A note on smooth transcendental approximation to $|x|$

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Abstract. In this review paper, we present a pellucid proof of how $x \tanh \left( \frac{x}{\mu} \right)$ approximates $|x|$ and is better than $\sqrt{x^2 + \mu}$ when we are concerned with accuracy.

1 Introduction

The following limits of hyperbolic tangent (see [2]):

$$\lim_{x \to -\infty} \tanh(x) = -1$$

and

$$\lim_{x \to \infty} \tanh(x) = 1$$

are known. It is easy to see that for $\mu > 0$,

$$\lim_{\mu \to 0} \tanh \left( \frac{x}{\mu} \right) = -1 \quad \text{for } x < 0$$

and

$$\lim_{\mu \to 0} \tanh \left( \frac{x}{\mu} \right) = 1 \quad \text{for } x > 0.$$  

As a consequence for $\mu \to 0$ one can write

$$x \tanh \left( \frac{x}{\mu} \right) \approx |x|.$$  

$x \tanh \left( \frac{x}{\mu} \right)$ being differentiable can be a good approximation for $|x|$. The following theorem [1] in this connection was recently proposed by first author.

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Theorem 1. ([1, Theorem 1]) The approximation \( h(x) = x \tanh \left( \frac{x}{\mu} \right) ; \mu > 0 \in \mathbb{R} \) to \(|x|\) satisfies

\[
x \tanh \left( \frac{x}{\mu} \right) - \mu < |x| < x \tanh \left( \frac{x}{\mu} \right) + \mu.
\] (1.1)

The proof of Theorem 1 in [1] is somewhat cumbersome and doesn’t sound much convincing. The initial goal of this paper is to provide new pellucid proof of Theorem 1 and then to show how \( x \tanh(x/\mu) \) is better approximation of \(|x|\) than \( \sqrt{x^2 + \mu^2} \) or \( \sqrt{x^2 + \mu} \) in terms of accuracy. The details about the approximations \( \sqrt{x^2 + \mu^2} \) and \( \sqrt{x^2 + \mu} \) can be found in [4] and [3] respectively.

2 Main Result

The following lemma needs for our promising proof.

Lemma 1. For \( x \in \mathbb{R} \) such that \( x \neq 0 \) we have

\[
|\tanh(x)| + \frac{1}{|x|} > 1.
\] (2.1)

Proof: We consider the following two cases:

Case(1): For \( x > 0 \), we introduce the function \( f(x) = \tanh(x) + \frac{1}{x} - 1 \) which on differentiation gives

\[
f'(x) = \frac{1}{\cosh^2(x)} - \frac{1}{x^2}.
\]

Therefore \( f'(x) < 0 \), since \( \cosh(x) > x \). Hence \( f(x) \) is decreasing on \((0, \infty)\) and we have that

\[
f(x) > f(\infty^-) \text{ for any } x > 0.
\]

So

\[
tanh(x) + \frac{1}{x} - 1 > 0.
\]

Case(2): For \( x < 0 \) we introduce the function \( g(x) = \tanh(x) + \frac{1}{x} + 1 \). As in Case 1, \( g'(x) < 0 \) and is decreasing in \((-\infty, 0)\).

Hence \( g(x) < g(-\infty^+) \) for any \( x < 0 \).
So we get
\[ \tanh(x) + \frac{1}{x} + 1 < 0, \]
which proves our lemma.

**Proof of Theorem 1:** Clearly for \( x = 0 \) the theorem holds. For \( x \neq 0 \) we prove (1.1) by making use of Lemma 1 as follows:

consider
\[
||x| - x \tanh \left( \frac{x}{\mu} \right) || = ||x| - |x \tanh \left( \frac{x}{\mu} \right) || \\
= |x| |1 - |\tanh \left( \frac{x}{\mu} \right) || \\
< |x| \left| \frac{\mu}{x} \right| = \mu
\]
by Lemma 1. This completes the proof. \( \square \)

In the same paper, it is claimed that the approximation \( x \tanh(x/\mu) \) to \( |x| \) is better than \( \sqrt{x^2 + \mu^2} \). The claim is supported by graphs; but is not proved. We prove this claim as follows:

**Comparison between two approximations:** All three functions being positive and
\[ x \tanh \left( \frac{x}{\mu} \right) < |x| < \sqrt{x^2 + \mu^2} \]
that is
\[ x^2 \tanh^2 \left( \frac{x}{\mu} \right) < x^2 < x^2 + \mu^2. \]

It is enough to prove that
\[ x^2 - x^2 \tanh^2 \left( \frac{x}{\mu} \right) < \mu^2 \]
which is equivalent to
\[ x^2 \text{sech}^2 \left( \frac{x}{\mu} \right) < \mu^2. \]
This follows immediately due to
\[ \cosh \left( \frac{x}{\mu} \right) > \frac{x}{\mu}. \]
Now as $\mu \to 0$, $\sqrt{x^2 + \mu^2} < \sqrt{x^2 + \mu}$, proving that $x \tanh(x/\mu)$ is far better than $\sqrt{x^2 + \mu}$ as far as accuracy is concerned.

This fact is illustrated in the following table by investigating global $L_2$ error which is given by

$$e(h) = \int_{-\infty}^{\infty} [|x| - h(x)]^2 \, dx$$

where $h(x)$ is approximation to $|x|$.

Table 1: Global $L_2$ errors $e(h)$ for the functions $h(x)$

<table>
<thead>
<tr>
<th>$h(x)$</th>
<th>$x \tanh(x/\mu)$</th>
<th>$\sqrt{x^2 + \mu}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e(h)$</td>
<td>$\approx 0.000158151$</td>
<td>$\approx 0.042164$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mu = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(x)$</td>
</tr>
<tr>
<td>$e(h)$</td>
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Again it is easy to verify the following by above formula: $e(x \tanh(x/\mu)) = 0.158151 \times \mu^3$, while $e(\sqrt{x^2 + \mu^2}) = \frac{4}{3} \times \mu^3$ supporting the claim.

3 Conclusion

A new crystal clear proof of old theorem is presented in a simple way and two approximations are compared analytically as well as by numerical illustrations.

References


[3] Carlos Ramirez, Reinaldo Sanchez, Vladik Kreinovich and Miguel Argaéz, \(\sqrt{x^2 + \mu}\) is the most computationally efficient smooth approximation to \(|x|\): a Proof, Journal of Uncertain Systems, Volume 8, Number 3, pp. 205-210, 2014.