = \emptyset$. In either case, it follows, then, that $\mathcal{S}_{\mathfrak{g}}$ have at least one $\mathfrak{T}_{\mathfrak{g}}$-accumulation point. Accordingly, $\mathfrak{T}_{\mathfrak{g}}$ is a $\mathfrak{T}^{[\mathrm{CA}]}_{\mathfrak{g}}$-space. For every $\alpha\in I^*_\infty$, set $\mathcal{U}_{\mathfrak{g},2\alpha-1} = \{2\alpha-1\}$ and $\mathcal{U}_{\mathfrak{g},2\alpha} = \{2\alpha\}$. Accordingly, $\mathcal{U}_{\mathfrak{g},2\alpha-1}, \mathcal{U}_{\mathfrak{g},2\alpha} \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ since, $\mathcal{U}_{\mathfrak{g},2\alpha-1}, \mathcal{U}_{\mathfrak{g},2\alpha} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},(2\alpha-1,2\alpha)}) \in \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\alpha\in I^*_\infty$. Observe that, $\mathcal{S}_{\mathfrak{g}}\cap(\mathcal{U}_{\mathfrak{g},2\alpha-1}\setminus\{2\alpha-1\}) = \emptyset$ and $\mathcal{S}_{\mathfrak{g}}\cap(\mathcal{U}_{\mathfrak{g},2\alpha}\setminus\{2\alpha\}) = \emptyset$ for every $\alpha\in I^*_\infty$. This proves the existence of an infinite

$\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ with no $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-accumulation point and, hence, $\mathfrak{T}_{\mathfrak{g}}$ is not a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{CA}]}$-space.

Further $\mathcal{T}_{\mathfrak{g}}$-properties amongst the $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$-spaces $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{A}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]})$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{CA}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{CA}]})$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{SA}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{SA}]})$, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}^{[\mathrm{LA}]} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{[\mathrm{LA}]})$ called, respectively, $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space*, *countably* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space*, *sequentially* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space*, and *locally* $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}^{[\mathrm{A}]}$*-space*, can be discussed in a similar way by slight modifications of some $\mathcal{T}_{\mathfrak{g}}$-properties found in those cases. In the following section, concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness are presented.

4.3. Concluding Remarks. In this paper, a new theory called *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Compactness* has been presented, the foundation of which was based on the theories of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-maps and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-separation axioms. A careful perusal of the Mathematical developments of the earlier sections will reveal that the proposed theory has, in its own rights, several advantages. The very first advantage is that the theory holds equally well when $(\Omega, \mathcal{T}_{\mathfrak{g}}) = (\Omega, \mathcal{T})$, and other characteristics adapted on this ground, in which case it might be called *Theory of* $\mathfrak{g}$-$\mathfrak{T}$*-Connectedness*.

Thus, in a $\mathcal{T}_{\mathfrak{g}}$-space the theoretical framework categorises such statements as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in terms of relatively open $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in terms of relatively semi-open $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in terms of relatively preopen $\mathfrak{T}_{\mathfrak{g}}$-sets, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in terms of relatively semi-preopen $\mathfrak{T}_{\mathfrak{g}}$-sets as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way; in a $\mathcal{T}$-space it categorises such statements as $\mathfrak{g}$-$\mathfrak{T}$-compactness in terms of relatively open $\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathfrak{T}$-compactness in terms of relatively semi-open $\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathfrak{T}$-compactness in terms of relatively preopen $\mathfrak{T}$-sets, and $\mathfrak{g}$-$\mathfrak{T}$-compactness in terms of relatively semi-preopen $\mathfrak{T}$-sets as $\mathfrak{g}$-$\mathfrak{T}$-compactness of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way.

It is an interesting topic for future research to develop the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of mixed categories. More precisely, for some pair $(\nu, \mu) \in I_3^0 \times I_3^0$ such that $\nu \neq \mu$, to develop the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-compactness in terms of relatively $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets belonging to the class $\{\mathcal{U}_{\mathfrak{g}} = \mathcal{U}_{\mathfrak{g},\nu} \cup \mathcal{U}_{\mathfrak{g},\mu} : (\mathcal{U}_{\mathfrak{g},\nu}, \mathcal{U}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\}$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, as [1] and [9] developed the theory of b-open and b-closed sets in a $\mathcal{T}$-space $\mathfrak{T}$. Such a theory is what we thought would certainly be worth considering, and the discussion of this paper ends here.

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