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On new quicker n-tupled fixed point implicit iterations for contractive-like mappings and comparison of their rate of convergence in hyperbolic metric spaces

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Abstract

In this paper, we study the convergence of new implicit iterations dealing with n-tupled fixed point results for nonlinear contractive-like mappings on W-hyperbolic metric spaces. Herein, we demonstrate that our newly implicit iteration schemes have faster rate of convergence than implicit S-iteration process, implicit Ishikawa and Mann type iteration processes. Furthermore, a numerical simulation to improve our theoretical results is obtained.

Key words: Implicit iterations, $\varphi-$contractive mappings, convergence rate.

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1 Introduction and Preliminaries

The classical Banach Contraction Principle is one of the most important results of analysis which considered as the main source of metric fixed point theory. It states that every contraction operator on a complete metric

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space has a unique fixed point. A great deal of expansions of this principle have been done, for the most part by generalizing the contraction operator and some of the time by expanding the necessity of completeness or even both. This principle is applied to prove the existence and uniqueness of solutions of nonlinear Volterra integral equations and nonlinear integro-differential equations in Banach spaces other than supporting the assembly of calculations in Computational Mathematics.

In 1987, Guo and Lakshmikantham [5] presented some results about coupled fixed point. Thereafter, in 2006, Bhaskar and Lakshmikantham [6] gave some fixed point results for weak contractivity type mappings on a partially ordered complete metric space, using a mixed monotone property. In 2009, Lakshmikantham and Ciric [7] defined the concept of mixed g-monotone mappings and introduced coupled coincidence and coupled common fixed point results for nonlinear contractive mappings in a metric space endowed with a partial ordering, which also extended the fixed point theorems due to Bhaskar and Lakshmikantham [5]. In 2013, Imdad et al. [14] gave the definition of n-tupled common fixed point as well as n-tupled common coincidence and utilized these two notions to obtain n-tupled coincidence as well as n-tupled common fixed point results for contraction operators. The notions given by Imdad et al. [14] are quite different from the notions of Roldán et al. [16].

On the other hand, there is one, may be more characteristic, approach to respect the idea of convexity. Convexity in a vector space is characterized utilizing lines between points; in Euclidean spaces or more generally in Banach spaces, there is a line of most brief length that joins two points, and the length of this line is the line segment between its two endpoints. However, metric spaces do not naturally have this convex structure. In 1970, Takahashi [18] defined the concept of convexity in metric spaces and gave some fixed point theorems for non-expansive mappings in such spaces. A convex metric space is more general than normed space and cone Banach space [18]. After some time, diverse raised convex structures have been presented on metric spaces. In 2004, Kohlenbach [11] gave the notion of W-hyperbolic spaces which represents a consolidated approach for both linear and nonlinear structures at the same time. It is worth noting that every hyperbolic space is a convex metric space defined by Takahashi [18] but converse may not be true in general [3].

Iterative methods for approximating fixed points in convex metric spaces have been studied by many authors (see, e.g., [3, 4, 8, 9]), using implicit iterative procedures which are incredible significance from numerical angle as they give precise estimate.

The following forms give a metric version of implicit Ishikawa and implicit Mann iteration schems defined by Ćirić et al. [1, 2] in the background of W-hyperbolic space.

Let E be a nonempty convex subset of a W-hyperbolic space X and $T : E \to E$. Choose $x_0 \in E$ and define the sequence $\{x_n\}$ as follows:

$$x_n = W(x_{n-1}, Ty_n, \alpha_n) \tag{1}$$
$$y_n = W(x_n, Tx_n, \beta_n), \quad n \in \mathbb{N},$$

and

$$x_n = W(x_{n-1}, Tx_n, \alpha_n), \quad n \in \mathbb{N}. \tag{2}$$

Recently, Yildirim and Abbas [13] introduced implicit S-iteration process with higher rate of convergence than Mann type (2) and Ishikawa type (1) implicit iterative processes.

Initiated with $x_0 \in E$, then we get the following sequence $\{x_n\}$:

$$x_n = W(Tx_{n-1}, Ty_n, \alpha_n) \tag{3}$$
$$y_n = W(x_n, Tx_n, \beta_n), \quad n \in \mathbb{N},$$

where (α_n) and (β_n) are certain real sequences in $[0, 1]$. In this paper, we introduce some new implicit iteration process and study their rate of convergence in hyperbolic metric spaces. Also, we give a numerical example to exhibit the utility of our proved results.

We need the following definitions and lemma in order to introduce our main results.

Definition 1 *[11] A W-hyperbolic space (X, d, W) is a metric space (X, d) together with a convex mapping $W : X^2 \times [0, 1] \to X$ satisfying the following properties:*

(i) $d(u, W(x, y, \alpha)) \leq (1 - \alpha)d(u, x) + \alpha d(u, y)$,

(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = \|\alpha - \beta\| d(x, y)$,

(iii) $W(x, y, \alpha) = W(y, x, 1 - \alpha)$,

(iv) $d(W(x,z,\alpha),W(y,w,\alpha)) \le (1-\alpha)d(x,y)+\alpha d(z,w)$,

for all $x,y,z,w \in X$ *and* $\alpha,\beta \in [0,1]$.

Definition 2 *[13] A self mapping* T *on* X *is called a contractive-like mapping if there exists a constant* $\delta \in [0,1)$ *and a strictly increasing and continuous function* $\varphi : [0,1) \to [0,1)$ *with* $\varphi(0)=0$ *such that for any* $x,y \in X$ *we have*

$$d(Tx,Ty) \le \delta d(x,y) + \varphi(d(x,Tx)). \tag{4}$$

Definition 3 *[5] An element* $(x,y) \in X \times X$ *is called a coupled fixed point of the mapping* $T : X \times X \to X$ *if*

$$T(x,y) = x \text{ and } T(y,x) = y.$$

Definition 4 *[14] Let* X *be a nonempty set.An element* $x^1,x^2,x^3,...,x^n \in X^n$ *is called an n-tupled fixed point of the mapping* T *if*

$$x^1 = T(x_2^1,x^2,x^3,...,x^n),$$

$$x^2 = T(x^2,x_2^3,x^3,...,x^n,x^1),$$

$$\vdots$$

$$x^n = T(x^n,x^1,x^2,...,x^{n-1}).$$

Lemma 5 *[17] Let* $\{c_n\}_{n=1}^{\infty} \subset [0,\infty)$*. If there exists an* $n_0 \in \mathbb{N}$ *such that for all* $n \ge n_0$*, we get*

$$c_{n+1} \le (1+\eta_n)c_n + \eta_n\theta_n,$$

where $\eta_n \in (0,1)$*,* $\sum\limits_{n=0}^{\infty} \eta = \infty$ *and* $\theta_n \ge 0$ *for all* $n \in \mathbb{N}$*. Then the following holds:*

$$0 \le \lim_{n\to\infty} sup\, c_n \le \lim_{n\to\infty} sup\, \theta_n.$$

2 Main Results

Definition 6 *Let* E *be a nonempty convex subset of a W-hyperbolic space* X *and* $T : \prod\limits_{i=1}^{n} E \to E$*. Choose* $\{x_0^i\} \in E$ *and define the two sequences* $\{x_m^i\}$ *and* $\{y_m^i\}$ *for each* $i = 1,2,3,...,n$ *as follows:*

$$x_m^1 = W(T^m(x_{m-1}^1,x_{m-1}^2,\ldots,x_{m-1}^n),T(y_m^1,y_m^2,\ldots,y_m^n),\alpha_m) \tag{5}$$

$$x_m^2 = W(T^m(x_{m-1}^2, x_{m-1}^3, \dots, x_{m-1}^n, x_{m-1}^1), T(y_m^2, y_m^3, \dots, y_m^n, y_m^1), \alpha_m)$$

$$\vdots$$

$$x_m^n = W(T^m(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, \dots, x_{m-1}^{n-1}), T(y_m^n, y_m^1, y_{m-1}^2, \dots, y_m^{n-1}), \alpha_m)$$

$$y_m^1 = W(x_m^1, T(x_m^1, x_m^2, \dots, x_m^n), \beta_m)$$

$$y_m^2 = W(x_m^2, T(x_m^2, x_m^3, \dots, x_m^n, x_m^1), \beta_m)$$

$$\vdots$$

$$y_m^n = W(x_m^n, T(x_m^n, x_m^1, x_m^2, \dots, x_m^{n-1}), \beta_m), \quad m, n \in \mathbb{N},$$

where $T^m(x^1, x^2, \dots, x^n) = T(T^{m-1}(x^1, x^2, \dots, x^n), T(x^2, x^3, \dots, x^n, x^1), \dots, T(x^n, x^1, \dots, x^{n-1}))$ *and* (α_m), (β_m) *are certain real sequences in* $[0, 1]$.

Definition 7 *Let* E *be a nonempty convex subset of a W-hyperbolic space* X *and* $T : \prod_{i=1}^{n} E \to E$. *Choose* $\{x_0^i\} \in E$ *and define the two sequences* $\{x_m^i\}$ *and* $\{y_m^i\}$ *for each* $i = 1, 2, 3, ..., n$ *as follows:*

$$x_m^1 = W(T(x_{m-1}^1, x_{m-1}^2, \dots, x_{m-1}^n), T(y_m^1, y_m^2, \dots, y_m^n), \alpha_m) \tag{6}$$

$$x_m^2 = W(T(x_{m-1}^2, x_{m-1}^3, \dots, x_{m-1}^n, x_{m-1}^1), T(y_m^2, y_m^3, \dots, y_m^n, y_{m-1}^1), \alpha_m)$$

$$\vdots$$

$$x_m^n = W(T(x_{m-1}^n, x_{m-1}^1, x_{m-1}^2, \dots, x_{m-1}^{n-1}), T(y_m^n, y_m^1, y_{m-1}^2, \dots, y_m^{n-1}), \alpha_m)$$

$$y_m^1 = W(x_m^1, T(x_m^1, x_m^2, \dots, x_m^n), \beta_m)$$

$$y_m^2 = W(x_m^2, T(x_m^2, x_m^3, \dots, x_m^n, x_m^1), \beta_m)$$

$$\vdots$$

$$y_m^n = W(x_m^n, T(x_m^n, x_m^1, x_m^2, \dots, x_m^{n-1}), \beta_m), \quad m, n \in \mathbb{N},$$

where (α_m) *and* (β_m) *are certain real sequences in* $[0, 1]$.

Definition 8 *Let* E *be a nonempty convex subset of a W-hyperbolic space* X *and* $T : \prod_{i=1}^{n} E \to E$. *Choose* $\{x_0^i\} \in E$ *and define the two sequences* $\{x_m^i\}$ *and* $\{y_m^i\}$ *for each* $i = 1, 2, 3, ..., n$ *as follows:*

$$x_m^1 = W(x_{m-1}^1, T(y_m^1, y_m^2, \dots, y_m^n), \alpha_m) \tag{7}$$

$$x_m^2 = W(x_{m-1}^2, T(y_m^2, y_m^3, \ldots, y_m^n, y_{m-1}^1), \alpha_m)$$
$$\vdots$$
$$x_m^n = W(x_{m-1}^n, T(y_m^n, y_m^1, y_{m-1}^2, \ldots, y_m^{n-1}), \alpha_m)$$
$$y_m^1 = W(x_m^1, T(x_m^1, x_m^2, \ldots, x_m^n), \beta_m)$$
$$y_m^2 = W(x_m^2, T(x_m^2, x_m^3, \ldots, x_m^n, x_m^1), \beta_m)$$
$$\vdots$$
$$y_m^n = W(x_m^n, T(x_m^n, x_m^1, x_m^2, \ldots, x_m^{n-1}), \beta_m), \quad m, n \in \mathbb{N},$$

where (α_m) *and* (β_m) *are certain real sequences in* $[0, 1]$.

Definition 9 *Let* E *be a nonempty convex subset of a W-hyperbolic space* X *and* $T : \prod_{i=1}^{n} E \to E$. *Choose* $\{x_0^i\} \in E$ *and define the two sequences* $\{x_m^i\}$ *and* $\{y_m^i\}$ *for each* $i = 1, 2, 3, ..., n$ *as follows:*

$$x_m^1 = W(x_{m-1}^1, T(x_m^1, x_m^2, \ldots, x_m^n), \alpha_m) \tag{8}$$
$$x_m^2 = W(x_{m-1}^2, T(x_m^2, x_m^3, \ldots, x_m^n, x_{m-1}^1), \alpha_m)$$
$$\vdots$$
$$x_m^n = W(x_{m-1}^n, T(x_m^n, x_m^1, x_{m-1}^2, \ldots, x_m^{n-1}), \alpha_m), \quad m, n \in \mathbb{N},$$

where (α_m) *and* (β_m) *are certain real sequences in* $[0, 1]$.

Definition 10 *A self mapping* T *on* $X \times X$ *is called a contractive-like mapping if there exists a constant* $\delta \in [0, 1)$ *and a strictly increasing and continuous function* $\varphi : [0, \infty) \to [0, \infty)$ *with* $\varphi(0) = 0$, *such that for any* $x, y \in X$ *we have*

$$\begin{aligned} d(T(x^1, x^2, \ldots, x^n), T((y^1, y^2, \ldots, y^n))) \quad &\leq \frac{\delta}{n}[d(x^1, y^1) + d(x^2, y^2) + \ldots + d(x^n, y^n)] + \varphi(d(x^1, T(x^1, x^2, \ldots, x^n)) + \\ &d(x^2, T(x^2, x^3, \ldots, x^n, x^1)) + \ldots + d(x^n, T(x^n, x^1, \ldots, x^{n-1}))). \end{aligned} \tag{9}$$

Theorem 11 *Let* $T : \prod_{i=1}^{n} E \to E$ *be a contractive-like mapping defined in* (9) *on a nonempty closed convex subset* E *of a W-hyperbolic metric space* (X, d, W) *with* $F(T) \neq \phi$ *(*$F(T)$ *denote the set of all fixed points of* T*). Then, for the sequence* $\{x_m\}$ *defined in* (5) *we have* $\lim_{m\to\infty} x_m^i = p^i$, *where* $(p^1, p^2, ..., p^i) \in F(T)$.

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Proof. Assume that $(p^1, p^2, ..., p^i) \in F(T)$. Using (5) and (9), we get

$$\begin{aligned}
& d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n) \\
&= d(W(T^m(x_{m-1}^1, x_{m-1}^2, \ldots, x_{m-1}^n), T(y_m^1, \ldots, y_{m-1}^n), \alpha_m), p^1) \\
&+ d(W(T^m(x_{m-1}^2, x_{m-1}^3, \ldots, x_{m-1}^n, x_{m-1}^1), T(y_m^2, \ldots, y_{m-1}^n, y_{m-1}^1), \alpha_m), p^2) \\
&\vdots \\
&+ d(W(T^m(x_{m-1}^n, x_{m-1}^1, \ldots, x_{m-1}^{n-1}), T(y_m^n, y_{m-1}^1, \ldots, y_{m-1}^{n-1}), \alpha_m), p^n) \\
&\leq \alpha_m d(T(T^{m-1}(x_{m-1}^1, x_{m-1}^2, \ldots, x_{m-1}^n), T^{m-1}(x_{m-1}^2, x_{m-1}^3, \ldots, x_{m-1}^1), \ldots, \\
& T^{m-1}(x_{m-1}^n, x_{m-1}^1, \ldots, x_{m-1}^{n-1})), T(p^1, p^2, \ldots, p^n)) \\
& + (1-\alpha_m) d(T(y_m^1, \ldots, y_{m-1}^n), T(p^1, p^2, \ldots, p^n)) \\
& + \alpha_m d(T(T^{m-1}(x_{m-1}^2, x_{m-1}^3, \ldots, x_{m-1}^1), T^{m-1}(x_{m-1}^3, x_{m-1}^4, \ldots, x_{m-1}^2), \ldots, \\
& T^{m-1}(x_{m-1}^1, x_{m-1}^2, \ldots, x_{m-1}^n)), T(p^2, p^3, \ldots, p^n, p^1)) \\
& + (1-\alpha_m) d(T(y_m^2, \ldots, y_{m-1}^1), T(p^2, p^3, \ldots, p^1)) \\
&\vdots \\
& + \alpha_m d(T(T^{m-1}(x_{m-1}^n, x_{m-1}^1, \ldots, x_{m-1}^{n-1}), T^{m-1}(x_{m-1}^1, x_{m-1}^2, \ldots, x_{m-1}^n), \ldots, \\
& T^{m-1}(x_{m-1}^{n-1}, x_{m-1}^n, x_{m-1}^1 \ldots, x_{m-1}^{n-2})), T(p^n, p^1, \ldots, p^{n-1})) \\
& + (1-\alpha_m) d(T(y_m^n, T(y_m^1, \ldots, y_{m-1}^{n-1}), T(p^n, p^1, \ldots, p^n)) \\
&\leq \alpha_m \delta^m [d(x_{m-1}^1, p^1) + d(x_{m-1}^2, p^2) + \ldots + d(x_{m-1}^n, p^n)] + (1-\alpha_n)\delta[d(y_m^1, p^1) + d(y_m^2, p^2) + \ldots + d(y_m^n, p^n)],
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
& d(y_m^1, p^1) + d(y_m^2, p^2) + \ldots + d(y_m^n, p^n) \\
&= d(W(x_m^1, T(x_m^1, x_m^2, \ldots, x_m^n), \beta_m)), p^1) \\
&+ d(W(x_m^2, T(x_m^2, x_m^3, \ldots, x_m^1), \beta_m)), p^2) \\
&\vdots \\
&+ d(W(x_m^n, T(x_m^n, x_m^1, \ldots, x_m^{n-1}), \beta_m)), p^n)
\end{aligned} \tag{11}$$

$$\leq \beta_m[d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n)]$$

$$+(1-\beta_m)[d(T(x_m^1, x_m^2, \ldots, x_m^n), T(p^1, p^2, \ldots, p^n)) + d(T(x_m^2, x_m^3, \ldots, x_m^1), T(p^2, p^3, \ldots, p^1))$$

$$\vdots$$

$$+d(T(x_m^n, x_m^1, \ldots, x_m^{n-1}), T(p^n, p^1, \ldots, p^{n-1}))]$$

$$\leq (\beta_m + (1-\beta_m)\delta)[d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n)].$$

By (10) and (11), we have

$$\begin{aligned} & d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n) \\ & \leq \alpha_m \delta^m [d(x_{m-1}^1, p^1) + d(x_{m-1}^2, p^2) + \ldots + d(x_{m-1}^n, p^n)] \\ & +(1-\alpha_m)\delta(\beta_m + (1-\beta_m)\delta)[d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n)] \\ & < \alpha_m \delta^m [d(x_{m-1}^1, p^1) + d(x_{m-1}^2, p^2) + \ldots + d(x_{m-1}^n, p^n)] \\ & +(1-\alpha_m)\delta[d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n)] \end{aligned} \tag{12}$$

which implies that

$$d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n) \quad < D_m[d(x_{m-1}^1, p^1) + d(x_{m-1}^2, p^2) + \ldots + d(x_{m-1}^n, p^n)], \tag{13}$$

where

$$D_m = \frac{\alpha_m \delta^m}{1-(1-\alpha_m)\delta}.$$

Observe that

$$\begin{aligned} 1 - D_m \quad & = 1 - \frac{\alpha_m \delta^m}{1-(1-\alpha_m)\delta} = \frac{1-(1-\alpha_m)\delta - \alpha_m\delta^m}{1-(1-\alpha_n)\delta} \\ & \geq 1-(1-\alpha_m)\delta - \alpha_m\delta^m \end{aligned}$$

implies that

$$D_m \quad \leq (1-\alpha_m)\delta + \alpha_m\delta^m < \delta \tag{14}$$

Now, in view of (13) and (14), we have

$$d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n) \quad < D_m[d(x_{m-1}^1, p^1) + d(x_{m-1}^2, p^2) + \ldots + d(x_{m-1}^n, p^n)] \tag{15}$$

$$< \delta[d(x^1_{m-1}, p^1) + d(x^2_{m-1}, p^2) + \ldots + d(x^n_{m-1}, p^n)]$$

$$\vdots$$

$$< \delta^m[d(x^1_0, p^1) + d(x^2_0, p^2) + \ldots + d(x^n_0, p^n)].$$

Taking limit on both sides of the above inequality, we have $\lim_{m\to\infty}[d(x^1_m, p^1) + d(x^2_m, p^2) + \ldots + d(x^n_m, p^n)] = 0$.

Theorem 12 *Let $T : \prod_{i=1}^{n} E \to E$ be a contractive-like mapping defined in* (9) *on a nonempty closed convex subset E of a W-hyperbolic metric space (X, d, W) with $F(T) \neq \phi$. Then, for the sequence $\{x_n\}$ defined in* (7) *we have $\lim_{m\to\infty} x^i_m = p^i$, where $(p^1, p^2, ..., p^i) \in F(T)$.*

Proof. Assume that $(p^1, p^2, ..., p^i) \in F(T)$. Using (7) and (9), we get

$$(x^1_m, p^1) + d(x^2_m, p^2) + \ldots + d(x^n_m, p^n) \tag{16}$$

$$= d(W(T(x^1_{m-1}, x^2_{m-1}, \ldots, x^n_{m-1}), T(y^1_m y^2_m, \ldots, y^n_m), \alpha_m)), p^1)$$

$$+d(W(T(x^2_{m-1}, x^3_{m-1}, \ldots, x^1_{m-1}), T(y^2_m, y^3_m, \ldots, y^1_m), \alpha_m)), p^2)$$

$$\vdots$$

$$+d(W(T(x^n_{m-1}, x^1_{m-1}, \ldots, x^{n-1}_{m-1}), T(y^n_m, y^1_m, \ldots, y^{n-1}_m), \alpha_m)), p^n)$$

$$\leq \alpha_m[d(T(x^1_{m-1}, x^2_{m-1}, \ldots, x^n_{m-1}), T(p^1, p^2, \ldots, p^n)) + \ldots + d(T(x^n_{m-1}, x^1_{m-1}, \ldots, x^{n-1}_{m-1}), T(p^n, p^1, \ldots, p^{n-1}))]$$

$$+(1-\alpha_m)[d(T(y^1_m, y^2_m, \ldots, y^n_m), T(p^1, p^2, \ldots, p^n)) + \ldots + d(T(y^n_m, y^1_m, \ldots, y^{n-1}_m), T(p^n, p^1, \ldots, p^{n-1}))]$$

$$\leq \alpha_m \delta[d(x^1_{m-1}, p^1) + d(x^2_{m-1}, p^2) + \ldots + d(x^n_{m-1}, p^n)]$$

$$+(1-\alpha_m)\delta[d(y^1_m, p^1) + d(y^2_m, p^2) + \ldots + d(y^n_m, p^n)]$$

and

$$d(y^1_m, p^1) + d(y^2_m, p^2) + \ldots + d(y^n_m, p^n) \tag{17}$$

$$= d(W(x^1_m, T(x^1_m, x^2_m, \ldots, x^n_m), \beta_m)), p^1)$$

$$+d(W(x^2_m, T(x^2_m, x^3_m, \ldots, x^1_m), \beta_m)), p^2)$$

$$\vdots$$

$$+d(W(x^n_m, T(x^n_m, x^1_m, \ldots, x^{n-1}_m), \beta_m)), p^n)$$

$$\leq \beta_m[d(x_m^1,p^1)+d(x_m^2,p^2)+\ldots+d(x_m^n,p^n)]$$

$$+(1-\beta_m)[d(T(x_m^1,x_m^2,\ldots,x_m^n),T(p^1,p^2,\ldots,p^n))+d(T(x_m^2,x_m^3,\ldots,x_m^1),T(p^2,p^3,\ldots,p^1))$$

$$\vdots$$

$$+d(T(x_m^n,x_m^1,\ldots,x_m^{n-1}),T(p^n,p^1,\ldots,p^{n-1}))]$$

$$\leq (\beta_m+(1-\beta_m)\delta)[d(x_m^1,p^1)+d(x_m^2,p^2)+\ldots+d(x_m^n,p^n)].$$

Therefore,

$$\begin{aligned} & d(x_m^1,p^1)+d(x_m^2,p^2)+\ldots+d(x_m^n,p^n) \\ & \leq \alpha_m\delta[d(x_{m-1}^1,p^1)+d(x_{m-1}^2,p^2)+\ldots+d(x_{m-1}^n,p^n)] \\ & +(1-\alpha_m)\delta(\beta_m+(1-\beta_m)\delta)[d(x_m^1,p^1)+d(x_m^2,p^2)+\ldots+d(x_m^n,p^n)] \\ & < \alpha_m\delta[d(x_{m-1}^1,p^1)+d(x_{m-1}^2,p^2)+\ldots+d(x_{m-1}^n,p^n)] \\ & +(1-\alpha_m)\delta[d(x_m^1,p^1)+d(x_m^2,p^2)+\ldots+d(x_m^n,p^n)], \end{aligned} \tag{18}$$

which implies that

$$d(x_m^1,p^1)+d(x_m^2,p^2)+\ldots+d(x_m^n,p^n) \quad < D_m[d(x_{m-1}^1,p^1)+d(x_{m-1}^2,p^2)+\ldots+d(x_{m-1}^n,p^n)], \tag{19}$$

where

$$D_m=\frac{\alpha_m\delta}{1-(1-\alpha_m)\delta}.$$

Observe that

$$1-D_m \quad =1-\frac{\alpha_m\delta}{1-(1-\alpha_m)\delta}=\frac{1-\delta}{1-(1-\alpha_m)\delta}\geq 1-\delta$$

implies that

$$D_m \quad \leq \delta \tag{20}$$

Due to (19) and (20), we have

$$d(x_m^1,p^1)+d(x_m^2,p^2)+\ldots+d(x_m^n,p^n) \quad < \delta[d(x_{m-1}^1,p^1)+d(x_{m-1}^2,p^2)+\ldots+d(x_{m-1}^n,p^n)] \tag{21}$$

$$< (\delta)^2[d(x^1_{m-2}, p^1) + d(x^2_{m-2}, p^2) + \ldots + d(x^n_{m-2}, p^n)]$$

$$\vdots$$

$$< (\delta)^m[d(x^1_0, p^1) + d(x^2_0, p^2) + \ldots + d(x^n_0, p^n)].$$

Taking limit on both sides of the above inequality, we obtain $\lim_{m\to\infty}[d(x^1_m, p^1)+d(x^2_m, p^2)+\ldots+d(x^n_m, p^n)] = 0$.

Theorem 13 *Let* $T : \prod_{i=1}^{n} E \to E$ *be a contractive-like mapping defined in* (9) *on a nonempty closed convex subset* E *of a W-hyperbolic metric space* (X, d, W) *with* $F(T) \neq \phi$. *Then, for the sequences* $\{x_n\}$, $\{y_n\}$ *defined in* (6) *with* $\sum(1 - \varphi_n) = \infty$, *we have* $\lim_{m\to\infty} x^i_m = p^i$, *where* $(p^1, p^2, ..., p^i) \in F(T)$.

Proof. Assume that $(p^1, p^2, ..., p^i) \in F(T)$. Using (7) and (9), we get

$$\begin{aligned} & d(x^1_m, p^1) + d(x^2_m, p^2) + \ldots + d(x^n_m, p^n) \qquad (22)\\ &= d(W(x^1_{m-1}, T(y^1_m, y^2_m, \ldots, y^n_m), \alpha_m), p^1) + d(W(x^2_{m-1}, T(y^2_m, y^3_m, \ldots, y^1_m), \alpha_m), p^2)\\ &+ \ldots + d(W(x^n_{m-1}, T(y^n_m, y^1_m, \ldots, y^{n-1}_m), \alpha_m), p^n)\\ &\leq \alpha_m[x^1_m, p^1) + d(x^2_m, p^2) + \ldots + d(x^n_m, p^n)] + (1-\alpha_m)\delta[d(y^1_m, p^1) + d(y^2_m, p^2) + \ldots + d(y^n_m, p^n)]. \end{aligned}$$

And

$$d(y^1_m, p^1) + d(y^2_m, p^2) + \ldots + d(y^n_m, p^n) \leq (\beta_n + (1-\beta_n)\delta)[d(x^1_m, p^1) + d(x^2_m, p^2) + \ldots + d(x^n_m, p^n)].(23)$$

Therefore

$$\begin{aligned} d(x^1_m, p^1) + d(x^2_m, p^2) + \ldots + d(x^n_m, p^n) &\leq \alpha_m[d(x^1_{m-1}, p^1) + d(x^2_{m-1}, p^2) + \ldots + d(x^n_{m-1}, p^n)] \qquad (24)\\ &+(1-\alpha_m)\delta(\beta_m + (1-\beta_m)\delta)[d(x^1_m, p^1) + d(x^2_m, p^2) + \ldots + d(x^n_m, p^n)]\\ &< \alpha_m[d(x^1_{m-1}, p^1) + d(x^2_{m-1}, p^2) + \ldots + d(x^n_{m-1}, p^n)]\\ &+(1-\alpha_m)\delta[d(x^1_m, p^1) + d(x^2_m, p^2) + \ldots + d(x^n_m, p^n)], \end{aligned}$$

which implies that

$$[d(x^1_m, p^1) + d(x^2_m, p^2) + \ldots + d(x^n_m, p^n)] < D_m[d(x^1_{m-1}, p^1) + d(x^2_{m-1}, p^2) + \ldots + d(x^n_{m-1}, p^n)], \quad (25)$$

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where

$$D_m = \frac{\alpha_m}{1-(1-\alpha_m)\delta}.$$

Observe that

$$1 - D_m \quad = 1 - \frac{\alpha_m}{1-(1-\alpha_m)\delta} = \frac{(1-\alpha_m)(1-\delta)}{1-(1-\alpha_m)\delta} \geq (1-\alpha_m)(1-\delta)$$

implies that

$$D_m \quad \leq 1-(1-\alpha_m)(1-\delta) \tag{26}$$

From (25) and (26), we have

$$\begin{aligned}
[d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n)] \quad &< 1-(1-\alpha_m)(1-\delta)[[d(x_{m-1}^1, p^1) + d(x_{m-1}^2, p^2) + \ldots + d(x_{m-1}^n, p^n)]] \qquad (27)\\
&< \prod_{i=1}^{m}(1-(1-\alpha_i)(1-\delta))[d(x_0^1, p^1) + d(x_0^2, p^2) + \ldots + d(x_0^n, p^n)]\\
&< \exp\{\sum_{i=1}^{m}(1-\alpha_i)(1-\delta)\}[d(x_0^1, p^1) + d(x_0^2, p^2) + \ldots + d(x_0^n, p^n)]\\
&< \exp\{\sum_{i=1}^{\infty}(1-\alpha_i)(1-\delta)\}[d(x_0^1, p^1) + d(x_0^2, p^2) + \ldots + d(x_0^n, p^n)]
\end{aligned}$$

Using the fact that $0 \leq \delta < 1$ and $\sum(1-\alpha_i) = \infty$, we conclude that

$$\lim_{m\to\infty}[d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n)] = 0.$$

Theorem 14 *Let* $T : \prod_{i=1}^{n} E \to E$ *be a contractive-like mapping defined in* (9) *on a nonempty closed convex subset* E *of a* W*-hyperbolic metric space* (X, d, W) *with* $F(T) \neq \phi$*. Then, for the sequences* $\{x_n\}$, $\{y_n\}$ *defined in* (8) *with* $\sum(1-\alpha_i) = \infty$, *we have* $\lim_{m\to\infty} x_m^i = p^i$, *where* $(p^1, p^2, ..., p^i) \in F(T)$.

Proof. Assume that $(p^1, p^2, ..., p^i) \in F(T)$. Using (8) and (9), we get

$$d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n) \tag{28}$$

$$= d(W(x_{m-1}^1, T(x_m^1, x_m^2, \ldots, x_m^n), \alpha_m), p^1) + d(W(x_{m-1}^2, T(x_m^2, x_m^3, \ldots, x_m^1), \alpha_m), p^2)$$

$$+ \ldots + d(W(x_{m-1}^n, T(x_m^n, x_m^1, \ldots, x_m^{n-1}), \alpha_m), p^n)$$

$$\leq \alpha_m[d(x^1_{m-1},p^1)+d(x^2_{m-1},p^2)+\ldots+d(x^n_{m-1},p^n)]$$

$$+(1-\alpha_m)\delta[d(x^1_m,p^1)+d(x^2_m,p^2)+\ldots+d(x^n_m,p^n)],$$

which implies that

$$d(x^1_m,p^1)+d(x^2_m,p^2)+\ldots+d(x^n_m,p^n) \quad \leq D_m[d(x^1_{m-1},p^1)+d(x^2_{m-1},p^2)+\ldots+d(x^n_{m-1},p^n)], \quad (29)$$

where

$$D_m=\frac{\alpha_m}{1-(1-\alpha_m)\delta}.$$

Observe that

$$1-D_m \quad =1-\frac{\alpha_m}{1-(1-\alpha_m)\delta}=\frac{(1-\alpha_m)(1-\delta)}{1-(1-\alpha_m)\delta}\geq(1-\alpha_m)(1-\delta)$$

implies that

$$D_m \quad \leq 1-(1-\alpha_m)(1-\delta) \quad (30)$$

From (29) and (30), we have

$$\begin{aligned} d(x^1_m,p^1)+d(x^2_m,p^2)+\ldots+d(x^n_m,p^n) &\leq (1-(1-\alpha_m)(1-\delta))[d(x^1_{m-1},p^1)+d(x^2_{m-1},p^2)+\ldots+d(x^n_{m-1},p^n)] \\ &< \prod_{i=1}^{m}(1-(1-\alpha_i)(1-\delta))[d(x^1_0,p^1)+d(x^2_0,p^2)+\ldots+d(x^n_0,p^n)] \\ &< \exp\{\sum_{i=1}^{m}(1-\alpha_i)(1-\delta)\}[d(x^1_0,p^1)+d(x^2_0,p^2)+\ldots+d(x^n_0,p^n)] \\ &< \exp\{\sum_{i=1}^{\infty}(1-\alpha_n)(1-\delta)\}[d(x^1_0,p^1)+d(x^2_0,p^2)+\ldots+d(x^n_0,p^n)] \quad (31)\end{aligned}$$

Using the fact that $0\leq\delta<1$ and $\sum(1-\alpha_i)=\infty$, we conclude that

$$\lim_{n\to\infty}[d(x^1_m,p^1)+d(x^2_m,p^2)+\ldots+d(x^n_m,p^n)]=0.$$

The following result deals with the rate of convergence of implicit S-n-tupled iteration process.

Theorem 15 *Let $T:\prod_{i=1}^{n}E\to E$ be a contractive-like mapping defined in (9) on a nonempty closed convex subset E of a W-hyperbolic metric space (X,d,W) with $F(T)\neq\phi$. Then, the sequences $\{x_m\}$, defined in (5) with $\sum(1-\alpha_m)=\infty$, converges to the $n-$fixed point of T faster than (6), (7) and (8) iterations.*

Proof. Let $(p^1, p^2, ..., p^i)$ be a fixed point of T. Using the implicit type iteration process given in (6), we have

$$\begin{aligned} & d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n) \\ & < \frac{\alpha_m \delta}{1-(1-\alpha_m)\delta}[d(x_{m-1}^1, p^1) + d(x_{m-1}^2, p^2) + \ldots + d(x_{m-1}^n, p^n)] \\ & < ... < I_m, \end{aligned} \tag{32}$$

where

$$I_m = (\frac{\alpha_m \delta}{1-(1-\alpha_m)\delta})^m [d(x_0^1, p^1) + d(x_0^2, p^2) + \ldots + d(x_0^n, p^n)]. \tag{33}$$

Now, using the implicit iteration defined in (7), we obtain that

$$\begin{aligned} & d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n) \\ & < \frac{\alpha_m}{1-(1-\alpha_m)\delta(\beta_m + (1-\beta_m)\delta)}[d(x_{m-1}^1, p^1) + d(x_{m-1}^2, p^2) + \ldots + d(x_{m-1}^n, p^n)] \\ & < ... < J_m, \end{aligned} \tag{34}$$

where

$$J_m = (\frac{\alpha_m}{1-(1-\alpha_m)\delta(\beta_m + (1-\beta_m)\delta)})^m [d(x_0^1, p^1) + d(x_0^2, p^2) + \ldots + d(x_0^n, p^n)]. \tag{35}$$

Next, using the implicit iteration given in (8), we obtain that

$$\begin{aligned} & d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n) \\ & < \frac{\alpha_m}{1-(1-\alpha_m)\delta}[d(x_{m-1}^1, p^1) + d(x_{m-1}^2, p^2) + \ldots + d(x_{m-1}^n, p^n)] \\ & < ... < K_m, \end{aligned} \tag{36}$$

where

$$K_m = (\frac{\alpha_m}{1-(1-\alpha_m)\delta})^m [d(x_0^1, p^1) + d(x_0^2, p^2) + \ldots + d(x_0^n, p^n)]. \tag{37}$$

Finally, using the iteration process (5), we have

$$\begin{aligned} & d(x_m^1, p^1) + d(x_m^2, p^2) + \ldots + d(x_m^n, p^n) \\ & < \frac{\alpha_m \delta^m}{1-(1-\alpha_m)\delta}[d(x_{m-1}^1, p^1) + d(x_{m-1}^2, p^2) + \ldots + d(x_{m-1}^n, p^n)] \\ & < ... < L_m, \end{aligned} \tag{38}$$

where

$$L_m = (\frac{\alpha_m \delta^m}{1-(1-\alpha_m)\delta})^m [d(x_0^1, p^1) + d(x_0^2, p^2) + \ldots + d(x_0^n, p^n)]. \tag{39}$$

Now, in view of (33), (35), (37) and (39), we have

$$\lim_{m\to\infty} \frac{L_m}{I_m} = 0, \quad \lim_{m\to\infty} \frac{L_m}{J_m} = 0 \ \text{ and } \ \lim_{m\to\infty} \frac{L_m}{K_m} = 0.$$

3 Numerical Simulations

In this section, we are interested in numerical simulations to support our analytical results by a numerical example using MATLAB.

Example 16 *Suppose that* $T : \prod_{i=1}^{n} [0,1] \to [0,1]$ *is a mapping defined by* $T(x_1, x_2, x_3) = \frac{3}{4}x_1$, $T(x_2, x_3, x_1) = \frac{3}{4}x_2$, $T(x_3, x_2, x_1) = \frac{3}{4}x_3$. *Choose* $\alpha_n = \beta_n = 1 - \frac{1}{n}$, $n \geq 2$ *and for* $n = 1, \alpha_n = \beta_n = 0$. *Then we have the following:*

(a) T *is a contractive type mapping,*

(b) $T^m(x_1, x_2, x_3) = (\frac{3}{4})^m x_1$, $T^m(x_2, x_3, x_1) = (\frac{3}{4})^m x_2$, $T^m(x_3, x_2, x_1) = (\frac{3}{4})^m x_3$,

(c) $x_m^1 = x_m^2 = x_m^3$ *as* $n = 1, 2, 3$ *in the implicit iterative processes* (5), (6), (7) *and* (8),

(d) $\lim_{n\to\infty} (x_m^1, x_m^2, x_m^3) = (0, 0, 0)$ *in the implicit iterative processes* (5), (6), (7) *and* (8).

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Table 1 shows the comparison of the rate of convergence of the implicit iterations (5), (6), (7) *and* (8) *to the tripled fixed point* $(0,0,0)$ *with the initial value* $(x_1^1, x_1^2, x_1^3) = (1,1,1)$ *for the mapping given in Example* (16).

Table 1: The values of (x_m^1, x_m^2, x_m^3) for Iterations (5), (6), (7), (8).

n	Iteration (8)	Iteration (7)	Iteration (6)	Iteration (5)
2	(1.0000, 1.0000,1.0000)	(1.0000, 1.0000,1.0000)	(1.0000, 1.0000,1.0000)	(1.0000, 1.0000,1.0000)
3	(0.8000 ,0.8000 ,0.8000)	(0.7442,0.7442,0.7442)	(0.5581,0.5581,0.5581)	(0.4186,0.4186,0.4186)
4	(0.7111,0.7111,0.7111)	(0.6436, 0.6436, 0.6436)	(0.3620,0.3620 ,0.3620)	(0.1527,0.1527,0.1527)
5	(0.6564,0.6564,0.6564)	(0.5857,0.5857,0.5857)	(0.2471,0.2471,0.2471)	(0.0440,0.0440,0.0440)
6	(0.6178,0.6178,0.6178)	(0.5464,0.5464,0.5464)	(0.1729,0.1729,0.1729)	(0.0097,0.0097,0.0097)
7	(0.5884,0.5884,0.5884)	(0.5173, 0.5173, 0.5173)	(0.1228,0.1228,0.1228)	(0.0016,0.0016,0.0016)
8	(0.5648,0.5648,0.5648)	(0.4945,0.4945,0.4945)	(0.0880,0.0880,0.0880)	(0.0002,0.0002,0.0002)
9	(0.5454,0.5454,0.5454)	(0.4759,0.4759,0.4759)	(0.0635,0.0635,0.0635)	(0.0000,0.0000,0.0000)
10	(0.5288,0.5288,0.5288)	(0.4603,0.4603,0.4603)	(0.0461,0.0461, 0.0461)	(0.0000,0.0000,0.0000)
11	(0.5145,0.5145,0.5145)	(0.4470,0.4470, 0.4470)	(0.0336,0.0336,0.0336)	(0.0000,0.0000,0.0000)
12	(0.5020,0.5020,0.5020)	(0.4353,0.4353,0.4353)	(0.0245,0.0245,0.0245)	(0.0000,0.0000,0.0000)
13	(0.4908,0.4908, 0.4908)	(0.4251, 0.4251, 0.4251)	(0.0180,0.0180,0.0180)	(0.0000,0.0000,0.0000)

Remark 17 *By the example (16), we note that the implicit iteration* (5) *is faster than the implicit iterations* (6), (7) *and* (8).

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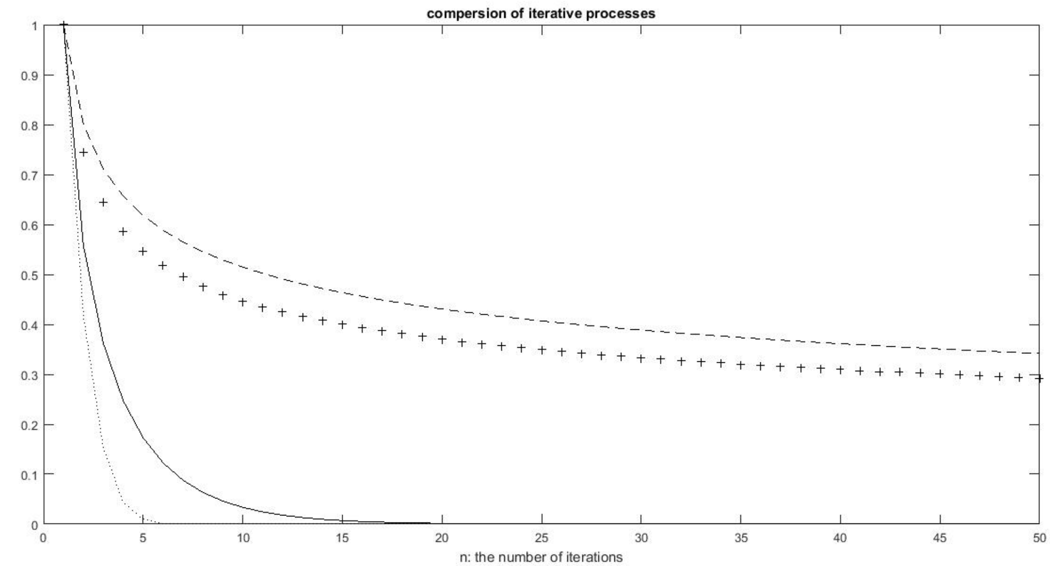
Figure 1 confirm that the iteration (5)) is converges faster than the iteration methods (6), (7) and (8).



.... Implicit iteration (5)
————————————Implicit iteration (6)
+++++++++++++++++++++++++Implicit iteration(7)
- -Implicit iteration (8)

Figure 1: Present the rate of convergence for the iterations (5), (6), (7) and (8).

Competing interests

The authors declare that they have no competing interests.

Data Availability

No data were used to support this study.

Authors' contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References

[1] Lj. B. Ćirić A. Rafiq, S. Radenović M. Rajović & Ume, J. S. On Mann implicit iterations for strongly accretive and strongly pseudo-contractive mappings. Applied Mathematics and Computation, 198:128-137, 2008.

[2] Lj. B. Ciric, J. S. M. Ume, and S. Khan, On the convergence of the Ishikawa iterates to a common fixed point of two mappings. Archivum Mathematicum (Brno) Tomus, 39:123-127, 2003.

[3] R. Chugh, P. Malik and V. Kumar. On analytical and numerical study of implicit fixed point iterations. Cogent Mathematics, 2:1021623, 2015.

[4] S. S. Chang, Y.J. Cho, H. Zhou Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings J. Korean Math. Soc., 38 (2001), pp. 1245-1260.

[5] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operations with applications, Nonlinear Anal. 11, 52, (1987), 525 -562.

[6] T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal, 65 (2006), 1379-1393.

[7] V. Lakshmikantham, L. Cirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal, 70 (2009) 4341-4349.

[8] S. Ishikawa Fixed points and iteration of nonexpansive mappings of in a Banach spaces Proc. Amer. Math. Soc., 73 (1967), pp. 61-71.

[9] M. O. Osilike, A. Udomene Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings Math. Comput. Modelling, 32 (2000), pp. 1181-1191.

[10] J. Schu Iterative construction of fixed points of asymptotically nonexpansive mappings J. Math. Anal. Appl., 158 (1991), pp. 407-413.

[11] U. Kohlenbach. Some logical metatherems with applications in functional analysis. Transactions of the American Mathematical Society, 357:89-128, 2004.

[12] I. Yildirim and M. Abbas, Convergence rate of implicit iteration process and a data dependence result, European Journal of pure and applied mathematics, Vol. 11, No. 1, 2018, 189-201.

[13] C. O. Imoru, M.O. Olantiwo. On the stability of Picard and Mann iteration processes, Carpath. J. Math. 19:155-160, 2003.

[14] M. Imdad, A. H. Soliman, B. S. Choudhury, and P. Das, On n-tupled coincidence and common fixed points results in metric spaces. J. Oper. (2013). Article ID 532867

[15] M. O. Osilike, A. Udomene. Short proofs of stability results for fixed point itera- tion procedures for a class of contractive-type mappings. Indian J. Pure Appl. Math. 30: 1229 - 1234, 1999.

[16] A. Roldan, J. Martínez-Moreno and C. Roldán, Multidimensional fixed point theorems in partially ordered metric spaces. J. Math. Anal. Appl. 396, 536–545 (2012)

[17] S. M. Soltuz, T. Grosan, Data dependence for Ishikawa iteration when dealing with contractive like operators, Fixed Point Theory Appl. Article ID 242916 (2008). doi:10.1155/2008/242916, 2008.

[18] W. Takahashi. A convexity in metric space and nonexpansive mappings. Kodai Math. Sem. Rep. 22:142-149, 1970.

[19] T. Zamfirescu. Fix point theorems in metric spaces. Arch. Math. 23:292-298, 1972.