Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 February 2019 doi:10.20944/preprints201902.0273.v1

MSC 03H05

On The Model Of Hyperrational Numbers With Selective Ultrafilter

A. Grigoryants
Moscow State University Yerevan Branch

February 25, 2019

Abstract

In standard construction of hyperrational numbers using ultrapower we assume that the ultrafilter is selective. It makes possible to assign real value to any finite hyperrational number. So, we can consider hyperrational numbers with selective ultrafilter as extension of traditional real numbers. Also proved the existence of strictly monotonic or stationary representing sequence for any hyperrational number.

Keywords: hyperrational number, selective ultrafilter, non-standard analysis, ultrapower

1 Notation and definitions

We use standard set theoretic notation (see [3]). Let us give some well-known definition for convenience.

Partition of a set S is a pairwise disjoint family $\{S_i\}_{i\in I}$ of nonempty subsets such that $\bigcup_{i\in I} S_i = S$.

Let $\mathfrak{F} \subset 2^\omega$ be non-principal ultrafilter on ω. We'll call elements in $\mathfrak{F}$ *big subsets* (relative to $\mathfrak{F}$) and elements not in $\mathfrak{F}$ *small subsets* (relative to $\mathfrak{F}$).

$\mathfrak{F}$ is called *selective ultrafilter* if for every partition $\{S_n\}_{n\in\omega}$ of ω into $\aleph_0$ pieces such that $S_n \notin \mathfrak{F}$ for all n (*small partition*) there exists $B \in \mathfrak{F}$ (*big selection*) such that $B \cap S_n$ is singleton for all $n \in \omega$. Equivalently, $\mathfrak{F}$ is selective if for every function $f : \omega \to \omega$ such

© 2019 by the author(s). Distributed under a Creative Commons CC BY license.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 February 2019 doi:10.20944/preprints201902.0273.v1

that $f^{-1}(i)$ is small for every i, there exists a big subset B such that restriction $f|_B$ is injective.

The restriction of $\mathfrak{F}$ to a big subset $B \subset \omega$ defined as $\mathfrak{F}_B = \{J \cap B \mid J \in \mathfrak{F}\} \subset 2^B$ is selective ultrafilter on B.

Continuum hypothesis (CH) implies existence of selective ultrafilters due to Galvin [3, theor. 7.8]. The result of Shelah [4] shows that existence of selective ultrafilters is unprovable in ZFC. So, we continue with ZFC & CH to ensure the existence of selective ultrafilter.

Let $[S]^k = \{X \subset S : |X| = k\}$ is the set of all subsets of S that have exactly k elements. If $\{X_i\}_{i\in I}$ is a partition of $[S]^k$ then a subset $H \subset S$ is *homogeneous* for the partition if for some $i :\ [H]^k \subset X_i$.

The following fact is special case of Kunen's theorem proven in [5, theor. 9.6]:

Theorem 1. *An ultrafilter on ω is selective if and only if for every partition of $[\omega]^2$ into two pieces there is a big homogeneous set.*

Due to this theorem selective ultrafilters are also called Ramsey ultrafilters. We'll give simplified proof of theorem 1.

We say that filter $\mathfrak{F} \subset 2^\omega$ is *normal* if for any collection $\{A_i\}_{i\in\omega} \subset \mathfrak{F}$ there exists $B \in \mathfrak{F}$ such that for any $i, j \in B : i < j \implies j \in A_i$. Equally, we can say that $\mathfrak{F}$ is normal if there exists $I \in \mathfrak{F}$ such that above definition holds for any collection $\{A_i\}_{i\in I} \subset \mathfrak{F}$. Indeed we can expand given collection adding $A_k = \omega$ for $k \notin I$, then apply definition to $\{A_i\}_{i\in\omega}$ and intersect obtained B with I.

We continue with fixed selective ultrafilter $\mathfrak{F}$ on ω.

Let us denote $\mathbb{Q}_* = \mathbb{Q}^\omega / \sim_{\mathfrak{F}}$ the quotient of $\mathbb{Q}^\omega = \{(x_i)_{i\in\omega} \mid \forall i : x_i \in \mathbb{Q}\}$ by the following equivalence relation:

$$(x_i) \sim_{\mathfrak{F}} (y_i) \Leftrightarrow \{i \in I \mid x_i = y_i\} \in \mathfrak{F} \ .$$

This is well-known ultrapower construction widely used in model theory and in Robinson's non-standard analysis (see [1], [2]). We only added the property of selectivity to $\mathfrak{F}$. So, we call elements of $\mathbb{Q}_*$ *hyperrational numbers*. There is natural embedding $\iota : \mathbb{Q} \to \mathbb{Q}_*$ where $\iota(q)$ is the equivalence class of constant sequence $(q, q, \ldots, q, \ldots)$. We'll identify $\mathbb{Q}$ with $\iota(\mathbb{Q})$. Also $\mathbb{Q}_*$ satisfies the transfer principle. So, all true first order statements about $\mathbb{Q}$ are also valid in $\mathbb{Q}_*$. In particular, $\mathbb{Q}_*$ is ordered field.

We call element $x \in \mathbb{Q}_*$ *infinitely large* if $|x| > |q|$ for all $q \in \mathbb{Q}$, *infinitesimal* if $|x| < |q|$ for all $q \in \mathbb{Q}$. Otherwise x is called *finite*. We write $x < \infty$ if x is finite or infinitesimal.

We use short notation x_n for some representative $(x_n)_{n\in\omega}$ of equivalence class $x = [(x_n)_{n\in\omega}] \in \mathbb{Q}_*$. So, we can make arbitrary changes to sequence x_n on arbitrary small set without changing appropriate

$x \in \mathbb{Q}_*$. Allowing some inaccuracy we'll talk about hyperrational number $x = x_n$. Let us call subsequence $x_{n_k} = x_J$ of x_n *big subsequence* if the subset $J = \{n_k : k \in \omega\}$ of indexes is big.

The hyperrational number $x = x_n$ is infinitely large if and only if any big subsequence x_{n_k} is unbounded, is infinitesimal if and only if $1/x$ is infinitely large.

2 Propositions

Theorem 2. *For any $x = x_n \in \mathbb{Q}_*$, there exists big subsequence x_{n_k}, which is strictly increasing or strictly decreasing or stationary. The cases are mutually exclusive. In case of $x < \infty$ the subsequence is fundamental and any two such subsequences x_{n_k} and x_{m_k} are equivalent in traditional metric sense $\lim\limits_{k\to\infty}(x_{n_k} - x_{m_k}) = 0$*

Hyperrational numbers given by increasing (decreasing) sequences we call *left* (*right*) numbers. Hyperrational numbers given by stationary sequences are exactly rational numbers.

Let $\mathbb{Q}_{*fin} = \{x \in \mathbb{Q}_* \mid x < \infty\}$ be the set of all finite hyperrational numbers and infinitesimals.

Theorem 3. *The set $\mathbb{Q}_{*fin}$ is local ring, which unique maximal ideal is the set of infinitesimals $\mathfrak{I}$. The factor ring of $\mathbb{Q}_{*fin}$ modulo $\mathfrak{I}$ is the field isomorphic to the field of real numbers:*

$$\mathbb{Q}_{*fin}/\mathfrak{I} \simeq \mathbb{R}$$

3 Proofs

We give the proof of the theorem 1 which is a bit more simple than the one from [5] in part of implication "selective $\Rightarrow$ normal".

Proof of theorem 1. We use following proof schema: selective $\Rightarrow$ normal $\Rightarrow$ Ramsey $\Rightarrow$ selective.

selective $\Rightarrow$ normal. Let $\{A_i\}_{i\in\omega} \subset \mathfrak{F}$ be arbitrary collection of big sets. We can assume with no loss of generality that $\forall i \in A_k \ : \ i > k$, because we can replace A_k with big subsets $A'_k = \{i \in A_k | i > k\}$. Let us define mapping $f : \omega \to \omega, f(i) = \min\{j \in \omega \mid i \notin A_j\}$. Thus $f(i) \leq i$ because $i \notin A_i$. Obviously, $f^{-1}(j) \cap A_j = \varnothing$ and, so, sets $f^{-1}(j)$ are small for all j. Then there exists big set B such that f is injective on B.

We'll construct big set A with the property: $\forall i, j \in A : i < j \Rightarrow i < f(j) \leq j$. Such a set satisfies the conditions of the statement.

Now let us construct subsets P_k of B as follows:

$$\begin{aligned} m_0 &= f(b_0) = \min f(B), P_0 = \{b_0\}, \\ S_k &= \{s \in f(B) \mid s > \max \bigcup_{i=0}^{k-1} P_i\},\ k \geq 1, \\ m_k &= f(b_k) = \min S_k,\ k \geq 1, \\ P_k &= \{b \in B \mid m_{k-1} < f(b) \leq m_k\},\ k \geq 1 \end{aligned}$$

All P_k are finite because $f|_B$ is injective. Obviously, $\cup_{i=0}^{k} P_i = \{b \in B \mid f(b) \leq m_k\}$. All S_k are infinite because $f(B)$ is infinite. Thus $S_k \neq \varnothing$ and all m_k are correctly defined. Note that $S_{k+1} \subseteq S_k$ and so $m_k \leq m_{k+1}$ for all k.

In fact $m_k = f(b_k) \leq b_k < m_{k+1}$ because $b_k \in P_k$ and $m_{k+1} > \max \cup_{i=0}^{k} P_i$. So, the sequence m_k is strictly increasing. The sets P_k are not empty because at least $b_k \in P_k$.

Now if $y \in P_{k+2}$ then $\max(\cup_{i=0}^{k} P_i) < m_{k+1} < f(y)$ and so $x < f(y) \leq y$ for all $x \in \cup_{i=0}^{k} P_i$.

We have $\coprod_k P_k = B$. One of two sets $\coprod_k P_{2k}$ and $\coprod_k P_{2k+1}$ is big and partitioned with small sets P_n where n is odd or even. Let A be big selection from this partition. Let $i < j$ be arbitrary elements of A. Then $i \in P_k$ and $j \in P_{k+2s}$ for some k and $s > 0$. So, $i < f(j) \leq j$ and $j \in A_i$.

normal ⇒ Ramsey. Let $[\omega]^2 = P \coprod Q$ be some partition of $[\omega]^2$. We consider following subsets of ω:

$$\begin{aligned} P_i &= \{j \in \omega \mid \{i,j\} \in P \text{ and } j > i\} \\ Q_i &= \{j \in \omega \mid \{i,j\} \in Q \text{ and } j > i\} \end{aligned}$$

Obviously, $\omega = P_i \coprod Q_i \coprod \{1, \ldots, i\}$ for all i. So, for fixed i one and only one of P_i and Q_i is big. Let $B = \{i \in \omega \mid P_i \in \mathfrak{F}\}$ and $C = \{i \in \omega \mid Q_i \in \mathfrak{F}\}$. One of B and C is big. Let it be B. So, we have family $\{P_i\}_{i \in B}$ of big sets. By the definition of normal ultrafilter there exists big set A such that for any two elements $i < j$ from A we have $j \in P_i$ which means $\{i,j\} \in P$. So, A is homogeneous.

Ramsey ⇒ selective. Let $\{S_i\}_{i \in \omega}$ be a small partition of ω. Consider $Q = \{\{i,j\} \in [\omega]^2 \mid \exists k : i \in S_k \text{ and } j \in S_k\}$ and $P = \omega - Q$. There exists big subset $H \subset \omega$ such that $[H]^2 \subset Q$ or $[H]^2 \subset P$. But $[H]^2 \subset Q$ implies $H \subset S_k$ for some k which is impossible because S_k is small. Thus, $[H]^2 \subset P$ and the intersection $H \cap S_k$ can not contain more than one element for any k. We can add elements to H if some of the intersections are empty. So, H is the desired big selection. ☐

Lemma 1. *For any injection $\pi : \omega \to \omega$ there exists big subset B such that $\pi|_B$ is increasing.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 February 2019 doi:10.20944/preprints201902.0273.v1

Proof. We define the partition $[\omega]^2 = P \coprod Q$ as follows $P = \{\{i, j\} \in [\omega]^2 \mid i < j \text{ and } \pi(i) < \pi(j)\}$ and $Q = [\omega]^2 - P$. There exists homogeneous big set B for this partition. But $[B]^2$ can not be subset of Q because there is no infinitely decreasing sequences in ω. So, $[B]^2 \subset P$ and $\pi|_B$ is increasing. ☐

Proof of theorem 2. Let $x = x_n \in \mathbb{Q}_*$ be arbitrary hyperrational number. First of all let us consider the equivalence relation on $\omega:\ n \sim k$ if and only if $x_n = x_k$. If the partition corresponding to this relation have big subset then there is big stationary subsequence of x_n.

Otherwise, we consider $D \subset \omega$ be big selection from this partition. So, for $k, n \in D$ if $k \neq n$ then $x_k \neq x_n$. For any $k \in \mathbb{Z}$ we define subsets $I_k = \{n \in D \mid x_n \in (k, k+1]\}$. It is clear that choosing nonempty subsets I_k we get the partition of D. If this partition is small then there exists big selection B. One and only one of two sets $B_1 = \{n \in B \mid x_n > 0\}$ and $B_2 = \{n \in B \mid x_n < 0\}$ is big. Note that the set $\{x_n \mid n \in B_i\}$ for big B_i has order type ω in case of B_1 and ω^* in case of B_2.

Otherwise, I_k is big for some k. Thus, x_{I_k} is big bounded subsequence and number x is finite or infinitesimal. Only one of two sets $E_1 = \{n \in B \mid x_n < x\}$ and $E_2 = \{n \in B \mid x_n > x\}$ is big.

Let show that if E_1 is big then there exists big subset $B_3 \subset E_1$ such that $\{x_n \mid n \in B_3\}$ has order type ω and x_{B_3} is fundamental. For this purpose let us define the partition of E_1 as follows: $J_s = \{i \in E_1 \mid x - \frac{1}{n} \leq x_i < x - \frac{1}{n+1}\}$. J_s subsets are small for all s because if J_s is big for some s then the number x' given by x_{J_s} is equal to x but on other hand $x' < x$. Contradiction. Thus, we can get big selection B_3 from the partition $\{J_s\}_{s\in\omega}$. The sequence x_{B_3} is obviously fundamental and the set $\{x_n \mid n \in B_3\}$ has order type ω.

Similarly If E_2 is big we can get big subset $B_4 \subset E_2$ such that the sequence x_{B_4} is fundamental and the set $\{x_n \mid n \in B_4\}$ has order type ω^*.

If x_K and x_L are big fundamental subsequences then $x_{K\cap L}$ is big fundamental subsequence of both x_K and x_L. Thus, x_K and x_L are equivalent in traditional metric sense.

Let C be the only big subset from subsets B_i and $X = \{x_n\}_{n\in C} \subset \mathbb{Q}$ subset of sequence elements. We define injection $\pi : \omega \to \omega$ as follows:

$$\pi(n) = \begin{cases} \text{index of } \min(X - \{x_{\pi(1)}, \dots, x_{\pi(n-1)}\}), \text{ if } C = B_1 \text{ or } C = B_3 \\ \text{index of } \max(X - \{x_{\pi(1)}, \dots, x_{\pi(n-1)}\}), \text{ if } C = B_2 \text{ or } C = B_4 \end{cases}$$

Thus, π is descending or ascending ordering of X and for any $i < j$ we have $x_{\pi(i)} < x_{\pi(j)}$ in first case and $x_{\pi(i)} > x_{\pi(j)}$ in second case.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 February 2019 doi:10.20944/preprints201902.0273.v1

According to lemma 1 there exists big subset $E \subset \omega$ such that π_E is increasing and subsequence $x_{E \cap C}$ is monotonic. $\square$

We call $\nu(x) \in \mathbb{R}$ the *value* of number $x \in \mathbb{Q}_{*fin}$.

Proof of theorem 3. Let us define mapping $\nu : \mathbb{Q}_{*fin} \to \mathbb{R}$. For any $x = x_n \in \mathbb{Q}_{*fin}$ we set $\nu(x)$ equal to limit of some big fundamental subsequence of x_n. This limit is uniquely defined as follows from the theorem 2. It is easy to see that ν is epimorphism of $\mathbb{Q}$-algebras and $\ker \nu = \mathfrak{I}$ is the ideal of all infinitesimals in $\mathbb{Q}_{*fin}$. All elements of compliment of $\mathfrak{I}$ in $\mathbb{Q}_{*fin}$ are invertible. Thus, $\mathfrak{I}$ is only maximal ideal in local ring $\mathbb{Q}_{*fin}$ and $\mathbb{Q}_{*fin}/\mathfrak{I} \simeq \mathbb{R}$. $\square$

References

[1] Robinson, Abraham *"Non-standard analysis"*, Proc. Roy. Acad. Sci. Amst., ser. A 64 (1961), 432-440

[2] Robinson, Abraham *"Non-standard analysis"*, Princeton University Press, 1996, ISBN: 978-0-691-04490-3

[3] Jech, Thomas *"Set theory"*, Springer, 2006, ISBN 3-540-44085-2

[4] Wimmers, Edward (March 1982), *"The Shelah P-point independence theorem"*, Israel Journal of Mathematics, Hebrew University Magnes Press, 43 (1): 28–48, doi:10.1007/BF02761683

[5] Comfort W. W., Negrepontis S., *"The theory of ultrafilters"*, Springer-Verlag Berlin Heidelberg, 1974, ISBN-10: 0387066047