 doi:10.20944/preprints201902.0200.v1

Article

Univalent Functions Defined by a Generalized Multipler Differential Operator

Adnan Ghazy Alamoush

Faculty of Science, Taibah University, Saudi Aarabia; adnan–omoush@yahoo.com

Abstract: In this paper, we investigate a new subclass of univalent functions defined by a generalized differential operator, and obtain some interesting properties of functions belonging to the class $R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$.

Keywords: analytic functions; univalent function; differential operator; differential subordination; convex functions; structural formula

---

1. Introduction

Let A denote the class of the functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $H(\mathbb{U})$ be the space of holomorphic functions in $\mathbb{U}$. By S and K we denote the subclasses of functions in A which are univalent and convex in $\mathbb{U}$, respectively. Let P be the well-known Caratheodory class of normalized functions with positive real part in $\mathbb{U}$.

The Hadamard product or convolution of the functions

$$f(z) = z + \textstyle\sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

is given by

$$(f * g)(z) = z + \textstyle\sum_{k=2}^{\infty} a_k b_k z^k, \quad (z \in \mathbb{U})$$

We now define a new generalized multiplier differential operator.

Definition 1. *Let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\alpha,\beta,\lambda,\mu,\rho \geq 0$, $0 \leq \gamma \leq \lambda$, $0 < \varphi \leq 1$, $\alpha+\beta > 0$. Then for $f \in A$, we define a new generalized multiplier operator $D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ by*

$$D^0_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z) = f(z),$$

$$D^1_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z) = \frac{[\alpha+\beta+\gamma-(2\vartheta-1)(\lambda+\mu)]f(z)+[(2\vartheta-1)(\lambda+\mu)-\gamma]zf'(z)+\gamma\lambda z^2 f''(z)}{\alpha+\beta},$$

$$\cdots,$$

$$D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z) = D_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)\left(D^{m-1}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)\right).$$

Remark 1. *If $f(z)$ is given by (1), then from Definition 1, we obtain*

$$D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z) = z + \sum_{k=2}^{\infty} \Omega^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)a_k z^k, \tag{2}$$

where

$$\Omega^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta) = \left[\frac{\alpha+[(2\vartheta-1)(\lambda+\mu)+\gamma(k\lambda-1)](k-1)+\beta}{\alpha+\beta}\right]^m. \tag{3}$$

 doi:10.20944/preprints201902.0200.v1

From (2) it follows that $D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z)$ can be written in terms of convolution as

$$D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z) = (f * g)(z), \tag{4}$$

where

$$g(z) = z + \sum_{k=2}^{\infty} \Omega^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)a_k z^k. \tag{5}$$

The class $D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ includes many earlier classes (see also [1]), which are mentioned below:

- $D^m_{1,0}(0,1,0,1) := D^k f(z)$, has been studied by Salagean [2].
- $D^m_{1,0}(1,1,0,1) := L^m f(z)$, has been studied by Uralegaddi and Somanatha [3].
- $D^m_{1,0}(\alpha,1,0,1) := L^m_\beta$, has been studied by Cho and Srivastava [4].
- $D^\rho_{\lambda,0}(0,1,0,1) := D^\rho_\lambda f(z)$, has been studied by Acu and Owa [5].
- $D^m_{\lambda,0}(0,1,0,1) := D^m_\lambda f(z)$, has been studied by Al-Oboudi [6].
- $D^\rho_{\lambda,0}(1,\beta,\gamma,1) := L_1(\rho,\lambda,\beta)f(z)$, has been studied by Catas et al. [7].
- $D^m_{\lambda,0}(\alpha,0,0,1) := D^m_\lambda(\alpha)$, has been studied by Aouf et al. [8].
- $D^m_{\lambda,0}(0,1,0,\frac{\alpha+\beta}{2}) := D^m_{\alpha,\beta,0,\lambda}f(z)$, has been studied AlAmoush and Darus [9].
- $D^m_{\lambda,0}(0,1,\gamma,1) := D^m_{\lambda,\gamma}f(z)$, has been studied by Răducanu and Orhan [10].
- $D^m_{\lambda,\mu}(\alpha,\beta,0,1) := D^m_\lambda(\alpha,\beta,\mu)$, has been studied by Darus and Faisal [11].
- $D^m_{\lambda,\mu}(\alpha,0,0,1) := D^m_\lambda(\alpha,\mu)$, has been studied by Darus and Faisal [12].

Definition 2. *Let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\varpi \in [0,1)$, $\alpha,\beta,\lambda,\mu \geq 0$, $0 \leq \gamma \leq \lambda$, $0 < \varphi \leq 1$, $\alpha+\beta > 0$. Then a function $f \in A$ is said to be in the class $R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$, if it satisfies the condition*

$$\Re\left[D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)\right]' > \varpi,\ z \in \mathbb{U}.$$

The main object of this paper is to present a systematic investigation for the class $R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$. In particular, for this function class, we derive an inclusion result, structural formula, extreme points and other interesting properties.

2. Preliminaries

In order to prove our results, we will make use of the following lemmas.

Lemma 1. *[13] Let $h \in K$, and Let $A \geq 0$. Suppose that $B(z)$ and $D(z)$ are analytic in $\mathbb{U}$, with $D(0) = 0$ and*

$$\Re\left(B(z)\right) \geq A + 4\left|\frac{D(z)}{h'(0)}\right|,\ z \in \mathbb{U}.$$

If an analytic function p with $p(0) = h(0)$ satisfies

$$Az^2p''(z) + B(z)zp'(z) + p(z) + D(z) \prec h(z),\ z \in \mathbb{U},$$

then

$$p(z) \prec h(z),\ z \in \mathbb{U}.$$

Lemma 2. *[14] Let q be a convex function in $\mathbb{U}$ and let*

$$h(z) = q(z) + \varpi zq'(z),$$

where $\varpi > 0$. If $p \in H(\mathbb{U})$ with

$$p(z) = q(0) + p_1z + p_2z^2 + ...\ \text{and}\ \ p(z) + \varpi zp'(z) \prec h(z),\ z \in \mathbb{U},$$

then

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 February 2019 doi:10.20944/preprints201902.0200.v1

$$p(z) \prec q(z),\ z \in \mathbb{U},$$

and this result is sharp.

Lemma 3. *[15] If $p(z)$ is analytic in $\mathbb{U}$, $p(0) = 1$ and $^{-1}(z)$ and , $\Re(p(z)) > \frac{1}{2}$, then for any function F analytic in $\mathbb{U}$, the function $F * p$ takes values in the convex hull of $F(\mathbb{U})$.*

Note that the symbol " $\prec$ " stands for subordination throughout this paper.

3. Main Results

Theorem 1. *Let $m \in N_0 = N \cup \{0\}$, $\varpi \in [0,1)$, $\alpha, \beta, \lambda, \mu \geq 0$, $0 \leq \gamma \leq \lambda$, $0 < \varphi \leq 1$, $\alpha + \beta > 0$, then $R^{m+1}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta) \subset R^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$.*

Proof. Let $f \in R^{m+1}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$. By using the properties of the operator $D^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$, we get

$$D^{m+1}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z) = \frac{[\alpha+\beta+\gamma-(2\vartheta-1)(\lambda+\mu)]\,D^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z) + [(2\vartheta-1)(\lambda+\mu)-\gamma]\,z(D^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z))' + \gamma\lambda z^2(D^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z))''}{\alpha+\beta}. \quad (6)$$

Differentiating (6) with respect to z, we obtain

$$(D^{m+1}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z))' = p(z) + \left[\frac{(2\vartheta-1)(\lambda+\mu)+2\lambda\gamma}{\alpha+\beta}\right]p'(z) + \left[\frac{\gamma\lambda}{\alpha+\beta}\right]p''(z), \quad (7)$$

where

$$p(z) = (D^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z))'.$$

Since $f \in R^{m+1}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$, by using Definition 2 and (7), we have

$$\Re\left\{p(z) + \left[\tfrac{(2\vartheta-1)(\lambda+\mu)+2\lambda\gamma}{\alpha+\beta}\right]p'(z) + \left[\tfrac{\gamma\lambda}{\alpha+\beta}\right]p''(z)\right\} > \varpi, z \in \mathbb{U},$$

which is equivalent to

$$\left\{p(z) + \left[\tfrac{(2\vartheta-1)(\lambda+\mu)+2\lambda\gamma}{\alpha+\beta}\right]p'(z) + \left[\tfrac{\gamma\lambda}{\alpha+\beta}\right]p''(z)\right\} \prec \tfrac{1+(2\varpi-1)z}{1-z} \equiv h(z).$$

From Lemma 1, with $A = \frac{\gamma\lambda}{\alpha+\beta}$, $B(z) = \frac{(2\vartheta-1)(\lambda+\mu)+2\lambda\gamma}{\alpha+\beta}$, and $D(z) = 0$, we have $p(z) \prec h(z)$, which implies that $\Re\left\{(D^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z))'\right\} > \varpi$, $z \in \mathbb{U}$. Hence $f \in R^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ and the proof of the theorem is complete.

Clearly $R^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta) \subset R^{m-1}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta) \subset ... \subset R^{0}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta) \subset S$ (see [16,17]).

Now we will show that the set $R^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ is convex (see [18]).

Theorem 2. *The set $R^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ is convex.*

Proof. Let the functions

$$f_i(z) = z + \textstyle\sum_{k=2}^{\infty} a_i z^k \quad (i = 1,2)$$

be in the class $R^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$. It is sufficient to show that the function $h(z) = \chi_1 f_1(z) + \chi_2 f_2(z)$, with χ_1 and χ_2 nonnegative and $\chi_1 + \chi_2 = 1$, is in the class $R^{m}_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$.

Since

$$h(z) = z + \textstyle\sum_{k=2}^{\infty}(\chi_1 a_{k_1} + \chi_2 a_{k_2})z^k,$$

then we have

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 February 2019 doi:10.20944/preprints201902.0200.v1

$$[D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)h(z)]' = 1 + \sum_{k=2}^{\infty} k(\chi_1 a_{k_1} + \chi_2 a_{k_2})[\varpi^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)]z^{k-1},$$

hence

$$\Re\left(D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)h(z)\right)' = \Re\left(1+\chi_1\sum_{k=2}^{\infty} k[\varpi^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)]a_{k_1}z^{k-1}\right) + \Re\left(1+\chi_2\sum_{k=2}^{\infty} k[\varpi^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)]a_{k_2}z^{k-1}\right) - 1 \quad (8)$$

Since $f_1, f_2 \in R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$, this implies that

$$\Re\left(1+\chi_i\sum_{k=2}^{\infty} k[\varpi^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)]a_{k_i}z^{k-1}\right) > 1+\chi_i(\varpi-1). \quad (9)$$

Using (9) in (8), we obtain

$$\Re\left(D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)h(z)\right)' > 1+\varpi(\chi_1+\chi_2)-(\chi_1+\chi_2)$$

and since $\chi_1+\chi_2=1$, the theorem is proved. □

Theorem 3. *Let q be convex function with $q(0)=1$ and let h be a function of the form $h(z)=q(z)+zq'(z)$, $z\in\mathbb{U}$. If $f\in A$ satisfies the differential subordination $(D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta))' \prec h(z)$, $z\in\mathbb{U}$, then $D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)/z \prec q(z)$ and the result is sharp.*

Proof. If we let $p(z) = D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)/z$, $z\in\mathbb{U}$, then we get $(D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta))' = p(z)+zp'(z)$. So the subordination $(D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta))' \prec h(z)$, $z\in\mathbb{U}$, becomes $p(z)+zp'(z) = q(z)+zq'(z)$, $z\in\mathbb{U}$. Hence from Lemma 2, we have $(D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta))/z \prec q(z)$. The result is sharp. □

4. Structural Formula

In this section, a structural formula, extreme points and coefficient bounds for functions in $R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ are obtained.

Theorem 4. *A function $f\in A$ is in the class $R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ if and only if it can be expressed as*

$$f(z) = \left[z+\sum_{k=2}^{\infty}\frac{1}{\Omega^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)}z^k\right] * \int_{|\zeta|=1}\left[z+2(1-\varpi)\bar{\zeta}\sum_{k=2}^{\infty}\frac{(\zeta z)^k}{k}\right]d\sigma(\zeta), \quad (10)$$

where $\Omega^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ is given by (3) and σ is a positive probability measure defined on the unit circle $\mathbb{T}=\{\zeta\in\mathbb{C}:|\zeta|<1\}$.

Proof. From (4) it follows that, $f\in R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ if and only if

$$\frac{(D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z))'-\varpi}{1-\varpi} \in P.$$

Using Hergoltz integral representation of functions in Caratheodory class P (see [19] and [20]), there exists a positive Borel probability measure σ such that

$$\frac{(D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z))'-\varpi}{1-\varpi} = \int_{|\zeta|=1}\frac{1+\zeta z}{1-\zeta z}d\sigma(\zeta),\ z\in\mathbb{U},$$

which is equivalent to

$$(D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z))' = \int_{|\zeta|=1}\frac{1+(1-2\varpi)\zeta z}{1-\zeta z}d\sigma(\zeta),\ z\in\mathbb{U}. \quad (11)$$

Integrating (11), we obtain

$$D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)f(z) = \int_0^z\left[\int_{|\zeta|=1}\frac{1+(1-2\varpi)\zeta u}{1-\zeta u}d\sigma(\zeta)\right]du$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 February 2019 doi:10.20944/preprints201902.0200.v1

$$= \int_{|\zeta|=1} \left[\int_0^z \frac{1+(1-2\omega)\zeta u}{1-\zeta u} du \right] d\sigma(\zeta)$$

that is

$$D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta) = \int_{|\zeta|=1} \left[z + 2(1-\omega)\bar{\zeta} \sum_{k=2}^{\infty} \frac{(\zeta z)^k}{k} \right] d\sigma(\zeta). \quad (12)$$

Equality (10) follows now, from (4), (5) and (12). Since the converse of this deductive process is also true, we have proved our theorem. □

Corollary 1. *The extreme points of the class* $D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ *are*

$$f(z)_\zeta = z + 2(1-\omega)\bar{\zeta} \sum_{k=2}^{\infty} \frac{(\zeta z)^k}{k\Omega^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)}, \; z \in \mathbb{U}, \; |\zeta| = 1. \quad (13)$$

Proof. Consider the functions

$$g_\zeta(z) = z + 2(1-\omega)\bar{\zeta} \sum_{k=2}^{\infty} \frac{(\zeta z)^k}{k} \text{ and } g_\sigma(z) = \int_{|\zeta|=1} g_\zeta(z) d\sigma(\zeta).$$

Since the map $\sigma \to g_\zeta$ is one-to-one, making use of (5), (6) and (12), the assertion follows from (10) (see [21]).
□

From Corollary 2, we can obtain coefficient bounds for the functions in the class $D^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$.

Corollary 2. *If* $f \in R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ *is given by (1), then*

$$|a_k| \le \frac{2(1-\omega)}{k\Omega^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)}, \; k \ge 2.$$

The result is sharp.

Proof. The coefficient bounds are maximized at an extreme point. Therefore, the result follows from (13). □

Corollary 3. *If* $f \in R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ *then, for* $|z| = r < 1$

$$r - 2(1-\omega)r^2 \sum_{k=2}^{\infty} \frac{1}{k\Omega^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)} \le |f(z)| \le r + 2(1-\omega)r^2 \sum_{k=2}^{\infty} \frac{1}{k\Omega^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)},$$

and

$$1 - 2(1-\omega)r \sum_{k=2}^{\infty} \frac{1}{k\Omega^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)} \le |f(z)| \le 1 + 2(1-\omega)r \sum_{k=2}^{\infty} \frac{1}{k\Omega^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)}.$$

5. Convolution Property

in this part, we prove the analogue of the Polya- Schoenberg conjecture for the class $R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$.

Theorem 5. *The class* $R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ *is closed under the convolution with a convex function. That is, if* $f \in R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ *and* $g \in C$ *then* $f * g \in R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$.

Proof. It is known that if g is convex univalent in U, then (see [13])

$$\Re\left\{\frac{g(z)}{z}\right\} > \frac{1}{2}.$$

Using convolution properties, we have

$$\Re\left(R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)(f*g)(z)\right)' = \Re\left([R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)(f)]' * \frac{g(z)}{z}\right) \tag{14}$$

and the result follows by application of Lemma 3. □

Corollary 4. *The class $R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$ is invariant under Bernardi integral operator.*

Proof. Let $R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$. The Bernardi integral operator is defined as (see [22]):

$$F_c(f)(z) = \tfrac{1+c}{z^k} \int_0^z t^{c-1} f(t)dt, \ (c \in A, \ c > -1).$$

It is easy to check that $F_c(f)(z) = (f*g)(z)$ where

$$g(z) = \textstyle\sum_{k=1}^{\infty} \frac{1+c}{k+c} z^k = \frac{1+c}{z^k} \int_0^z \frac{t^c}{1-t} f(t)dt, \ (z \in \mathbb{U}, \ c > -1).$$

Since the function $\phi(z) = \frac{z}{1-z}$, $z \in \mathbb{U}$ is convex, it follows (see [23]) that the function g is also convex. From Theorem 5, we obtain $F_c(f) \in R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$. Therefore, $F_c[R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)] \subset R^m_{\lambda,\mu}(\alpha,\beta,\gamma,\vartheta)$. □

Conflicts of Interest: The author declares no conflict of interest.

References

1. AlAmoush, A.; Darus, M. On certain subclasses of analytic functions defined by generalized differential operators. *ROMAI J.* **2015**, *11*, 17–31.
2. Salagean, G.S. Subclasses of univalent functions, Complex analysis-fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981). *Lect. Notes Math* **1983**, *1*, 362–372.
3. Uralegaddi, B.A.; Somanatha, C. Certain classes of univalent functions. In *Current Topics in Analytic Function Theory*; World Scientific: River Edge, NJ, USA, 1992; pp. 371–374.
4. Cho, N.E.; Kim, T.H. Multiplier transformations and strongly close-to-convex functions. *Bull. Korean Math. Soc.* **2003**, *40*, 399–410.
5. Acu, M.; Owa, S. Note on a class of starlike functions. *RIMS* **2006**, 1–10.
6. Al-Oboudi, F.M. On univalent functions defined by a generalized Salagean derivative operator. *Int. J. Math. Math. Sci.* **2004**, *2004*, 1429–1436.
7. Catas, A.; Oros, G.I.; Oros, G. Differential subordinations associated with multiplier transformations. *Abstr. Appl. Anal.* **2008**, *2008*, 1–11.
8. Aouf, M.K.; El-Ashwah, R.M.; El-Deeb, S.M. Some inequalities for certain p-valent functions involving extended multiplier transformations. *Proc. Pak. Acad. Sci.* **2009**, *46*, 217–221.
9. AlAmoush, A.; Darus, M. New criteria for certain classes containing generalised differential operator. *J. Qual. Meas. Anal.* **2013**, *9*, 59–71.
10. Răducanu, D.; Orhan, H. Subclasses of analytic functions defined by a generalized differential operator. *Int. J. Math. Anal.* **2010**, *4*, 1–15.
11. Darus, M.; Faisal, I. Problems concerning subclasses of analytic functions. *Acta Universitatis Palackianae Olomucensis Facultas Rerum Naturalium. Mathematica* **2012**, *51*, 35–42.
12. Darus, M.; Faisal, I. Characrerization properties for a class of analytic aunctions defined by generalized Cho and Srivastava operator. *Proc. 2nd Inter. Conf. Math. Sci.* **2010**, 1106–1113.
13. Miller, S.S.; Mocanu, P. T. *Differential Subordinations: Theory and Applications*; Marcel-Dekker: New York, NY, USA, 2000.
14. Miller, S.S.; Mocanu, P.T. On some classes of first order differential subordinations. *Mich. Math. J.* **1985**, *32*, 185–195.
15. Singh, R.; Singh, S. Convolution properties of a class of starlike functions. *Proc. Am. Math. Soc.* **1989**, *106*, 145–152.

 doi:10.20944/preprints201902.0200.v1

16. Goodman, A.W. *Univalent Functions*; Mariner Publishing Company Inc.: Tampa, FL, USA, 1984.
17. Kaplan, W. Close-to-convex schlicht function. *Mich. Math. J.* **1953**, *1*, 169–185.
18. Li, Z.; X, Q.-H. On univalent functions defined by the multiplier differential operator. *Int. J. Math. Anal.* **2012**, *6*, 735–742.
19. Hallenbeck, D.J.; MacGregor, T.H. *Linear Problems and Convexity Techniques in Geometric Function Theory*; Pitman: London, UK, **1984**.
20. Keopf, W. A uniqueness theorem for functions of positive real part. *J. Math. Sci.* **1994**, *28*, 78–90.
21. Brickman, L.; Mac-Gregor, T.H.; Wilken, D.R. Convex hull of some classical families of univalent functions. *Trans. Am. Math. Soc.* **1971**, *156*, 91–107.
22. Bernardi, S.D. Convex and starlike functions. *Trans. Am. Math. Soc.* **1969**, *135*, 429–446.
23. Lewandowski, Z.; Miller, S.S.; Zlotkiewicz, E. Generating functions for some classes of univalent functions. *Proc. Am. Math. Soc.* **1976**, *56*, 111–117.