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The Fundamental Theorems of Hyper-Operations

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Abstract

We examine the extensions of the basic arithmetical operations of addition and multiplication on the natural numbers into higher-rank hyper-operations also on the natural numbers. We go on to define the concepts of prime and composite numbers under these hyper-operations and derive some results about factorisation, resulting in fundamental theorems analogous to the Fundamental Theorem of Arithmetic.

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1 Introduction

Again-Again!

Teletubbies

Multiplication is repeated addition. Exponentiation is repeated multiplication. We can continue applying operations again and again, to come up with higher-rank operations, known as hyper-operations.

Definition 1.1. Hyper-operation: For $a, b, n \in \mathbb{N}$ we define the hyper-operation $a\,[n]\,b$ recursively as follows:

$$a\,[n]\,b = \begin{cases} b+1 & \text{if } n = 0 \\ a & \text{if } n = 1 \text{ and } b = 0 \\ 0 & \text{if } n = 2 \text{ and } b = 0 \\ 1 & \text{if } n \geq 3 \text{ and } b = 0 \\ a\,[n-1]\,(a\,[n]\,(b-1)) & \text{otherwise} \end{cases}$$

Taken from [1].

The reader may find it useful to refer to the Wikipedia article [1] to gain familiarisation with hyper-operations and their results.

Definition 1.2. n-operation: a hyper-operation $a\,[k]\,b$ when $k = n$. We say that n is the rank of the hyper-operation, and that a and b are its operands.

Definition 1.3. n-product: the result of an n-operation.

Definition 1.4. n-factorisation: for $m > 1$, an n-factorisation is an expression of the form

$$m = a_1\,[n]\,(a_2\,[n]\,(a_3\,[n]\,(\cdots [n]\,a_k)))$$

with $k > 0$. The individual numbers a_i $(i = 1 \dots k)$ are said to be the n-factors in the expression. The innermost (i.e. right-most) n-factor a_k is said to be the final n-factor in the n-factorisation.

Unless stated otherwise, in this paper all values are taken to be natural numbers, i.e.$\in \mathbb{N} = \{0, 1, 2, \dots\}$. In this light, statements such as these are equivalent:

$$a > 1$$
$$a \geq 2$$

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Note that in this paper the hyper-operation in definition 1.1 is defined on natural numbers *only*; in some of the proofs we will have to pause and check that neither of the arguments have crept outside of the natural numbers (i.e. are not fractions nor negative integers).

Note that the n-operation has priority over multiplication, i.e.

$$ab\,[n]\,c \equiv a\,(b\,[n]\,c) \not\equiv (ab)\,[n]\,c$$

very much similar to exponentiation having priority over multiplication:

$$ab^c \equiv a\,(b^c) \qquad \not\equiv (ab)^c$$

The 0-operation ($a\,[0]\,b \equiv b+1$) is referred to as *succession*. For other small n, we will often use standard mathematical notation for n-operations:

$$\begin{aligned} a\,[1]\,b &\equiv a+b && \text{addition} \\ a\,[2]\,b &\equiv a \times b \equiv ab && \text{multiplication} \\ a\,[3]\,b &\equiv a^b && \text{exponentiation} \end{aligned} \qquad (1)$$

and this nicely-compact notation for the 4-operation, referred to as *tetration*:

$$a\,[4]\,b \equiv {}^{b}a \qquad \text{tetration}$$

We shall often use these named operations (addition, multiplication, exponentiation and tetration) in examples. We shall switch between notations whenever it suits our purpose. The notation $a\,[n]\,b$ is more cumbersome but is better suited to results which apply to n-operations in general.

Regarding (1), there is one small difference between the 3-operation and traditional exponentiation, in that (depending on the branch of mathematics)

$$\begin{aligned} &0^0 \text{ is undefined} && \text{but} \\ &0\,[3]\,0 = 1 && \text{by definition 1.1} \end{aligned}$$

This small difference in behaviour will not concern us.

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2 Some Basic Equalities

Let us now prove our first results. The proofs are by induction, which is the approach taken for many of the proofs in this paper; this is to be expected with results that concern natural numbers and a function defined using recursion (definition 1.1).

Lemma 2.1. *Suppose* $n > 1$. *Then* $a\,[n]\,1 = a$.

Proof. Proof is by induction on n, starting at $n = 2$ (since we are given $n > 1$).

For $n = 2$, the n-operation is ordinary multiplication, so we have:

$$a\,[n]\,1 = a \times 1 = a$$

and so the result holds true for $n = 2$.

Suppose the result holds true for $n = k$, i.e. that $a\,[k]\,1 = a$. Then for the $(k+1)$-operation we have:

$$\begin{aligned} a\,[k+1]\,1 &= a\,[k]\,(a\,[k+1]\,(1-1)) && \text{by definition 1.1} \\ &= a\,[k]\,(a\,[k+1]\,0) && \\ &= a\,[k]\,1 && \text{by definition 1.1,} \\ & && \text{since } n > 1 \text{ gives } k+1 > 2 \\ &= a && \text{by the induction hypothesis} \end{aligned}$$

and so the result holds for $n = k+1$ and our proof by induction is complete. □

Lemma 2.2. *Suppose* $n > 2$. *Then* $1\,[n]\,b = 1$.

Proof. Proof is by induction on n, starting at $n = 3$ (since we are given $n > 2$).

For $n = 3$, the n-operation is ordinary exponentiation, so we have:

$$1\,[n]\,b = 1^b = 1$$

and so the result holds true for $n = 3$.

Suppose the result holds true for $n = k$, i.e. that $1\,[k]\,b = 1$.

For the $(k+1)$-operation we have these cases according to the value of b:

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If $b = 0$:

$$1\,[k+1]\,b = 1 \qquad \text{by definition 1.1}$$

Otherwise $b > 0$:

$$\begin{aligned} 1\,[k+1]\,b &= 1\,[k]\,(1\,[k+1]\,(b-1)) && \text{by definition 1.1} \\ &= 1 && \text{by the induction hypothesis} \end{aligned}$$

and so for all values of b the result holds for $n = k+1$ and our proof by induction is complete. □

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3 Some Basic Inequalities

> Big fleas have little fleas upon
> their backs to bite 'em,
> And little fleas have lesser fleas,
> and so, ad infinitum.
> And the great fleas, themselves,
> in turn, have greater fleas to go
> on;
> While these again have greater
> still, and greater still, and so on.
>
> *Augustus De Morgan,*
> *Siphonaptera*

In this section we look at the relative sizes of various n-products - i.e. how the value $a\,[n]\,b$ grows in relation to the values of a, b and n.

First we establish a tiny result about natural numbers.

Lemma 3.1. *Suppose $c > b > a$. Then $c > a + 1$.*

Proof. Since we are restricted to $\mathbb{N}$, we have:

$$\begin{aligned} b &\geq a + 1 && \text{since } b > a \\ c &> a + 1 && \text{substituting into } c > b \end{aligned}$$

□

Now we turn to hyper-operations.

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3.1 A hyper-product is larger than its operands

We confirm the intuitive result that combining a and b in an n-operation yields an n-product which is bigger than both a and b. For example:

$$\begin{aligned} 2+3 &> 3 \\ 2\times 3 &> 3 \\ 2^3 &> 3 \\ 3^2 &> 3 \\ {}^{3}2 &> 3 \\ {}^{2}3 &> 3 \qquad \text{etc.} \end{aligned}$$

We shall now examine the following inequalities:

$$\begin{aligned} a\,[n]\,b &> a \\ a\,[n]\,b &> b \end{aligned}$$

We shall examine these inequalities in separate lemmas, as we shall find that different constraints apply to the ranges of b and n for the inequalities to hold.

Lemma 3.2. *Suppose $a > 1, b > 1, n > 0$. Then $a\,[n]\,b > a$.*

Proof. We prove using double induction, the outer induction being on n.

For $n = 1$, we have $a\,[n]\,b = a + b$, and so trivially

$$a + b > a \qquad \text{we are given } b > 1$$

Suppose the inequality holds true for $n = k$, i.e. $a\,[k]\,b > a$. Then for the $(k+1)$-operation we have:

$$a\,[k+1]\,b = a\,[k]\,(a\,[k+1]\,(b-1)) \qquad \text{by definition 1.1}$$

We now perform an inner induction on b. We start with $b = 2$ (since we are given $b > 1$):

$$\begin{aligned} a\,[k+1]\,2 &= a\,[k]\,(a\,[k+1]\,(2-1)) && \text{by definition 1.1} \\ &= a\,[k]\,(a\,[k+1]\,1) \\ &= a\,[k]\,a && \text{by lemma 2.1,} \\ & && \text{since } n > 0 \text{ gives } k+1 > 1 \\ &> a && \text{by the induction hypothesis on } n \end{aligned}$$

and so the inequality $a\,[k+1]\,b > a$ holds when $b = 2$.

We continue the inner induction by assuming it holds true when $b = j$, i.e. $a\,[k+1]\,j > a$. Then, for $b = j+1$ we have:

$$
\begin{aligned}
a\,[k+1]\,(j+1) &= a\,[k]\,(a\,[k+1]\,j) && \text{by definition 1.1}\\
&> a && \text{by the induction hypothesis on } n,\\
& && \text{since } a\,[k+1]\,j > a > 1
\end{aligned}
$$

and so the inequality $a\,[k+1]\,(j+1) > a$ holds for $b = j+1$, completing our inner induction on b.

Returning to our outer induction, we have now shown that

$$a\,[k+1]\,b > a$$

for every value $b > 1$, and thus our outer induction on n is complete.

□

We now turn to the inequality $a\,[n]\,b > b$, where we can prove a stronger result than for lemma 3.2, in that the ranges of values for b and n are unrestricted.

Lemma 3.3. *Suppose $a > 1$. Then $a\,[n]\,b > b$.*

Proof. We prove using double induction, the outer induction being on n.

For $n = 0$, we have $a\,[n]\,b = b+1$, and so trivially $b+1 > b$ and the result holds.

(In fact, for $n = 0$ we do not require the restriction $a > 1$ but we have no need in this paper for such a slightly stronger result. Similarly, for $n = 1$ we would only require $a > 0$.)

Suppose the inequality holds true for $n = k$, i.e. $a\,[k]\,b > b$. We examine the $(k+1)$-operation:

$$a\,[k+1]\,b = a\,[k]\,(a\,[k+1]\,(b-1)) \quad \text{by definition 1.1}$$

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We now perform an inner induction on b. We start with $b = 0$:

$$\begin{aligned} a\,[k+1]\,(b+1) &= a\,[k+1]\,(0+1) \\ &= a\,[k+1]\,1 \\ &= a && \text{by definition 1.2 when } k+1=1, \\ & && \text{or by lemma 2.1 when } k+1>1 \\ &> 0 && \text{since we are given } a>1 \\ &= b \end{aligned}$$

and so the inequality $a\,[k+1]\,(b+1) > b$ holds when $b = 0$.

We continue the inner induction by assuming it holds true when $b = j$, i.e. that $a\,[k+1]\,j > j$. Then, for $b = j+1$ we have:

$$\begin{aligned} a\,[k+1]\,(j+1) &= a\,[k]\,(a\,[k+1]\,j) && \text{by definition 1.1} \\ &> a\,[k+1]\,j && \text{by the induction hypothesis on } n \\ &> j && \text{by the induction hypothesis on } b \\ a\,[k+1]\,(j+1) &> j+1 && \text{by lemma 3.1} \end{aligned}$$

and so the inequality holds for $b = j+1$, completing our inner induction on b.

Returning to our outer induction, we have now shown that

$$a\,[k+1]\,b > b$$

for every value of b, and thus our outer induction on n is complete.

□

We confirm the intuitive notion that the value of $a\,[n]\,b$ increases if we increase the value of any of a, b or n, except for some corner cases.

Firstly we look at increases in a or b.

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3.2 A hyper-product grows with its operands

We show that increasing either operand also increases the result of an n-operation. For example:

$$\begin{aligned} 4+3 < 5+3 &< 5+4 \\ 4\times 3 < 5\times 3 &< 5\times 4 \\ 4^3 < 5^3 &< 5^4 \\ {}^3 4 < {}^3 5 &< {}^4 5 \\ \ldots < \ldots &< \ldots \\ 4\,[n]\,3 < 5\,[n]\,3 &< 5\,[n]\,4 \end{aligned}$$

and so on.

We shall examine the inequalities

$$a\,[n]\,(b+1) > a\,[n]\,b$$
$$(a+1)\,[n]\,b > a\,[n]\,b$$

We first prove the result for increasing b - not only is it a simpler result to prove than the result for increasing a, but the result for increasing b is used in the proof for increasing a.

Theorem 3.4. *Suppose $a > 1$. Then $a\,[n]\,(b+1) > a\,[n]\,b$.*

Proof.

Proof is by induction on n. For $n = 0$:

$$a\,[0]\,(b+1) = (b+1)+1 > b+1 = a\,[0]\,b$$

Suppose the inequality holds true for $n = k$, i.e. $a\,[k]\,(b+1) > a\,[k]\,b$. Then for the $(k+1)$-operation we have:

$$\begin{aligned} a\,[k+1]\,(b+1) &= a\,[k]\,(a\,[k+1]\,b) && \text{by definition 1.1 with } n = k+1 \\ &> a\,[k+1]\,b && \text{by lemma 3.3} \end{aligned}$$

and so the inequality holds for $n = k+1$, completing our proof by induction. □

Corollary 3.5. *Suppose $a > 1$ and $d > b$. Then $a\,[n]\,d > a\,[n]\,b$.*

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Proof. By theorem 3.4, this sequence is strictly increasing:

$$\{a\,[n]\,b, a\,[n]\,(b+1)\,, \ldots, a\,[n]\,(d-1)\,, a\,[n]\,d\}\,.$$

□

Corollary 3.6. *Suppose* $a\,[n]\,b = a\,[n]\,d$ *with* $a > 1$*. Then* $b = d$*.*

Proof. Suppose $b \neq d$, and without loss of generality assume $b < d$. Then by corollary 3.5, we have $a\,[n]\,b < a\,[n]\,d$, contradicting $a\,[n]\,b = a\,[n]\,d$.

Hence $b = d$ as required. □

Theorem 3.7. *Suppose* $a > 0, b > 0, n > 0$*. Then* $(a+1)\,[n]\,b > a\,[n]\,b$*.*

Proof. We first show the result for $a = 1$. We check the results for the various values of $n > 0$.

For $n = 1$ we have:

$$\begin{aligned}(a+1)\,[n]\,b &= 2\,[1]\,b\\ &= 2+b\\ &> 1+b\\ &= 1\,[1]\,b\\ &= a\,[n]\,b\end{aligned}$$

For $n = 2$ we have:

$$\begin{aligned}(a+1)\,[n]\,b &= 2\,[2]\,b\\ &= 2b\\ &> 1b && \text{since } b > 0\\ &= 1\,[2]\,b\\ &= a\,[n]\,b\end{aligned}$$

For $n \geq 3$ we have:

$$\begin{aligned}(a+1)\,[n]\,b &= 2\,[n]\,b\\ &> 2\,[n]\,0 && \text{by corollary 3.5 since } b > 0\\ &= 1 && \text{by definition 1.1}\\ &= a\,[n]\,b && \text{by lemma 2.2}\end{aligned}$$

and so the result $(a+1)\,[n]\,b > a\,[n]\,b$ holds for all values of n when $a = 1$.

Now we deal with larger values of a. For $a > 1$ we can lean on previous results and we prove using double induction, the outer induction being on n.

For $n = 1$, we have trivially that

$$(a+1)\,[n]\,b = (a+1) + b > a + b = a\,[n]\,b$$

Suppose the inequality holds true for $n = k$, i.e. $(a+1)\,[k]\,b > a\,[k]\,b$. Then for the $(k+1)$-operation we have:

$$(a+1)\,[k+1]\,b = (a+1)\,[k]\,((a+1)\,[k+1]\,(b-1))$$

We now perform an inner induction on b. We start with $b = 1$ (since we are given $b > 0$):

$$\begin{aligned}
(a+1)\,[k+1]\,1 &= a+1 && \text{by lemma 2.1, since } n>0 \text{ gives } k+1>1 \\
&> a \\
&= a\,[k+1]\,1 && \text{by lemma 2.1}
\end{aligned}$$

and so the inequality $(a+1)\,[k+1]\,b > a\,[k+1]\,b$ holds when $b = 1$.

We continue the inner induction by assuming it holds true when $b = j$, i.e. $(a+1)\,[k+1]\,j > a\,[k+1]\,j$. Then, for $b = j+1$ we have:

$$(a+1)\,[k+1]\,(j+1) = (a+1)\,[k]\,((a+1)\,[k+1]\,j) \tag{2}$$

We examine the RHS of (2) to find

$$\begin{aligned}
(a+1)\,[k]\,((a+1)\,[k+1]\,j) &> a\,[k]\,((a+1)\,[k+1]\,j) && \text{by the induction hypothesis on } n \\
(a+1)\,[k+1]\,j &> a\,[k+1]\,j && \text{by the induction hypothesis on } b \\
a\,[k]\,((a+1)\,[k+1]\,j) &> a\,[k]\,(a\,[k+1]\,j) && \text{by corollary 3.5, since } a>1 \\
&= a\,[k+1]\,(j+1) && \text{by definition 1.1, with } n=k+1 \\
(a+1)\,[k+1]\,(j+1) &> a\,[k+1]\,(j+1) && \text{combining the above with (2)}
\end{aligned}$$

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and so the inequality $(a+1)\,[k+1]\,b > a\,[k+1]\,b$ holds for $b = j+1$, completing our inner induction on b.

Returning to our outer induction, we have now shown that

$$(a+1)\,[k+1]\,b > a\,[k+1]\,b$$

for every value $b > 0$, and thus our outer induction on n is complete.

We have thus proved the result for the cases $a = 1$ and $a > 1$. Hence the overall result holds.

□

Corollary 3.8. *Suppose $c > a > 0, b > 0, n > 0$. Then $c\,[n]\,b > a\,[n]\,b$.*

Proof. By theorem 3.7, this sequence is strictly increasing

$$\{a\,[n]\,b, (a+1)\,[n]\,b, \ldots, (c-1)\,[n]\,b, c\,[n]\,b\}\,.$$

□

3.3 A higher-rank hyper-product is greater than a lower-rank hyper-product on the same operands

We show that an $(n+1)$-operation yields a larger result than an n-operation on the same operands.

For example,

$$^{5}3 > 3^5 > 3 \times 5 > 3 + 5$$

However, there is a particular corner case that causes us trouble. Observe that

$$2 + 2 = 2 \times 2 = 2^2 = {}^{2}2 = \cdots = 4$$

We formalise this pattern in the following lemma.

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> Freedom is the freedom to say that two plus two make four. If that is granted, all else follows.
>
> *George Orwell, 1984*

Lemma 3.9. *Suppose* $n > 0$. *Then* $2\,[n]\,2 = 4$.

Proof. We prove by induction on n.

For $n = 1$, the n-operation is addition, and so we have trivially that $2+2 = 4$.

Suppose the result holds for $n = k$, i.e. $2\,[k]\,2 = 4$. Then we have:

$$\begin{aligned}
2\,[k+1]\,2 &= 2\,[k]\,(2\,[k+1]\,(2-1)) && \text{by definition 1.1 with } n = k+1 \\
&= 2\,[k]\,(2\,[k+1]\,1) && \\
&= 2\,[k]\,2 && \text{by lemma 2.1 since } k+1 > 1 \\
&= 4 && \text{by the induction hypothesis}
\end{aligned}$$

and so the result holds for $n = k+1$ and the proof by induction is complete. □

Now we show that an $(n+1)$-operation yields a larger result than an n-operation on the same operands when the corner case above is excluded.

Theorem 3.10. *Suppose* $a > 1$ *and* $b > 1$ *with not both* $a = 2$ *and* $b = 2$. *Then* $a\,[n+1]\,b > a\,[n]\,b$.

Proof.

Proof is by induction on b. We start with $b = 2$ since we are given $b > 1$.

We are given that when $b = 2$ then $a \neq 2$. We are also given $a > 1$ so we deduce that $a > 2$. We check the various cases for n.

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For $n = 0$ we have:

$$\begin{aligned} a\,[n+1]\,2 &= a\,[1]\,2 \\ &= a + 2 \\ &> 3 && \text{since } a > 2 \\ &= 2 + 1 \\ &= a\,[0]\,2 \\ &= a\,[n]\,2 \end{aligned}$$

For $n = 1$ we have:

$$\begin{aligned} a\,[n+1]\,2 &= a\,[2]\,2 \\ &= 2a \\ &= a + a \\ &> a + 2 && \text{since } a > 2 \\ &= a\,[n]\,2 \end{aligned}$$

For $n > 1$ we have:

$$\begin{aligned} a\,[n+1]\,2 &= a\,[n]\,(a\,[n+1]\,(2-1)) && \text{by definition 1.1} \\ &= a\,[n]\,(a\,[n+1]\,1) \\ &= a\,[n]\,a && \text{by lemma 2.1 since } n > 1 \\ &> a\,[n]\,2 && \text{by corollary 3.5 with } a > 2 \end{aligned}$$

and so the result holds for $b = 2$ for all values of n.

Suppose the result holds for $b = j \geq 2$, i.e. $a\,[n+1]\,j > a\,[n]\,j$. Then for $j + 1$ we have:

$$a\,[n+1]\,(j+1) = a\,[n]\,(a\,[n+1]\,j)) \qquad \text{by definition 1.1} \tag{3}$$

$$\begin{aligned} a\,[n+1]\,j &> a\,[n]\,j && \text{by the induction hypothesis} \\ &> j && \text{by lemma 3.3 since } a > 1 \end{aligned}$$

$$a\,[n+1]\,j > j + 1 \qquad \text{by lemma 3.1} \tag{4}$$

$$a\,[n+1]\,(j+1) > a\,[n]\,(j+1) \qquad \text{applying corollary 3.5 to (3) using (4)}$$

and so the result holds for $b = j+1$ and the proof by induction is complete. □

Corollary 3.11. *Suppose $m > n$, $a > 1$ and $b > 1$ with not both $a = 2$ and $b = 2$. Then $a\,[m]\,b > a\,[n]\,b$.*

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Proof. By theorem 3.10, this sequence is strictly increasing

$$\{a\,[n]\,b, a\,[n+1]\,b, \ldots, a\,[m-1]\,b, a\,[m]\,b\}\,.$$

□

Corollary 3.12. *Suppose* $a\,[m]\,b = a\,[n]\,b$ *with* $m > n$*,* $a > 1$ *and* $b > 1$*. Then* $n > 0$*,* $a = 2$ *and* $b = 2$*.*

Proof. This is an immediate consequence of corollary 3.11, since if not both $a = 2$ and $b = 2$ then $a\,[m]\,b > a\,[n]\,b$, contradicting $a\,[m]\,b = a\,[n]\,b$.

We must have $n > 0$ since $2\,[0]\,2 = 3$ but $2\,[m]\,2 = 4$ by lemma 3.9 since $m > 0$. □

4 Hyper-Prime and Hyper-Composite Numbers

Under multiplication, we have the familiar notion of prime and composite numbers. We say that $m > 1$ is *composite* if $m = ab$ for some $a > 1$ and $b > 1$; otherwise m is said to be *prime*. The number 1 is said to be a *unit*, and is neither prime nor composite.

We extend these definitions to higher-rank hyper-operations. We ignore succession and addition as they do not yield any interesting results.

Definition 4.1. n-composite: For $n > 1$, a number $m > 1$ is said to be n-composite if $m = a\,[n]\,b$ for some $a > 1$ and $b > 1$.

Definition 4.2. n-prime: For $n > 1$, a number $m > 1$ is said to be n-prime if m is not n-composite.

Under these definitions, the familiar notions of prime and composite become those of 2-prime and 2-composite.

We now investigate some properties of n-prime and n-composite numbers.

4.1 More n-primes as n increases

We will show that once a number $m > 1$ is n-prime, it stays hyper-prime for all operations of higher rank than n.

In particular, we will show that for $n > 1$:

- all 2-prime numbers are n-prime - they are *forever prime*
- one particular number, 4, is never n-prime - it is *forever composite*
- all other numbers $m > 1$ are *eventually prime* - each m is n-composite for $n < k$ and n-prime for $n \geq k$, for some k depending on m.

Theorem 4.3. *Suppose $m > 1$ is n-prime (for $n > 1$). Then m is also $(n+1)$-prime.*

Proof. We prove by contradiction.

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Suppose m is n-prime but $(n+1)$-composite. Then for some $a > 1$ and $b > 1$, we have:

$$\begin{aligned} m &= a\,[n+1]\,b && \text{by definition 4.1} \\ &= a\,[n]\,(a\,[n+1]\,(b-1)) && \text{by definition 1.1} \end{aligned} \tag{5}$$

We are given $a > 1$ and we now check that $a\,[n+1]\,(b-1) > 1$ for the possible values of b.

When $b = 2$:

$$\begin{aligned} a\,[n+1]\,(b-1) &= a\,[n+1]\,(2-1) \\ &= a\,[n+1]\,1 \\ &= a && \text{by lemma 2.1} \\ &> 1 && \text{given} \end{aligned}$$

When $b > 2$, we have $b - 1 > 1$ and so:

$$\begin{aligned} a\,[n+1]\,(b-1) &> a && \text{by lemma 3.2} \\ &> 1 && \text{given} \end{aligned}$$

Hence (5) is an n-product of two numbers, each of which is > 1. By definition 1.4, (5) provides an n-factorisation of m, which contradicts the assumption that m is n-prime. □

Corollary 4.4. *Suppose p is 2-prime. Then p is forever prime.*

Proof. Trivial from theorem 4.3 with $n = 2$. □

Theorem 4.5. *4 is forever composite.*

Proof. Already shown in lemma 3.9. □

In our consideration of n-prime and n-composite numbers, we have excluded the number 1, dealt with the 2-primes 2 and 3, and mentioned the special case of the number 4. We now turn to larger numbers.

We now show that every number larger than 4 eventually becomes n-prime, for some high-enough ranked n-operation.

For example:

- $15 = 3 \times 5$ is 2-composite, but is 3-prime because 15 is not a perfect power of any natural number. It follows that 15 is also 4-prime, 5-prime and so on.

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- $27 = 3 \times 9 = 3^3 = {}^{2}3$ is 2-composite, 3-composite and 4-composite, but is 5-prime, 6-prime and so on.

Theorem 4.6. *Suppose $m > 4$. Then there is an $n > 1$ such that m is n-prime.*

Proof. Given the k-operation, firstly we look for the smallest k-composite number larger than 4, and observe that all number between 4 and it must be k-prime. Then we establish that that the smallest $(k+1)$-composite number larger than 4 is bigger than the smallest k-composite number larger than 4, which gives us that eventually for some n-operation, the smallest n-composite number larger than 4 is larger than m, thus proving that m is n-prime.

Under the k-operation, consider the k-products $u_k = 2\,[k]\,3$ and $v_k = 3\,[k]\,2$.

We see that $u_k > 4$ and $v_k > 4$ since:

$$u_k = 2\,[k]\,3 > 2\,[k]\,2 = 4 \qquad \text{by theorem 3.4}$$
$$v_k = 3\,[k]\,2 > 2\,[k]\,2 = 4 \qquad \text{by theorem 3.7}$$

We set $c_k = \min(u_k, v_k)$ and show that c_k is the least k-composite number larger than 4.

(Note that when $k = 2$ the two values 2×3 and 3×2 are equal, but for higher-rank operations this will not be the case and one will be smaller than the other, e.g. $2^3 < 3^2$).

We see that c_k is trivially k-composite and larger than 4 because both of u_k and v_k are k-composite and larger than 4, and c_k is equal to one (or both) of u_k, v_k.

All other k-composite numbers larger than 4 are of the form $a\,[k]\,b$ with $a > 1$ and $b > 1$, with not both $a = 2$ and $b = 2$.

When $a = 2$, we have $b \geq 3$, and so $a\,[k]\,b = 2\,[k]\,b \geq 2\,[k]\,3 = u_k \geq c_k$.

When $a \geq 3$, we have $b \geq 2$, and so $a\,[k]\,b \geq 3\,[k]\,b \geq 3\,[k]\,2 = v_k \geq c_k$.

Thus we have that c_k is indeed the least k-composite number larger than 4. Hence every number d in the range $4 < d < c_k$ is k-prime.

Now we consider the $(k+1)$-operation, and see that:

$$u_{k+1} > u_k \qquad \text{by theorem 3.10}$$
$$v_{k+1} > v_k \qquad \text{by theorem 3.10}$$
$$\min(u_{k+1}, v_{k+1}) > \min(u_k, v_k)$$
$$c_{k+1} > c_k$$

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and so the infinite sequence $\{c_2, c_3, \ldots, c_k, \ldots\}$ is strictly increasing and therefore the values in the sequence are eventually greater than any value m.

Hence there must be some n for which $4 < m < c_n$, and since every number d in the range $4 < d < c_n$ is n-prime, then m is n-prime, as required. □

4.2 Infinitely many n-prime and n-composite numbers

We confirm the intuitive result that for any $n > 1$ there are infinitely many n-prime and n-composite numbers.

Theorem 4.7. *Suppose $n > 1$. Then there are infinitely many n-prime numbers.*

Proof. By Euclid [2], there are infinitely many 2-prime numbers.

By theorem 4.3 each 2-prime is also n-prime for any $n > 2$, hence there are infinitely many n-primes for any $n > 1$. □

Theorem 4.8. *Suppose $n > 1$. Then there are infinitely many n-composite numbers.*

Proof. Consider the infinite sequence $\{2\,[n]\,2, 2\,[n]\,3, \ldots, 2\,[n]\,j, \ldots\}$.

Each element in the sequence is n-composite and the sequence is strictly increasing since $2\,[n]\,(j+1) > 2\,[n]\,j$ by theorem 3.4. Hence all the elements of the sequence are distinct, giving that there are infinitely many n-composite numbers. □

Clearly the sequence in the above proof does not contain *all* the n-composite numbers - for example, it does not contain $3\,[n]\,2$. It is sufficient merely to show that there are infinitely many values in the sequence.

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5 Hyper-Products and Equality

We are used to the elementary properties of addition and multiplication that allow the same result to be obtained in more than one way, e.g.:

$$1 + 4 = 2 + 3 = 3 + 2 = 5$$

$$1 \times 12 = 3 \times 4 = 6 \times 2 = 12$$

It is also possible for different exponentiation operations to yield the same results, e.g.:

$$3^4 = 9^2 = 81$$

$$2^6 = 4^3 = 8^2 = 64$$

To formalise, the above examples show that for $n \in \{1, 2, 3\}$ and $a, b, c, d > 1$, it is possible for $a\,[n]\,b = c\,[n]\,d$ for non-trivial cases (i.e. where $a \neq c$).

We observe that in the equality $4^3 = 8^2$, the bases 4 and 8 are not perfect powers of each other. Indeed, $8 = 4^{\frac{3}{2}}$ and $4 = 8^{\frac{2}{3}}$. However, both $4 = 2^2$ and $8 = 2^3$ *are* perfect powers of 2. This relationship will feature in the proofs later in this section.

We now ask whether non-trivial equalities are possible for higher-rank operations, such as tetration.

Let's look at a couple of examples. Firstly, does ${}^2 4 = {}^b 3$ for some b?

We see that ${}^2 4 = 4^4$ is a power of 4, but ${}^b 3 = 3^{{}^{b-1}3}$ is a power of 3. By 2-prime factorisation, no non-trivial power of 4 can equal a power of 3, and so we can see that there are no solutions to ${}^2 4 = {}^b 3$.

So let us try to use a base which shares its 2-prime factors with 4, namely 2. Does ${}^2 4 = {}^b 2$ for some b?

Expanding the LHS gives

$${}^2 4 = 4^4 = \left(2^2\right)^{\left(2^2\right)} = 2^{2 \times 2^2} = 2^{2^1 \times 2^2} = 2^{2^{1+2}} = 2^{2^3} \tag{6}$$

We can see that the final 4-factor of 3 cannot be expressed as a tetration of 2 (since 3 is not even a power of 2, let alone a tetration of 2) and thus there is no b for which ${}^2 4 = {}^b 2$.

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Note that the somewhat laboured steps in (6) give a flavour of the steps in the proof that we will see below.

As a further confirmation, we can check the size of ${}^{2}4$ against the sizes of these tetrations of 2:

$$\begin{aligned} 16 &= 2^{2^2} = {}^{3}2 \\ &< 256 = 4^4 = {}^{2}4 \\ &< 65,536 = 2^{16} = 2^{2^{2^2}} = {}^{4}2 \end{aligned}$$

Informally, we can see there is no room between the neighbouring tetrations ${}^{3}2$ and ${}^{4}2$ to fit another tetration of 2 that equals the required value of ${}^{2}4$.

So, we cannot express ${}^{2}4$ as a tetration of 2. We shall now extend this example into a theorem which shows that it is impossible to express a non-trivial tetration of two numbers as a non-trivial tetration of two other numbers.

Before we prove the main theorem, we assemble some tools that we shall need to do the job.

Lemma 5.1. *Suppose $r > 1$. Then $r^u + u < r^{u+1}$.*

Proof. We prove by induction on u.

For $u = 0$, we have:

$$\begin{aligned} r^u + u &= r^0 + 0 \\ &= 1 \\ &< r && \text{since } r > 1 \\ &= r^{u+1} \end{aligned}$$

and so the result holds for $u = 0$.

Suppose the result holds for $u = k$, i.e. $r^k + k < r^{k+1}$, then for $k + 1$ we have:

$$\begin{aligned} r^{k+1} + (k+1) &\le r^{k+1} + \left(k + r^k\right) && \text{since } r > 0 \text{ gives us } r^k \ge 1 \\ &< r^{k+1} + r^{k+1} && \text{by the induction hypothesis } k + r^k < r^{k+1} \\ &= 2r^{k+1} \\ &\le r\left(r^{k+1}\right) && \text{since } r \ge 2 \\ &= r^{(k+1)+1} \end{aligned}$$

Thus, the result holds for $u = k+1$, and the proof by induction is complete. ☐

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Lemma 5.2. *Suppose $r^u < s < r^{u+1}$ with $r > 1$. Then s is not a power of r.*

Proof. By theorem 3.4 the powers of r strictly increase and we deduce that between the two neighbouring powers r^u and r^{u+1} there cannot lie any other power of r. □

Definition 5.3. Least root: Given $m > 1$, then r is said to be the least root of m if r is the least value such that $m = r^g$.

Lemma 5.4. *Suppose $m > 1$. Then there is a unique r which is the least root of m. Furthermore, $1 < r \leq m$.*

Proof. We calculate r as follows. Let the 2-prime factorisation of m be

$$m = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$$

for distinct 2-primes p_i and indices $b_i > 0$ $(i = 1 \ldots k)$, with $p_i < p_j$ for $i < j$.

Such a factorisation of m exists and is unique by the Fundamental Theorem of Arithmetic. Observe that $k > 0$ because $m > 1$ and so m has at least one prime factor.

We set g as the greatest common divisor of the indices:

$$g = \gcd(b_1, b_2, \ldots, b_k)$$

We now specify the value of r:

$$r = p_1^{\frac{b_1}{g}} p_2^{\frac{b_2}{g}} \cdots p_k^{\frac{b_k}{g}}$$

We now check that r has the properties we require.

We have that $r \in \mathbb{N}$ and $r > 1$ because r is a product of positive integers (since we know that each $\frac{b_i}{g}$ is an integer, by the definition of g as a common divisor).

We have that $r \leq m$ because we know that $g \geq 1$ by the properties of a greatest common divisor [4].

We have that r is a root of m, since

$$r^g = \left(p_1^{\frac{b_1}{g}} p_2^{\frac{b_2}{g}} \cdots p_k^{\frac{b_k}{g}} \right)^g = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k} = m$$

(and in the trivial case where $g = 1$, then $m = r$, i.e. m is its own least root).

Lastly, we show that r is the least of all roots of m.

Assume that there is a smaller root $u < r$ with $m = u^v$. Then $v > g$ and

$$u = m^{\frac{1}{v}} = p_1^{\frac{b_1}{v}} p_2^{\frac{b_2}{v}} \cdots p_k^{\frac{b_k}{v}}$$

We have that u is a natural number and so each index $\frac{b_i}{v}$ must be a natural number, and thus we have $v|b_i$ for each $i = 1 \ldots k$. Thus v is a common divisor of $(b_1, b_2, \ldots, b_k)$ greater than g, contradicting the definition of g as greatest common divisor.

Thus there can be no root of m less than r, confirming that r is the least root of m. □

We now show that any root of any non-trivial power of m must be a natural power of the least root of m.

Lemma 5.5. *Suppose $m > 1, d > 0$, and u is a root of m^d. Then $u = r^v$ for some $v \in \mathbb{N}$, where r is the least root of m.*

Proof. We have that r is the least root of m, so by continuing to use the notation in lemma 5.4 we have $r^g = m$.

We have that u is a root of m^d, so $u^w = m^d$ for some w, and thus

$$u^w = m^d = (r^g)^d = r^{gd}$$

$$u = m^{\frac{d}{w}} = r^{\frac{gd}{w}}$$

We set $v = \frac{gd}{w}$ and show that v is a natural number, which will prove that $u = r^v$ is a natural power of r.

As in the proof of lemma 5.4, we factorise m into its prime factors:

$$m = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$$

and thus

$$u = p_1^{\frac{b_1 d}{w}} p_2^{\frac{b_2 d}{w}} \cdots p_k^{\frac{b_k d}{w}}$$

with $\frac{b_i d}{w} \in \mathbb{N}$ for $i = 1 \ldots k$ and thus $w|b_i d$ for $i = 1 \ldots k$.

By the properties of gcd [4] we have:

$$w|\gcd(b_1 d, b_2 d, \ldots, b_k d) = d \gcd(b_1, b_2, \ldots, b_k) = dg$$

We thus have $w|dg$ giving $v = \frac{dg}{w} \in \mathbb{N}$ as required. □

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Lemma 5.6. *Suppose $m > 1$ with least root r. Then r is 3-prime.*

Proof. Let r be the least root of $m = r^g$.

Trivially we have $r > 1$, and so r is not the unit 1. Thus r must be either 3-prime or 3-composite (by definition 4.1 and definition 4.2).

Suppose r is 3-composite. Then by definition 4.1, $r = u^v$ for some u and v, with $1 < u < r$ and $v > 1$.

Thus we have $m = r^g = (u^v)^g = u^{vg}$ and so u is a root of m. But $u < r$, contradicting the definition of r as least root of m.

Thus r cannot be 3-composite and must instead be 3-prime. □

So, with our tools laid out, we can now prove our theorem that all non-trivial tetrations result in different 4-products.

Theorem 5.7. *Suppose ${}^ba = {}^dc$ with $a, b, c, d > 1$. Then $a = c$ and $b = d$.*

Proof. If $a = c$, then we have $b = d$ by corollary 3.6.

Otherwise, $a \neq c$, and we shall show that this eventually leads to a contradiction.

Without loss of generality, assume $a < c$. This gives $b > d$, since if $b \leq d$ then by lemmas 3.8 and 3.5 we would have ${}^ba < {}^dc$ contradicting ${}^ba = {}^dc$.

Given c, what values are possible for a?

We are given $b > 1$ and thus $b - 1 > 0$ and ${}^{b-1}a > 0$. We deduce that ${}^ba = a^{{}^{b-1}a}$ is a non-trivial power of a.

Similarly, ${}^dc = c^{{}^{d-1}c}$ is a non-trivial power of c.

We now set r to be the least root of $c = r^h$ (which exists and is unique by lemma 5.4).

From lemma 5.5, a must also be a power of r (with the possibility that $a = r$). We set $a = r^g$ for some $g > 0$, and we have:

$$a = r^g$$
$$c = r^h$$
$$g < h \qquad \text{since } a < c \tag{7}$$

Now let us return to the equation in the statement of this theorem:

$$
\begin{aligned}
{}^{b}a &= {}^{d}c && \text{given}\\
{}^{b}\left(r^{g}\right) &= {}^{d}\left(r^{h}\right) && \text{substituting for } a, c\\
\left(r^{g}\right)^{{}^{b-1}\left(r^{g}\right)} &= \left(r^{h}\right)^{{}^{d-1}\left(r^{h}\right)} && \text{expanding according to definition 1.1}\\
&= \left(r^{g\times\frac{h}{g}}\right)^{{}^{d-1}\left(r^{h}\right)} && \text{rewriting } h \text{ as } g\times\frac{h}{g}\\
&= \left(r^{g}\right)^{{}^{d-1}\left(r^{h}\right)\frac{h}{g}}\\
{}^{b-1}\left(r^{g}\right) &= {}^{d-1}\left(r^{h}\right)\frac{h}{g} && \text{by corollary 3.6,}\\
& && \text{since } r>1 \text{ and } g>0 \text{ gives } r^{g}>1\\
\frac{h}{g} &= \frac{{}^{b-1}\left(r^{g}\right)}{{}^{d-1}\left(r^{h}\right)}\\
&= \frac{\left(r^{g}\right)^{{}^{b-2}\left(r^{g}\right)}}{\left(r^{h}\right)^{{}^{d-2}\left(r^{h}\right)}} && \text{expanding according to definition 1.1} \qquad (8)\\
&= \frac{r^{{}^{b-2}\left(r^{g}\right)g}}{r^{{}^{d-2}\left(r^{h}\right)h}}\\
&= r^{{}^{b-2}\left(r^{g}\right)g-{}^{d-2}\left(r^{h}\right)h} && (9)
\end{aligned}
$$

We pause to check that step (8) is valid, i.e. that both $b-2$ and $d-2$ are natural numbers; we are given that $d>1$ and have deduced that $b>d$, hence $b-2>d-2\geq 0$ and thus step (8) is indeed valid.

From (9) we have that $\frac{h}{g}$ is an integer power of r, and (7) gives us that $\frac{h}{g}>1$. We thus have that $\frac{h}{g}$ is a natural power of r.

We set

$$
\begin{aligned}
u &= {}^{b-2}\left(r^{g}\right)g-{}^{d-2}\left(r^{h}\right)h && (10)\\
\frac{h}{g} &= r^{u} && \text{substituting into (9)}
\end{aligned}
$$

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and so $u > 0$ (since $h > g$ gives $\frac{h}{g} > 1$), and we see from (10) that u is a result of integer operations, hence $u \in \mathbb{N}$.

$$\begin{aligned}
h &= gr^u && \text{rearranging} \\
u &= {}^{b-2}(r^g)g - {}^{d-2}\left(r^h\right)gr^u && \text{substituting for } h \text{ in (10)} \\
\frac{u}{g} &= {}^{b-2}(r^g) - {}^{d-2}\left(r^h\right)r^u && \text{dividing by } g \\
{}^{b-2}(r^g) &= {}^{d-2}\left(r^h\right)r^u + \frac{u}{g} && \text{rearranging} \qquad (11)
\end{aligned}$$

We recognise that ${}^{d-2}\left(r^h\right)$ is a power of r, i.e. ${}^{d-2}\left(r^h\right) = r^v, v \geq 0$.

If $d = 2$, then ${}^{d-2}\left(r^h\right) = 1$, giving $v = 0$.

Otherwise $d > 2$, and ${}^{d-2}\left(r^h\right) = \left(r^h\right)^{{}^{d-3}\left(r^h\right)} = r^v$ gives us $v = {}^{d-3}\left(r^h\right)h$.

Similarly, ${}^{b-2}(r^g)$ is also a power of r, i.e. ${}^{b-2}(r^g) = r^w, w > 0$. We know that w is strictly positive because $b - 2 > d - 2 \geq 0$ gives us $b - 3 \geq 0$ and so ${}^{b-2}(r^g) = (r^g)^{{}^{b-3}(r^g)} = r^w$ gives us $w = {}^{b-3}(r^g)g$.

We can now write equation (11) as

$$\begin{aligned}
r^w &= r^v r^u + \frac{u}{g} \\
r^{w-v} &= r^u + \frac{u}{gr^v} && \text{dividing by } r^v \qquad (12)
\end{aligned}$$

We now show that (12) results in a contradiction.

The RHS of (12) satisfies these inequalities:

$$\begin{aligned}
r^u < r^u + \frac{u}{gr^v} &< r^u + u && \text{since } u > 0, \text{ and } gr^v > 1 \\
&< r^{u+1} && \text{by lemma 5.1}
\end{aligned}$$

and so we have that $r^u + \frac{u}{gr^v}$ lies strictly between the neighbouring powers r^u and r^{u+1} and thus by lemma 5.2 the RHS of (12) cannot be a power of r.

However, the LHS of (12) clearly *is* a power of r, and so we arrive at a contradiction.

Hence, there are no non-trivial solutions to ${}^b a = {}^d c$, as required. □

Having established the result for tetrations, we can easily extend it to higher-rank operations.

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Corollary 5.8. *Suppose* $a\,[n]\,b = c\,[n]\,d$ *with* $a, b, c, d > 1$ *and* $n \geq 4$. *Then* $a = c$ *and* $b = d$.

Proof. Proof is by induction on n.

The result for $n = 4$ is proved in theorem 5.7.

Assume the result holds for $n = k$, i.e. there are no non-trivial solutions to $a\,[k]\,g = c\,[k]\,h$ for any $a, g, c, h > 1$.

Suppose we have an equation using the $[k+1]$ operation for $a, b, c, d > 1$:

$$a\,[k+1]\,b = c\,[k+1]\,d \tag{13}$$

$$a\,[k]\,(a\,[k+1]\,(b-1)) = c\,[k]\,(c\,[k+1]\,(d-1)) \qquad \text{by definition 1.1} \tag{14}$$

We put

$$g = a[k+1](b-1)$$
$$h = c[k+1](d-1)$$

to arrive at

$$a\,[k]\,g = c\,[k]\,h \tag{15}$$

We pause to check that the expansion in (14) is valid: we are given $b > 0$ and $d > 0$, hence $b - 1 > 0$ and $d - 1 > 0$ and the expansion in (14) is indeed valid.

We check that $g > 1$. If $b = 2$, then by lemma 2.1 we have $g = a\,[k+1]\,1 = a > 1$. Otherwise $b > 2$ and $b - 1 > 1$, and by lemma 3.2 we have $g > a > 1$.

Similarly we have $h > 1$. We already have $a > 1$ and $c > 1$, so it is valid to apply the induction hypothesis to (15) to show that $a = c$. We then apply corollary 3.6 to (13) to show that $b = d$.

Thus the result holds for $n = k + 1$, and the proof by induction is complete. □

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6 Commutativity and Associativity

I'm playing all the right notes, but not necessarily in the right order.

Eric Morecambe

The commutative and associative properties of addition and multiplication are well known, i.e. for any a, b and c:

$$\begin{aligned} a + b &= b + a \\ ab &= ba \\ (a + b) + c &= a + (b + c) \\ (ab)c &= a(bc) \end{aligned}$$

We take a look at other hyper-operations.

6.1 Commutativity

Here we look at whether commutativity holds for other hyper-operations. We ignore the trivial cases where $a = b$ since clearly any operation which takes in two identical operands will yield the same result if those identical operands are swapped. We also ignore cases where either operand is 0 or 1.

First we show, as a mere curiosity, that the succession operation ($n = 0$) is not commutative:

Theorem 6.1. *Suppose* $a\,[0]\,b = b\,[0]\,a$. *Then* $b = a$.

Proof.

$$\begin{aligned} a\,[0]\,b &= b\,[0]\,a \\ b + 1 &= a + 1 && \text{by definition 1.1 with } n = 0 \\ b &= a \end{aligned}$$

so there are non-trivial commutative 0-operations. □

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The proofs for commutativity under addition and multiplication (for all values a and b) are well known and are not repeated here.

For the moment we skip the 3-operation and look at higher-rank operations.

Theorem 6.2. *Suppose* $a\,[n]\,b = b\,[n]\,a$ *with* $a > 1, b > 1, n \geq 4$. *Then* $b = a$.

Proof. It follows directly from corollary 5.8 that $b = a$. □

Returning now to the 3-operation, let's see if there are any non-trivial commutative operations. We can see at once that not every 3-operation is commutative by way of a simple counter-example, $2^3 \neq 3^2$.

However, after some searching, we do find a non-trivial commutative 3-operation:

$$2^4 = 16 = 4^2$$

Are there any others? The answer is no, which we shall now prove.

Theorem 6.3. *Suppose* $a^b = b^a$ *with* $1 < a < b$. *Then* $a = 2$ *and* $b = 4$.

Proof. We set r to be the least root of $a = r^g$ for some $r > 1, g > 0$ by lemma 5.4.

Then by lemma 5.5, r is also the least root of $b = r^h$ for some $h > 0$. We also have that $h > g$ since we are given $b > a$.

$$\begin{aligned}
a^b &= b^a && \text{given} \\
(r^g)^{\left(r^h\right)} &= \left(r^h\right)^{\left(r^g\right)} && \text{substituting for } a \text{ and } b \\
r^{gr^h} &= r^{hr^g} \\
gr^h &= hr^g && \text{by corollary 3.6, since } r > 1 \\
\frac{h}{g} &= r^{h-g} && \text{rearranging}
\end{aligned}$$

(which shows that $\frac{h}{g}$ is a natural number > 1, since $h > g$ gives that the RHS is a non-zero power of r)

$$\begin{aligned}
r\left(\frac{h}{g}\right) &= r^{h-g+1} && \text{multiplying by } r \\
&= r^{\left(\frac{h}{g}-1\right)g+1} && (16) \\
&\geq r^{\left(\frac{h}{g}-1\right)+1} && \text{since } g \geq 1 \\
&= r^{\left(\frac{h}{g}\right)}
\end{aligned}$$

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resulting in

$$r\left(\frac{h}{g}\right) \geq r^{\left(\frac{h}{g}\right)} \tag{17}$$

However, applying corollary 3.5 with $n = 2$ gives us that

$$r^{\left(\frac{h}{g}\right)} > r\left(\frac{h}{g}\right)$$

contradicting (17), unless $r = 2$ and $\frac{h}{g} = 2$.

We duly set $r = 2$ and $\frac{h}{g} = 2$ and substitute into (16) to give

$$\begin{aligned} 2\,(2) &= 2^{(2-1)g+1} \\ 4 &= 2^{g+1} \\ g &= 1 \\ h &= 2 \\ a &= r^g = 2 \\ b &= r^h = 4 \qquad\qquad \text{as required.} \end{aligned}$$

□

6.2 Associativity

Now we turn to whether associativity holds for other hyper-operations. Again we ignore the trivial cases where $a = b = c$ and also ignore cases where any of the operands is 0 or 1.

Theorem 6.4. $(a\,[0]\,b)\,[0]\,c \neq a\,[0]\,(b\,[0]\,c)$ *for any* a, b *and* c.

Proof.

Suppose

$$(a\,[0]\,b)\,[0]\,c = a\,[0]\,(b\,[0]\,c) \tag{18}$$

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Then:

$$
\begin{aligned}
(a\,[0]\,b)\,[0]\,c &= c+1 && \text{expanding LHS of (18), by definition 1.1}\\
a\,[0]\,(b\,[0]\,c) &= (b\,[0]\,c)+1 && \text{expanding RHS of (18), by definition 1.1}\\
&= (c+1)+1 && \text{by definition 1.1}\\
c+1 &= c+2 && \text{substituting back into (18)}
\end{aligned}
\tag{19}
$$

giving a contradiction, so there are no associative 0-operations (even trivial ones). □

As for commutativity, the proofs for associativity under addition and multiplication (for all values a and b) are well known and are not repeated here.

Theorem 6.5. *Suppose* $\left(a^b\right)^c = a^{\left(b^c\right)}$ *with* $a > 1, b > 1, c > 1$*. Then* $b = 2$ *and* $c = 2$*.*

Proof.

$$
\begin{aligned}
\left(a^b\right)^c &= a^{\left(b^c\right)} && \text{given}\\
a^{bc} &= a^{\left(b^c\right)} && \\
bc &= b^c && \text{by corollary 3.6}\\
b &= 2, c = 2 && \text{by corollary 3.12}
\end{aligned}
$$

□

Theorem 6.6. *Suppose* $(a\,[n]\,b)\,[n]\,c = a\,[n]\,(b\,[n]\,c)$ *with* $a > 1, n \geq 4$*. Then* $b = 1$ *and* $c = 1$*.*

Proof. By corollary 5.8 we have:

$$
\begin{aligned}
a\,[n]\,b &= a && \text{and} && (20)\\
c &= b\,[n]\,c && && (21)\\
b &= 1 && \text{applying lemma 2.1 to (20)}\\
c &= 1\,[n]\,c && \text{substituting } b = 1\\
c &= 1 && \text{applying lemma 2.2 to (21)}
\end{aligned}
$$

and so there are no associative n-operations for $n \geq 4$ with $a > 1, b > 1, c > 1$. □

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7 Unique n-Factorisation: The Fundamental Theorems

There are often many ways of using multiplication to arrive at the same result. For example,

$$12 = 1 \times 12 = 2 \times 6 = 2 \times 3 \times 2 = 3 \times 4 = 6 \times 1 \times 2 \times 1$$

It is natural to ask whether there is a unique canonical representation of a number, arrived at through multiplication of other numbers.

The Fundamental Theorem of Arithmetic [3] is a well-known result that every $m > 1$ can be factorised as a product of powers of 2-primes, i.e.

$$m = {p_1}^{b_1} {p_2}^{b_2} \dots {p_k}^{b_k}$$

with $b_i > 0$ for $i = 1 \dots k$ and $k > 0$.

Due to the commutativity of multiplication, these powers of primes can be put in any order to arrive at the product m. In order to provide a *unique* factorisation of m, we choose a canonical order for the primes, the most natural being to place them in ascending order, i.e. $p_i < p_j$ for $i < j$.

Note that there may be gaps in the 2-primes, for example $20 = 2^2 \times 5^1$, which makes no mention of the 2-prime 3. In this paper (for the purposes of our theorems) we do not permit an index of 0, and so we cannot write $20 = 2^2 \times 3^0 \times 5^1$.

We now explore factorisation for higher-rank operations, arriving at results that we shall name as the Fundamental Theorems of Hyper-Operations. We shall prove two distinct but related theorems, one for the 3-operation and another for $\geq$ 4-operations.

7.1 The Fundamental Theorem of the 3-operation

It is easy to come up examples of multiple ways of using the 3-operation to arrive at the same result.

For instance,

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$$43,046,721 = 9^8 = 81^4 = 3^{16} = 3^{4^2}$$

and each of these expressions use the 3-operation (but no other operations) to arrive at the same value, $43,046,721$.

We now show that there is a canonical representation of any number $m > 1$ using only the 3-operation.

> It's no use, Mr. James — it's turtles all the way down.
>
> *J. R. Ross, Constraints on Variables in Syntax*

Definition 7.1. All-3-prime factorisation: a 3-factorisation where all of the 3-factors are 3-prime.

Theorem 7.2. *Suppose $m > 1$. Then there is a unique all-3-prime factorisation of m.*

Proof. We construct an all-3-prime factorisation of m, and then show that this all-3-prime factorisation is unique.

Let r be the least root of m, with $m = r^g$. Then r is 3-prime by lemma 5.6.

We put $a_1 = r$ and $b_1 = g$ to give $m = {a_1}^{b_1}$, with a_1 a 3-prime and $b_1 > 0$.

If $b_1 = 1$, then m is its own least root, and we have that $m = r = a_1$ is a 3-factorisation with its single 3-factor a_1 being a 3-prime. Thus we have a (trivial) all-3-prime factorisation of m.

Otherwise, $b_1 > 1$ and we repeat our procedure to find the least root a_2 of b_1 to give us a 3-factorisation of $b_1 = {a_2}^{b_2}$ for some $b_2 > 0$.

Thus we have $m = {a_1}^{b_1} = {a_1}^{{a_2}^{b_2}}$, with a_1, a_2 both 3-prime and $b_2 > 0$.

It is clear from lemma 3.3 that $b_2 < b_1 < m$.

We repeat the procedure on each b_i, giving

$$m = {a_1}^{b_1} = {a_1}^{{a_2}^{b_2}} = \cdots = a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot^{a_i^{b_i}}}}}}}$$

with the values of sequence of $b_i > 0$ strictly decreasing, i.e. $b_j < b_i$ for $j > i$.

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After a finite number of iterations, $b_k = 1$ for some $k > 0$, halting the procedure, and giving us an all-3-prime factorisation $m = a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot^{a_k}}}}}}$.

Finally, we show that this all-3-prime factorisation of m is unique.

Suppose $m = c_1^{c_2^{c_3^{\cdot^{\cdot^{\cdot^{c_t}}}}}}$ is an all-3-prime factorisation of m.

We see that c_1 is a root of m (which is a natural power of a_1), and by lemma 5.5, c_1 is a natural power of the least root r of m. We chose a_1 to be the least root of m and so $c_1 = {a_1}^u$ for some $u > 0$.

However, by definition 7.1, c_1 is 3-prime which forces $u = 1$ and thus $c_1 = a_1$, giving $c_2^{c_3^{\cdot^{\cdot^{\cdot^{c_t}}}}} = a_2^{a_3^{\cdot^{\cdot^{\cdot^{a_k}}}}}$ by corollary 3.6.

Repeating this check for each c_i gives us that $c_i = a_i$ for $i = 1 \ldots k$, and also forces $t = k$. Thus the all-3-prime factorisations $a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot^{a_k}}}}}}$ and $c_1^{c_2^{c_3^{\cdot^{\cdot^{\cdot^{c_t}}}}}}$ are identical, proving our uniqueness requirement. $\square$

Returning to the example shown earlier, we can now apply the procedure from the proof to create an all-3-prime factorisation of 43, 046, 721 as follows:

$$43,046,721^1 = 3^{16} = 3^{2^4} = 3^{2^{2^2}}$$

with this last expression containing 3-factors which are all 3-prime.

We observe that, in comparison to a 2-prime factorisation, the order of the factors in an all-3-prime factorisation is critically important.

For example, consider the 6 permutations of the 2-primes 2, 3 and 5.

Under multiplication, these permutations all produce the same result:

$$\begin{aligned} 30 &= 2 \times 3 \times 5 \\ &= 2 \times 5 \times 3 \\ &= 3 \times 2 \times 5 \\ &= 3 \times 5 \times 2 \\ &= 5 \times 2 \times 3 \\ &= 5 \times 3 \times 2 \end{aligned}$$

and we choose the permutation with ascending primes $(2 \times 3 \times 5)$ as the canonical unique representation of the 2-prime factorisation of 30.

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By contrast, under exponentiation, the 6 permutations each produce a number which is different from the other permutations:

$$2^{3^5} \neq 2^{5^3} \neq 3^{2^5} \neq 3^{5^2} \neq 5^{2^3} \neq 5^{3^2}$$

This difference in the effect of ordering between a 2-prime factorisation and an all-3-prime factorisation is related to commutativity (see section 6.1).

7.2 The Fundamental Theorem of the (≥ 4)-operations

Having established a unique factorisation theorem for the 3-operation, we now turn to higher-rank operations.

Using tetration as an example, we have:

$$^{1}18,014,398,509,481,984 = {}^{27}4 = {}^{(3^3)}4 = {}^{^{2}3}4$$

Observe that in the expression ${}^{^{2}3}4$, not all the 4-factors are 4-prime. In particular, 4 is 4-composite, since it can be 4-factored as $4 = {}^{2}2$. However, due to theorem 5.7, we know we cannot equate a tetration of 4 with a tetration of any other number, and so we are stuck with the number 4 being the first 4-factor in any non-trivial 4-factorisation of $18,014,398,509,481,984$.

Clearly then, we cannot prove a unique factorisation theorem which insists that every 4-factor is a 4-prime. This contrasts strongly with theorem 7.2 for all-3-prime factorisation where every 3-factor *is* a 3-prime.

However, we were able express the index 27 as ${}^{2}3$, so that both ${}^{27}4$ and ${}^{^{2}3}4$ are expressions that use only the tetration operation, but evaluate to the same result.

Observe that in ${}^{27}4$, the final 4-factor 27 is 4-composite, whereas in ${}^{^{2}3}4$ the final 4-factor 2 is 4-prime. It is this final factor that is important in our uniqueness theorem.

Definition 7.3. Final-n-prime factorisation: an n-factorisation where the final n-factor is n-prime.

Theorem 7.4. *Suppose $m > 1$ and $n \geq 4$. Then there is a unique final-n-prime factorisation of m.*

Proof. We take a similar approach to that used in the proof for theorem 7.2.

We construct a final-n-prime factorisation of m, and then show that this final-n-prime factorisation is unique.

Consider m; it is either n-prime or n-composite (it is not a unit since $m > 1$).

If m is n-prime, then we set $a_1 = m$, and so m is its own final-n-prime factorisation. Its final (indeed, only) n-factor a_1 is n-prime.

Otherwise m is n-composite, then we have $m = a_1 \, [n] \, b_1$ for some $a_1 > 1$ and $b_1 > 1$. It is clear from lemma 3.3 that $b_1 < m$.

We repeat our procedure to find an n-factorisation of b_1, setting $b_1 = a_2 \, [n] \, b_2$ for some $b_2 > 0$.

Thus we have $m = a_1 \, [n] \, b_1 = a_1 \, [n] \, (a_2 \, [n] \, b_2)$.

It is clear from lemma 3.3 that $b_2 < b_1 < m$.

We repeat the procedure on each b_i, giving

$$m = a_1 \, [n] \, b_1 = a_1 \, [n] \, (a_2 \, [n] \, b_2) = \cdots = a_1 \, [n] \, (a_2 \, [n] \, (a_3 \, [n] \, (\cdots [n] \, (a_i \, [n] \, b_i))))$$

with the sequence of $b_i > 1$ strictly decreasing. After a finite number of iterations, b_{k-1} is n-prime for some $k > 0$, halting the procedure, and giving us an n-factorisation where the final factor $a_k = b_{k-1}$ is n-prime, as required:

$$m = a_1 \, [n] \, (a_2 \, [n] \, (a_3 \, [n] \, (\cdots [n] \, (a_{k-1} \, [n] \, a_k))))$$

Finally, we show that this final-n-prime factorisation of m is unique.

Suppose $m = c_1 \, [n] \, (c_2 \, [n] \, (c_3 \, [n] \, (\cdots [n] \, c_t)))$ is a final-n-prime factorisation of m.

If $k = 1$, then $m = a_1$ is n-prime, forcing $t = 1$ and $c_1 = a_1$, and so the (trivial) final-n-prime factorisations of $m = a_1 = c_1$ are identical.

Otherwise, $k > 1$, and $m = a_1 \, [n] \, b_1$, where

$$b_1 = a_2 \, [n] \, (a_3 \, [n] \, (\cdots [n] \, a_k)) \, .$$

If $t = 1$, then $c_1 = m$ is n-composite, contradicting definition 7.3 which requires the final n-factor c_1 to n-prime.

Thus we have $t > 1$ and $m = a_1 \, [n] \, b_1 = c_1 \, [n] \, d_1$, where

$$d_1 = c_2 \, [n] \, (c_3 \, [n] \, (\cdots [n] \, c_t)) \, .$$

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By corollary 5.8, we must have $c_1 = a_1$ and $d_1 = b_1$. We repeat the check on the n-factorisation of b_1 and d_1 to deduce that $c_i = a_i$ for $i = 1 \ldots k$, also forcing $t = k$.

This shows that the final-n-prime factorisation

$$m = c_1 \, [n] \, (c_2 \, [n] \, (c_3 \, [n] \, (\cdots [n] \, (c_t))))$$

is identical to

$$m = a_1 \, [n] \, (a_2 \, [n] \, (a_3 \, [n] \, (\cdots [n] \, (a_k))))$$

thus proving our uniqueness requirement. $\square$

Returning to our example, we have the unique final-4-prime factorisation

$$18,014,398,509,481,984 = {}^{{}^{2}3}4.$$

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