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Bicomplex Tetranacci and Tetranacci-Lucas Quaternions

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Abstract. In this paper, we introduce the bicomplex Tetranacci and Tetranacci-Lucas quaternions. Moreover, we present Binet's formulas, generating functions, and the summation formulas for those bicomplex quaternions.

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1. Introduction

In this paper, we define bicomplex Tetranacci and bicomplex Tetranacci-Lucas quaternions by combining bicomplex numbers and Tetranacci, Tetranacci-Lucas numbers and give some properties of them. Before giving their definition, we present some information on bicomplex numbers and also on Tetranacci and Tetranacci-Lucas numbers.

The bicomplex numbers (quaternions) are defined by the four bases elements $1, i, j, ij$ where i, j and ij satisfy the following properties:

$$i^2 = -1, \ j^2 = -1, \ ij = ji.$$

A bicomplex number can be expressed as follows:

$$q = a_0 + ia_1 + ja_2 + ija_3 = (a_0 + ia_1) + j(a_2 + ia_3) = z_0 + jz_1$$

where a_0, a_1, a_2, a_3 are real numbers and z_0, z_1 are complex numbers. So the set of bicomplex number is

$$\mathbb{BC} = \{z_0 + jz_1 : z_0, z_1 \in \mathbb{C}, j^2 = -1\}.$$

Moreover, for any bicomplex numbers $q = a_0 + ia_1 + ja_2 + ija_3$ and $p = b_0 + ib_1 + jb_2 + ijb_3$ and skaler $\lambda \in \mathbb{R}$, the addition, substraction and multiplication with scalar are defined as componentwise, i.e

$$\begin{aligned} q+p &= (a_0+b_0)+i(a_1+b_1)+j(a_2+b_2)+ij(a_3+b_3), \\ q-p &= (a_0-b_0)+i(a_1-b_1)+j(a_2-b_2)+ij(a_3-b_3), \\ \lambda q &= \lambda a_0 + i\lambda a_1 + j\lambda a_2 + ij\lambda a_3 \end{aligned}$$

respectively, and product (multiplication) is defined as follows:

$$\begin{aligned} q \times p &= (a_0b_0 - a_1b_1 - a_2b_2 + a_3b_3) \\ &\quad +i(a_0b_1 + a_1b_0 - a_2b_3 - a_3b_2) \\ &\quad +j(a_0b_2 - a_1b_3 + a_2b_0 - a_3b_1) \\ &\quad +ij(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0). \end{aligned}$$

Multiplication of basis elements of bicomplex numbers can be done according to following Table 1:

Table 1. Multiplication Table

$\times$	1	i	j	ij
1	1	i	j	ij
i	i	-1	ij	$-j$
j	j	ij	-1	$-i$
ij	ij	$-j$	$-i$	-1

There are three different conjugations (involutions) for bicomplex numbers, namely

$$\begin{aligned} q_i^* &= a_0 - ia_1 + ja_2 - ija_3 = \overline{z_0} + j\overline{z_1}, \\ q_j^* &= a_0 + ia_1 - ja_2 - ija_3 = z_0 - jz_1, \\ q_{ij}^* &= a_0 - ia_1 - ja_2 + ija_3 = \overline{z_0} - j\overline{z_1}, \end{aligned}$$

for $q = a_0 + ia_1 + ja_2 + ija_3$. The squares of norms of the bicomplex numbers which arise from the definitions of conjugations are given by

$$\begin{aligned} N_i^2(q) &= |q_i \times q_i^*| := \left|a_0^2 + a_1^2 - a_2^2 - a_3^2 + 2j(a_0a_2 + a_1a_3)\right|, \\ N_j^2(q) &= \left|q_j \times q_j^*\right| := \left|a_0^2 + a_1^2 - a_2^2 - a_3^2 + 2i(a_0a_1 + a_2a_3)\right|, \\ N_{ij}^2(q) &= \left|q_{ij} \times q_{ij}^*\right| := \left|a_0^2 + a_1^2 + a_2^2 + a_3^2 + 2ij(a_0a_3 - a_2a_1)\right|. \end{aligned}$$

We now give some basic informations on quaternions. Quaternions were formally invented by Irish mathematician W. R. Hamilton (1805-1865) as an extension to the complex numbers and for some background about this type of hypercomplex numbers we refer the works, for example, in [3, 6, 23]. The field $\mathbb{H}$ of quaternions is a four-dimensional non-commutative $\mathbb{R}$-field generated by four base elements $1, i, j$ and k that satisfy the following rules:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1.1}$$

and

$$ij = k = -ji, \; jk = i = -kj, \; ki = j = -ik. \tag{1.2}$$

Briefly $\mathbb{BC}$, the set of bicomplex numbers, has the following properties:

- Quaternions and bicomplex numbers are generalizations of complex numbers, but one difference between them is that quaternions are non-commutative, whereas bicomplex numbers are commutative.
- Real quaternions are non-commutative, and don't have zero divisors and non-trivial idempotent elements. But bicomplex numbers are commutative, have zero divisors and non-trivial idempotent elements:

$$\begin{aligned} ij &= ji, \\ (i+j)(i-j) &= i^2 - ij + ji - j^2 = 0, \\ \left(\frac{1+ij}{2}\right)^2 &= \frac{1+ij}{2}. \end{aligned}$$

- All above norms are isotropic. For example, for N_i, we calculate $N_i(q)$ for $q = 1 + ij$ as

$$N_i^2(1+ij) = (1+ij)(1-ij) = 1^2 - ij + ij - (ij)^2 = 0.$$

- $\mathbb{BC}$ is a real vector space with the addition of bicomplex numbers and the multiplication of a bicomplex number by a real scalar.
- $\mathbb{BC}$ forms a commutative ring with unity which contains $\mathbb{C}$.
- $\mathbb{BC}$ forms a two-dimensional algebra over $\mathbb{C}$, and since $\mathbb{C}$ is of dimension two over $\mathbb{R}$, the bicomplex numbers are an algebra over $\mathbb{R}$ of dimension four.
- $\mathbb{BC}$ is a real associative algebra with the bicomplex number product $\times$.

For more details about these type of numbers (quaternions), we refer to, for example, the works [8,18], among others.

Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ are defined by the fourth-order recurrence relations

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}, \qquad M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2 \tag{1.3}$$

and

$$R_n = R_{n-1} + R_{n-2} + R_{n-3} + R_{n-4}, \qquad R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7 \tag{1.4}$$

respectively. More detail on these sequences can be found, for example, in [11], [14], [15], [19], [21] and [22].

The sequences $\{M_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = -M_{-(n-1)} - M_{-(n-2)} - M_{-(n-3)} + M_{-(n-4)}$$

and

$$R_{-n} = -R_{-(n-1)} - R_{-(n-2)} - R_{-(n-3)} + R_{-(n-4)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.3) and (1.4) hold for all integer n.

The following Table 2 presents the first few values of the Tetranacci and Tetranacci-Lucas numbers with positive and negative subscripts:

Table 2. Tetranacci and Tetranacci-Lucas Numbers with non-negative and negative indices

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
M_n	0	1	1	2	4	8	15	29	56	108	208	401	773	1490	...
M_{-n}	0	0	0	1	-1	0	0	2	-3	1	0	4	-8	5	...
R_n	4	1	3	7	15	26	51	99	191	367	708	1365	2631	5071	...
R_{-n}	4	-1	-1	-1	7	-6	-1	-1	15	-19	4	-1	31	-53	...

It is well known that for all integers n, usual Tetranacci and Tetranacci-Lucas numbers can be expressed using Binet's formulas

$$M_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

(see for example [24] or [11])

or

$$M_n = \frac{\alpha-1}{5\alpha-8}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-1} + + \frac{\delta-1}{5\delta-8}\delta^{n-1} \tag{1.5}$$

(see for example [7]) and

$$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n$$

respectively, where α, β, γ and δ are the roots of the equation $x^4 - x^3 - x^2 - x - 1 = 0$. Moreover,

$$\begin{aligned}
\alpha &= \frac{1}{4} + \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}}, \\
\beta &= \frac{1}{4} + \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}}, \\
\gamma &= \frac{1}{4} - \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}}, \\
\delta &= \frac{1}{4} - \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},
\end{aligned}$$

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where

$$\omega = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}} \right)^{1/3} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}} \right)^{1/3}}.$$

Note that the Binet form of a sequence satisfying (1.3) and (1.4) for non-negative integers is valid for all integers n. This result of Howard and Saidak [12] is even true in the case of higher-order recurrence relations.

The generating functions for the Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ are

$$\sum_{n=0}^{\infty} M_n x^n = \frac{x}{1-x-x^2-x^3-x^4} \quad \text{and} \quad \sum_{n=0}^{\infty} R_n x^n = \frac{4-3x-2x^2-x^3}{1-x-x^2-x^3-x^4},$$

respectively.

2. The Bicomplex Tetranacci and Tetranacci-Lucas Quaternions and their Generating Functions, Binet's Formulas and Summations Formulas

In this section we define the bicomplex Tetranacci and Tetranacci-Lucas quaternions and give generating functions and Binet formulas for them. First, we give some information about bicomplex type quaternion sequences from the literature.

Nurkan and Güven [16] (see also [17]) introduced nth bicomplex Fibonacci and nth bicomplex Lucas numbers (quaternions) as

$$BF_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}ij$$

and

$$BL_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}ij$$

respectively, where F_n and L_n are the nth Fibonacci and Lucas numbers respectively. Various families of bicomplex number (quaternion) sequences have been defined and studied by a number of authors. See, for example, [1, 2, 4, 9, 10] for second order bicomplex quaternion sequences and [5, 13] for third order bicomplex quaternion sequences.

We now define bicomplex Tetranacci and Tetranacci-Lucas quaternions over the algebra $\mathbb{BC}$.

Definition 1. *The nth bicomplex Tetranacci quaternion is*

$$\mathbb{BC}M_n = M_n + iM_{n+1} + jM_{n+2} + ijM_{n+3} \tag{2.1}$$

and the nth Tetranacci-Lucas quaternion is

$$\mathbb{BC}R_n = R_n + iR_{n+1} + jR_{n+2} + ijR_{n+3}. \tag{2.2}$$

It can be easily shown that $\{\mathbb{BC}M_n\}_{n\geq 0}$ and $\{\mathbb{BC}R_n\}_{n\geq 0}$ can also be defined by the recurrence relations:

$$\mathbb{BC}M_n = \mathbb{BC}M_{n-1} + \mathbb{BC}M_{n-2} + \mathbb{BC}M_{n-3} + \mathbb{BC}M_{n-4} \tag{2.3}$$

and

$$\mathbb{BC}R_n = \mathbb{BC}R_{n-1} + \mathbb{BC}R_{n-2} + \mathbb{BC}R_{n-3} + \mathbb{BC}R_{n-4} \tag{2.4}$$

with the intial conditions

$$\mathbb{BC}M_0 = i + j + 2ij,\ \mathbb{BC}M_1 = 1 + i + 2j + 4ij,\ \mathbb{BC}M_2 = 1 + 2i + 4j + 8ij,\ \mathbb{BC}M_3 = 2 + 4i + 8j + 15ij$$

and

$$\mathbb{BC}R_0 = 4+i+3j+7ij,\ \mathbb{BC}R_1 = 1+3i+7+15ij,\ \mathbb{BC}R_2 = 3+7i+15j+26ij,\ \mathbb{BC}R_3 = 7+15i+26j+51ij.$$

The sequences $\{\mathbb{BC}M_n\}_{n\geq 0}$ and $\{\mathbb{BC}R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathbb{BC}M_{-n} = -\mathbb{BC}M_{-(n-1)} - \mathbb{BC}M_{-(n-2)} - \mathbb{BC}M_{-(n-3)} + \mathbb{BC}M_{-(n-4)}$$

and

$$\mathbb{BC}R_{-n} = -\mathbb{BC}R_{-(n-1)} - \mathbb{BC}R_{-(n-2)} - \mathbb{BC}R_{-(n-3)} + \mathbb{BC}R_{-(n-4)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.3) and (2.4) hold for all integer n.

The first few bicomplex Tetranacci and Tetranacci-Lucas quaternions with positive subscript and negative subscript are given in the following Table 1 and Table 2:

Table 1 bicomplex Tetranacci quaternions

n	$\mathbb{BC}M_n$	$\mathbb{BC}M_{-n}$
0	$i + j + 2ij$	$i + j + 2ij$
1	$1 + i + 2j + 4ij$	$j + ij$
2	$1 + 2i + 4j + 8ij$	ij
3	$2 + 4i + 8j + 15ij$	1
4	$4 + 8i + 15j + 29ij$	$-1 + i$
5	$8 + 15i + 29j + 56ij$	$-i + j$
6	$15 + 29i + 56j + 108ij$	$-j + ij$
7	$29 + 56i + 108j + 208ij$	$2 - ij$

Table 2 bicomplex Tetranacci-Lucas quaternions

n	$\mathbb{BC}R_n$	$\mathbb{BC}R_{-n}$
0	$4 + i + 3j + 7ij$	$4 + i + 3j + 7ij$
1	$1 + 3i + 7 + 15ij$	$-1 + 4i + j + 3ij$
2	$3 + 7i + 15j + 26ij$	$-1 - i + 4j + ij$
3	$7 + 15i + 26j + 51ij$	$-1 - i - j + 4ij$
4	$15 + 26i + 51j + 99ij$	$7 - i - j - ij$
5	$26 + 51i + 99j + 191ij$	$-6 + 7i - j - ij$
6	$51 + 99i + 191j + 367ij$	$-1 - 6i + 7j - ij$
7	$99 + 191i + 367j + 708ij$	$-1 - i - 6j + 7ij$

For two bicomplex Tetranacci quaternions $\mathbb{BC}M_n$ and $\mathbb{BC}M_k$ and for skaler $\lambda \in \mathbb{R}$, the addition, substraction and multiplication with scalar are defined as componentwise, i.e.,

$$\begin{aligned}
\mathbb{BC}M_n + \mathbb{BC}M_k &= (M_n + M_k) + i(M_{n+1} + M_{k+1}) + j(M_{n+2} + M_{k+2}) + ij(M_{n+3} + M_{k+3}), \\
\mathbb{BC}M_n - \mathbb{BC}M_k &= (M_n - M_k) + i(M_{n+1} - M_{k+1}) + j(M_{n+2} - M_{k+2}) + ij(M_{n+3} - M_{k+3}), \\
\lambda\mathbb{BC}M_n &= \lambda M_n + i\lambda M_{n+1} + j\lambda M_{n+2} + ij\lambda M_{n+3}
\end{aligned}$$

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respectively, and product (multiplication) is defined as follows:

$$\begin{aligned}
\mathbb{BC}M_n \times \mathbb{BC}M_k &= (M_nM_k - M_{n+1}M_{k+1} - M_{n+2}M_{k+2} + M_{n+3}M_{k+3}) \\
&\quad +i(M_nM_{k+1} + M_{n+1}M_k - M_{n+2}M_{k+3} - M_{n+3}M_{k+2}) \\
&\quad +j(M_nM_{k+2} - M_{n+1}M_{k+3} + M_{n+2}M_k - M_{n+3}M_{k+1}) \\
&\quad +ij(M_nM_{k+3} + M_{n+1}M_{k+2} + M_{n+2}M_{k+1} + M_{n+3}M_k) \\
&= \mathbb{BC}M_k \times \mathbb{BC}M_n.
\end{aligned}$$

Similarly, for two bicomplex Tetranacci-Lucas quaternions $\mathbb{BC}R_n$ and $\mathbb{BC}R_k$ and for skaler $\lambda \in \mathbb{R}$, the addition, substraction and multiplication with scalar are defined as componentwise, i.e.,

$$\begin{aligned}
\mathbb{BC}R_n \pm \mathbb{BC}R_k &= (R_n \pm R_k) + i(R_{n+1} \pm R_{k+1}) + j(R_{n+2} \pm R_{k+2}) + ij(R_{n+3} \pm R_{k+3}), \\
\lambda\mathbb{BC}R_n &= \lambda R_n + i\lambda R_{n+1} + j\lambda R_{n+2} + ij\lambda R_{n+3}
\end{aligned}$$

respectively, and product (multiplication) is defined as follows:

$$\begin{aligned}
\mathbb{BC}R_n \times \mathbb{BC}R_k &= (R_nR_k - R_{n+1}R_{k+1} - R_{n+2}R_{k+2} + R_{n+3}R_{k+3}) \\
&\quad +i(R_nR_{k+1} + R_{n+1}R_k - R_{n+2}R_{k+3} - R_{n+3}R_{k+2}) \\
&\quad +j(R_nR_{k+2} - R_{n+1}R_{k+3} + R_{n+2}R_k - R_{n+3}R_{k+1}) \\
&\quad +ij(R_nR_{k+3} + R_{n+1}R_{k+2} + R_{n+2}R_{k+1} + R_{n+3}R_k) \\
&= \mathbb{BC}R_k \times \mathbb{BC}R_n.
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbb{BC}M_n \times \mathbb{BC}M_n &= (M_n^2 - M_{n+1}^2 - M_{n+2}^2 + M_{n+3}^2) + 2i(M_nM_{n+1} - M_{n+2}M_{n+3}) \\
&\quad +2j(M_nM_{n+2} - M_{n+1}M_{n+3}) + 2ij(M_nM_{n+3} + M_{n+1}M_{n+2})
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{BC}R_n \times \mathbb{BC}R_n &= (R_n^2 - R_{n+1}^2 - R_{n+2}^2 + R_{n+3}^2) + 2i(R_nR_{n+1} - R_{n+2}R_{n+3}) \\
&\quad +2j(R_nR_{n+2} - R_{n+1}R_{n+3}) + 2ij(R_nR_{n+3} + R_{n+1}R_{n+2}).
\end{aligned}$$

Moreover, three different conjugations for the bicomplex Tribonacci quaternion $\mathbb{BC}M_n = M_n + iM_{n+1} + jM_{n+2} + ijM_{n+3}$ are given as

$$\begin{aligned}
(\mathbb{BC}M_n)_i^* &= M_n - iM_{n+1} + jM_{n+2} - ijM_{n+3}, \\
(\mathbb{BC}M_n)_j^* &= M_n + iM_{n+1} - jM_{n+2} - ijM_{n+3}, \\
(\mathbb{BC}M_n)_{ij}^* &= M_n - iM_{n+1} - jM_{n+2} + ijM_{n+3},
\end{aligned}$$

and the squares of norms of the bicomplex Tribonacci quaternion are given by

$$\begin{aligned}
N_i^2(\mathbb{BC}M_n) &= |(\mathbb{BC}M_n)_i \times (\mathbb{BC}M_n)_i^*| := \left|M_n^2 + M_{n+1}^2 - M_{n+2}^2 - M_{n+3}^2 + 2j(M_n M_{n+2} + M_{n+1}M_{n+3})\right|, \\
N_j^2(\mathbb{BC}M_n) &= \left|(\mathbb{BC}M_n)_j \times (\mathbb{BC}M_n)_j^*\right| := \left|M_n^2 + M_{n+1}^2 - M_{n+2}^2 - M_{n+3}^2 + 2i(M_n M_{n+1} + M_{n+2}M_{n+3})\right|, \\
N_{ij}^2(\mathbb{BC}M_n) &= \left|(\mathbb{BC}M_n)_{ij} \times (\mathbb{BC}M_n)_{ij}^*\right| := \left|M_n^2 + M_{n+1}^2 + M_{n+2}^2 + M_{n+3}^2 + 2ij(M_n M_{n+3} - M_{n+2}M_{n+1})\right|.
\end{aligned}$$

Similarly, we can give three different conjugations and the squares of norms for the bicomplex Tribonacci-Lucas quaternion $\mathbb{BC}R_n = R_n + iR_{n+1} + jR_{n+2} + ijR_{n+3}$.

Now, we will state Binet's formula for the bicomplex Tetranacci and Tetranacci-Lucas quaternions and in the rest of the paper we fix the following notations.

$$\begin{aligned}
\widehat{\alpha} &= 1 + i\alpha + j\alpha^2 + ij\alpha^3, \\
\widehat{\beta} &= 1 + i\beta + j\beta^2 + ij\beta^3, \\
\widehat{\gamma} &= 1 + i\gamma + j\gamma^2 + ij\gamma^3, \\
\widehat{\delta} &= 1 + i\delta + j\delta^2 + ij\delta^3.
\end{aligned}$$

THEOREM 2. *(Binet's Formulas) For any integer n, the nth bicomplex Tetranacci quaternion is*

$$\begin{aligned}
\mathbb{BC}M_n &= \frac{\widehat{\alpha}\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\widehat{\beta}\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\
&\quad + \frac{\widehat{\gamma}\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\widehat{\delta}\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \qquad (2.5) \\
&= \frac{\alpha-1}{5\alpha-8}\widehat{\alpha}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\widehat{\beta}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\widehat{\gamma}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\widehat{\delta}\delta^{n-1} \qquad (2.6)
\end{aligned}$$

and the nth bicomplex Tetranacci-Lucas quaternion is

$$\mathbb{BC}R_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{\delta}\delta^n. \qquad (2.7)$$

Proof. Using Binet's formula of the Tetranacci-Lucas numbers we have

$$\begin{aligned}
\mathbb{BC}R_n &= R_n + iR_{n+1} + jR_{n+2} + ijR_{n+3} \\
&= (\alpha^n + \beta^n + \gamma^n + \delta^n) + i(\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1} + \delta^{n+1}) \\
&\quad + j(\alpha^{n+2} + \beta^{n+2} + \gamma^{n+2} + \delta^{n+2}) + ij(\alpha^{n+3} + \beta^{n+3} + \gamma^{n+3} + \delta^{n+3}) \\
&= \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{\delta}\delta^n.
\end{aligned}$$

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Note that using Binet's formula (1.5) of the Tetranacci numbers we have

$$\begin{aligned}
\mathbb{BC}M_n &= M_n + iM_{n+1} + jM_{n+2} + ijM_{n+3} \\
&= (\frac{\alpha-1}{5\alpha-8}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\delta^{n-1}) \\
&\quad +i(\frac{\alpha-1}{5\alpha-8}\alpha^{n} + \frac{\beta-1}{5\beta-8}\beta^{n} + \frac{\gamma-1}{5\gamma-8}\gamma^{n} + \frac{\delta-1}{5\delta-8}\delta^{n}) \\
&\quad +j(\frac{\alpha-1}{5\alpha-8}\alpha^{n+1} + \frac{\beta-1}{5\beta-8}\beta^{n+1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n+1} + \frac{\delta-1}{5\delta-8}\delta^{n+1}) \\
&\quad +ij(\frac{\alpha-1}{5\alpha-8}\alpha^{n+2} + \frac{\beta-1}{5\beta-8}\beta^{n+2} + \frac{\gamma-1}{5\gamma-8}\gamma^{n+2} + \frac{\delta-1}{5\delta-8}\delta^{n+2}) \\
&= \frac{\alpha-1}{5\alpha-8}\widehat{\alpha}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\widehat{\beta}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\widehat{\gamma}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\widehat{\delta}\delta^{n-1}.
\end{aligned}$$

This proves (2.6). Similarly, we can obtain (2.5).

Next, we present generating functions.

THEOREM 3. *The generating functions for the bicomplex Tetranacci and Tetranacci-Lucas quaternions are*

$$\sum_{n=0}^{\infty} \mathbb{BC}M_n x^n = \frac{(i+j+2ij) + (1+j+2ij)x + (j+2ij)x^2 + (j+ij)x^3}{1-x-x^2-x^3-x^4} \tag{2.8}$$

and

$$\sum_{n=0}^{\infty} \mathbb{BC}R_n x^n = \frac{(4+i+3j+7ij) + (-3+2i+4j+8ij)x + (-2+3i+5j+4ij)x^2 + (-1+4i+j+3ij)x^3}{1-x-x^2-x^3-x^4} \tag{2.9}$$

respectively.

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \mathbb{BC}M_n x^n$$

be generating function of the bicomplex Tetranacci quaternions. Then using the definition of the bicomplex Tetranacci quaternions, and substracting $xg(x)$, $x^2g(x)$, $x^3g(x)$ and $x^4g(x)$ from $g(x)$ and using the recurrence relation $\mathbb{BC}M_n = \mathbb{BC}M_{n-1} + \mathbb{BC}M_{n-2} + \mathbb{BC}M_{n-3} + \mathbb{BC}M_{n-4}$, we obtain

$$\begin{aligned}
(1-x-x^2-x^3-x^4)g(x) &= \mathbb{BC}M_0 + (\mathbb{BC}M_1 - \mathbb{BC}M_0)x + (\mathbb{BC}M_2 - \mathbb{BC}M_1 - \mathbb{BC}M_0)x^2 \\
&\quad +(\mathbb{BC}M_3 - \mathbb{BC}M_2 - \mathbb{BC}M_1 - \mathbb{BC}M_0)x^3.
\end{aligned}$$

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Now using

$$\begin{aligned}
\mathbb{BC}M_{-1} &= j+ij \\
\mathbb{BC}M_{0} &= i+j+2ij \\
\mathbb{BC}M_{1} &= 1+i+2j+4ij \\
\mathbb{BC}M_{2} &= 1+2i+4j+8ij \\
\mathbb{BC}M_{3} &= 2+4i+8j+15ij
\end{aligned}$$

we obtain

$$g(x)=\frac{(i+j+2ij)+(1+j+2ij)x+(j+2ij)x^2+(j+ij)x^3}{1-x-x^2-x^3-x^4}.$$

Similarly, we can obtain (2.9).

Next we present some summation formulas of Tetranacci numbers.

Lemma 4. *For $n\geq 1$ we have the following formulas:*

- **(a):** $\sum\limits_{p=1}^{n} M_p=\frac{1}{3}(M_{n+2}+2M_n+M_{n-1}-1)$
- **(b):** $\sum\limits_{p=1}^{n} M_{2p+1}=\frac{1}{3}(2M_{2n+2}+M_{2n}-M_{2n-1}-2)$
- **(c):** $\sum\limits_{p=1}^{n} M_{2p}=\frac{1}{3}(2M_{2n+1}+M_{2n-1}-M_{2n-2}-2)$.

The above Lemma is given in Soykan [20, Corollary 2.7]. It now follows that for every integer $n\geq 0$,

$$\sum_{p=0}^{n} M_p = M_0+\sum_{p=1}^{n} M_p=\frac{1}{3}(M_{n+2}+2M_n+M_{n-1}-1), \tag{2.10}$$

$$\sum_{p=0}^{n} M_{2p+1} = M_1+\sum_{p=1}^{n} M_{2p+1}=\frac{1}{3}(2M_{2n+2}+M_{2n}-M_{2n-1}+1) \tag{2.11}$$

$$\sum_{p=0}^{n} M_{2p} = M_0+\sum_{p=1}^{n} M_{2p}=\frac{1}{3}(2M_{2n+1}+M_{2n-1}-M_{2n-2}-2). \tag{2.12}$$

In the following Lemma we present some summation formulas of Tetranacci-Lucas numbers.

Lemma 5. *For $n\geq 1$ we have the following formulas:*

- **(a):** $\sum\limits_{p=1}^{n} R_p=\frac{1}{3}(R_{n+2}+2R_n+R_{n-1}-10)$
- **(b):** $\sum\limits_{p=1}^{n} R_{2p+1}=\frac{1}{3}(2R_{2n+2}+R_{2n}-R_{2n-1}-11)$
- **(c):** $\sum\limits_{p=1}^{n} R_{2p}=\frac{1}{3}(2R_{2n+1}+R_{2n-1}-R_{2n-2}-2)$.

The above Lemma is given in Soykan [20, Corollary 2.8]. It now follows that for every integer $n \geq 0$,

$$\sum_{p=0}^{n} R_p = R_0 + \sum_{p=1}^{n} R_p = \frac{1}{3}(R_{n+2} + 2R_n + R_{n-1} + 2), \tag{2.13}$$

$$\sum_{p=0}^{n} R_{2p+1} = R_1 + \sum_{p=1}^{n} R_{2p+1} = +\frac{1}{3}(2R_{2n+2} + R_{2n} - R_{2n-1} - 8) \tag{2.14}$$

$$\sum_{p=0}^{n} R_{2p} = R_0 + \sum_{p=1}^{n} R_{2p} = \frac{1}{3}(2R_{2n+1} + R_{2n-1} - R_{2n-2} + 10). \tag{2.15}$$

Next we present some summation formulas of bicomplex Tetranacci quaternions.

Theorem 6. *For $n \geq 0$ we have the following formulas:*

(a):

$$\sum_{p=0}^{n} \mathbb{BC}M_p = \frac{1}{3}(\mathbb{BC}M_{n+2} + 2\mathbb{BC}M_n + \mathbb{BC}M_{n-1} - (1 + i + 4j + 7ij)) \tag{2.16}$$

(b):

$$\sum_{p=0}^{n} \mathbb{BC}M_{2p+1} = \frac{1}{3}(2\mathbb{BC}M_{2n+2} + \mathbb{BC}M_{2n} - \mathbb{BC}M_{2n-1} + (1 - 2i - 2j - 5ij))$$

(c):

$$\sum_{p=0}^{n} \mathbb{BC}M_{2p} = \frac{1}{3}(2\mathbb{BC}M_{2n+1} + \mathbb{BC}M_{2n-1} - \mathbb{BC}M_{2n-2} - (2 - i + 2j + 2ij)).$$

Proof.

(a): Using (2.1) and (2.10), we obtain

$$\begin{aligned}\sum_{p=0}^{n} \mathbb{BC}M_p &= \sum_{p=0}^{n} M_p + i\sum_{p=0}^{n} M_{p+1} + j\sum_{p=0}^{n} M_{p+2} + ij\sum_{p=0}^{n} M_{p+3} \\ &= (M_0 + ... + M_n) + i(M_1 + ... + M_{n+1}) \\ &\quad +j(M_2 + ... + M_{n+2}) + ij(M_3 + ... + M_{n+2}).\end{aligned}$$

and so

$$\begin{aligned}3\sum_{p=0}^{n} \mathbb{BC}M_p &= (M_{n+2} + 2M_n + M_{n-1} - 1) \\ &\quad +i(M_{n+3} + 2M_{n+1} + M_n - 1 - 3M_0) \\ &\quad +j(M_{n+4} + 2M_{n+2} + M_{n+1} - 1 - 3(M_0 + M_1)) \\ &\quad +ij(M_{n+5} + 2M_{n+3} + M_{n+2} - 1 - 3(M_0 + M_1 + M_2)) \\ &= \mathbb{BC}M_{n+2} + 2\mathbb{BC}M_n + \mathbb{BC}M_{n-1} + c\end{aligned}$$

where

$$\begin{aligned} c &= -1+i(-1-3M_0)+j(-1-3(M_0+M_1))+ij(-1-3(M_0+M_1+M_2)) \\ &= -1-i-4j-7ij. \end{aligned}$$

Hence

$$\sum_{p=0}^{n} \mathbb{BC}M_p = \frac{1}{3}(\mathbb{BC}M_{n+2}+2\mathbb{BC}M_n+\mathbb{BC}M_{n-1}-(1+i+4j+7ij)).$$

This proves (2.16).

(b): and (c) follows from the identities (2.11) and (2.12).

In the following Theorem, we give some summation formulas of bicomplex Tetranacci-Lucas quaternions.

THEOREM 7. *For $n \geq 0$ we have the following formulas:*

(a):

$$\sum_{p=0}^{n} \mathbb{BC}R_p = \frac{1}{3}(\mathbb{BC}R_{n+2}+2\mathbb{BC}R_n+\mathbb{BC}R_{n-1}+(2-10i-13j-22ij)). \tag{2.17}$$

(b):

$$\sum_{p=0}^{n} \mathbb{BC}R_{2p+1} = \frac{1}{3}(2\mathbb{BC}R_{2n+2}+\mathbb{BC}R_{2n}-\mathbb{BC}R_{2n-1}-(8+2i+11j+11ij))$$

(c):

$$\sum_{p=0}^{n} \mathbb{BC}R_{2p} = \frac{1}{3}(2\mathbb{BC}R_{2n+1}+\mathbb{BC}R_{2n-1}-\mathbb{BC}R_{2n-2}+(10-8i-2j-11ij)).$$

Proof.

(a): Using (2.3) and (2.13), we obtain (2.17).

(b): and (c) follows from the identities (2.14) and (2.15).

3. Matrices and Determinants related with Tetranacci and Tetranacci-Lucas Quaternions

We define the square matrix B of order 4 as:

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det B = -1$.

Induction proof may be used to establish

$$B^n = \begin{pmatrix} M_{n+1} & M_n + M_{n-1} + M_{n-2} & M_n + M_{n-1} & M_n \\ M_n & M_{n-1} + M_{n-2} + M_{n-3} & M_{n-1} + M_{n-2} & M_{n-1} \\ M_{n-1} & M_{n-2} + M_{n-3} + M_{n-4} & M_{n-2} + M_{n-3} & M_{n-2} \\ M_{n-2} & M_{n-3} + M_{n-4} + M_{n-5} & M_{n-3} + M_{n-4} & M_{n-3} \end{pmatrix}. \tag{3.1}$$

Matrix formulation of M_n and R_n can be given as

$$\begin{pmatrix} M_{n+3} \\ M_{n+2} \\ M_{n+1} \\ M_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} M_3 \\ M_2 \\ M_1 \\ M_0 \end{pmatrix} \tag{3.2}$$

and

$$\begin{pmatrix} R_{n+3} \\ R_{n+2} \\ R_{n+1} \\ R_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} R_3 \\ R_2 \\ R_1 \\ R_0 \end{pmatrix}. \tag{3.3}$$

Induction proofs may be used to establish the matrix formulations M_n and R_n.

Now we define the matrices B_M and B_R as

$$B_M = \begin{pmatrix} \mathbb{BC}M_5 & \mathbb{BC}M_4 + \mathbb{BC}M_3 + \mathbb{BC}M_2 & \mathbb{BC}M_4 + \mathbb{BC}M_3 & \mathbb{BC}M_4 \\ \mathbb{BC}M_4 & \mathbb{BC}M_3 + \mathbb{BC}M_2 + \mathbb{BC}M_1 & \mathbb{BC}M_3 + \mathbb{BC}M_2 & \mathbb{BC}M_3 \\ \mathbb{BC}M_3 & \mathbb{BC}M_2 + \mathbb{BC}M_1 + \mathbb{BC}M_0 & \mathbb{BC}M_2 + \mathbb{BC}M_1 & \mathbb{BC}M_2 \\ \mathbb{BC}M_2 & \mathbb{BC}M_1 + \mathbb{BC}M_0 + \mathbb{BC}M_{-1} & \mathbb{BC}M_1 + \mathbb{BC}M_0 & \mathbb{BC}M_1 \end{pmatrix}$$

and

$$B_R = \begin{pmatrix} \mathbb{BC}R_5 & \mathbb{BC}R_4 + \mathbb{BC}R_3 + \mathbb{BC}R_2 & \mathbb{BC}R_4 + \mathbb{BC}R_3 & \mathbb{BC}R_4 \\ \mathbb{BC}R_4 & \mathbb{BC}R_3 + \mathbb{BC}R_2 + \mathbb{BC}R_1 & \mathbb{BC}R_3 + \mathbb{BC}R_2 & \mathbb{BC}R_3 \\ \mathbb{BC}R_3 & \mathbb{BC}R_2 + \mathbb{BC}R_1 + \mathbb{BC}R_0 & \mathbb{BC}R_2 + \mathbb{BC}R_1 & \mathbb{BC}R_2 \\ \mathbb{BC}R_2 & \mathbb{BC}R_1 + \mathbb{BC}R_0 + \mathbb{BC}R_{-1} & \mathbb{BC}R_1 + \mathbb{BC}R_0 & \mathbb{BC}R_1 \end{pmatrix}$$

These matrices B_M and B_R can be called bicomplex Tetranacci quaternion matrix and bicomplex Tetranacci-Lucas quaternion matrix, respectively.

THEOREM 8. *For $n \geq 0$, the followings are valid:*

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(a):

(3.4)
$$B_M\begin{pmatrix}1&1&1&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}^n=\begin{pmatrix}\mathbb{BC}M_{n+5}&\mathbb{BC}M_{n+4}+\mathbb{BC}M_{n+3}+\mathbb{BC}M_{n+2}&\mathbb{BC}M_{n+4}+\mathbb{BC}M_{n+3}&\mathbb{BC}M_{n+4}\\\mathbb{BC}M_{n+4}&\mathbb{BC}M_{n+3}+\mathbb{BC}M_{n+2}+\mathbb{BC}M_{n+1}&\mathbb{BC}M_{n+3}+\mathbb{BC}M_{n+2}&\mathbb{BC}M_{n+3}\\\mathbb{BC}M_{n+3}&\mathbb{BC}M_{n+2}+\mathbb{BC}M_{n+1}+\mathbb{BC}M_{n}&\mathbb{BC}M_{n+2}+\mathbb{BC}M_{n+1}&\mathbb{BC}M_{n+2}\\\mathbb{BC}M_{n+2}&\mathbb{BC}M_{n+1}+\mathbb{BC}M_{n}+\mathbb{BC}M_{n-1}&\mathbb{BC}M_{n+1}+\mathbb{BC}M_{n}&\mathbb{BC}M_{n+1}\end{pmatrix}$$

(b):

(3.5)
$$B_R\begin{pmatrix}1&1&1&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}^n=\begin{pmatrix}\mathbb{BC}R_{n+5}&\mathbb{BC}R_{n+4}+\mathbb{BC}R_{n+3}+\mathbb{BC}R_{n+2}&\mathbb{BC}R_{n+4}+\mathbb{BC}R_{n+3}&\mathbb{BC}R_{n+4}\\\mathbb{BC}R_{n+4}&\mathbb{BC}R_{n+3}+\mathbb{BC}R_{n+2}+\mathbb{BC}R_{n+1}&\mathbb{BC}R_{n+3}+\mathbb{BC}R_{n+2}&\mathbb{BC}R_{n+3}\\\mathbb{BC}R_{n+3}&\mathbb{BC}R_{n+2}+\mathbb{BC}R_{n+1}+\mathbb{BC}R_{n}&\mathbb{BC}R_{n+2}+\mathbb{BC}R_{n+1}&\mathbb{BC}R_{n+2}\\\mathbb{BC}R_{n+2}&\mathbb{BC}R_{n+1}+\mathbb{BC}R_{n}+\mathbb{BC}R_{n-1}&\mathbb{BC}R_{n+1}+\mathbb{BC}R_{n}&\mathbb{BC}R_{n+1}\end{pmatrix}$$

Proof. We prove (a) by mathematical induction on n. If $n = 0$ then the result is clear. Now, we assume it is true for $n = k$, that is

$$B_MB^k=\begin{pmatrix}\mathbb{BC}M_{k+5}&\mathbb{BC}M_{k+4}+\mathbb{BC}M_{k+3}+\mathbb{BC}M_{k+2}&\mathbb{BC}M_{k+4}+\mathbb{BC}M_{k+3}&\mathbb{BC}M_{k+4}\\\mathbb{BC}M_{k+4}&\mathbb{BC}M_{k+3}+\mathbb{BC}M_{k+2}+\mathbb{BC}M_{k+1}&\mathbb{BC}M_{k+3}+\mathbb{BC}M_{k+2}&\mathbb{BC}M_{k+3}\\\mathbb{BC}M_{k+3}&\mathbb{BC}M_{k+2}+\mathbb{BC}M_{k+1}+\mathbb{BC}M_{k}&\mathbb{BC}M_{k+2}+\mathbb{BC}M_{k+1}&\mathbb{BC}M_{k+2}\\\mathbb{BC}M_{k+2}&\mathbb{BC}M_{k+1}+\mathbb{BC}M_{k}+\mathbb{BC}M_{k-1}&\mathbb{BC}M_{k+1}+\mathbb{BC}M_{k}&\mathbb{BC}M_{k+1}\end{pmatrix}.$$

If we use (2.3), then we have $\mathbb{BC}M_{k+4} = \mathbb{BC}M_{k+3} + \mathbb{BC}M_{k+2} + \mathbb{BC}M_{k+1} + \mathbb{BC}M_k$. Then by induction hypothesis, we obtain

$$\begin{aligned}B_MB^{k+1}&=(B_MB^k)B\\&=\begin{pmatrix}\mathbb{BC}M_{k+5}&\mathbb{BC}M_{k+4}+\mathbb{BC}M_{k+3}+\mathbb{BC}M_{k+2}&\mathbb{BC}M_{k+4}+\mathbb{BC}M_{k+3}&\mathbb{BC}M_{k+4}\\\mathbb{BC}M_{k+4}&\mathbb{BC}M_{k+3}+\mathbb{BC}M_{k+2}+\mathbb{BC}M_{k+1}&\mathbb{BC}M_{k+3}+\mathbb{BC}M_{k+2}&\mathbb{BC}M_{k+3}\\\mathbb{BC}M_{k+3}&\mathbb{BC}M_{k+2}+\mathbb{BC}M_{k+1}+\mathbb{BC}M_{k}&\mathbb{BC}M_{k+2}+\mathbb{BC}M_{k+1}&\mathbb{BC}M_{k+2}\\\mathbb{BC}M_{k+2}&\mathbb{BC}M_{k+1}+\mathbb{BC}M_{k}+\mathbb{BC}M_{k-1}&\mathbb{BC}M_{k+1}+\mathbb{BC}M_{k}&\mathbb{BC}M_{k+1}\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}\\&=\begin{pmatrix}\mathbb{BC}M_{k+6}&\mathbb{BC}M_{k+5}+\mathbb{BC}M_{k+4}+\mathbb{BC}M_{k+3}&\mathbb{BC}M_{k+5}+\mathbb{BC}M_{k+4}&\mathbb{BC}M_{k+5}\\\mathbb{BC}M_{k+5}&\mathbb{BC}M_{k+4}+\mathbb{BC}M_{k+3}+\mathbb{BC}M_{k+2}&\mathbb{BC}M_{k+4}+\mathbb{BC}M_{k+3}&\mathbb{BC}M_{k+4}\\\mathbb{BC}M_{k+4}&\mathbb{BC}M_{k+3}+\mathbb{BC}M_{k+2}+\mathbb{BC}M_{k+1}&\mathbb{BC}M_{k+3}+\mathbb{BC}M_{k+2}&\mathbb{BC}M_{k+3}\\\mathbb{BC}M_{k+3}&\mathbb{BC}M_{k+2}+\mathbb{BC}M_{k+1}+\mathbb{BC}M_{k}&\mathbb{BC}M_{k+2}+\mathbb{BC}M_{k+1}&\mathbb{BC}M_{k+2}\end{pmatrix}\end{aligned}$$

Thus, (3.4) holds for all non-negative integers n.

(3.5) can be similarly proved .

Corollary 9. *For $n \geq 0$, the followings hold:*

(a): $\mathbb{BC}M_{n+3} = \mathbb{BC}M_3M_{n+1} + (\mathbb{BC}M_2 + \mathbb{BC}M_1 + \mathbb{BC}M_0)M_n + (\mathbb{BC}M_1 + \mathbb{BC}M_2)M_{n-1} + \mathbb{BC}M_2M_{n-2}$

(b): $\mathbb{BC}R_{n+3} = \mathbb{BC}R_3M_{n+1} + (\mathbb{BC}R_2 + \mathbb{BC}R_1 + \mathbb{BC}R_0)M_n + (\mathbb{BC}R_1 + \mathbb{BC}R_2)M_{n-1} + \mathbb{BC}R_2M_{n-2}$.

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Proof. The proof of (a) can be seen by the coefficient of the matrix B_M and (3.1). The proof of (b) can be seen by the coefficient of the matrix B_R and (3.1).

4. Five-Diagonal Matrix with Fourth Order Sequences and Applications

In this section we give another way to obtain nth term of the bicomplex Tetranacci and Tetranacci-Lucas quaternions. For this we need the following theorem.

THEOREM 10. *Let $\{x_n\}$ be any fourth-order linear sequence defined recursively as follows:*

$$x_n = rx_{n-1} + sx_{n-2} + tx_{n-3} + ux_{n-4}, \qquad n \geq 4$$

with the initial conditions $x_0 = a$, $x_1 = b$, $x_2 = c$, $x_3 = d$. Then for all $n \geq 0$, we have

$$x_n = \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & u & t & s & r & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & s & r & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & t & s & r & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & u & t & s & r \end{vmatrix}_{(n+1)\times(n+1)}.$$

Proof. We proceed by induction on n. Since

$$x_0 = |a|_{1\times 1} = a,\ x_1 = \begin{vmatrix} a & -1 \\ b & 0 \end{vmatrix}_{2\times 2} = b,\ x_2 = \begin{vmatrix} a & -1 & 0 \\ b & 0 & -1 \\ c & 0 & 0 \end{vmatrix}_{3\times 3} = c,\ x_3 = \begin{vmatrix} a & -1 & 0 & 0 \\ b & 0 & -1 & 0 \\ c & 0 & 0 & -1 \\ d & 0 & 0 & 0 \end{vmatrix}_{4\times 4} = d,$$

the equality holds for $n = 0, 1, 2, 3$. Now we assume that the equality is true for $4 \leq k \leq n$. Then we will complete the inductive step $n+1$ as follows: Note that

$$x_{n+1} = \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & r & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & s & r & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & t & s & r \end{vmatrix}_{(n+2)\times(n+2)}$$

and

$$\begin{aligned} x_{n+1} &= r(-1)^{(n+2)+(n+2)} \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & r & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & s & r \end{vmatrix}_{(n+1)\times(n+1)} \\ &+(-1)(-1)^{(n+2)+(n+1)} \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & r & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & t & s \end{vmatrix}_{(n+1)\times(n+1)} \end{aligned}$$

and so

$$
\begin{aligned}
x_{n+1} &= r x_n + s \begin{vmatrix}
a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\
d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\
0 & u & t & s & r & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & r
\end{vmatrix}_{n\times n} \\
&+ \begin{vmatrix}
a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\
d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\
0 & u & t & s & r & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & t
\end{vmatrix}_{n\times n} .
\end{aligned}
$$

Now it follows that

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$$
\begin{aligned}
x_n &= rx_n + sx_{n-1} + t\begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & t \end{vmatrix}_{(n-1)\times(n-1)} \\
&+ \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & u \end{vmatrix}_{(n-1)\times(n-1)} \\
&= rx_n + sx_{n-1} + tx_{n-2} + \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & u \end{vmatrix}_{(n-1)\times(n-1)} \\
&= rx_n + sx_{n-1} + tx_{n-2} + u(-1)^{(n-1)+(n-1)}\begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -1 \end{vmatrix}_{(n-2)\times(n-2)} \\
&= rx_n + sx_{n-1} + tx_{n-2} + ux_{n-3}.
\end{aligned}
$$

This completes the inductive step and the proof of the theorem.

Note that in our cases $r = s = t = u = 1$. As a corallary of the above theorem, in the following we present another way to obtain nth term of the bicomplex Tetranacci and Tetranacci-Lucas quaternions.

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Corollary 11. *For all $n \geq 0$, we have*

(a):

$$\mathbb{C}M_n = \begin{vmatrix} \mathbb{BC}M_0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}M_1 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}M_2 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}M_3 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 & 1 \end{vmatrix}_{(n+1)\times(n+1)}.$$

(b):

$$\mathbb{BC}R_n = \begin{vmatrix} \mathbb{BC}R_0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}R_1 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}R_2 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}R_3 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 & 1 \end{vmatrix}_{(n+1)\times(n+1)}.$$

Proof. (a) follows from (2.3) and Theorem 10. (b) follows from (2.4) and Theorem 10.

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