OSTROWSKI TYPE INEQUALITIES PERTAINING STRONGLY CONVEX FUNCTIONS VIA CONFORMABLE FRACTIONAL INTEGRALS AND THEIR APPLICATIONS

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ABSTRACT. In the article, by applied the concept of strongly convex function and one known identity, we establish several Ostrowski type inequalities involving conformable fractional integrals. As applications, some new error estimations for the midpoint formula are provided as well.

1. INTRODUCTION

The subsequent inequality is known as Ostrowski inequality.

Theorem 1. *Let $h : I \longrightarrow \mathbb{R}$ be a mapping differentiable on I° and let $a_1, a_2 \in I^\circ$ with $a_1 < a_2$. If $|h'(x)| \le M$ for all $x \in [a_1, a_2]$, then*

$$\left| h(x) - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} h(x)dx \right| \le M(a_2 - a_1) \left[\frac{1}{4} + \frac{\left(x - \frac{a_1+a_2}{2}\right)^2}{(a_2 - a_1)^2} \right], \quad \forall x \in [a_1, a_2]. \tag{1}$$

Ostrowski inequality is playing a very important role in all the fields of mathematics, especially in the theory of approximations. Thus such inequalities were studied extensively by many researches and numerous generalizations, extensions, variants and applications can be found in the literature [[1]-[4],[7]-[13],[15],[16],[18]-[20],[23]-[27],[29],[30]].

Definition 1. [17] Given a function $h : [0, \infty) \longrightarrow \mathbb{R}$. Conformable fractional derivative of h of order α is defined by

$$\mathtt{D}_\alpha(h)(t) = \lim_{\epsilon \to 0} \frac{h(t + \epsilon t^{1-\alpha}) - h(t)}{\epsilon}, \tag{2}$$

for all $t > 0$ and $\alpha \in (0, 1]$. If the conformable fractional derivative of h of order α exists, then we say that h is α-differentiable.

Let h be α-differentiable in $(0, a)$, and $\lim_{t\to 0^+} h^\alpha(t)$ exists, then define

$$h^\alpha(0) = \lim_{t\to 0^+} h^\alpha(t). \tag{3}$$

We will sometimes write $h^\alpha(t)$ and $\frac{d_\alpha}{d_\alpha t}(h)$ for $\mathtt{D}_\alpha(h)(t)$, to denote the conformable fractional derivatives of h of order α.

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Theorem 2. *[17] Let $\alpha \in (0,1]$ and ϕ, ψ be α-differentiable at a point $t > 0$. Then*

i. $\frac{d_\alpha}{d_\alpha t}(t^n) = nt^{n-\alpha}$, *for all* $n \in \mathbb{R}$.
ii. $\frac{d_\alpha}{d_\alpha t}(b) = 0$, *for all constant functions* $\phi(t) = b$.
iii. $\frac{d_\alpha}{d_\alpha t}(a_1\phi(t) + a_2\psi(t)) = a_1\frac{d_\alpha}{d_\alpha t}(\phi(t)) + a_2\frac{d_\alpha}{d_\alpha t}(\psi(t))$, *for all* $a_1, a_2 \in \mathbb{R}$.
iv. $\frac{d_\alpha}{d_\alpha t}(\phi(t)\psi(t)) = \phi(t)\frac{d_\alpha}{d_\alpha t}(\psi(t)) + \psi(t)\frac{d_\alpha}{d_\alpha t}(\phi(t))$.
v. $\frac{d_\alpha}{d_\alpha t}\left(\frac{\phi(t)}{\psi(t)}\right) = \frac{\psi(t)\frac{d_\alpha}{d_\alpha t}(\phi(t)) - \phi(t)\frac{d_\alpha}{d_\alpha t}(\psi(t))}{(\psi(t))^2}$.
vi. $\frac{d_\alpha}{d_\alpha t}((\phi \circ \psi)(t)) = \phi'(\psi(t))\frac{d_\alpha}{d_\alpha t}(\psi(t))$, *for* ϕ *differentiable at* $\psi(t)$.

If, in addition, the function ϕ is differentiable, then

$$\frac{d_\alpha}{d_\alpha t}(\phi(t)) = t^{1-\alpha}\frac{d}{dt}(\phi(t)). \tag{4}$$

Now, let us recall some basic definitions of various convex functions and conformable fractional integral.

Definition 2. A function $h : \mathtt{I} \longrightarrow \mathbb{R}$, $\mathtt{I} \subseteq \mathbb{R}$, is said to be convex on $\mathtt{I}$ if the inequality

$$h(ta_1 + (1-t)a_2) \leq th(a_1) + (1-t)h(a_2) \tag{5}$$

holds for all $a_1, a_2 \in \mathtt{I}$ and $t \in [0,1]$. Also, we say that h is concave, if the inequality (5) is reversed.

Definition 3. A function $h : \mathtt{I} \longrightarrow \mathbb{R}$ is called strongly convex with modulus $c > 0$, if

$$h(ta_1 + (1-t)a_2) \leq th(a_1) + (1-t)h(a_2) - ct(1-t)(a_2 - a_1)^2 \tag{6}$$

holds for all $a_1, a_2 \in \mathtt{I}$ and $t \in [0,1]$.

Strongly convex functions play important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature [[6],[14],[21],[22],[28]].

Definition 4. [5] (Conformable fractional integral). Let $\alpha \in (0,1)$ and $0 \leq a_1 < a_2$. A function $h : [a_1, a_2] \longrightarrow \mathbb{R}$ is α-fractional integrable on $[a_1, a_2]$ if the integral

$$\int_{a_1}^{a_2} h(t)d_\alpha t = \int_{a_1}^{a_2} h(t)t^{\alpha-1}dt \tag{7}$$

exists and is finite. All α-fractional integrable functions on $[a_1, a_2]$ is indicated by $\mathtt{L}^1_\alpha([a_1, a_2])$.

Remark 1.

$$\mathtt{I}^{a_1}_\alpha(h)(t) = \mathtt{I}^{a_1}_1(t^{\alpha-1}h) = \int_{a_1}^{x} \frac{h(t)}{t^{1-\alpha}}dt, \tag{8}$$

where the integral is the usual Riemann improper integral, $\alpha \in (0,1]$.

The main purpose of the article is to find several Ostrowski type inequalities involving conformable fractional integrals using the concept of strongly convex functions and one known identity. At the end of the paper we give some error estimations for the midpoint formula.

2. Main Results

In order to prove our main results we need the following lemma.

Lemma 1. *[1] Let $0 < \alpha \leq 1$, $0 \leq a_1 < a_2$ and $h : [a_1, a_2] \longrightarrow \mathbb{R}$ be an α- fractional differentiable function. Then the identity*

$$\begin{aligned} & h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(s) d_\alpha s \\ = & \frac{x-a_1}{a_2^\alpha - a_1^\alpha}\int_0^1 \left(((1-t)a_1 + tx)^{2\alpha-1} - a_1^\alpha \left((1-t)a_1 + tx\right)^{\alpha-1}\right) \\ \times & D_\alpha(h)\left((1-t)a_1 + tx\right) t^{1-\alpha} d_\alpha t \\ + & \frac{a_2 - x}{a_2^\alpha - a_1^\alpha}\int_0^1 \left(((1-t)a_2 + tx)^{2\alpha-1} - a_2^\alpha \left((1-t)a_2 + tx\right)^{\alpha-1}\right) \\ \times & D_\alpha(h)\left((1-t)a_2 + tx\right) t^{1-\alpha} d_\alpha t \end{aligned}$$

holds if $D_\alpha(h) \in L_\alpha^1([a_1, a_2])$.

Theorem 3. *Let $0 \leq a_1 < a_2$ and $h : [a_1, a_2] \longrightarrow \mathbb{R}$ be an α-fractional differentiable function for $\alpha \in (0, 1]$. If $D_\alpha(h) \in L_\alpha^1([a_1, a_2])$ and $|h'(x)|$ is strongly convex function with modulus $c > 0$, then*

$$\begin{aligned} & \left| h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(s) d_\alpha s \right| \\ \leq & \frac{x-a_1}{a_2^\alpha - a_1^\alpha}\Delta_1 + \frac{a_2-x}{a_2^\alpha - a_1^\alpha}\Delta_2 - c\frac{(x-a_1)^3}{a_2^\alpha - a_1^\alpha}\Delta_3 - c\frac{(a_2-x)^3}{a_2^\alpha - a_1^\alpha}\Delta_4, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \Delta_1 &= \frac{1}{6}a_1^{\alpha-1}x|h'(a_1)| + \frac{1}{12}x^{\alpha-1}a_1|h'(a_1)| + \frac{1}{12}x|h'(a_1)| - \frac{1}{4}a_1^\alpha|h'(a_1)| \\ &+ \frac{1}{12}a_1|h'(x)| + \frac{1}{12}x^{\alpha-1}a_1|h'(x)| + \frac{1}{4}x|h'(x)| - \frac{1}{2}a_1^\alpha|h'(x)|, \\ \Delta_2 &= \frac{1}{6}a_2^\alpha|h'(a_2)| - \frac{1}{6}x^\alpha|h'(a_2)| + \frac{1}{3}a_2^\alpha|h'(x)| - \frac{1}{3}x^\alpha|h'(x)|, \\ \Delta_3 &= \frac{a_1^\alpha + x^\alpha}{20} + \frac{a_1x^{\alpha-1} + xa_1^{\alpha-1}}{30} - \frac{a_1^\alpha}{6}, \\ \Delta_4 &= \frac{a_2^\alpha}{6} - \frac{a_2^\alpha + x^\alpha}{12}. \end{aligned}$$

Proof. By Lemma 1, the fact that $x^{\alpha-1}$ and $-x^\alpha$ are both convex for $x > 0$, properties of the modulus and since the function $|h'(x)|$ is strongly convex with modulus $c > 0$, we

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have

$$
\begin{aligned}
&\left|h(x)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(s)d_\alpha s\right|\\
\leq\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\int_0^1\left(((1-t)a_1+tx)^\alpha-a_1^\alpha\right)\left|h'((1-t)a_1+tx)\right|dt\\
+\ & \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\int_0^1\left(a_2^\alpha-((1-t)a_2+tx)^\alpha\right)\left|h'((1-t)a_2+tx)\right|dt\\
\leq\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\int_0^1\left(((1-t)a_1+tx)^{\alpha-1}\left((1-t)a_1+tx\right)-a_1^\alpha\right)\left|h'((1-t)a_1+tx)\right|dt\\
+\ & \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\int_0^1\left(a_2^\alpha-((1-t)a_2^\alpha+tx^\alpha)\right)\left|h'((1-t)a_2+tx)\right|dt\\
\leq\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\int_0^1\left(\left((1-t)a_1^{\alpha-1}+tx^{\alpha-1}\right)\left((1-t)a_1+tx\right)-a_1^\alpha\right)\left|h'((1-t)a_1+tx)\right|dt\\
+\ & \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\int_0^1\left(a_2^\alpha-((1-t)a_2^\alpha+tx^\alpha)\right)\left|h'((1-t)a_2+tx)\right|dt\\
\leq\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\int_0^1\left(\left((1-t)a_1^{\alpha-1}+tx^{\alpha-1}\right)\left((1-t)a_1+tx\right)-a_1^\alpha\right)\\
\times\ & \left[(1-t)|h'(a_1)|+t|h'(x)|-ct(1-t)(x-a_1)^2\right]dt\\
+\ & \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\int_0^1\left(a_2^\alpha-((1-t)a_2^\alpha+tx^\alpha)\right)\\
\times\ & \left[(1-t)|h'(a_2)|+t|h'(x)|-ct(1-t)(a_2-x)^2\right]dt\\
=\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\Delta_1+\frac{a_2-x}{a_2^\alpha-a_1^\alpha}\Delta_2-c\frac{(x-a_1)^3}{a_2^\alpha-a_1^\alpha}\Delta_3-c\frac{(a_2-x)^3}{a_2^\alpha-a_1^\alpha}\Delta_4.
\end{aligned}
$$

Hence, we have the result in (9). □

Corollary 1. *If we take $c \longrightarrow 0^+$ in Theorem 3, we obtain (see [1], Theorem 2.2).*

Corollary 2. *If we take $x = (a_1 + a_2)/2$ in Theorem 3, we get*

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$$\begin{aligned}
&\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(s)d_\alpha s\right| \\
\leq\ & \frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\left[\left(\frac{2a_1^{\alpha-1}a_2-10a_1^\alpha+a_1+a_2}{24}\right)|h'(a_1)|+\frac{a_1}{12}\left(\frac{a_1+a_2}{2}\right)^{\alpha-1}|h'(a_1)|\right. \\
+\ & \left(\frac{5a_1+3a_2-12a_1^\alpha}{24}\right)\left|h'\left(\frac{a_1+a_2}{2}\right)\right|+\frac{a_1}{12}\left(\frac{a_1+a_2}{2}\right)^{\alpha-1}\left|h'\left(\frac{a_1+a_2}{2}\right)\right| \\
+\ & \frac{a_2^\alpha}{6}|h'(a_2)|-\frac{1}{6}\left(\frac{a_1+a_2}{2}\right)^{\alpha}|h'(a_2)| \\
+\ & \left.\frac{a_2^\alpha}{3}\left|h'\left(\frac{a_1+a_2}{2}\right)\right|-\frac{1}{3}\left(\frac{a_1+a_2}{2}\right)^{\alpha}\left|h'\left(\frac{a_1+a_2}{2}\right)\right|\right] \\
-\ & \frac{c}{8}\frac{(a_2-a_1)^3}{(a_2^\alpha-a_1^\alpha)}\left[\frac{a_1^\alpha+\left(\frac{a_1+a_2}{2}\right)^\alpha}{20}-\frac{a_2^\alpha+\left(\frac{a_1+a_2}{2}\right)^\alpha}{12}\right. \\
+\ & \left.\frac{a_2^\alpha-a_1^\alpha}{6}+\frac{a_1\left(\frac{a_1+a_2}{2}\right)^{\alpha-1}+a_1^{\alpha-1}\left(\frac{a_1+a_2}{2}\right)}{30}\right].
\end{aligned}$$

Remark 2. If $\alpha=1$, then Corollary 2 becomes

$$\begin{aligned}
&\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}h(s)ds\right| \\
\leq\ & \left(\frac{a_2-a_1}{24}\right)\left[|h'(a_1)|+4\left|h'\left(\frac{a_1+a_2}{2}\right)\right|+|h'(a_2)|-\frac{c}{4}(a_2-a_1)^2\right].
\end{aligned}$$

Theorem 4. *Let $0\leq a_1<a_2$ and $h:[a_1,a_2]\longrightarrow\mathbb{R}$ be an α-fractional differentiable function for $\alpha\in(0,1]$. If $D_\alpha(h)\in L_\alpha^1([a_1,a_2])$ and $|h'(x)|^q$ is strongly convex function with modulus $c>0$ for $q>1$ and $p^{-1}+q^{-1}=1$, then*

$$\begin{aligned}
&\left|h(x)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(s)d_\alpha s\right| \\
\leq\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\left(A_1(\alpha,p)\right)^{\frac{1}{p}}\left[\frac{|h'(a_1)|^q+|h'(x)|^q}{2}-\frac{c}{6}(x-a_1)^2\right]^{\frac{1}{q}} \qquad (10)\\
+\ & \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\left(A_2(\alpha,p)\right)^{\frac{1}{p}}\left[\frac{|h'(a_2)|^q+|h'(x)|^q}{2}-\frac{c}{6}(a_2-x)^2\right]^{\frac{1}{q}},
\end{aligned}$$

$$\text{where}\quad \mathtt{A}_1(\alpha,p) = \int_0^1 (((1-t)a_1+tx)^{\alpha}-a_1^{\alpha})^p\,dt = \frac{1}{(x-a_1)}\int_{a_1}^{x}(t^{\alpha}-a_1^{\alpha})^p\,dt,$$
$$\mathtt{A}_2(\alpha,p) = \int_0^1 (a_2^{\alpha}-((1-t)a_2+tx)^{\alpha})^p\,dt = \frac{1}{(a_2-x)}\int_{x}^{a_2}(a_2^{\alpha}-t^{\alpha})^p\,dt.$$

Proof. Using Lemma 1, properties of the modulus, Hölder's inequality and since the function $|h'(x)|^q$ is strongly convex with modulus $c>0$, we have

$$\begin{aligned}
&\left|h(x)-\frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2}h(s)d_{\alpha}s\right|\\
\le\ & \frac{x-a_1}{a_2^{\alpha}-a_1^{\alpha}}\int_0^1 (((1-t)a_1+tx)^{\alpha}-a_1^{\alpha})\left|h'((1-t)a_1+tx)\right|dt\\
+\ & \frac{a_2-x}{a_2^{\alpha}-a_1^{\alpha}}\int_0^1 (a_2^{\alpha}-((1-t)a_2+tx)^{\alpha})\left|h'((1-t)a_2+tx)\right|dt\\
\le\ & \frac{x-a_1}{a_2^{\alpha}-a_1^{\alpha}}\left(\int_0^1 (((1-t)a_1+tx)^{\alpha}-a_1^{\alpha})^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|h'((1-t)a_1+tx)\right|^q dt\right)^{\frac{1}{q}}\\
+\ & \frac{a_2-x}{a_2^{\alpha}-a_1^{\alpha}}\left(\int_0^1 (a_2^{\alpha}-((1-t)a_2+tx)^{\alpha})^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|h'((1-t)a_2+tx)\right|^q dt\right)^{\frac{1}{q}}\\
\le\ & \frac{x-a_1}{a_2^{\alpha}-a_1^{\alpha}}(\mathtt{A}_1(\alpha,p))^{\frac{1}{p}}\left(\int_0^1\left[(1-t)|h'(a_1)|^q+t|h'(x)|^q-ct(1-t)(x-a_1)^2\right]dt\right)^{\frac{1}{q}}\\
+\ & \frac{a_2-x}{a_2^{\alpha}-a_1^{\alpha}}(\mathtt{A}_2(\alpha,p))^{\frac{1}{p}}\left(\int_0^1\left[(1-t)|h'(a_2)|^q+t|h'(x)|^q-ct(1-t)(a_2-x)^2\right]dt\right)^{\frac{1}{q}}\\
=\ & \frac{x-a_1}{a_2^{\alpha}-a_1^{\alpha}}(\mathtt{A}_1(\alpha,p))^{\frac{1}{p}}\left[\frac{|h'(a_1)|^q+|h'(x)|^q}{2}-\frac{c}{6}(x-a_1)^2\right]^{\frac{1}{q}}\\
+\ & \frac{a_2-x}{a_2^{\alpha}-a_1^{\alpha}}(\mathtt{A}_2(\alpha,p))^{\frac{1}{p}}\left[\frac{|h'(a_2)|^q+|h'(x)|^q}{2}-\frac{c}{6}(a_2-x)^2\right]^{\frac{1}{q}}.
\end{aligned}$$

Hence, we have the result in (10). □

Corollary 3. *If we take $c\longrightarrow 0^+$ in Theorem 4, we get the following inequality*

$$\begin{aligned}
&\left|h(x)-\frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2}h(s)d_{\alpha}s\right|\\
\le\ & \frac{x-a_1}{a_2^{\alpha}-a_1^{\alpha}}(\mathtt{A}_1(\alpha,p))^{\frac{1}{p}}\left[\frac{|h'(a_1)|^q+|h'(x)|^q}{2}\right]^{\frac{1}{q}}\\
+\ & \frac{a_2-x}{a_2^{\alpha}-a_1^{\alpha}}(\mathtt{A}_2(\alpha,p))^{\frac{1}{p}}\left[\frac{|h'(a_2)|^q+|h'(x)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

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Corollary 4. *If we take $x = (a_1 + a_2)/2$ in Theorem 4, we get*

$$\begin{aligned}
&\left|h\left(\frac{a_1+a_2}{2}\right) - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(s)d_\alpha s\right| \\
\leq\ & \frac{a_2 - a_1}{2(a_2^\alpha - a_1^\alpha)}\left\{(B_1(\alpha,p))^{\frac{1}{p}}\left[\frac{|h'(a_1)|^q + \left|h'\left(\frac{a_1+a_2}{2}\right)\right|^q}{2} - \frac{c}{24}(a_2-a_1)^2\right]^{\frac{1}{q}}\right. \\
+\ & \left.(B_2(\alpha,p))^{\frac{1}{p}}\left[\frac{|h'(a_2)|^q + \left|h'\left(\frac{a_1+a_2}{2}\right)\right|^q}{2} - \frac{c}{24}(a_2-a_1)^2\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

$$\begin{aligned}
\textit{where}\ \ B_1(\alpha,p) &= \frac{2}{(a_2-a_1)}\int_{a_1}^{\frac{a_1+a_2}{2}} (t^\alpha - a_1^\alpha)^p\, dt, \\
B_2(\alpha,p) &= \frac{2}{(a_2-a_1)}\int_{\frac{a_1+a_2}{2}}^{a_2} (a_2^\alpha - t^\alpha)^p\, dt.
\end{aligned}$$

Remark 3. If $\alpha = 1$, then Corollary 4 becomes

$$\begin{aligned}
&\left|h\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} h(s)ds\right| \\
\leq\ & \left(\frac{a_2-a_1}{4}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \times \left\{\left[\frac{|h'(a_1)|^q + \left|h'\left(\frac{a_1+a_2}{2}\right)\right|^q}{2} - \frac{c}{24}(a_2-a_1)^2\right]^{\frac{1}{q}}\right. \\
+\ & \left.\left[\frac{|h'(a_2)|^q + \left|h'\left(\frac{a_1+a_2}{2}\right)\right|^q}{2} - \frac{c}{24}(a_2-a_1)^2\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 5. *Let $M > 0$, $0 \leq a_1 < a_2$ and $h : [a_1, a_2] \longrightarrow \mathbb{R}$ be an α-fractional differentiable function for $\alpha \in (0, 1]$. If $D_\alpha(h) \in L^1_\alpha([a_1, a_2])$ and $|h'(x)|^q$ is strongly convex function with modulus $c > 0$ for $q \geq 1$ and $|h'(x)| \leq M$, $\forall x \in [a_1, a_2]$, then*

$$\begin{aligned}
&\left|h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(s)d_\alpha s\right| \\
\leq\ & M\left(\frac{x-a_1}{a_2^\alpha - a_1^\alpha}\right)(A_1(\alpha))^{1-\frac{1}{q}}\left[A_2(\alpha) + A_3(\alpha) - c(x-a_1)^2 G_1(\alpha)\right]^{\frac{1}{q}} \qquad (11) \\
+\ & M\left(\frac{a_2-x}{a_2^\alpha - a_1^\alpha}\right)(B_1(\alpha))^{1-\frac{1}{q}}\left[B_2(\alpha) + B_3(\alpha) - c(a_2-x)^2 G_2(\alpha)\right]^{\frac{1}{q}},
\end{aligned}$$

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$$\begin{aligned}
\textit{where}\ \ A_1(\alpha) &= \frac{x^{\alpha+1}-a_1^{\alpha+1}}{(\alpha+1)(x-a_1)}-a_1^{\alpha},\\
B_1(\alpha) &= a_2^{\alpha}-\frac{x^{\alpha+1}-a_2^{\alpha+1}}{(\alpha+1)(a_2-x)},\\
A_2(\alpha) &= -\frac{a_1^{\alpha+1}}{(\alpha+1)(x-a_1)}\frac{(\alpha+2)(x-a_1)+a_1}{(\alpha+2)(x-a_1)}+\frac{x^{\alpha+2}}{(\alpha+1)(x-a_1)^2(\alpha+2)}-\frac{a_1^{\alpha}}{2},\\
B_2(\alpha) &= \frac{a_2^{\alpha}}{2}+\frac{a_2^{\alpha+1}}{(\alpha+1)(a_2-x)}\frac{(\alpha+2)(a_2-x)+a_2}{(\alpha+2)(a_2-x)}-\frac{x^{\alpha+2}}{(\alpha+1)(a_2-x)^2(\alpha+2)},\\
A_3(\alpha) &= \frac{x^{\alpha+1}}{(\alpha+1)(x-a_1)}\frac{(\alpha+2)(x-a_1)-x}{(\alpha+2)(x-a_1)}+\frac{a_1^{\alpha+2}}{(\alpha+1)(x-a_1)^2(\alpha+2)}-\frac{a_1^{\alpha}}{2},\\
B_3(\alpha) &= \frac{a_2^{\alpha}}{2}-\frac{x^{\alpha+1}}{(\alpha+1)(a_2-x)}\frac{(\alpha+2)(a_2-x)-x}{(\alpha+2)(a_2-x)}-\frac{a_2^{\alpha+2}}{(\alpha+1)(a_2-x)^2(\alpha+2)},\\
G_1(\alpha) &= \frac{1}{(x-a_1)^3}\left[\frac{x}{\alpha+2}\left(x^{\alpha+2}-a_1^{\alpha+2}\right)-\frac{a_1x}{\alpha+1}\left(x^{\alpha+1}-a_1^{\alpha+1}\right)\right.\\
&- \left.\frac{1}{\alpha+3}\left(x^{\alpha+3}-a_1^{\alpha+3}\right)-\frac{a_1}{\alpha+2}\left(x^{\alpha+2}-a_1^{\alpha+2}\right)\right]-\frac{a_1^{\alpha}}{6},\\
G_2(\alpha) &= \frac{a_2^{\alpha}}{6}-\frac{1}{(a_2-x)^3}\left[\frac{a_2}{\alpha+2}\left(a_2^{\alpha+2}-x^{\alpha+2}\right)-\frac{1}{\alpha+3}\left(a_2^{\alpha+3}-x^{\alpha+3}\right)\right.\\
&- \left.\frac{a_2x}{\alpha+1}\left(a_2^{\alpha+1}-x^{\alpha+1}\right)+\frac{x}{\alpha+2}\left(a_2^{\alpha+2}-x^{\alpha+2}\right)\right].
\end{aligned}$$

Proof. Using Lemma 1, properties of the modulus, the well-known power mean inequality, $|h'(x)| \leq M$, $\forall x \in [a_1, a_2]$ and since the function $|h'(x)|^q$ is strongly convex with modulus $c > 0$, we have

$$
\begin{aligned}
&\left|h(x)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(s)d_\alpha s\right|\\
\leq\;& \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\int_0^1\left(((1-t)a_1+tx)^\alpha-a_1^\alpha\right)\left|h'((1-t)a_1+tx)\right|dt\\
+\;& \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\int_0^1\left(a_2^\alpha-((1-t)a_2+tx)^\alpha\right)\left|h'((1-t)a_2+tx)\right|dt\\
\leq\;& \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\left(\int_0^1\left(((1-t)a_1+tx)^\alpha-a_1^\alpha\right)dt\right)^{1-\frac{1}{q}}\\
\times\;& \left(\int_0^1\left(((1-t)a_1+tx)^\alpha-a_1^\alpha\right)\left|h'((1-t)a_1+tx)\right|^q dt\right)^{\frac{1}{q}}\\
+\;& \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\left(\int_0^1\left(a_2^\alpha-((1-t)a_2+tx)^\alpha\right)dt\right)^{1-\frac{1}{q}}\\
\times\;& \left(\int_0^1\left(a_2^\alpha-((1-t)a_2+tx)^\alpha\right)\left|h'((1-t)a_2+tx)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\;& \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\left(\mathtt{A}_1(\alpha)\right)^{1-\frac{1}{q}}\left[\int_0^1\left(((1-t)a_1+tx)^\alpha-a_1^\alpha\right)\right.\\
\times\;& \left.\left[(1-t)|h'(a_1)|^q+t|h'(x)|^q-ct(1-t)(x-a_1)^2\right]dt\right]^{\frac{1}{q}}\\
+\;& \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\left(\mathtt{A}_2(\alpha)\right)^{1-\frac{1}{q}}\left[\int_0^1\left(a_2^\alpha-((1-t)a_2+tx)^\alpha\right)\right.\\
\times\;& \left.\left[(1-t)|h'(a_2)|^q+t|h'(x)|^q-ct(1-t)(a_2-x)^2\right]dt\right]^{\frac{1}{q}}\\
\leq\;& M\left(\frac{x-a_1}{a_2^\alpha-a_1^\alpha}\right)\left(\mathtt{A}_1(\alpha)\right)^{1-\frac{1}{q}}\left[\mathtt{A}_2(\alpha)+\mathtt{A}_3(\alpha)-c(x-a_1)^2\mathtt{G}_1(\alpha)\right]^{\frac{1}{q}}\\
+\;& M\left(\frac{a_2-x}{a_2^\alpha-a_1^\alpha}\right)\left(\mathtt{B}_1(\alpha)\right)^{1-\frac{1}{q}}\left[\mathtt{B}_2(\alpha)+\mathtt{B}_3(\alpha)-c(a_2-x)^2\mathtt{G}_2(\alpha)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Hence, we have the result in (11). □

Corollary 5. *If we take $c \longrightarrow 0^+$ in Theorem 5, we obtain (see [1], Theorem 2.5).*

Corollary 6. *If we take $x = (a_1 + a_2)/2$ in Theorem 5, we get*

$$\begin{aligned}
&\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(s)d_\alpha s\right|\\
\leq\ &\frac{M}{2}\left(\frac{a_2-a_1}{a_2^\alpha-a_1^\alpha}\right)\left\{(C_1(\alpha))^{1-\frac{1}{q}}\left[C_2(\alpha)+C_3(\alpha)-\frac{c}{4}(a_2-a_1)^2E_1(\alpha)\right]^{\frac{1}{q}}\right.\\
+\ &\left.(D_1(\alpha))^{1-\frac{1}{q}}\left[D_2(\alpha)+D_3(\alpha)-\frac{c}{4}(a_2-a_1)^2E_2(\alpha)\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

$$\begin{aligned}
\text{where } C_1(\alpha) &= \frac{\left(\frac{a_1+a_2}{2}\right)^{\alpha+1} - a_1^{\alpha+1}}{(\alpha+1)\left(\left(\frac{a_1+a_2}{2}\right)-a_1\right)} - a_1^{\alpha},\\
D_1(\alpha) &= a_2^{\alpha} - \frac{\left(\frac{a_1+a_2}{2}\right)^{\alpha+1} - a_2^{\alpha+1}}{(\alpha+1)\left(a_2-\left(\frac{a_1+a_2}{2}\right)\right)},\\
C_2(\alpha) &= -\frac{a_1^{\alpha+1}}{(\alpha+1)\left(\left(\frac{a_1+a_2}{2}\right)-a_1\right)}\frac{(\alpha+2)\left(\left(\frac{a_1+a_2}{2}\right)-a_1\right)+a_1}{(\alpha+2)\left(\left(\frac{a_1+a_2}{2}\right)-a_1\right)}\\
&+ \frac{\left(\frac{a_1+a_2}{2}\right)^{\alpha+2}}{(\alpha+1)\left(\left(\frac{a_1+a_2}{2}\right)-a_1\right)^2(\alpha+2)} - \frac{a_1^{\alpha}}{2},\\
D_2(\alpha) &= \frac{a_2^{\alpha}}{2} + \frac{a_2^{\alpha+1}}{(\alpha+1)\left(a_2-\left(\frac{a_1+a_2}{2}\right)\right)}\frac{(\alpha+2)\left(a_2-\left(\frac{a_1+a_2}{2}\right)\right)+a_2}{(\alpha+2)\left(a_2-\left(\frac{a_1+a_2}{2}\right)\right)}\\
&- \frac{\left(\frac{a_1+a_2}{2}\right)^{\alpha+2}}{(\alpha+1)\left(a_2-\left(\frac{a_1+a_2}{2}\right)\right)^2(\alpha+2)},\\
C_3(\alpha) &= \frac{\left(\frac{a_1+a_2}{2}\right)^{\alpha+1}}{(\alpha+1)\left(\left(\frac{a_1+a_2}{2}\right)-a_1\right)}\frac{(\alpha+2)\left(\left(\frac{a_1+a_2}{2}\right)-a_1\right)-\left(\frac{a_1+a_2}{2}\right)}{(\alpha+2)\left(\left(\frac{a_1+a_2}{2}\right)-a_1\right)}\\
&+ \frac{a_1^{\alpha+2}}{(\alpha+1)\left(\left(\frac{a_1+a_2}{2}\right)-a_1\right)^2(\alpha+2)} - \frac{a_1^{\alpha}}{2},\\
D_3(\alpha) &= \frac{a_2^{\alpha}}{2} - \frac{\left(\frac{a_1+a_2}{2}\right)^{\alpha+1}}{(\alpha+1)\left(a_2-\left(\frac{a_1+a_2}{2}\right)\right)}\frac{(\alpha+2)\left(a_2-\left(\frac{a_1+a_2}{2}\right)\right)-\left(\frac{a_1+a_2}{2}\right)}{(\alpha+2)\left(a_2-\left(\frac{a_1+a_2}{2}\right)\right)}\\
&- \frac{a_2^{\alpha+2}}{(\alpha+1)\left(a_2-\left(\frac{a_1+a_2}{2}\right)\right)^2(\alpha+2)},\\
E_1(\alpha) &= \frac{1}{\left(\left(\frac{a_1+a_2}{2}\right)-a_1\right)^3}\\
&\times \left[\frac{\left(\frac{a_1+a_2}{2}\right)}{\alpha+2}\left(\left(\frac{a_1+a_2}{2}\right)^{\alpha+2}-a_1^{\alpha+2}\right) - \frac{a_1\left(\frac{a_1+a_2}{2}\right)}{\alpha+1}\left(\left(\frac{a_1+a_2}{2}\right)^{\alpha+1}-a_1^{\alpha+1}\right)\right.\\
&- \left.\frac{1}{\alpha+3}\left(\left(\frac{a_1+a_2}{2}\right)^{\alpha+3}-a_1^{\alpha+3}\right) - \frac{a_1}{\alpha+2}\left(\left(\frac{a_1+a_2}{2}\right)^{\alpha+2}-a_1^{\alpha+2}\right)\right] - \frac{a_1^{\alpha}}{6},\\
E_2(\alpha) &= \frac{a_2^{\alpha}}{6} - \frac{1}{\left(a_2-\left(\frac{a_1+a_2}{2}\right)\right)^3}\\
&\times \left[\frac{a_2}{\alpha+2}\left(a_2^{\alpha+2}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+2}\right) - \frac{1}{\alpha+3}\left(a_2^{\alpha+3}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+3}\right)\right.\\
&- \left.\frac{a_2\left(\frac{a_1+a_2}{2}\right)}{\alpha+1}\left(a_2^{\alpha+1}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+1}\right) + \frac{\left(\frac{a_1+a_2}{2}\right)}{\alpha+2}\left(a_2^{\alpha+2}-\left(\frac{a_1+a_2}{2}\right)^{\alpha+2}\right)\right].
\end{aligned}$$

Remark 4. If $\alpha = 1$, then Corollary 6 becomes

$$
\begin{aligned}
&\left| h\left(\frac{a_1 + a_2}{2}\right) - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} h(s)ds \right| \\
\leq\ & \frac{M}{2}\left\{ (\mathtt{C}_1(1))^{1-\frac{1}{q}} \left[\mathtt{C}_2(1) + \mathtt{C}_3(1) - \frac{c}{4}(a_2 - a_1)^2 \mathtt{E}_1(1)\right]^{\frac{1}{q}} \right. \\
+\ & \left. (\mathtt{D}_1(1))^{1-\frac{1}{q}} \left[\mathtt{D}_2(1) + \mathtt{D}_3(1) - \frac{c}{4}(a_2 - a_1)^2 \mathtt{E}_2(1)\right]^{\frac{1}{q}} \right\},
\end{aligned}
$$

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$$
\begin{aligned}
\text{where}\ \ \mathtt{C}_1(1) &= \frac{\left(\frac{a_1+a_2}{2}\right)^2 - a_1^2}{2\left(\left(\frac{a_1+a_2}{2}\right) - a_1\right)} - a_1,\\
\mathtt{D}_1(1) &= a_2 - \frac{\left(\frac{a_1+a_2}{2}\right)^2 - a_2^2}{2\left(a_2 - \left(\frac{a_1+a_2}{2}\right)\right)},\\
\mathtt{C}_2(1) &= -\frac{a_1^2}{6\left(\left(\frac{a_1+a_2}{2}\right) - a_1\right)} \frac{3\left(\left(\frac{a_1+a_2}{2}\right) - a_1\right) + a_1}{\left(\left(\frac{a_1+a_2}{2}\right) - a_1\right)}\\
&+ \frac{\left(\frac{a_1+a_2}{2}\right)^3}{6\left(\left(\frac{a_1+a_2}{2}\right) - a_1\right)^2} - \frac{a_1}{2},\\
\mathtt{D}_2(1) &= \frac{a_2}{2} + \frac{a_2^2}{6\left(a_2 - \left(\frac{a_1+a_2}{2}\right)\right)} \frac{3\left(a_2 - \left(\frac{a_1+a_2}{2}\right)\right) + a_2}{\left(a_2 - \left(\frac{a_1+a_2}{2}\right)\right)}\\
&- \frac{\left(\frac{a_1+a_2}{2}\right)^3}{6\left(a_2 - \left(\frac{a_1+a_2}{2}\right)\right)^2},\\
\mathtt{C}_3(1) &= \frac{\left(\frac{a_1+a_2}{2}\right)^2}{6\left(\left(\frac{a_1+a_2}{2}\right) - a_1\right)} \frac{3\left(\left(\frac{a_1+a_2}{2}\right) - a_1\right) - \left(\frac{a_1+a_2}{2}\right)}{\left(\left(\frac{a_1+a_2}{2}\right) - a_1\right)}\\
&+ \frac{a_1^3}{6\left(\left(\frac{a_1+a_2}{2}\right) - a_1\right)^2} - \frac{a_1}{2},\\
\mathtt{D}_3(1) &= \frac{a_2}{2} - \frac{\left(\frac{a_1+a_2}{2}\right)^2}{6\left(a_2 - \left(\frac{a_1+a_2}{2}\right)\right)} \frac{3\left(a_2 - \left(\frac{a_1+a_2}{2}\right)\right) - \left(\frac{a_1+a_2}{2}\right)}{\left(a_2 - \left(\frac{a_1+a_2}{2}\right)\right)}\\
&- \frac{a_2^3}{6\left(a_2 - \left(\frac{a_1+a_2}{2}\right)\right)^2},\\
\mathtt{E}_1(1) &= \frac{1}{\left(\left(\frac{a_1+a_2}{2}\right) - a_1\right)^3}\\
&\times \left[\frac{\left(\frac{a_1+a_2}{2}\right)}{3}\left(\left(\frac{a_1+a_2}{2}\right)^3 - a_1^3\right) - \frac{a_1\left(\frac{a_1+a_2}{2}\right)}{2}\left(\left(\frac{a_1+a_2}{2}\right)^2 - a_1^2\right)\right.\\
&\left. - \frac{1}{4}\left(\left(\frac{a_1+a_2}{2}\right)^4 - a_1^4\right) - \frac{a_1}{3}\left(\left(\frac{a_1+a_2}{2}\right)^3 - a_1^3\right)\right] - \frac{a_1}{6},\\
\mathtt{E}_2(1) &= \frac{a_2}{6} - \frac{1}{\left(a_2 - \left(\frac{a_1+a_2}{2}\right)\right)^3}\\
&\times \left[\frac{a_2}{3}\left(a_2^3 - \left(\frac{a_1+a_2}{2}\right)^3\right) - \frac{1}{4}\left(a_2^4 - \left(\frac{a_1+a_2}{2}\right)^4\right)\right.\\
&\left. - \frac{a_2\left(\frac{a_1+a_2}{2}\right)}{2}\left(a_2^2 - \left(\frac{a_1+a_2}{2}\right)^2\right) + \frac{\left(\frac{a_1+a_2}{2}\right)}{3}\left(a_2^3 - \left(\frac{a_1+a_2}{2}\right)^3\right)\right].
\end{aligned}
$$

Theorem 6. *Let $0 \leq a_1 < a_2$ and $h : [a_1, a_2] \longrightarrow \mathbb{R}$ be an α-fractional differentiable function for $\alpha \in (0,1]$. If $D_\alpha(h) \in L^1_\alpha([a_1,a_2])$ and $|h'(x)|^q$ is strongly convex function with modulus $c > 0$ and $q \geq 1$, then*

$$\begin{aligned} &\left| h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) d_\alpha s \right| \\ \leq\ & \frac{x-a_1}{a_2^\alpha - a_1^\alpha} \left(E_1(\alpha)\right)^{1-\frac{1}{q}} \left[|h'(a_1)|^q F_1(\alpha) + |h'(x)|^q H_1(\alpha) - c(x-a_1)^2 \Delta_3 \right]^{\frac{1}{q}} \\ +\ & \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} \left(E_2(\alpha)\right)^{1-\frac{1}{q}} \left[|h'(a_2)|^q F_2(\alpha) + |h'(x)|^q H_2(\alpha) - c(a_2-x)^2 \Delta_4 \right]^{\frac{1}{q}}, \end{aligned} \tag{12}$$

$$\begin{aligned} \text{where } E_1(\alpha) &= \int_0^1 \left(\left((1-t)a_1^{\alpha-1} + tx^{\alpha-1}\right) \left((1-t)a_1 + tx\right) - a_1^\alpha \right) dt \\ &= \frac{x^\alpha + a_1^\alpha}{3} + \frac{a_1 x^{\alpha-1} + x a_1^{\alpha-1}}{6} - a_1^\alpha, \\ E_2(\alpha) &= \int_0^1 \left(a_2^\alpha - ((1-t)a_2^\alpha + tx^\alpha)\right) dt = \frac{a_2^\alpha - x^\alpha}{2}, \\ F_1(\alpha) &= \frac{x^\alpha - 15a_1^\alpha + a_1 x^{\alpha-1} + x a_1^{\alpha-1}}{12}, \\ H_1(\alpha) &= \frac{3x^\alpha - 5a_1^\alpha + a_1 x^{\alpha-1} + x a_1^{\alpha-1}}{12}, \\ F_2(\alpha) &= \frac{a_2^\alpha - x^\alpha}{6}, \\ H_2(\alpha) &= \frac{a_2^\alpha - x^\alpha}{3}. \end{aligned}$$

and Δ_3, Δ_4 are defined as in Theorem 3.

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Proof. Using Lemma 1, properties of the modulus, the well-known power mean inequality and since the function $|h'(x)|^q$ is strongly convex with modulus $c>0$, we have

$$
\begin{aligned}
&\left|h(x)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(s)d_\alpha s\right|\\
\le\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\int_0^1\left(((1-t)a_1+tx)^\alpha-a_1^\alpha\right)\left|h'((1-t)a_1+tx)\right|dt\\
+\ & \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\int_0^1\left(a_2^\alpha-((1-t)a_2+tx)^\alpha\right)\left|h'((1-t)a_2+tx)\right|dt\\
\le\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\int_0^1\left(((1-t)a_1+tx)^{\alpha-1}\left((1-t)a_1+tx\right)-a_1^\alpha\right)\left|h'((1-t)a_1+tx)\right|dt\\
+\ & \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\int_0^1\left(a_2^\alpha-((1-t)a_2^\alpha+tx^\alpha)\right)\left|h'((1-t)a_2+tx)\right|dt\\
\le\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\int_0^1\left(\left((1-t)a_1^{\alpha-1}+tx^{\alpha-1}\right)\left((1-t)a_1+tx\right)-a_1^\alpha\right)\left|h'((1-t)a_1+tx)\right|dt\\
+\ & \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\int_0^1\left(a_2^\alpha-((1-t)a_2^\alpha+tx^\alpha)\right)\left|h'((1-t)a_2+tx)\right|dt\\
\le\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\left(\int_0^1\left(\left((1-t)a_1^{\alpha-1}+tx^{\alpha-1}\right)\left((1-t)a_1+tx\right)-a_1^\alpha\right)dt\right)^{1-\frac{1}{q}}\\
\times\ & \left(\int_0^1\left(\left((1-t)a_1^{\alpha-1}+tx^{\alpha-1}\right)\left((1-t)a_1+tx\right)-a_1^\alpha\right)\left|h'((1-t)a_1+tx)\right|^q dt\right)^{\frac{1}{q}}\\
+\ & \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\left(\int_0^1\left(a_2^\alpha-((1-t)a_2^\alpha+tx^\alpha)\right)dt\right)^{1-\frac{1}{q}}\\
\times\ & \left(\int_0^1\left(a_2^\alpha-((1-t)a_2^\alpha+tx^\alpha)\right)\left|h'((1-t)a_2+tx)\right|^q dt\right)^{\frac{1}{q}}\\
\le\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\left(\mathtt{E}_1(\alpha)\right)^{1-\frac{1}{q}}\left[\int_0^1\left(\left((1-t)a_1^{\alpha-1}+tx^{\alpha-1}\right)\left((1-t)a_1+tx\right)-a_1^\alpha\right)\right.\\
\times\ & \left.\left[(1-t)|h'(a_1)|^q+t|h'(x)|^q-ct(1-t)(x-a_1)^2\right]dt\right]^{\frac{1}{q}}\\
+\ & \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\left(\mathtt{E}_2(\alpha)\right)^{1-\frac{1}{q}}\left[\int_0^1\left(a_2^\alpha-((1-t)a_2^\alpha+tx^\alpha)\right)\right.\\
\times\ & \left.\left[(1-t)|h'(a_2)|^q+t|h'(x)|^q-ct(1-t)(a_2-x)^2\right]dt\right]^{\frac{1}{q}}\\
=\ & \frac{x-a_1}{a_2^\alpha-a_1^\alpha}\left(\mathtt{E}_1(\alpha)\right)^{1-\frac{1}{q}}\left[|h'(a_1)|^q\mathtt{F}_1(\alpha)+|h'(x)|^q\mathtt{H}_1(\alpha)-c(x-a_1)^2\Delta_3\right]^{\frac{1}{q}}\\
+\ & \frac{a_2-x}{a_2^\alpha-a_1^\alpha}\left(\mathtt{E}_2(\alpha)\right)^{1-\frac{1}{q}}\left[|h'(a_2)|^q\mathtt{F}_2(\alpha)+|h'(x)|^q\mathtt{H}_2(\alpha)-c(a_2-x)^2\Delta_4\right]^{\frac{1}{q}}.
\end{aligned}
$$

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Hence, we have the result in (12). □

Corollary 7. *If we take $q = 1$ in Theorem 6, we obtain Theorem 3.*

Corollary 8. *If we take $c \longrightarrow 0^+$ in Theorem 6, we get the following inequality*

$$\begin{aligned} &\left| h(x) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) d_\alpha s \right| \\ &\leq \frac{x - a_1}{a_2^\alpha - a_1^\alpha} (E_1(\alpha))^{1-\frac{1}{q}} \left[|h'(a_1)|^q F_1(\alpha) + |h'(x)|^q H_1(\alpha) \right]^{\frac{1}{q}} \\ &+ \frac{a_2 - x}{a_2^\alpha - a_1^\alpha} (E_2(\alpha))^{1-\frac{1}{q}} \left[|h'(a_2)|^q F_2(\alpha) + |h'(x)|^q H_2(\alpha) \right]^{\frac{1}{q}}. \end{aligned}$$

Corollary 9. *If we take $x = (a_1 + a_2)/2$ in Theorem 6, we get*

$$\begin{aligned} &\left| h\left(\frac{a_1 + a_2}{2}\right) - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \int_{a_1}^{a_2} h(s) d_\alpha s \right| \\ &\leq \frac{a_2 - a_1}{2(a_2^\alpha - a_1^\alpha)} \left\{ (L_1(\alpha))^{1-\frac{1}{q}} \left[|h'(a_1)|^q M_1(\alpha) + \left| h'\left(\frac{a_1 + a_2}{2}\right) \right|^q N_1(\alpha) - \frac{c}{4}(a_2 - a_1)^2 P_1(\alpha) \right]^{\frac{1}{q}} \right. \\ &+ \left. (L_2(\alpha))^{1-\frac{1}{q}} \left[|h'(a_2)|^q M_2(\alpha) + \left| h'\left(\frac{a_1 + a_2}{2}\right) \right|^q N_2(\alpha) - \frac{c}{4}(a_2 - a_1)^2 P_2(\alpha) \right]^{\frac{1}{q}} \right\}, \end{aligned}$$

$$\begin{aligned} \textit{where } L_1(\alpha) &= \frac{\left(\frac{a_1+a_2}{2}\right)^\alpha + a_1^\alpha}{3} + \frac{a_1 \left(\frac{a_1+a_2}{2}\right)^{\alpha-1} + \left(\frac{a_1+a_2}{2}\right) a_1^{\alpha-1}}{6} - a_1^\alpha, \\ L_2(\alpha) &= = \frac{a_2^\alpha - \left(\frac{a_1+a_2}{2}\right)^\alpha}{2}, \\ M_1(\alpha) &= \frac{\left(\frac{a_1+a_2}{2}\right)^\alpha - 15a_1^\alpha + a_1 \left(\frac{a_1+a_2}{2}\right)^{\alpha-1} + \left(\frac{a_1+a_2}{2}\right) a_1^{\alpha-1}}{12}, \\ N_1(\alpha) &= \frac{3\left(\frac{a_1+a_2}{2}\right)^\alpha - 5a_1^\alpha + a_1 \left(\frac{a_1+a_2}{2}\right)^{\alpha-1} + \left(\frac{a_1+a_2}{2}\right) a_1^{\alpha-1}}{12}, \\ M_2(\alpha) &= \frac{a_2^\alpha - \left(\frac{a_1+a_2}{2}\right)^\alpha}{6}, \\ N_2(\alpha) &= \frac{a_2^\alpha - \left(\frac{a_1+a_2}{2}\right)^\alpha}{3}, \\ P_1(\alpha) &= \frac{a_1^\alpha + \left(\frac{a_1+a_2}{2}\right)^\alpha}{20} + \frac{a_1 \left(\frac{a_1+a_2}{2}\right)^{\alpha-1} + \left(\frac{a_1+a_2}{2}\right) a_1^{\alpha-1}}{30} - \frac{a_1^\alpha}{6}, \\ P_2(\alpha) &= \frac{a_2^\alpha}{6} - \frac{a_2^\alpha + \left(\frac{a_1+a_2}{2}\right)^\alpha}{12}. \end{aligned}$$

Remark 5. If $\alpha = 1$, then Corollary 9 becomes

$$\begin{aligned}
&\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}h(s)ds\right| \\
\leq\ & \frac{1}{2}\Bigg\{(\mathtt{L}_1(1))^{1-\frac{1}{q}}\left[|h'(a_1)|^q\mathtt{M}_1(1)+\left|h'\left(\frac{a_1+a_2}{2}\right)\right|^q\mathtt{N}_1(1)-\frac{c}{4}(a_2-a_1)^2\mathtt{P}_1(1)\right]^{\frac{1}{q}} \\
+\ & (\mathtt{L}_2(1))^{1-\frac{1}{q}}\left[|h'(a_2)|^q\mathtt{M}_2(1)+\left|h'\left(\frac{a_1+a_2}{2}\right)\right|^q\mathtt{N}_2(1)-\frac{c}{4}(a_2-a_1)^2\mathtt{P}_2(1)\right]^{\frac{1}{q}}\Bigg\},
\end{aligned}$$

$$\begin{aligned}
\text{where }\ \mathtt{L}_1(1) &= \frac{\left(\frac{a_1+a_2}{2}\right)+a_1}{3}+\frac{a_1+\left(\frac{a_1+a_2}{2}\right)}{6}-a_1, \\
\mathtt{L}_2(1) &= =\frac{a_2-\left(\frac{a_1+a_2}{2}\right)}{2}, \\
\mathtt{M}_1(1) &= \frac{\left(\frac{a_1+a_2}{2}\right)-15a_1+a_1+\left(\frac{a_1+a_2}{2}\right)}{12}, \\
\mathtt{N}_1(1) &= \frac{3\left(\frac{a_1+a_2}{2}\right)-5a_1+a_1+\left(\frac{a_1+a_2}{2}\right)}{12}, \\
\mathtt{M}_2(1) &= \frac{a_2-\left(\frac{a_1+a_2}{2}\right)}{6}, \\
\mathtt{N}_2(1) &= \frac{a_2-\left(\frac{a_1+a_2}{2}\right)}{3}, \\
\mathtt{P}_1(1) &= \frac{a_1+\left(\frac{a_1+a_2}{2}\right)}{20}+\frac{a_1+\left(\frac{a_1+a_2}{2}\right)}{30}-\frac{a_1}{6}, \\
\mathtt{P}_2(1) &= \frac{a_2}{6}-\frac{a_2+\left(\frac{a_1+a_2}{2}\right)}{12}.
\end{aligned}$$

3. Applications to midpoint formula

Let $\mathtt{P}$ be the partition of the points $a_1 = x_0 < x_1 < ... < x_{n-1} < x_n = a_2$ of the interval $[a_1, a_2]$ and consider the quadrature formula

$$\int_{a_1}^{a_2} h(x)d_\alpha x = \mathtt{T}_\alpha(h,\mathtt{P}) + \mathtt{E}_\alpha(h,\mathtt{P}), \tag{13}$$

where

$$\mathtt{T}_\alpha(h,\mathtt{P}) = \sum_{i=0}^{n-1} h\left(\frac{x_i+x_{i+1}}{2}\right)\frac{\left(x_{i+1}^\alpha - x_i^\alpha\right)}{\alpha} \tag{14}$$

is the midpoint version and $\mathtt{E}_\alpha(h,\mathtt{P})$ denotes the associated approximation error. Here, we are going to derive some new estimates for the midpoint formula.

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Proposition 1. *Let $0 \le x_0 < x_n$ and $h : [x_0, x_n] \longrightarrow \mathbb{R}$ be an α-fractional differentiable function for $\alpha \in (0,1]$. If $D_\alpha(h) \in L^1_\alpha([x_0, x_n])$ and $|h'(x)|$ is strongly convex function with modulus $c > 0$, then*

$$\begin{aligned}
\left|E_\alpha(h,P)\right| \le\ & \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)}{2\alpha}\Bigg[\left(\frac{2x_i^{\alpha-1}x_{i+1} - 10x_i^\alpha + x_i + x_{i+1}}{24}\right)|h'(x_i)| \\
+\ & \frac{x_i}{12}\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1}|h'(x_i)| + \left(\frac{5x_i + 3x_{i+1} - 12x_i^\alpha}{24}\right)\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right| \\
+\ & \frac{x_i}{12}\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1}\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + \frac{x_{i+1}^\alpha}{6}|h'(x_{i+1})| - \frac{1}{6}\left(\frac{x_i+x_{i+1}}{2}\right)^\alpha |h'(x_{i+1})| \\
+\ & \frac{x_{i+1}^\alpha}{3}\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right| - \frac{1}{3}\left(\frac{x_i+x_{i+1}}{2}\right)^\alpha\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\Bigg] \\
-\ & \frac{c}{8\alpha}(x_{i+1}-x_i)^3\left[\frac{x_i^\alpha + \left(\frac{x_i+x_{i+1}}{2}\right)^\alpha}{20} - \frac{x_{i+1}^\alpha + \left(\frac{x_i+x_{i+1}}{2}\right)^\alpha}{12}\right. \\
+\ & \left.\frac{x_{i+1}^\alpha - x_i^\alpha}{6} + \frac{x_i\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1} + x_i^{\alpha-1}\left(\frac{x_i+x_{i+1}}{2}\right)}{30}\right].
\end{aligned}$$

Proof. Applying Corollary 2 of Theorem 3 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the partition P, we have

$$\begin{aligned}
& \left|h\left(\frac{x_i+x_{i+1}}{2}\right)\frac{(x_{i+1}^\alpha - x_i^\alpha)}{\alpha} - \int_{x_i}^{x_{i+1}} h(x)d_\alpha x\right| \\
\le\ & \frac{(x_{i+1}-x_i)}{2\alpha}\Bigg[\left(\frac{2x_i^{\alpha-1}x_{i+1} - 10x_i^\alpha + x_i + x_{i+1}}{24}\right)|h'(x_i)| + \frac{x_i}{12}\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1}|h'(x_i)| \\
+\ & \left(\frac{5x_i + 3x_{i+1} - 12x_i^\alpha}{24}\right)\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + \frac{x_i}{12}\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1}\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right| \\
+\ & \frac{x_{i+1}^\alpha}{6}|h'(x_{i+1})| - \frac{1}{6}\left(\frac{x_i+x_{i+1}}{2}\right)^\alpha |h'(x_{i+1})| \\
+\ & \frac{x_{i+1}^\alpha}{3}\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right| - \frac{1}{3}\left(\frac{x_i+x_{i+1}}{2}\right)^\alpha\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\Bigg] \\
-\ & \frac{c}{8\alpha}(x_{i+1}-x_i)^3\left[\frac{x_i^\alpha + \left(\frac{x_i+x_{i+1}}{2}\right)^\alpha}{20} - \frac{x_{i+1}^\alpha + \left(\frac{x_i+x_{i+1}}{2}\right)^\alpha}{12}\right. \\
+\ & \left.\frac{x_{i+1}^\alpha - x_i^\alpha}{6} + \frac{x_i\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1} + x_i^{\alpha-1}\left(\frac{x_i+x_{i+1}}{2}\right)}{30}\right].
\end{aligned}$$

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Hence from above

$$
\begin{aligned}
\left|\mathrm{E}_{\alpha}(h,\mathrm{P})\right| &= \left|\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}} h(x)d_{\alpha}x - h\left(\frac{x_i+x_{i+1}}{2}\right)\frac{(x_{i+1}^{\alpha}-x_i^{\alpha})}{\alpha}\right\}\right| \\
&\le \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} h(x)d_{\alpha}x - h\left(\frac{x_i+x_{i+1}}{2}\right)\frac{(x_{i+1}^{\alpha}-x_i^{\alpha})}{\alpha}\right| \\
&\le \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)}{2\alpha}\left[\left(\frac{2x_i^{\alpha-1}x_{i+1}-10x_i^{\alpha}+x_i+x_{i+1}}{24}\right)|h'(x_i)| + \frac{x_i}{12}\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1}|h'(x_i)|\right. \\
&+ \left(\frac{5x_i+3x_{i+1}-12x_i^{\alpha}}{24}\right)\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right| + \frac{x_i}{12}\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1}\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right| \\
&+ \frac{x_{i+1}^{\alpha}}{6}|h'(x_{i+1})| - \frac{1}{6}\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}|h'(x_{i+1})| \\
&+ \left.\frac{x_{i+1}^{\alpha}}{3}\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right| - \frac{1}{3}\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right] \\
&- \frac{c}{8\alpha}(x_{i+1}-x_i)^3\left[\frac{x_i^{\alpha}+\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}}{20} - \frac{x_{i+1}^{\alpha}+\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}}{12}\right. \\
&+ \left.\frac{x_{i+1}^{\alpha}-x_i^{\alpha}}{6} + \frac{x_i\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1}+x_i^{\alpha-1}\left(\frac{x_i+x_{i+1}}{2}\right)}{30}\right].
\end{aligned}
$$

□

Proposition 2. *Let $0 \le x_0 < x_n$ and $h : [x_0, x_n] \longrightarrow \mathbb{R}$ be an α-fractional differentiable function for $\alpha \in (0,1]$. If $D_{\alpha}(h) \in L_{\alpha}^{1}([x_0,x_n])$ and $|h'(x)|^q$ is strongly convex function with modulus $c > 0$ for $q > 1$ and $p^{-1}+q^{-1} = 1$, then*

$$
\begin{aligned}
\left|E_{\alpha}(h,P)\right| &\le \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)}{2\alpha}\left\{(S_{i,1}(\alpha,p))^{\frac{1}{p}}\left[\frac{|h'(x_i)|^q+\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{2} - \frac{c}{24}(x_{i+1}-x_i)^2\right]^{\frac{1}{q}}\right. \\
&+ \left.(S_{i,2}(\alpha,p))^{\frac{1}{p}}\left[\frac{|h'(x_{i+1})|^q+\left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{2} - \frac{c}{24}(x_{i+1}-x_i)^2\right]^{\frac{1}{q}}\right\},
\end{aligned}
$$

$$
\begin{aligned}
\text{where}\quad S_{i,1}(\alpha,p) &= \frac{2}{(x_{i+1}-x_i)}\int_{x_i}^{\frac{x_i+x_{i+1}}{2}}(t^{\alpha}-x_i^{\alpha})^p\,dt, \\
S_{i,2}(\alpha,p) &= \frac{2}{(x_{i+1}-x_i)}\int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}}(x_{i+1}^{\alpha}-t^{\alpha})^p\,dt.
\end{aligned}
$$

Proof. The proof is analogous to that of Proposition 1 only by using Corollary 4 of Theorem 4. □

Proposition 3. *Let $M > 0$, $0 \leq x_0 < x_n$ and $h : [x_0, x_n] \longrightarrow \mathbb{R}$ be an α-fractional differentiable function for $\alpha \in (0,1]$. If $D_\alpha(h) \in L^1_\alpha([x_0, x_n])$ and $|h'(x)|^q$ is strongly convex function with modulus $c > 0$ for $q \geq 1$ and $|h'(x)| \leq M$, $\forall x \in [x_0, x_n]$, then*

$$\begin{aligned}\left|E_\alpha(h,P)\right| &\leq \frac{M}{2\alpha}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\Bigg\{\left(C_{i,1}(\alpha)\right)^{1-\frac{1}{q}}\left[C_{i,2}(\alpha)+C_{i,3}(\alpha)-\frac{c}{4}(x_{i+1}-x_i)^2E_{i,1}(\alpha)\right]^{\frac{1}{q}}\\ &+ \left(D_{i,1}(\alpha)\right)^{1-\frac{1}{q}}\left[D_{i,2}(\alpha)+D_{i,3}(\alpha)-\frac{c}{4}(x_{i+1}-x_i)^2E_{i,2}(\alpha)\right]^{\frac{1}{q}}\Bigg\},\end{aligned}$$

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where
$$C_{i,1}(\alpha) = \frac{\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+1} - x_i^{\alpha+1}}{(\alpha+1)\left(\left(\frac{x_i+x_{i+1}}{2}\right) - x_i\right)} - x_i^{\alpha},$$

$$D_{i,1}(\alpha) = x_{i+1}^{\alpha} - \frac{\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+1} - x_{i+1}^{\alpha+1}}{(\alpha+1)\left(x_{i+1} - \left(\frac{x_i+x_{i+1}}{2}\right)\right)},$$

$$\begin{aligned} C_{i,2}(\alpha) &= -\frac{x_i^{\alpha+1}}{(\alpha+1)\left(\left(\frac{x_i+x_{i+1}}{2}\right) - x_i\right)} \frac{(\alpha+2)\left(\left(\frac{x_i+x_{i+1}}{2}\right) - x_i\right) + x_i}{(\alpha+2)\left(\left(\frac{x_i+x_{i+1}}{2}\right) - x_i\right)} \\ &+ \frac{\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+2}}{(\alpha+1)\left(\left(\frac{x_i+x_{i+1}}{2}\right) - x_i\right)^2(\alpha+2)} - \frac{x_i^{\alpha}}{2}, \end{aligned}$$

$$\begin{aligned} D_{i,2}(\alpha) &= \frac{x_{i+1}^{\alpha}}{2} + \frac{x_{i+1}^{\alpha+1}}{(\alpha+1)\left(x_{i+1} - \left(\frac{x_i+x_{i+1}}{2}\right)\right)} \frac{(\alpha+2)\left(x_{i+1} - \left(\frac{x_i+x_{i+1}}{2}\right)\right) + x_{i+1}}{(\alpha+2)\left(x_{i+1} - \left(\frac{x_i+x_{i+1}}{2}\right)\right)} \\ &- \frac{\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+2}}{(\alpha+1)\left(x_{i+1} - \left(\frac{x_i+x_{i+1}}{2}\right)\right)^2(\alpha+2)}, \end{aligned}$$

$$\begin{aligned} C_{i,3}(\alpha) &= \frac{\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+1}}{(\alpha+1)\left(\left(\frac{x_i+x_{i+1}}{2}\right) - x_i\right)} \frac{(\alpha+2)\left(\left(\frac{x_i+x_{i+1}}{2}\right) - x_i\right) - \left(\frac{x_i+x_{i+1}}{2}\right)}{(\alpha+2)\left(\left(\frac{x_i+x_{i+1}}{2}\right) - x_i\right)} \\ &+ \frac{x_i^{\alpha+2}}{(\alpha+1)\left(\left(\frac{x_i+x_{i+1}}{2}\right) - x_i\right)^2(\alpha+2)} - \frac{x_i^{\alpha}}{2}, \end{aligned}$$

$$\begin{aligned} D_{i,3}(\alpha) &= \frac{x_{i+1}^{\alpha}}{2} - \frac{\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+1}}{(\alpha+1)\left(x_{i+1} - \left(\frac{x_i+x_{i+1}}{2}\right)\right)} \frac{(\alpha+2)\left(x_{i+1} - \left(\frac{x_i+x_{i+1}}{2}\right)\right) - \left(\frac{x_i+x_{i+1}}{2}\right)}{(\alpha+2)\left(x_{i+1} - \left(\frac{x_i+x_{i+1}}{2}\right)\right)} \\ &- \frac{x_{i+1}^{\alpha+2}}{(\alpha+1)\left(x_{i+1} - \left(\frac{x_i+x_{i+1}}{2}\right)\right)^2(\alpha+2)}, \end{aligned}$$

$$\begin{aligned} E_{i,1}(\alpha) &= \frac{1}{\left(\left(\frac{x_i+x_{i+1}}{2}\right) - x_i\right)^3} \\ &\times \left[\frac{\left(\frac{x_i+x_{i+1}}{2}\right)}{\alpha+2}\left(\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+2} - x_i^{\alpha+2}\right) - \frac{x_i\left(\frac{x_i+x_{i+1}}{2}\right)}{\alpha+1}\left(\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+1} - x_i^{\alpha+1}\right) \right. \\ &- \left. \frac{1}{\alpha+3}\left(\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+3} - x_i^{\alpha+3}\right) - \frac{x_i}{\alpha+2}\left(\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+2} - x_i^{\alpha+2}\right)\right] - \frac{x_i^{\alpha}}{6}, \end{aligned}$$

$$\begin{aligned} E_{i,2}(\alpha) &= \frac{x_{i+1}^{\alpha}}{6} - \frac{1}{\left(x_{i+1} - \left(\frac{x_i+x_{i+1}}{2}\right)\right)^3} \\ &\times \left[\frac{x_{i+1}}{\alpha+2}\left(x_{i+1}^{\alpha+2} - \left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+2}\right) - \frac{1}{\alpha+3}\left(x_{i+1}^{\alpha+3} - \left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+3}\right)\right. \\ &- \left.\frac{x_{i+1}\left(\frac{x_i+x_{i+1}}{2}\right)}{\alpha+1}\left(x_{i+1}^{\alpha+1} - \left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+1}\right) + \frac{\left(\frac{x_i+x_{i+1}}{2}\right)}{\alpha+2}\left(x_{i+1}^{\alpha+2} - \left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha+2}\right)\right]. \end{aligned}$$

Proof. The proof is analogous to that of Proposition 1 only by using Corollary 6 of Theorem 5. □

Proposition 4. *Let $0 \leq x_0 < x_n$ and $h : [x_0, x_n] \longrightarrow \mathbb{R}$ be an α-fractional differentiable function for $\alpha \in (0, 1]$. If $D_\alpha(h) \in L_\alpha^1([x_0, x_n])$ and $|h'(x)|^q$ is strongly convex function with modulus $c > 0$ and $q \geq 1$, then*

$$\begin{aligned} \left|E_\alpha(h, P)\right| &\leq \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)}{2\alpha} \\ &\times \left\{ (L_{i,1}(\alpha))^{1-\frac{1}{q}} \left[|h'(x_i)|^q M_{i,1}(\alpha) + \left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q N_{i,1}(\alpha) - \frac{c}{4}(x_{i+1} - x_i)^2 P_{i,1}(\alpha)\right]^{\frac{1}{q}} \right. \\ &+ \left. (L_{i,2}(\alpha))^{1-\frac{1}{q}} \left[|h'(x_{i+1})|^q M_{i,2}(\alpha) + \left|h'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q N_{i,2}(\alpha) - \frac{c}{4}(x_{i+1} - x_i)^2 P_{i,2}(\alpha)\right]^{\frac{1}{q}} \right\}, \end{aligned}$$

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$$\begin{aligned}
\textit{where}\;\; L_{i,1}(\alpha) &= \frac{\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}+x_i^{\alpha}}{3}+\frac{x_i\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1}+\left(\frac{x_i+x_{i+1}}{2}\right)x_i^{\alpha-1}}{6}-x_i^{\alpha},\\
L_{i,2}(\alpha) &= =\frac{x_{i+1}^{\alpha}-\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}}{2},\\
M_{i,1}(\alpha) &= \frac{\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}-15x_i^{\alpha}+x_i\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1}+\left(\frac{x_i+x_{i+1}}{2}\right)x_i^{\alpha-1}}{12},\\
N_{i,1}(\alpha) &= \frac{3\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}-5x_i^{\alpha}+x_i\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1}+\left(\frac{x_i+x_{i+1}}{2}\right)x_i^{\alpha-1}}{12},\\
M_{i,2}(\alpha) &= \frac{x_{i+1}^{\alpha}-\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}}{6},\\
N_{i,2}(\alpha) &= \frac{x_{i+1}^{\alpha}-\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}}{3},\\
P_{i,1}(\alpha) &= \frac{x_i^{\alpha}+\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}}{20}+\frac{x_i\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha-1}+\left(\frac{x_i+x_{i+1}}{2}\right)x_i^{\alpha-1}}{30}-\frac{x_i^{\alpha}}{6},\\
P_{i,2}(\alpha) &= \frac{x_{i+1}^{\alpha}}{6}-\frac{x_{i+1}^{\alpha}+\left(\frac{x_i+x_{i+1}}{2}\right)^{\alpha}}{12}.
\end{aligned}$$

Proof. The proof is analogous to that of Proposition 1 only by using Corollary 9 of Theorem 6. □

4. Conclusion

In this paper, using the concept of strongly convex functions and one known identity, we found several Ostrowski type inequalities pertaining conformable fractional integrals. Also, we give some error estimations for the midpoint formula.

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