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Lukewarm blackhole in a Nash-Greene framework

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In this work, we investigate the embedding of a four-dimensional spherically symmetric metric in a six dimensional bulk. By using the Nash-Greene theorem, the additional $SO(2)$ symmetry of the two space-like extra-dimensions induces the appearance of horizons of a lukewarm charged black hole. Accordingly, an emerged mass-dependent cosmological constant is obtained with a prediction with a bound for the mass and minimal charge.

I. INTRODUCTION

The possibility that the universe might be embedded in extra dimensions has been explored in the last decades as a tentative to explain the hierarchy of the fundamental interactions that may drive answers to the cosmological era with their capital problems of dark matter, dark energy and the nature of the cosmological constant. Most of these models have been Kaluza-Klein or/and string inspired, such as the seminal works of the Arkani-Hamed, Dvali and Dimopolous (ADD) model [1] that predicted an unification of the fundamental interactions in six dimensional bulk for large extradimensions named latter as braneworlds in accordance with experiment colliders [2] suggesting a sub-millimeter gravity; and the Randall-sundrum model [3, 4] in which the fixed three-brane is embedded in a five dimensional anti-de Sitter (AdS_5) space-time where the Israel condition applies [5]. The DvaliGabadadzePorrati model (DPG) was another interesting approach where the 3+1 Minkowski space-time is fixed and embedded in a flat five dimensional bulk, predicting no need for a small but non-zero vacuum energy density [6]. In all those models and variants, the embedding was a rising issue not completed worked as a theoretical background since its generally fixed to a boundary and specific conditions are needed to obtain its dynamics. Till then, several authors explored embedding as a prior mathematical structure for a physical theory with the embedding equations as a fundamental mathematical guidance [7–15, 17–19]. The fundamentals of the model presented in this paper were originally proposed in [8–10].

This paper aims at investigating some consequences of spherically symmetric metric in a six dimensional bulk in the context of the embedding. For instance, in the case of a Schwarzschild solution the embedding would be compromised in Randall-Sundrum scheme and evinces the necessity of a more general framework, once the Schwarzschild geometry is completely embedded in six-dimensions [18, 20–22]. In recent years, mostly on the impact of the direct evidence of gravitational waves [23], the black hole studies have turned to a huge arena of investigation that reflects on the problems of standard model of particles, unification of the standard interactions and early universe. Thus, the study of higher dimensional spacetimes have been the focus on active research [24–27].

The paper is organized as follows: in the second section, we make a brief mathematical review on embedding. The third section present the development of a calculation for a four dimensional metric embedded in a six dimensional bulk and charge black hole horizons mass and charge are calculated. In the fourth section, we present the emergent cosmological constant related to the thick the embedded space-time and related quantities. In the final section, we present our remarks and prospects.

II. THE INDUCED EMBEDDED DIMENSIONAL EQUATIONS

The gravitational action functional in the presence of confined matter field on a four-dimensional embedded space with thickness l embedded in a D-dimensional ambient space (bulk) has the form

$$S=-\frac{1}{2\kappa_D^2}\int\sqrt{|\mathcal{G}|}\mathcal{R}d^Dx-\int\sqrt{|\mathcal{G}|}\mathcal{L}^*_m d^Dx\ , \qquad (1)$$

where κ_D^2 is the fundamental energy scale on the embedded space, $\mathcal{R}$ denotes de Ricci scalar of the bulk and $\mathcal{L}^*_m$ is the confined matter lagrangian. In this model, the matter energy momentum tensor occupies a finite hypervolume with constant radius l along the extra-dimensions. The variation of Einstein-Hilbert action in eq.(1) with respect to the bulk metric $\mathcal{G}_{AB}$ leads to the Einstein equations for the bulk

$$\mathcal{R}_{AB}-\frac{1}{2}\mathcal{G}_{AB}=\alpha^\star\mathcal{T}_{AB}\ , \qquad (2)$$

where $\alpha^\star$ is energy scale parameter and $\mathcal{T}_{AB}$ is the energy-momentum tensor for the bulk [9, 10, 13]. To generate a thick embedded space-time is important to perturb the related background and it should be done in accordance with the confinement hypothesis that depends only on the four-dimensionality of the space-time [28, 29]. Even though any gauge theory can be mathe-

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matically constructed in a higher dimensional space, the observed phenomenology imposes the fourth dimensionality of space-time [30].

In order to obtain a more general theory based on embeddings to elaborate a physical model, Nash's original embedding theorem [31] used a flat D-dimensional Euclidean space, later generalized to any Riemannian manifold including non-positive signatures by Greene [32] with independent orthogonal perturbations. This choice of perturbations facilitates to get to a differentiable smoothness of the embedding between the manifolds, which is a primary concern of Nash's theorem and satisfies the Einstein-Hilbert principle, where the variation of the Ricci scalar is the minimum as possible. Hence, it guarantees that the embedded geometry remains smooth (differentiable) after smooth (differentiable) perturbations. With all these concepts, let us consider a Riemannian manifold V_4 with a non-perturbed metric $\bar{g}_{\mu\nu}$ being locally and isometrically embedded in a n-dimensional Riemannian manifold V_n given by a differentiable and regular map $\mathcal{X} : V_4 \to V_n$ satisfying the embedding equations

$$\mathcal{X}^A_{,\mu}\mathcal{X}^B_{,\nu}\mathcal{G}_{AB} = \bar{g}_{\mu\nu} \ , \tag{3}$$

$$\mathcal{X}^A_{,\mu}\bar{\eta}^B_a\mathcal{G}_{AB} = 0 \ , \tag{4}$$

$$\bar{\eta}^A_a\bar{\eta}^B_b\mathcal{G}_{AB} = \bar{g}_{ab} \ , \tag{5}$$

where we have denoted by $\mathcal{G}_{AB}$ the metric components of V_n in arbitrary coordinates, $\bar{\eta}$ denotes a non-perturbed unit vector field orthogonal to V_4. Concerning notation, capital Latin indices run from 1 to n. Small case Latin indices refer to the only one extra dimension considered. All Greek indices refer to the embedded space-time counting from 1 to 4. Those set of equations represent the isometry condition eq.(3), orthogonality between the embedding coordinates $\mathcal{X}$ and $\bar{\eta}$ in eq.(4), and also, the vector normalization $\bar{\eta}$ and $\bar{g}_{ab} = \epsilon_a\delta_{ab}$ with $\epsilon_a = \pm 1$ in which the signs represent the signatures of the extra-dimensions. Hence, the integration of the system of equations eqs.(3), (4) and (5) assures the configuration of the embedding map $\mathcal{X}$.

The second fundamental form, or more commonly, the non-perturbed extrinsic curvature $\bar{k}_{\mu\nu}$ of V_4 is by definition the projection of the variation of $\bar{\eta}$ onto the tangent plane :

$$\bar{k}_{\mu\nu} = -X^A_{,\mu}\bar{\eta}^B_{,\nu}\mathcal{G}_{AB} = X^A_{,\mu\nu}\bar{\eta}^B\mathcal{G}_{AB} \ , \tag{6}$$

where the comma denotes the ordinary derivative.

If one defines a geometric object $\bar{\omega}$ in V_4, its Lie transport along the flow for a small distance δy is given by $\Omega = \bar{\Omega} + \delta y \pounds_{\bar{\eta}}\bar{\Omega}$, where $\pounds_{\bar{\eta}}$ denotes the Lie derivative with respect to $\bar{\eta}$. In particular, the Lie transport of the Gaussian frame $\{X^A_\mu, \bar{\eta}^A_a\}$, defined on V_4 gives straightforwardly

$$\mathcal{Z}^A_{,\mu} = X^A_{,\mu} + \delta y \ \pounds_{\bar{\eta}} X^A_{,\mu} = X^A_{,\mu} + \delta y \ \bar{\eta}^A_{,\mu} \tag{7}$$

$$\eta^A = \bar{\eta}^A + \delta y \ [\bar{\eta}, \bar{\eta}]^A \qquad = \ \bar{\eta}^A \ . \tag{8}$$

It is worth mentioning that Eq.(8) shows that the normal vector η^A does not change under orthogonal perturbations. However, from Eq.(6), we note that in general $\eta_{,\mu} \neq \bar{\eta}_{,\mu}$. A similar situation occurs with the so-called third geometrical form, or more commonly, the twist vector, $A_{\mu ba}$. Taking equations in Eq.(12), we can rewrite Eq.(4) as

$$g_{\mu b} = \ \mathcal{Z}^A_{,\mu}\eta^B_b\mathcal{G}_{AB} = \ \delta y^a A_{\mu ba} \ , \tag{9}$$

where $\mathcal{Z}^A$ are a set of perturbed coordinates. The Eq.(9) results from a generalization of the Gauss-Weingartein equations

$$\eta^A_{a,\mu} = \ A_{\mu ac}g^{cb}\eta^A_b - \bar{k}_{\mu\rho a}\bar{g}^{\rho\nu}\mathcal{Z}^A_{,\nu} \ . \tag{10}$$

then,

$$A_{\mu ba} = \ \eta^A_{a,\mu}\eta^B_b\mathcal{G}_{AB} = \ \bar{\eta}^A_{a,\mu}\bar{\eta}^B_b\mathcal{G}_{AB} = \ \bar{A}_{\mu ba} \ , \tag{11}$$

that shows that the twist vector also does not alter under perturbations. In geometric language, the presence of a twist potential tilts the embedded family of submanifolds with respect to the normal vector η^A. If the bulk has certain killing vectors then $A_{\mu ba}$ transform as the component of a gauge field under the group of isometries of the bulk [8, 33, 34]. It is worth noting that in our model, the gauge potential can only be present if the dimension of the bulk space is equal or greater than six ($n \geq 2$) in accordance with eq.(11) the twist vector field $A_{\mu ab}$ are antisymmetric under the exchange of extra coordinate a and b.

To describe the would-be perturbed embedded geometry, we set a perturbed coordinates $\mathcal{Z}^A$ needed to satisfy the embedding equations similar to Eqs.(3), (4) and (5) as

$$\mathcal{Z}^A_{,\mu}Z^B_{,\nu}\mathcal{G}_{AB} = g_{\mu\nu}, \ \mathcal{Z}^A_{,\mu}\eta^B_b\mathcal{G}_{AB} = g_{\mu b}, \ \eta^A_a\eta^B_b\mathcal{G}_{AB} = \ \epsilon_a\delta_{ab} \ . \tag{12}$$

Thus, with the Eqs.(12) and using the definition from Eq.(6), one obtains the perturbed metric and extrinsic curvature of the new manifold as written as

$$g_{\mu\nu} = \ \bar{g}_{\mu\nu} - 2y^a\bar{k}_{\mu\nu a} + \delta y^a\delta y^b\left[\bar{g}^{\sigma\rho}\bar{k}_{\mu\sigma a}\bar{k}_{\nu\rho b} + g^{cd}A_{\mu ca}A_{\nu db}\right] \ , \tag{13}$$

and the related perturbed extrinsic curvature

$$k_{\mu\nu a} = \ \bar{k}_{\mu\nu a} - \delta y^b\left(g^{cd}A_{\mu ca}A_{\nu db} + \bar{g}^{\sigma\rho}\bar{k}_{\mu\sigma a}\bar{k}_{\nu\rho b}\right) \ . \tag{14}$$

Taking the derivative of Eq.(13) with respect to y coordinate, one obtains Nash's deformation condition

$$k_{\mu\nu a} = \ -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial y^a} \ . \tag{15}$$

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The meaning of this expression can be realized in a pictorial view under the elementary theory of curves. For instance, one can construct an one-parameter group of diffeomorphisms defined by a map $h_y(p) : V_D \rightarrow V_D$, describing a continuous curve $\alpha(y) = h_y(p)$ that passes through the point $p \in V_4$, with unit normal vector $\alpha'(p) = \eta(p)$. The group is characterized by the composition $h_y \circ h_{\pm y'}(p) \overset{def}{=} h_{y \pm y'}(p)$, $h_0(p) \overset{def}{=} p$. With the diffeomorphism mapping all points of a small neighborhood of p, one gets a congruence of curves (or orbits) orthogonal to V_4 [35] which consists the action of the extrinsic curvature. Thus, it is not important if the parameter y is time-like or not, nor the sign of its signature. A similar expression was obtained years later in the ADM formulation by Choquet-Bruhat and York [36].

Moreover, the integrability condition for equations in Eq.(12) are given by the non-trivial components of the Riemann tensor of the embedding space expressed in the Gaussian frame $\{Z^A_\mu, \eta^A\}$ known as the Gauss-Codazzi equations. This guarantees to reconstruct the embedded geometry and understand its properties from the dynamics of the four-dimensional embedded space-time. Consequently, we can define a Gaussian coordinate system $\{\mathcal{Z}^A_{,\mu}, \eta^A\}$ for the bulk in the vicinity of V_4 in such a way

$$\mathcal{G}_{AB} = \begin{pmatrix} \bar{g}_{\mu\nu} + g^{ab} A_{\mu a} A_{\nu b} & A_{\mu a} \\ A_{\nu b} & g_{ab} \end{pmatrix} \tag{16}$$

which resembles the Kaluza-Klein metric and $A_{\mu a}$ plays the role of the Yang-Mills potentials. One obtains the induced field equations taking Eq.(2) in the frame defined in Eq.(16) in the background as

$$G_{\mu\nu} + Q_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} + T^{(YM)}_{\mu\nu} \right) , \tag{17}$$

$$k_{a,\nu} - k^\mu_{a\nu,\mu} = 8\pi G_N \Theta_{\mu\nu} , \tag{18}$$

$$\frac{G_N}{\beta} \left(F^{\mu\nu}_{am} F^m_{\mu\nu b} + \frac{1}{2} \eta_{ab} F^{lm}_{\mu\nu} F^{\mu\nu}_{lm} \right) \tag{19}$$
$$-\frac{1}{2} \eta_{ab} (R + k_{\mu\nu m} k^{\mu\nu m} - k_a k^a) = 8\pi G_N \Theta_{ab}$$

$$\nabla_\nu F^{\mu\nu ab} = 0 , \tag{20}$$

where G_{ab} is the induced Einstein tensor. The quantity $Q_{\mu\nu}$ is denoted by

$$Q_{\mu\nu} = g^{cd} \left(g^{\rho\sigma} k_{\mu\rho c} k_{\nu\sigma d} - k_{\mu\nu d} g^{\alpha\beta} k_{\alpha\beta c} \right) \tag{21}$$
$$-\frac{1}{2} \left(k_{\lambda\phi c} k^{\lambda\phi}_{\ \ d} - g^{\alpha\beta} k_{\alpha\beta d} g^{\gamma\delta} k_{\gamma\delta c} \right) g_{\mu\nu} , \tag{22}$$

and is independently conserved in the sense that $Q_{\mu\nu\ ;\nu} = 0$, where the semicolon sign denotes a covariant derivative. The tensors $\Theta_{\mu\nu}$ and Θ_{ab} are the projection tensors of the bulk space in such a way that

$$\beta = \begin{cases} 2(n-2), & \text{n} \neq 2; \\ 2, & \text{n=2} . \end{cases}$$

The tensor $T^{(YM)}_{\alpha\beta}$ is the Maxwell-Yang-Mills tensor denoted by

$$T^{(YM)}_{\mu\nu} = \frac{1}{4\pi\beta} \left(F^{\sigma}_{\mu \ \ lm} F^{lm}_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\mu\nu lm} F^{\mu\nu lm} \right) . \tag{23}$$

Also, the following fundamental relation between normal curvature radii L with a thickness l of the embedded space-time and the gauging coupling g_i, we consider that the four-dimensional Planck mass applies in such a way

$$L = \frac{n\beta}{4(n+2)} l^2 M^2_{pl} \frac{g_i^2}{4\pi} ,$$

where normal radii is the smallest value of the curvature radii obtained from

$$det(g_{\mu\nu} - l^a k_{\mu\nu a}) = 0 . \tag{24}$$

III. CHARGED BLACK HOLE IN SIX DIMENSIONS

In this section, we consider a static and symmetric solution of induced field equations eq.(2) given by the following induced metric

$$ds^2 = -e^{-2a(r)} dt^2 + e^{2b(r)} dt^2 + r^2 d\Omega^2 , \tag{25}$$

where $d\Omega$ is the ordinary two-sphere element.

The equations in Eq.(20) reduce to the usual Maxwell equations $\nabla_\nu F^{\mu\nu} = 0$. Therefore, the field strength in the Lorentz gauge $\nabla_\mu A^\mu = 0$ is given by

$$F = -\frac{q}{\sqrt{4\pi}} \frac{e^{a+b}}{r^2} dt \wedge dr - \frac{g}{\sqrt{4\pi}} \sin(\theta) d\theta \wedge d\varphi , \tag{26}$$

where q and g denote electric and magnetic charges, respectively, in which, due to the embedding, they are located in the center of a Kruskal spacetime. The related energy-momentum tensor components of eq.(23) are found to be

$$T^{EM}_{00} = \frac{Z^2 e^{2a}}{8\pi r^4} , \ T^{EM}_{11} = -\frac{Z^2 e^{2b}}{8\pi r^4} , \tag{27}$$
$$T^{EM}_{22} = \frac{Z^2}{8\pi r^2} , \ T^{EM}_{33} = \sin^2(\theta) T^{EM}_{22} ,$$

where all other $T^(EM)_{\mu\nu} = 0$, and $Z = \frac{1}{4\pi}(q^2 + g^2)$ Equations in eq.(19) give the restriction on the electric and magnetic charges $q^2 = g^2$. Moreover, Eq.(18) gives the components of extrinsic curvature as

$$k_{\mu\nu a} = \phi_a g_{\mu\nu} , \tag{28}$$

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where ϕ_a are constants. Hence, Eq.(24) leads to $L^{-2} = \eta_{ab}\phi^a\phi^b$. Therefore, there is not any direction in which any normal curvature has an extreme value and consequently the four-dimensional sub-manifold is umbilic [37] that leads to

$$Q_{\mu\nu} = \frac{3}{L} g_{\mu\nu} \, . \tag{29}$$

We stress that the y-coordinate is not necessary on the induced metric once the embedding equations are correctly applied. Since Nash's idea on embedding of manifolds using smooth deformations are applied to the embedding, the y coordinate, commonly used in rigid embedded models, e.g, Randall-Sundrum and variants [3, 4], can be omitted in the line element [8–15].

In six dimensions, there is an additional $SO(2)$ symmetry generated by two spacelike killing vectors of the two extradimensions ($n = 2$). Moreover, one assumes that $A_{ab\mu} = \epsilon_{ab}A_\mu$, where the antisymmetric symbols is defined as $\epsilon_{12} = \epsilon_{21} = -1$. Hence, the components G_{00} and G_{11} become

$$\frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} - \frac{2b'}{r}\right) = \frac{3}{L^2} + \frac{Z^2 G}{r^4} \tag{30}$$

$$\frac{1}{r^2} - e^{-2b}\left(\frac{1}{r^2} + \frac{2a'}{r}\right) = \frac{3}{L^2} + \frac{Z^2 G}{r^4} \tag{31}$$

By substraction, we see that $a + b = 0$ and consequently

$$\left(re^{-2b}\right)' = 1 - \frac{3}{L^2}r^2 - \frac{Z^2 G}{r^2}$$

The metric functions are thus given by

$$e^{2a} = e^{-2b} = 1 - \frac{2GM}{r} + \frac{Z^2 G}{r^2} - \frac{r^2}{L^2} \, . \tag{32}$$

The other components of Einstein equations are also satisfied. Hence, in a general sense, the D-dimensional Einstein vacuum equations eq.(2) induce a four dimensional Reissner-Nordström-de Sitter (RNdS) space-time, where the induced charge is the influence of a non-compact space-like extra-dimensions and an induced cosmological constant $\Lambda_{ind} = \frac{3}{L^2}$ is a consequence of the extrinsic shape of the black hole.

As largely known, according to the Einstein field equations, the distribution of matter determines the intrinsic geometric properties. On the other hand, in extrinsic gravity with the corresponding field equations eqs.(17 and (18) determine both intrinsic ($G_{\mu\nu}$) and extrinsic ($Q_{\mu\nu}$) geometric properties of the space-time. Consequently, in the absence of any matter fields, the embedded space-time will be a trivial flat space-time (with both intrinsically and extrinsic features). The last one means that the extrinsic radii of the embedded space-time will be globally infinite.

IV. THE RISING OF AN EMERGENT COSMOLOGICAL CONSTANT

The minimum measurable length over which the masses can be localized is about of the order of their compton wavelengths. Hence, let us assume that the with of the four dimensional embedded space-time has the same order of the compton wavelength of the black hole that we set $l = M^{-1}$. Therefore, the curvature radii defined in eq.(24) turns to be

$$L = \frac{1}{4}\frac{M_{pl}^2}{M^3} Z^2 = 2Z^2 \frac{L_{pl}^4}{R_{sch}^3} \, , \tag{33}$$

where L_{pl}^4 is the Planck's length and R_{sch} denotes the Schwarzchild radius. Hence, taking eqs.(32) and (33), we find an emergent cosmological constant Λ_e as

$$\Lambda_e = 48G^2 \frac{M^6}{Z^4} \, . \tag{34}$$

In the absence of a black hole ($M \to 0$), according to eq.(33), the normal curvature radii will be infinite and obtain the flat Minkowski space-time consistent with Maxwell-Lorenztian symmetry. Interestingly, the "cosmological constant" as a mass-dependent quantity was previously conjectured by Zeldovich [38, 39].

Eventually, the horizons can be found using the algebraic equation $e^{2a} = 0$ that has three positive roots. The outer horizon is located at r_{++}, the black hole horizon at r_+ and the Cauchy horizon at r_i. If we set $M^2 G = Z^2$, then the RNdS spacetime is called a lukewarm black hole [40, 41]. In the naive picture of a black hole evaporation, in the lukewarm solution, the cosmological constant Λ comes from the background universe and consequently this solution is thermodynamically stable and is the end-points of the evaporation process: if $M/M_{pl} > |Z|$ then the black hole is hotter than the de Sitter horizon and will evaporate until it reaches $M/M_{pl} = |Z|$. If $M/M_{pl} < |Z|$ then the de Sitter horizon is hotter and the black hole will accrete radiation until it reaches $M/M_{pl} = |Z|$. A similar process applies to our model, since the black hole will evaporate until it reaches $M/M_{pl} = |Z|$. Then, an emergent Λ_e will rise in the black hole with a value

$$\Lambda_e = 48 Z^2 M_{pl}^2 \, , \tag{35}$$

and the three related horizons will have the form

$$r_i = \frac{1}{8M_{pl}|Z|}(-1 + \sqrt{1 + 16Z^2}) \tag{36}$$

$$r_+ = \frac{1}{8M_{pl}|Z|}(1 - \sqrt{1 - 16Z^2}) \tag{37}$$

$$r_{++} = \frac{1}{8M_{pl}|Z|}(1 + \sqrt{1 - 16Z^2}) \, , \tag{38}$$

in which show that the lukewarm black holes are possible if $|Z| < \frac{1}{4}$ or $M < \frac{1}{4}M_{pl}$. In this case, both event and

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outer horizon has the same surface gravity

$$\kappa_+ = \kappa_{++} = 4|Z|M_{pl}\sqrt{1-16Z^2} \tag{39}$$

which means they have the same temperature $T = \frac{|\kappa_+|}{2\pi}$. Hence, for $Z = \mathcal{O}(1)$ will be an unstable black hole. Therefore, the lukewarm black holes in this model will reach a maximum temperature at

$$Z = \frac{1}{2\sqrt{2}} \tag{40}$$

$$M = \frac{1}{2\sqrt{2}}M_{pl} \tag{41}$$

$$\Lambda_e = 6M_{pl}^2 \sim \Lambda_{QFT} \tag{42}$$

$$T = \frac{M_{pl}}{2\pi} \sim T_{pl} = 3.5\times 10^{32}\ K \tag{43}$$

The emergent Λ_e is of the order of the vacuum energy of quantum field theory and consequently hotter than the cosmological horizon and this solutions are not thermodinamically stable. Since the total gravitational entropy is given by the sum of the area of the black holes and cosmological event horizons, the entropy is extremized for $\kappa_{bh} = -\kappa_{universe}$ that coincides with the condition that the black hole and de Sitter temperature must be equal. Thus, the final stable state of the black hole will be

$$\Lambda_e \;=\; \Lambda_{universe} \sim 3\times 10^{-56}\ cm^{-2} \tag{44}$$

$$Z \;=\; \frac{1}{4M_{pl}}\sqrt{\frac{\Lambda}{3}} \sim 10^{-60} \tag{45}$$

$$M \;=\; \frac{1}{4}\sqrt{\frac{\Lambda}{3}} \sim 1.5\times 10^{-66}g \tag{46}$$

$$r_i \sim r_+ \sim \frac{1}{4M_{pl}}\sqrt{\frac{\Lambda}{3}} \sim 10^{-93}cm \tag{47}$$

$$r_{++} \sim \sqrt{\frac{3}{\Lambda}} \sim 10^{28}cm \tag{48}$$

$$T = \frac{2}{\pi}M = \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}} \sim 10^{-60} \tag{49}$$

Hence, it means that the black hole reaches the universal lower bound of mass $M \sim 10^{-66}$g and minimum charge $q \sim 10^{-60}e$. This bound for mass has been verified by several authors [42–44].

V. REMARKS

In this paper, we have discussed the embedding of a four dimensional symmetric metric in a bulk of six dimensions. Applying the Nash-Greene embedding theorem to a static spherical symmetric metric, we have found a modification induced by the extrinsic curvature using a dynamical embedding with an appearance of the twist vector $A_{\mu ab}$ as gauge group of rotations $SO(2)$ in the extra-dimensions. Since a spherically symmetric metric, just like Schwarzschild geometry, is completely embedded in a six dimensional space-time, and with a calculations of the related horizons, a Lukewarm back holes were obtained and the rising of new elements like an emergent cosmological constant Λ_e has turned this model a worth framework to investigate further, differently from we obtained in a five-dimensional bulk, where the restrictions to embedding induced a serious constraints on the black hole stability [18]. The present results may open a new arena to thinking the Cosmological Constant problem and its variants [45–47] starting from its underlying origin. Hopefully, the understanding of an emergent Λ_e may would explain why the measured CC is not precisely zero and has a nonzero but very small value. As future prospects, the study of the related quasi-normal modes applied to the same framework in the analysis of Signal-to Noise ratio, which are in due course and will be reported elsewhere.

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