Nonlinear Schrödingers equations with cubic nonlinearity: M-derivative soliton solutions by $\exp(-\Phi(\xi))$-Expansion method

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Abstract This paper uses the $\exp(-\Phi(\xi))$-Expansion method to investigate solitons to the M-fractional nonlinear Schrödingers equation with cubic nonlinearity. The results obtained are dark solitons, trigonometric function solutions, hyperbolic solutions and rational solutions. Thus, the constraint relations between the model coefficients and the traveling wave frequency coefficient for the existence of solitons solutions are also derived.
Key words: Solitons; M-Fractional, Integrability.

1 Introduction

During the past two decades, fractional calculus have advanced in analytical solution of nonlinear partial differential equation. On this way, a lot off attention has been place for investigation an exact traveling wave solutions of fractional models which yields to fractional differential equations. In addition, fractional calculus can provide mathematical formulas to transform the nonlinear partial differential equation to the nonlinear ordinary equation to handle them by some tractable integration tools. Also, it is very important to use the fractional derivatives which can provides an excellent implementation for the description of memory and hereditary properties[1]. Moreover, conformable fractional versions of some nonlinear system were investigate[2–4]. Thus, investigation of optical soliton with fractional time evolution, become very important due to its application in secure communication system of analog and digital signals, and to carry out hight speed data transmission over distance of several thousands of kilometers[5–12]. Recently, some effective integration methods have been used to construct exact solutions for PDEs, such as semi-inverse variational principle[21], the simplest equation approach[22], the first integral method[23], ansatz scheme[24] and the generalized tanh method[26] and so on. On this way, exact optical solitons in metamaterials with different nonlinearities have been reported[13–15]. The present paper will consider the M-fractional nonlinear Schrödingers equation with cubic nonlinearity. To construct soliton solutions, the $\exp(-\Phi(\xi))$-Expansion method is used to derived the ordinary differential equation obtained.

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2 M-fractional preliminaries

During the last decade, several definitions of fractional derivatives have been used in literature such as Atangana-Baleanu derivative in Caputo direction, Atangana-Baleanu fractional derivative in Riemann-Liouville sense, the new truncated M-fractional derivative of Sousa and Oliveira[16–18], just to name a few. This section will highlight some basic definitions and theorem of M-derivative of order $\alpha \in (0,1)$.

Definition 1: Let g: $[0,\infty) \to \mathbb{R}$.

$$D_M^{\alpha,\beta} g(t) = \lim_{\varepsilon \to +\infty} \frac{g(t\mathbb{E}_\beta(\varepsilon t^{1-\alpha})) - g(t)}{\varepsilon}. \quad \forall t > 0, \quad \beta > 0. \tag{1}$$

Here $\mathbb{E}_\beta(\cdot)$ is the Mittag-Leffler function of one parameter [28].

Theorem 1: Let $0 < \alpha < 1$, $\beta > 0$, a, $b \in \mathbb{R}$ and g, f α-differentiable at a point $t > 0$. Hence,

1 . $D_M^{\alpha,\beta}[(ag+bf)(t)] = aD_M^{\alpha,\beta}[g(t)] + bD_M^{\alpha,\beta}[f(t)]$.
2 . $D_M^{\alpha,\beta}[(g \cdot f)(t)] = g(t)D_M^{\alpha,\beta}[f(t)] + f(t)D_M^{\alpha,\beta}[g(t)]$.
3 . $D_M^{\alpha,\beta}[\frac{g}{f}(t)] = \frac{f(t)D_M^{\alpha,\beta}[g(t)] - g(t)D_M^{\alpha,\beta} f(t)}{[f(t)]^2}$.
4 . $D_M^{\alpha,\beta}[c] = 0$.
5 . If g is differentiable, then $D_M^{\alpha,\beta}[g(t)] = \frac{t^{1-\alpha}}{\Gamma(\beta+1)} \frac{dg(t)}{dt}$.

3 Application

3.1 M-fractional nonlinear Schrödingers equation with cubic nonlinearity

The proposed equation has been studied and many exact solutions were obtained [19,20]

$$iD_{M,t}^{\alpha,\beta}\psi + D_{M,x}^{2\alpha,\beta}\psi + \Omega|\psi|^2\psi = 0, \quad t > 0, \quad 0 < \alpha < 1. \tag{2}$$

Where ψ is a complex valued function of the spatial coordinate x and time t, while Ω is the coefficient of the nonlinearity To establish solutions of (2), we surmise that $\psi = \psi(x,t)$ can be expressed as follows

$$\psi(x,t) = v(x,t) + iu(x,t), \tag{3}$$

Substitute (3) in (2), the following system is obtained

$$\begin{aligned} &D_{M,t}^{\alpha,\beta} v + D_{M,x}^{2\alpha,\beta} u + \Omega(v^2 u + u^3) = 0, \\ &-D_{M,t}^{\alpha,\beta} u + D_{M,x}^{2\alpha,\beta} v + \Omega(v^3 + u^2 v) = 0. \end{aligned} \tag{4}$$

To transform the system of equation obtained, we used the following variable

$$\xi = \frac{\Gamma(\beta+1)}{\alpha}(\kappa x^\alpha + \omega t^\alpha) \tag{5}$$

where κ and ω are real constants. Considering $v(x,t) = V(\xi)$ and $u(x,t) = U(\xi)$, it is obtained the following ODE

$$\begin{aligned} &\omega V' + \kappa^2 U'' + \Omega(V^2 U + U^3) = 0, \\ &-\omega U' + \kappa^2 V'' + \Omega(V^3 + U^2 V) = 0. \end{aligned} \tag{6}$$

3.2 $\exp(-\Phi(\xi))$-Expansion method

This section will be used the $\exp(-\Phi(\xi))$-Expansion method to construct solutions to (6). Thereby, solution of (6) can be expressed as follows

$$V(\xi)=A_0+\sum_{i=1}^{N}A_i\exp(-\Phi(\xi))^i,$$
$$U(\xi)=B_0+\sum_{i=1}^{M}B_i\exp(-\Phi(\xi))^i, \tag{7}$$

and $\Phi(\xi)$ satisfies the following ODE

$$\Phi'(\xi)=\exp(-\Phi(\xi))+\mu\exp(\Phi(\xi))+\lambda. \tag{8}$$

By using the homogeneous balance principle, it is recovered from (6) N=M=1.
Hence, (7) gives

$$V(\xi)=A_0+A_1\exp(-\Phi(\xi)),$$
$$U(\xi)=B_0+B_1\exp(-\Phi(\xi)), \tag{9}$$

Substitute (9) and (8) into (6), it is obtained a system of algebraic equations. After solving the set of algebraic equations by aid of MAPLE, we get the following results.

Result 1 : $A_0=-\frac{1}{2}B_1(-\lambda\pm\sqrt{-\lambda^2+4\mu})$, $A_1=B_1$, $\omega=\pm\frac{-\kappa^2\sqrt{-\lambda^2+4\mu}\Gamma(\beta+1)}{\alpha}$, $\Omega=\frac{-\kappa^2\Gamma(\beta+1)^2}{\alpha^2B_1^2}$.

Result 2 : $B_0=\frac{1}{2}(\lambda\pm\sqrt{-\lambda^2+4\mu})B_1$, $B_1=B_1$, $\omega=\pm\frac{-\kappa^2\sqrt{-\lambda^2+4\mu}\Gamma(\beta+1)}{\alpha}$, $\Omega=\frac{-\kappa^2\Gamma(\beta+1)^2}{\alpha^2B_1^2}$.

Using the five solutions of the auxiliary ODE(8) as in [29], we can obtained five exact solutions of (6) and the M-fractional soliton solutions of (2).

Case 1: If $-\lambda^2+4\mu>0$ and $\mu\neq 0$,

$$u_1(x,t)=\left[-\frac{1}{2}B_1(-\lambda\pm\sqrt{-\lambda^2+4\mu})+\frac{2B_1\mu}{-\sqrt{-4\mu+\lambda^2}\tanh\left(\frac{1}{2}\sqrt{-8\mu+2\lambda^2}(\xi+\xi_0)\right)-\lambda}\right]$$
$$\times\exp[i(\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha)]. \tag{10}$$

and

$$v_1(x,t)=\left[\frac{1}{2}(\lambda\pm\sqrt{-\lambda^2+4\mu})B_1+\frac{2B_1\mu}{-\sqrt{-4\mu+\lambda^2}\tanh\left(\frac{1}{2}\sqrt{-8\mu+2\lambda^2}(\xi+\xi_0)\right)-\lambda}\right]$$
$$\times\exp[i(\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha+\omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha)]. \tag{11}$$

Case 2: If $-\lambda^2+4\mu<0$ and $\mu\neq 0$,

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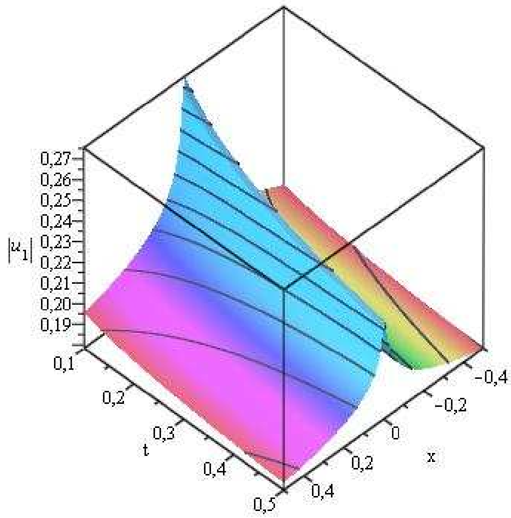

Fig. 1: Analytical solution of $u_1(x,t)$, at $B_1 = 0.2$, $\lambda = 0.44$, $\alpha = 0.45$, $\mu = 0.23$ and $\kappa = .02$

$$u_2(x,t) = \left[-\frac{1}{2}B_1(-\lambda \pm \sqrt{-\lambda^2+4\mu}) + \frac{2B_1\mu}{\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{-8\mu+2\lambda^2}\,(\xi+\xi_0)\right)-\lambda} \right]$$
$$\times \exp[i(\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha)]. \quad (12)$$

$$v_2(x,t) = \left[\frac{1}{2}(\lambda \pm \sqrt{-\lambda^2+4\mu})B_1 + \frac{2B_1\mu}{\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{-8\mu+2\lambda^2}\,(\xi+\xi_0)\right)-\lambda} \right]$$
$$\times \exp[i(\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha)]. \quad (13)$$

Case 3: If $-\lambda^2+4\mu > 0$ and $\mu = 0$,

$$u_3(x,t) = \left[-\frac{1}{2}B_1(-\lambda \pm \sqrt{-\lambda^2+4\mu}) + \frac{B_1\lambda}{\cosh(\lambda(\xi+\xi_0)+\sinh(\lambda(\xi+\xi_0))-1)} \right]$$
$$\times \exp[i(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha)]. \quad (14)$$

and

$$v_3(x,t) = \left[\frac{1}{2}(\lambda \pm \sqrt{-\lambda^2+4\mu})B_1 + \frac{B_1\lambda}{\cosh(\lambda(\xi+\xi_0)+\sinh(\lambda(\xi+\xi_0))-1)} \right]$$
$$\times \exp[i(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha)]. \quad (15)$$

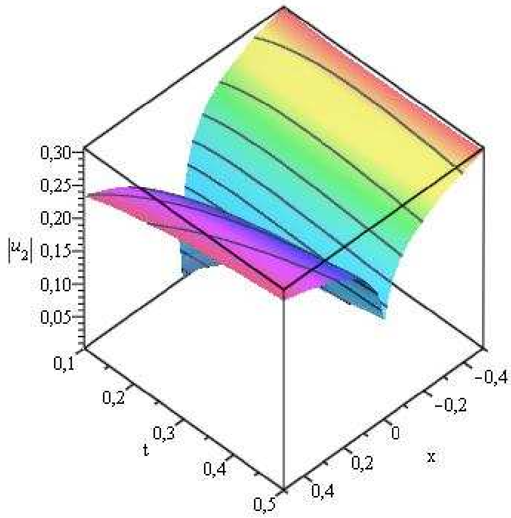

Fig. 2: Analytical solution of $u_2(x,t)$, at $B_1 = 0.032$, $\lambda = 0.034$,$\alpha = 0.35$, $\mu = 0.23$ and $\kappa = .02$

Case 4: If $-\lambda^2+4\mu = 0$ and $\mu \neq 0$ and $\lambda = 0$,

$$u_4(x,t) = \left[-\frac{1}{2}B_1(-\lambda \pm \sqrt{-\lambda^2+4\mu}) + \frac{B_1\lambda^2\,(\xi+\xi_0)}{-2\lambda\,(\xi+\xi_0)+2}\right] \times \exp\left[i(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha)\right]. \tag{16}$$

and

$$v_4(x,t) = \left[\frac{1}{2}(\lambda \pm \sqrt{-\lambda^2+4\mu})B_1 + \frac{B_1\lambda^2\,(\xi+\xi_0)}{-2\lambda\,(\xi+\xi_0)+2}\right] \times \exp\left[i(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha)\right]. \tag{17}$$

Case 5: If $-\lambda^2+4\mu = 0$ and $\mu = 0$ and $\lambda = 0$,

$$u_5(x,t) = \left[-\frac{1}{2}B_1(-\lambda \pm \sqrt{-\lambda^2+4\mu}) + \frac{B_1}{\xi+\xi_0}\right] \times \exp\left[i(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha)\right]. \tag{18}$$

$$v_5(x,t) = \left[\frac{1}{2}(\lambda \pm \sqrt{-\lambda^2+4\mu})B_1 + \frac{B_1}{\xi+\xi_0}\right] \times \exp\left[i(-\kappa\frac{\Gamma(\beta+1)}{\alpha}x^\alpha + \omega\frac{\Gamma(\beta+1)}{\alpha}t^\alpha)\right]. \tag{19}$$

4 Conclusion and remarks

In this paper, we investigate soliton solutions to the M-fractional nonlinear Schrödingers equation with cubic nonlinearity. The $\exp(-\Phi(\xi))$-Expansion method is used derived the couple of nonlinear ordinary differential equation obtained. As a result, Dark solitons, trigonometric function solutions, hyperbolic function solutions and rational solutions have been obtained. These results obtained may be helpful in explaining communication system and nonlinear complex system. Figures (1) and (2) illustrated the (3D) plot of dark (10) solitary waves and trigonometric function solutions (12).

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