Density Elimination for Semilinear Substructural Logics

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Abstract

We present a uniform method of density elimination for several semilinear substructural logics. Especially, the density elimination for the involutive uninorm logic IUL is proved. Then the standard completeness of IUL follows as a lemma by virtue of previous work by Metcalfe and Montagna.

Keywords: Density elimination, Involutive uninorm logic, Standard completeness of IUL, Semilinear substructural logics, Fuzzy logic

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Contents

1. Introduction .................................................................................................................. 2
2. Preliminaries .................................................................................................................. 5
3. Proof of the main theorem: A computational example ................................................... 7
4. Preprocessing of Proof Tree ........................................................................................ 15
5. The generalized density rule (D) for GLΩ .................................................................... 22
6. Extraction of Elimination Rules .................................................................................. 26
7. Separation of one branch ............................................................................................. 31
8. Separation algorithm of multiple branches ................................................................... 35
9. The proof of Main theorem ........................................................................................ 45
10. Final remarks and open problems ............................................................................ 46
References ....................................................................................................................... 47
Appendices ....................................................................................................................... 47
A.1. Why do we adopt Avron-style hypersequent calculi? ................................................ 47
A.2. Why do we need the constrained external contraction rule? ....................................... 48
A.3. Why do we need the separation of branches? ............................................................. 49
A.4. Some questions about Theorem 8.2 ......................................................................... 49
A.5. Illustrations of notations and algorithms ................................................................ 50
A.5.1. Illustration of two cases of (COM) in the proof of Lemma 5.6 ............................... 50
A.5.2. Illustration of Construction 6.1 ............................................................................. 51
A.5.3. Illustration of Notation 6.10 and Construction 6.11 .............................................. 51
A.5.4. Illustration of Theorem 8.2 .................................................................................. 52

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Notation

\( G_1 \equiv G_2 \ldots \) The symbol \( G_1 \) denotes a complex hypersequent \( G_2 \) temporarily for convenience.
\( X \equiv Y \) \ldots \ldots Define \( X \) as \( Y \) for two hypersequents (sets or derivations) \( X \) and \( Y \).
\( G_0 \ldots \ldots \) The upper hypersequent of strong density rule in Theorem 1.1, Page 4
\( \tau \) \ldots \ldots A cut-free proof of \( G_0 \) in \( GL_1 \), in Theorem 1.1, Page 4
\( \mathcal{P}(H) \) \ldots \ldots The position of \( H \in \tau \), Def. 2.13, Construction 6.1, Page 7.26
\( \langle H \rangle_{H}^{\epsilon} \) and \( \epsilon_{H}^{\epsilon} \langle H \rangle_{H}^{\epsilon} \) \ldots \ldots Construction 4.7, Page 17
\( G_{i}^{\epsilon} \) and \( \mathcal{C}_{i}^{\epsilon} \) \ldots \ldots Notation 4.10, Page 18
\( G_{i}^{\epsilon} \) \ldots \ldots The proof of \( G_{i}^{\epsilon} \) in \( GL_{\Delta} \) resulting from preprocessing of \( \tau \), Notation 4.13, Page 20
\( \tau \) \ldots \ldots The proof of \( G_{i}^{\epsilon} \) \ldots \ldots The root of \( \tau \) corresponding to the root \( G_0 \) of \( \tau \), Notation 4.13, Page 20
\( S_{i}^{\epsilon} \) \ldots \ldots The \( i \)-th \((pEC)\)-node in \( \tau \), the superscript \( \epsilon \) means contraction, Notation 4.14, Page 20
\( S_{i}^{\epsilon} \) and \( S_{i}^{\epsilon} \) \ldots \ldots Notation 4.14, Page 20
\( \{ S_{i}^{\epsilon}, H_{k}^{\epsilon} \} \) \ldots \ldots The set of all \((pEC)\)-nodes in \( \tau \), Notation 4.14, Page 20
\( GL_{\Delta} \) \ldots \ldots A restricted subsystem of \( GL_1 \), Definition 4.16, Page 20
\( [S_{i}^{\epsilon}], [G_{i}^{\epsilon}] \) \ldots \ldots The minimal closed unit of \( S \) and \( G_{i}^{\epsilon} \) in \( G \), respectively, Definition 5.1, Page 22
\( (D) \) \ldots \ldots The generalized density rule of \( GL_{\Delta} \), Definition 5.4, Page 22
\( \tau_{i}^{\epsilon} \) and \( G_{i}^{\epsilon} \) \ldots \ldots Notation 6.5, Page 28
\( H_{i}^{\epsilon} \sim H_{j}^{\epsilon} \) \ldots \ldots Notation 6.10, Page 29
\( I = \{ H_{i}^{\epsilon}, \ldots, H_{k}^{\epsilon} \} \) \ldots \ldots A subset of \( \{ H_{i}^{\epsilon}, \ldots, H_{k}^{\epsilon} \} \), Notation 6.10, Page 29
\( H_{i}^{\epsilon}, H_{j}^{\epsilon} \) \ldots \ldots The intersection nodes of \( I \) and, that of \( H_{i}^{\epsilon} \) and \( H_{j}^{\epsilon} \), Notation 6.10, Page 29
\( I = \{ S_{1}^{\epsilon}, \ldots, S_{l}^{\epsilon} \} \) \ldots \ldots A subset of \((pEC)\)-sequences to \( I \), Definition 6.14, Page 30
\( V = \{ G_{i}^{\epsilon}, S_{i}^{\epsilon}, \ldots, G_{n}^{\epsilon}, S_{n}^{\epsilon} \} \) \ldots \ldots A set of closed hypersequents to \( I \), Def. 6.14, Page 30
\( \langle H \rangle_{\tau}^{\epsilon}, \tau_{\tau}^{\epsilon} \) and \( G_{\tau}^{\epsilon} \) \ldots \ldots The elimination derivation, Construction 6.11, Lemma 6.13, Page 29, 30
\( \tau_{\tau}^{\epsilon} \) \ldots \ldots The elimination rule, Definition 6.14, Page 30
\( [S_{i}^{\epsilon}]_{l} \) \ldots \ldots A branch of \( H_{i}^{\epsilon} \) to \( I \), Definition 7.2, Page 31
\( G_{i}^{\epsilon}(\lambda) \) \ldots \ldots Construction 7.3, Page 31
\( G_{i}^{\epsilon}(\lambda) \) \ldots \ldots Construction 7.3, Page 31
\( G_{i}^{\epsilon}(\lambda) \) \ldots \ldots Construction 7.7, Theorem 8.2, Page 34, 36
\( \tau_{\tau}^{\epsilon} \) \ldots \ldots The skeleton of \( \tau_{\tau}^{\epsilon} \), Definition 7.13, Page 35
\( \partial_{\tau}^{\epsilon} (H) \) \ldots \ldots Theorem 8.2 (ii), Page 36
\( \partial_{\tau}^{\epsilon} (H) \) \ldots \ldots The module of \( \tau_{\tau}^{\epsilon} \) at \( G_{\tau}^{\epsilon} \), Definition 8.7, Page 38

1. Introduction

The problem of the completeness of Łukasiewicz infinite-valued logic (Ł for short) was posed by Łukasiewicz and Tarski in the 1930s. It was twenty-eight years later that it was syntactically solved by Rose and Rosser [20]. Chang [4] developed at almost the same time a theory of algebraic systems for Ľ, which is called MV-algebras, with an attempt to make MV-algebras correspond to Ľ as Boolean algebras to the classical two-valued logic. Chang [5] subsequently finished another proof for the completeness of Ľ by virtue of his MV-algebras.

It was Chang who observed that the key role in the structure theory of MV-algebras is not locally finite MV-algebras but linearly ordered ones. The observation was formalized by Hájek
who showing the completeness for his basic fuzzy logic (BL for short) with respect to linearly ordered BL-algebras. Starting with the structure of BL-algebras, Hájek [13] reduced the problem of the standard completeness of BL to two formulas to be provable in BL. Here and thereafter, by the standard completeness we mean that logics are complete with respect to algebras with lattice reduct [0, 1]. Cignoli et al. [6] subsequently proved the standard completeness of BL, i.e., BL is the logic of continuous t-norms and their residua.

Hájek’s approach toward fuzzy logic has been extended by Esteva and Godo in [9], where the authors introduced the logic MTL which aims at capturing the tautologies of left-continuous t-norms and their residua. The standard completeness of MTL was proved by Jenei and Montagna in [15], where the major step is to embed linearly ordered MTL-algebras into the dense ones under the situation that the structure of MTL-algebras has been unknown as yet.

Esteva and Godo’s work was further promoted by Metcalfe and Montagna [16] who introduced the uninorm logic UL and involutive uninorm logic IUL which aim at capturing tautologies of left-continuous uninorms and their residua and those of involutive left-continuous ones, respectively. Recently, Cintula and Noguera [8] introduced substructural logics which are substructural logics complete with respect to linearly ordered models. Almost all well-known families of fuzzy logics such as \( \mathbf{L} \), BL, MTL, UL and IUL belong to the class of semilinear substructural logics.

Metcalfe and Montagna’s method to prove standard completeness for UL and its extensions is of proof theory in nature and consists of two key steps. Firstly, they extended UL with the density rule of Takeuti and Titani [21]:

\[
\frac{\Gamma \vdash (A \rightarrow p) \lor (p \rightarrow B) \lor C}{\Gamma \vdash (A \rightarrow B) \lor C} (D)
\]

where \( p \) does not occur in \( \Gamma, A, B \) or \( C \), and then prove the logics with \( (D) \) are complete with respect to algebras with lattice reduct \([0,1]\). Secondly, they give a syntactic elimination of \( (D) \) that was formulated as a rule of the corresponding hypersequent calculus.

Hypersequents are a natural generalization of sequents which were introduced independently by Avron [1] and Pottinger [19] and have proved to be particularly suitable for logics with prelinearity [2, 16]. Following the spirit of Gentzen’s cut elimination, Metcalfe and Montagna succeeded to eliminate the density rule for GUL and several extensions of GUL by induction on the height of a derivation of the premise and shifting applications of the rule upwards, but failed for GIUL and therefore left it as an open problem.

There are several relevant works about the standard completeness of IUL as follows. With an attempt to prove the standard completeness of IUL, we generalized Jenei and Montagna’s method for IMTL in [22], but our effort was only partially successful. It seems that the subtle reason why it does not work for UL and IUL is the failure of FMP of these systems [23]. Jenei [14] constructed several classes of involutive FL\(_e\)-algebras, as he said, in order to gain a better insight into the algebraic semantic of the substructural logic IUL, and also to the longstanding open problem about its standard completeness. Ciabattoni and Metcalfe [7] introduced the method of density elimination by substitutions which is applicable to a general classes of (first-order) hypersequent calculi but fails to the case of GIUL.

We reconsidered Metcalfe and Montagna’s proof-theoretic method to investigate the standard completeness of IUL, because they have proved the standard completeness of UL by their method and we can’t prove such a result by the Jenei and Montagna’s model-theoretic method. In order to prove the density elimination for GUL, they prove that the following generalized density rule \((D_1)\):

\[
\frac{\Gamma \vdash (A \rightarrow p) \lor (p \rightarrow B) \lor C}{\Gamma \vdash (A \rightarrow B) \lor C} (D)
\]
Let \( n \geq 1 \) and \( \Lambda_i \geq 1 \) for some \( 1 \leq i \leq n \); (ii) \( p \) does not occur in \( \Gamma_i, \Delta_i, \Pi_j, \Sigma_k \) for \( i = 1 \cdots n, j = 1 \cdots m, k = 1 \cdots o \).

We may regard \((D_1)\) as a procedure whose input and output are the premise and conclusion of \((D_1)\), respectively. We denote the conclusion of \((D_1)\) by \(D_1(G_0)\) when its premise is \(G_0\). Observe that Metcalfe and Montagna had succeeded to define the suitable conclusion for almost arbitrary premise in \((D_1)\), but it seems impossible for \(\text{GIUL}\) (See Section 3 for an example). We then define the following generalized density rule \((D_0)\) for

\[
\text{GL} \in \{\text{GUL, GIUL, GMTL, GIMTL}\}
\]

and prove its admissibility in Section 9.

**Theorem 1.1 (Main theorem).** Let \( n, m \geq 1 \). Then the strong density rule

\[
\frac{G_0 \equiv \{\Gamma_i, \Lambda_i p \Rightarrow \Delta_i\}_{i=1\cdots n} \{\{\Pi_j \Rightarrow p, \Sigma_j\}_{j=1\cdots m}(D_1)}
\]

is admissible in \(\text{GL}\).

In proving the admissibility of \((D_1)\), Metcalfe and Montagna made some restriction on the proof \(\tau\) of \(G_0\), i.e., converted \(\tau\) into an \(r\)-proof. The reason why they need an \(r\)-proof is that they set the constraint (i) to \(G_0\). We may imagine the restriction on \(\tau\) and the constraints to \(G_0\) as two pallets of a balance, i.e., one is strong if another is weak and vice versa. Observe that we select the weakest form of \(G_0\) in \((D_0)\) that guarantees the validity of \((D)\). Then it is natural that we need make the strongest restriction on the proof \(\tau\) of \(G_0\). But it seems extremely difficult to follow such a way to prove the admissibility of \((D_0)\).

In order to overcome such a difficulty, we first of all choose Avron-style hypersequent calculi as the underlying systems (See A.1). Let \(\tau\) be a cut-free proof of \(G_0\) in \(\text{GL}\). Starting with \(\tau\), we construct a proof \(\tau^*\) of \(G\) in a restricted subsystem \(\text{GL}_{\tau^*}\) of \(\text{GL}\) by a systematic novel manipulations in Section 4. Roughly speaking, each sequent of \(G\) is a copy of some sequent of \(G_0\), and each sequent of \(G^*\) is a copy of some contraction sequent in \(\tau\). In Section 5, we define the generalized density rule \((D)\) in \(\text{GL}_{\tau^*}\) and prove that it is admissible.

Now, starting with \(G\) and its proof \(\tau^*\), we construct a proof \(\tau^*_0\) of \(G_0\) in \(\text{GL}_{\tau^*}\) such that each sequent of \(G^*\) is a copy of some sequent of \(G_0\). Then \(\tau^*_0\) \(D(G_0^*)\) by the admissibility of \((D)\). Then \(\tau^*_0\) by Lemma 9.1. Hence the density elimination theorem holds in \(\text{GL}\). Especially, the standard completeness of \(\text{IUL}\) follows from Theorem 62 of [16].

\(G^*\) is constructed by eliminating \((pEC)\)-sequents in \(G\) one by one. In order to control the process, we introduce the set \(I = \{H'_{n_1}, \cdots, H'_{n_m}\}\) of \((pEC)\)-nodes of \(\tau^*\) and the set \(I\) of the branches relative to \(I\) and construct \(G^*_I\) such that \(G^*_I\) doesn’t contain \((pEC)\)-sequents lower than any node in \(I\), i.e., \(S_j \in G^*_I\) implies \(H_j^*\) for all \(H_j^* \in I\). The procedure is called the separation algorithm of branches in which we introduce another novel manipulation and call it derivation-grafting operation in Section 7, 8.
2. Preliminaries

In this section, we recall the basic definitions and results involved, which are mainly from [16]. Substructural fuzzy logics are based on a countable propositional language with formulas \( \Pi \), \( \Theta \), \( \land \), \( \vee \), and constants \( \perp, \top, t, f \) with definable connective \( \neg A := A \rightarrow f \).

**Definition 2.1.** ([1, 16]) A sequent is an ordered pair \( (\Gamma, \Delta) \) of finite multisets (possibly empty) of formulas, which we denote by \( \Gamma \Rightarrow \Delta \). \( \Gamma \) and \( \Delta \) are called the antecedent and succedents, respectively, of the sequent and each formula in \( \Gamma \) and \( \Delta \) is called a component of \( \Gamma \) or \( \Delta \), respectively.

A hypersequent \( G \) is a finite multiset of the form \( \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \), where each \( \Gamma_i \Rightarrow \Delta_i \) is a sequent and is called a component of \( G \) for each \( 1 \leq i \leq n \). If \( \Delta_i \) contains at most one formula for \( i = 1 \cdots n \), then the hypersequent is single-conclusion, otherwise it is multiple-conclusion.

**Definition 2.2.** Let \( S \) be a sequent and \( G = S_1 \mid \cdots \mid S_m \) a hypersequent. We say that \( S \in G \) if \( S \) is one of \( S_1, \cdots, S_m \).

**Notation 2.3.** Let \( G_1 \) and \( G_2 \) be two hypersequents. We will assume from now on that all set terminology refers to multisets, adopting the conventions of writing \( \Gamma, \Delta \) for the multiset union of \( \Gamma \) and \( \Delta \), \( A \) for the singleton multiset \( \{A\} \), and \( \lambda \Gamma \) for the multiset union of \( \lambda \) copies of \( \Gamma \) for \( \lambda \in \mathbb{N} \). By \( G_1 \subseteq G_2 \) we mean that \( S \in G_2 \) for all \( S \in G_1 \) and the multiplicity of \( S \) in \( G_1 \) is not more than that of \( S \) in \( G_2 \). We will use \( G_1 = G_2, G_1 \cap G_2, G_1 \cup G_2, G_1 \setminus G_2 \) by their standard meaning for multisets by default and we will declare when we use them for sets. We sometimes write \( S \mid \cdots \mid S \) as \( \{S_1, \cdots, S_m\} \), \( G[S] \) (or \( G[S] \)), respectively.

**Definition 2.4.** ([1]) A hypersequent rule is an ordered pair consisting of a sequent \( G_1 \) of lower hypersequents \( G_1 \), \( \cdots \), \( G_n \) called the premises (upper hypersequents) of the rule, and a hypersequent \( G \) called the conclusion (lower hypersequent), written by \( G_1; \cdots; G_n \). If \( n = 0 \), then the rule has no premise and is called an initial sequent. The single-conclusion version of a rule adds the restriction that both the premises and conclusion must be single-conclusion; otherwise the rule is multiple-conclusion.

**Definition 2.5.** ([16]) \( \text{GUL} \) and \( \text{GIUL} \) consist of the single-conclusion and multiple-conclusion versions of the following initial sequents and rules, respectively:

**Initial Sequents**

\[
\frac{}{A \Rightarrow A} \quad (ID) \quad \frac{\Gamma \Rightarrow \top, \Delta}{\Gamma, \bot \Rightarrow \Delta} \quad (\top) \quad \frac{\Gamma, \bot \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad (\bot) \quad \frac{}{f \Rightarrow f} \quad (f)
\]

**Structural Rules**

\[
\frac{G[\Gamma] \Rightarrow A[\Gamma] \Rightarrow A}{G[\Gamma] \Rightarrow A} \quad (EC) \quad \frac{G[\Gamma] \Rightarrow A}{G} \quad (EW)
\]

\[
\frac{G_1[\Gamma_1, \Pi_1] \Rightarrow \Sigma_1, \Delta_1 \quad G_2[\Gamma_2, \Pi_2] \Rightarrow \Sigma_2, \Delta_2}{G_1[\Gamma_2][\Gamma_1, \Pi_1] \Rightarrow \Delta_1, \Delta_2 \mid \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2} \quad (COM)
\]

5
Logical Rules

\[
\frac{G\mid \Gamma \Rightarrow \Delta}{G\mid \Gamma, t \Rightarrow \Delta (t)}\quad \frac{G\mid \Gamma \Rightarrow \Delta}{G\mid \Gamma \Rightarrow f, \Delta (f)}
\]
\[
\frac{G_1\mid \Gamma_1 \Rightarrow A, \Delta_1 \quad G_2\mid \Gamma_2, B \Rightarrow \Delta_2}{G_1\mid G_2\mid \Gamma_1, \Gamma_2, A \Rightarrow B \Rightarrow \Delta_1, \Delta_2 (\rightarrow_1)}\quad \frac{G\mid \Gamma, A \Rightarrow B, \Delta}{G\mid \Gamma \Rightarrow A, \Delta (\rightarrow_r)}
\]
\[
\frac{G\mid \Gamma, A \Rightarrow B \Rightarrow \Delta (\odot_i)}{G\mid \Gamma, A \Rightarrow \Delta (\wedge_i)}\quad \frac{G\mid \Gamma, A \Rightarrow \Delta (\lor_i)}{G\mid \Gamma, A \Rightarrow \Delta (\vee_i)}\quad \frac{G\mid \Gamma \Rightarrow B, \Delta}{G\mid \Gamma \Rightarrow A \vee B, \Delta (\lor_r)}
\]

Cut Rule

\[
\frac{G_1\mid \Gamma_1, A \Rightarrow \Delta_1 \quad G_2\mid \Gamma_2 \Rightarrow A, \Delta_2}{G_1\mid G_2\mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 (CUT)}
\]

Definition 2.6. ([16]) GMTL and GIMTL are GUL and GIUL plus the single conclusion and multiple-conclusion versions, respectively, of:

\[
\frac{G\mid \Gamma \Rightarrow \Delta}{G\mid \Gamma, A \Rightarrow \Delta (WL)}\quad \frac{G\mid \Gamma \Rightarrow \Delta}{G\mid \Gamma \Rightarrow A, \Delta (WR)}
\]

Definition 2.7. (i) \((I) \in \{(t), (f), (\rightarrow_r), (\odot_i), (\wedge_i), (\lor_i), (\vee_i), (COM)\}\); and

\((II) \in \{(-), (\odot_i), (\wedge_i), (\vee_i)\}\)

(ii) By \(\frac{G\mid G_1\mid H'}{G\mid G_2\mid H'}(II)\) (or \(\frac{G\mid G_2\mid H'}{G\mid G_1\mid H'}(I)\)) we denote an instance of a two-premise rule (II) (or one-premise rule (I)) of GL, where \(S'\) and \(S''\) are its focus sequents and \(H'\) is its principle sequent (for \((-), (\odot_i), (\wedge_i)\) and \((\vee_i)\)) or hypersequent (for \((COM)\), \((\wedge_m)\) and \((\vee_m)\), see Definition 4.2).

Definition 2.8. ([16]) \(GL^D\) is GL extended with the following density rule:

\[
\frac{G\mid \Gamma_1, p \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow p, \Delta_2}{G\mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 (D)}
\]

where \(p\) does not occur in \(G, \Gamma_1, \Gamma_2, \Delta_1\) or \(\Delta_2\).

Definition 2.9. ([11]) A derivation \(\tau\) of a hypersequent \(G\) from hypersequents \(G_1, \ldots, G_n\) in a hypersequent calculus GL is a labeled tree with the root labeled by \(G\), leaves labeled initial sequents or some \(G_1, \ldots, G_n, \) and for each node labeled \(G_0\) with parent nodes labeled \(G_1', \ldots, G_m'\)

(where possibly \(m = 0\), \(\frac{G_1' \ldots G_m'}{G_0}\) is an instance of a rule of GL.
Notation 2.10. (i) $\frac{G_1 \cdots G_n}{G_0} (\tau)$ denotes that $\tau$ is a derivation of $G_0$ from $G_1, \cdots, G_n$;

(ii) Let $H$ be a hypersequent. $H \in \tau$ denotes that $H$ is a node of $\tau$. We call $H$ a leaf hypersequent if $H$ is a leaf of $\tau$, the root hypersequent if it is the root of $\tau$. $\frac{G'_1 \cdots G'_m}{G'_0}$ $\in \tau$ denotes that $G'_0 \in \tau$ and its parent nodes are $G'_1, \cdots, G'_m$;

(iii) Let $H \in \tau$ then $\tau(H)$ denotes the subtree of $\tau$ rooted at $H$;

(iv) $\tau$ determines a partial order $\leq_\tau$ with the root as the least element. $H_k \parallel H_2$ denotes $H_1 \leq_\tau H_2$ and $H_2 \nleq_\tau H_1$ for any $H_1, H_2 \in \tau$. By $H_1 =_\tau H_2$ we mean that $H_1$ is the same node as $H_2$ in $\tau$. We sometimes write $\leq_\tau$ as $\leq$;

(v) An inference of the form $\frac{G'|S^n}{G'|S} \in \tau$ is called the full external contraction and denoted by $(EC^*)$, if $n \geq 2$, $G'|S^n$ is not a lower hypersequent of an application of $(EC)$ whose contraction sequent is $S$, and $G'|S$ not an upper one in $\tau$.

Definition 2.11. Let $\tau$ be a derivation of $G$ and $H \in \tau$. The thread $Th_\tau(H)$ of $\tau$ at $H$ is a sequence $H_0, \cdots, H_n$ of node hypersequents of $\tau$ such that $H_0 =_\tau H$, $H_n =_\tau G$, $H_k \parallel H_{k+1}$ in $\tau$ or there exists $G' \in \tau$ such that $\frac{H_k}{H_{k+1}}$ or $\frac{G'}{H_{k+1}}$ in $\tau$ for all $0 \leq k \leq n - 1$.

Proposition 2.12. Let $H_1, H_2 \in \tau$. Then

(i) $H_1 \leq H_2$ if and only if $H_1 \in Th_\tau(H_2)$;

(ii) $H_1 \parallel H_2$ and $H_1 \leq H_2$ imply $H_2 \in H_1$;

(iii) $H_1 \leq H_3$ and $H_2 \leq H_3$ imply $H_1 \parallel H_2$;

We need the following definition to give each node of $\tau$ an identification number, which is used in Construction 6.1 to differentiate sequents in a hypersequent in a proof.

Definition 2.13. ([A.5.2]) Let $H \in \tau$ and $Th_\tau(H) = (H_0, \cdots, H_n)$. Let $b_n := 1$,

$$b_k := \begin{cases} 1 & \text{if } \frac{G'}{H_{k+1}} \in \tau, \\ 0 & \text{if } \frac{G'}{H_{k+1}} \not\in \tau \text{ or } \frac{H_k}{H_{k+1}} \in \tau \end{cases}$$

for all $0 \leq k \leq n - 1$. Then $\mathcal{P}(H) := \sum_{k=0}^{n} 2^k b_k$ and call it the position of $H$ in $\tau$.

Definition 2.14. A rule is admissible for a calculus GL if whenever its premises are derivable in GL, then so is its conclusion.

Lemma 2.15. ([16]) Cut-elimination holds for GL, i.e., proofs using (CUT) can be transformed syntactically into proofs not using (CUT).

3. Proof of the main theorem: A computational example

In this section, we present an example to illustrate the proof of Main theorem.

Let $G_0 \equiv \varphi, B \varphi \Rightarrow \varphi, \neg A \odot \neg A \Rightarrow A \odot C, p \Rightarrow A \odot A$. $G_0$ is a theorem of IUL and a cut-free proof $\tau$ of $G_0$ is shown in Figure 1, where we use an additional rule $\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$ for simplicity.
Note that we denote three applications of (EC) in \( \tau \) respectively by (EC)\(_1\), (EC)\(_2\), (EC)\(_3\) and three (\( \otimes \)) by (\( \otimes_1 \)), (\( \otimes_2 \)) and (\( \otimes_3 \)).

\[
\begin{align*}
p \Rightarrow p & \quad A \Rightarrow A \\
A \Rightarrow p | p \Rightarrow A & \Rightarrow A | p \Rightarrow A \quad (\text{COM}) \\
p \Rightarrow p & \quad A \Rightarrow A \\
A \Rightarrow p | A & \Rightarrow p | p, p \Rightarrow A \otimes A \\
A \Rightarrow p | p, p \Rightarrow A & \Rightarrow A \otimes A \\
\Rightarrow p, \neg A | p, p & \Rightarrow A \otimes A
\end{align*}
\]

(continued)

\[
\begin{align*}
p \Rightarrow p & \quad A \Rightarrow A \\
A \Rightarrow p | p \Rightarrow A & \Rightarrow A | p \Rightarrow A \quad (\text{COM}) \\
p \Rightarrow p | A & \Rightarrow p | p, p \Rightarrow A \otimes A \\
A \Rightarrow p | p, p \Rightarrow A & \Rightarrow A \otimes A \\
\Rightarrow p, \neg A | p, p & \Rightarrow A \otimes A
\end{align*}
\]

FIGURE 1  A proof \( \tau \) of \( G_0 \)

By applying (D) to free combinations of all sequents in \( p, B | B \Rightarrow p, \neg A \otimes \neg A \) and in \( p \Rightarrow C | C, p \Rightarrow A \otimes A \), we get that \( H_0 \equiv B, C | C \Rightarrow A \otimes A, B | B \Rightarrow C, \neg A \otimes \neg A | C, B \Rightarrow A \otimes A, \neg A \otimes \neg A \). \( H_0 \) is a theorem of IUL and a cut-free proof \( \rho \) of \( H_0 \) is shown in Figure 2. It supports the validity of the generalized density rule (\( D_0 \)) in Section 1, as an instance of (\( D_0 \)).

\[
\begin{align*}
A & \Rightarrow A \\
A & \Rightarrow A \\
A & \Rightarrow A \\
A, A & \Rightarrow A \otimes A \\
A & \Rightarrow \neg A, A \otimes A \\
A & \Rightarrow \neg A, A \otimes A \\
B & \Rightarrow B \\
B & \Rightarrow B \\
B & \Rightarrow B \\
A, A & \Rightarrow \neg A \otimes \neg A, A \otimes A \\
A, A & \Rightarrow \neg A \otimes \neg A, A \otimes A \\
C & \Rightarrow C \\
C & \Rightarrow C \\
C & \Rightarrow C \\
C & \Rightarrow C \\
H_1 & \equiv A \Rightarrow C | A, B \Rightarrow A \otimes A, \neg A \otimes \neg A | C \Rightarrow A \otimes A, B \\
H_1 & \equiv A \Rightarrow C | A, B \Rightarrow A \otimes A, \neg A \otimes \neg A | C \Rightarrow A \otimes A, B \\
A & \Rightarrow \neg A, A \otimes A \\
A & \Rightarrow \neg A, A \otimes A \\
A & \Rightarrow \neg A, A \otimes A \\
\neg A, C | A, B & \Rightarrow A \otimes A, \neg A \otimes \neg A | C \Rightarrow A \otimes A, B \\
A, A, B & \Rightarrow A \otimes A, \neg A \otimes \neg A | A, B \Rightarrow A \otimes A, \neg A \otimes \neg A | A, B \Rightarrow A \otimes A, \neg A \otimes \neg A | A, B \Rightarrow A \otimes A, \neg A \otimes \neg A | A, B \\
A & \Rightarrow C | A \Rightarrow C \\
A & \Rightarrow C | A \Rightarrow C \\
A & \Rightarrow C | A \Rightarrow C \\
A & \Rightarrow C | A \Rightarrow C \\
H_2 & \equiv A \Rightarrow C | B, C \Rightarrow A \otimes A, \neg A \otimes \neg A | C \Rightarrow A \otimes A, B \\
H_2 & \equiv A \Rightarrow C | B, C \Rightarrow A \otimes A, \neg A \otimes \neg A | C \Rightarrow A \otimes A, B
\end{align*}
\]
Our task is to construct $\rho$, starting from $\tau$. The tree structure of $\rho$ is more complicated than that of $\tau$. Compared with UL, MTL, and IMTL, there is no one-to-one correspondence between nodes in $\tau$ and $\rho$.

Following the method given by G. Metcalfe and F. Montagna, we need to define a generalized density rule for IUL. We denote such an expected unknown rule by $(D_\ast)$ for convenience. Then $D_\ast(h)$ must be definable for all $h \in \tau$. Naturally, $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. 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Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$. Let $D_\ast(p \Rightarrow p) = \tau$.

Notice that two upper hypersequents $p, \neg A \Rightarrow p, \neg A \Rightarrow p$, $p \Rightarrow A \Rightarrow p$ are permissible inputs of $(D_2)$. Why is $H_{xx}$ an invalid input? One reason is that, two applications $(EC)_1$ and $(EC)_2$ cut off two sequents $A \Rightarrow p$ such that two $p$'s disappear in all nodes lower than upper hypersequent of $(EC)_1$ or $(EC)_2$, including $H_{xx}$. These make occurrences of $p$'s to be incomplete in $H_{xx}$. We then perform the following operation in order to get complete occurrences of $p$'s in $H_{xx}$.

**Step 1 (Preprocessing of $\tau$).** Firstly, we replace $H$ with $H[S']$ for all $H \leq G[S']$ for all $\forall k \in \tau$. $G[S']$, then replace $G[S']$ with $G[S']$ for all $k = 1, 2, 3$. Then we construct a proof without $(EC)$, which we denote by $\tau_1$, as shown in Figure 3. We call such manipulations
sequent-inserting operations, which eliminate applications of (EC) in \( \tau \).

![Figure 3](https://www.preprints.org)  
A proof \( \tau_1 \)

However, we also can’t define \( \mathcal{D}_1(H'_{\nu_x}) \) for \( H'_{\nu_x} \in \tau_1 \) in that \( \Rightarrow p, p, \neg A \circ \neg A | p, p \Rightarrow A \circ A \subseteq H'_{\nu_x} \). The reason is that the origins of \( p' \)'s in \( H'_{\nu_x} \) are indistinguishable if we regard all leaves in the form \( p \Rightarrow p \) as the origins of \( p' \)'s which occur in the inner node. For example, we don’t know which \( p \) comes from the left subtree of \( \tau_1(H'_{\nu_x}) \) and which from the right subtree in two occurrences of \( p' \)'s in \( \Rightarrow p, p, \neg A \circ \neg A \in H'_{\nu_x} \). We then perform the following operation in order to make all occurrences of \( p' \)'s in \( H'_{\nu_x} \) distinguishable.

We assign the unique identification number to each leaf in the form \( p \Rightarrow p \in \tau_1 \) and transfer these identification numbers from leaves to the root, as shown in Figure 4. We denote the proof of \( G|G^* \) resulting from this step by \( \tau^* \), where \( G \equiv \Rightarrow p_1, B \Rightarrow p_4, \neg A \circ \neg A | p_1 \Rightarrow C, p_2 \Rightarrow A \circ A \) in which each sequent is a copy of some sequent in \( G_0 \) and \( G^* \equiv \Rightarrow p_1 | A \Rightarrow p_2, p_3 \Rightarrow A \circ A \) in which each sequent is a copy of some external contraction sequent in \( (EC) \)-node of \( \tau \). We call such manipulations eigenvariable-labeling operations, which make us to trace eigenvariables in \( \tau \).

![Figure 4](https://www.preprints.org)  
A proof \( \tau^* \) of \( G|G^* \)

Then all occurrences of \( p \) in \( \tau^* \) are distinguishable and we regard them as distinct eigenvariables (See Definition 4.16 (i)). Firstly, by selecting \( p_1 \) as the eigenvariable and applying (D) to \( G|G^* \), we get

\[
G' \equiv A \Rightarrow C \Rightarrow p_2, B \Rightarrow p_4, \neg A \circ \neg A | C, p_2 \Rightarrow A \circ A | A \Rightarrow p_3, p_4 \Rightarrow A \circ A.
\]

Secondly, by selecting \( p_2 \) and applying (D) to \( G' \), we get

\[
G'' \equiv A \Rightarrow C | B \Rightarrow p_4, \neg A \circ \neg A | C \Rightarrow B, A \circ A | A \Rightarrow p_3, p_4 \Rightarrow A \circ A.
\]

Repeatedly, we get

\[
G''' \equiv A \Rightarrow C | A, B \Rightarrow A \circ A, \neg A \circ \neg A | C \Rightarrow A \circ A, B.
\]
We define such iterative applications of (D) as D-rule (See Definition 5.4). Lemma 5.8 shows that G ≡ G∗ if G ≡ G∗. Then we obtain G ≡ G∗, i.e., G ≡ G∗.

A miracle happens here! The difficulty that we encountered in GIUL is overcome by converting H'₁ = A ⇒ p₁ ⇒ p₂, p₁, A ⇒ A ⊩ A into A ⇒ p₁ ⇒ p₂, p₁, A ⇒ A ⊩ A and using (D) to replace it.

Why do we assign the unique identification number to each p ⇒ p ∈ τ₁? Would we return to the same situation as that of τ₁ if we assign the same indices to all p ⇒ p ∈ τ₁ or, replace p₁ ⇒ p₁ and p₂ ⇒ p₂ by p₁ ⇒ p₂ in τ∗?

Note that D(G) = H₁. So we have built up a one-one correspondence between the proof τ∗ of G∗ and that of H₁. Observe that each sequent in G∗ is not a copy of any sequent in G₀.

In the following steps, we work on eliminating these sequents in G∗.

**Step 2 (Extraction of Elimination Rules).** We select A ⇒ p₂ as the focus sequent in H'₁ in τ∗ and keep A ⇒ p₁ unchanged from H'₁ downward to G∗ (See Figure 4). So we extract a derivation from A ⇒ p₂ by pruning some sequents (or hypersequents) in τ∗, which we denote by τ'₁: A ⇒ p₂, as shown in Figure 5.

![Figure 5](image)

A derivation τ'₁: A ⇒ p₁ from A ⇒ p₁ is constructed by replacing p₂ with p₁, p₃ with p₅ and p₄ with p₆ in τ'₁: A ⇒ p₂, as shown in Figure 6. Notice that we assign new identification numbers to new occurrences of p in τ'₁: A ⇒ p₁.

![Figure 6](image)

Next, we apply τ'₁: A ⇒ p₁ to A ⇒ p₁ in G|G∗. Then we construct a proof τ|G(G∗), as shown in Figure 7, where G' ≡ G|G∗\{A ⇒ p₁}. 

11
However, $G'_{H_1,G^*} \Rightarrow p_2, B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A$ contains more copies of sequents from $G^*$ and seems more complex than $G|G^*$. We will present a unified method to tackle with it in the following steps. Other derivations are shown in Figures 8, 9, 10, 11.

\begin{align*}
A \Rightarrow p_1 | p_2, \neg A|p_1, p_2 \Rightarrow A \odot A
\end{align*}

\begin{align*}
B \Rightarrow B
\end{align*}

\begin{align*}
C \Rightarrow C
\end{align*}

\begin{align*}
A \Rightarrow p_3 | p_6, \neg A|p_5, p_6 \Rightarrow A \odot A
\end{align*}

\begin{align*}
B \Rightarrow B
\end{align*}

\begin{align*}
C \Rightarrow C
\end{align*}

\begin{align*}
A \Rightarrow p_5 | p_6, p_3, \neg A \odot A|p_5, p_6 \Rightarrow A \odot A
\end{align*}

\begin{align*}
B \Rightarrow B
\end{align*}

\begin{align*}
C \Rightarrow C
\end{align*}

\begin{align*}
A \Rightarrow p_3 \quad A \Rightarrow p_4
\end{align*}

\begin{align*}
B \Rightarrow B
\end{align*}

\begin{align*}
C \Rightarrow C
\end{align*}

\begin{align*}
A \Rightarrow p_5 \quad A \Rightarrow p_3
\end{align*}

\begin{align*}
B \Rightarrow B
\end{align*}

\begin{align*}
C \Rightarrow C
\end{align*}

\begin{align*}
p_5 \Rightarrow p_3, p_4 \Rightarrow A \odot \neg A
\end{align*}

\begin{align*}
p_2, B \Rightarrow p_4, \neg A \odot \neg A
\end{align*}

\begin{align*}
p_3, p_4 \Rightarrow A \odot \neg A
\end{align*}

\begin{align*}
p_3 \Rightarrow p_3, p_4 \Rightarrow A \odot \neg A
\end{align*}

\begin{align*}
p_3, p_4 \Rightarrow A \odot \neg A
\end{align*}
Step 3 (Separation of one branch). A proof $\tau^{(2)}_{G^{\oplus*}_G}$ is constructed by applying sequentially

$$\tau^{(2)}_{G^{\oplus*}_G}$$

to $p_3, p_4 \Rightarrow A \odot A$ and $p_5, p_6 \Rightarrow A \odot A$ in $G^{\oplus(1)}_{H^{*}G^{*}}$, as shown in Figure 12, where $G^{\oplus(1)}_{H^{*}G^{*}} \equiv G^{\oplus(1)}_{H^{*}G^{*}} \setminus \{p_3, p_4 \Rightarrow A \odot A, p_5, p_6 \Rightarrow A \odot A\}$

![Figure 12](image-url)

A proof $\tau^{(2)}_{G^{\oplus*}_G}$ of $G^{\oplus(2)}_{H^{*}G^{*}}$

$G^{\oplus(2)}_{H^{*}G^{*}} \equiv G^{\oplus(2)}_{H^{*}G^{*}} \setminus \{p_3, p_4 \Rightarrow A \odot A, p_5, p_6 \Rightarrow A \odot A\}$

Notice that

$$D(B \Rightarrow p_6, \neg A \odot \neg A|A \Rightarrow p_4 \Rightarrow p_3, p_5 \Rightarrow C|C, p_2 \Rightarrow A \odot A|A \Rightarrow p_1 \Rightarrow p_1, B|)$$

Then it is permissible to cut off the part

$$B \Rightarrow p_6, \neg A \odot \neg A|A \Rightarrow p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A$$

of $G^{\oplus(2)}_{H^{*}G^{*}}$, which corresponds to applying $(EC)$ to $D(G^{\oplus(2)}_{H^{*}G^{*}})$. We regard such a manipulation as a constrained contraction rule applied to $G^{\oplus(2)}_{H^{*}G^{*}}$, and denote it by $(EC_\Omega)$. Define $G^{\oplus*}_{H^{*}G^{*}}$ to be

$$\Rightarrow p_3, B|B \Rightarrow p_4, \neg A \odot \neg A|A \Rightarrow p_3, p_4 \Rightarrow C|C, p_2 \Rightarrow A \odot A$$

Then we construct a proof of $G^{\oplus*}_{H^{*}G^{*}}$ by $D(G^{\oplus*}_{H^{*}G^{*}}) \setminus (EC_\Omega)$, which guarantees the validity of $$\Rightarrow p_3, B|B \Rightarrow p_4, \neg A \odot \neg A|A \Rightarrow p_3, p_4 \Rightarrow C|C, p_2 \Rightarrow A \odot A.$$
under the condition

$$\vdash_{\text{GLUL}} \mathcal{D}(G_{G^*}^{*})$$.

A change happens here! There is only one sequent which is a copy of a sequent in $G^*$ in $G_{G^*}^{*}$. It is simpler than $G|G^*$. So we are moving forward. The above procedure is called the separation of $G|G^*$ as a branch of $H^*_1$ and reformulated as follows (See Section 7 for details).

$$\frac{G(G^*)}{G_{G^*}^{*}} \left( \tau_{H^*_1:A\leftarrow p_1} \right) \quad \frac{G_{G^*}^{*}}{G_{G^*}^{*}} \left( \tau_{H^*_2:p_3\leftarrow A\leftarrow A} \right) \quad \frac{G_{G^*}^{*}}{G_{G^*}^{*}} \left( \tau_{H^*_3:p_5\leftarrow A\leftarrow A} \right) \quad (EC_\Omega)$$

The separation of $G|G^*$ as a branch of $H^*_2$ is constructed by a similar procedure as follows.

$$\frac{G|G^*}{G_{G^*}^{*}} \left( \tau_{H^*_2:A\leftarrow p_1} \right) \quad \frac{G_{G^*}^{*}}{G_{G^*}^{*}} \left( \tau_{H^*_1:p_3\leftarrow A\leftarrow A} \right) \quad \frac{G_{G^*}^{*}}{G_{G^*}^{*}} \left( \tau_{H^*_3:p_5\leftarrow A\leftarrow A} \right) \quad (EC_\Omega)$$

Note that $\mathcal{D}(G_{G^*}^{*}) = H_2$ and $\mathcal{D}(G_{G^*}^{*}) = H_3$. So we have built up one-one correspondences between proofs of $G_{G^*}^{*}, G_{G^*}^{*}$ and those of $H_2, H_3$.

**Step 3 (Separation algorithm of multiple branches).** We will prove $\vdash_{\text{GLUL}} \mathcal{D}_0(G_0)$ in a direct way, i.e., only the major step of Theorem 8.2 is presented in the following. (See A.5.4 for a complete illustration.) Recall that

$$G_{H^*_1}^{*} \equiv \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \land \neg A|p_1 \Rightarrow C|p_2 \Rightarrow A \land A$$

$$\Rightarrow p_1 \Rightarrow p_3, B|p_3 \Rightarrow C|p_4 \Rightarrow A \land A.$$  

$$G_{H^*_2}^{*} = A \Rightarrow p_1 \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \land \neg A|p_1 \Rightarrow C|p_2 \Rightarrow A \land A$$

$$\Rightarrow p_3, \neg A \land \neg A|p_3 \Rightarrow C|p_4 \Rightarrow A \land A.$$  

By reassigning identification numbers to occurrences of $p'$s in $G_{H^*_2}^{*},$

$$G_{H^*_3}^{*} = A \Rightarrow p_3 \Rightarrow p_6, B|B \Rightarrow p_8, \neg A \land \neg A|p_5 \Rightarrow C|p_6 \Rightarrow A \land A$$

$$\Rightarrow p_7, \neg A \land \neg A|p_7 \Rightarrow C|p_8 \Rightarrow A \land A.$$  

By applying $\tau_{H^*_1:A\leftarrow p_1, H^*_2:A\leftarrow p_3}$ to $A \Rightarrow p_3$ in $G_{H^*_1}^{*}$, and $A \Rightarrow p_5$ in $G_{H^*_3}^{*}$, we get $\vdash_{\text{GLUL}} G'$, where

$$G' \equiv \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \land \neg A|p_1 \Rightarrow C|p_2 \Rightarrow A \land A \Rightarrow p_1, B$$

$$\Rightarrow p_3 \Rightarrow C|p_4 \Rightarrow A \land A \Rightarrow p_6, B|B \Rightarrow p_8, \neg A \land \neg A|p_5 \Rightarrow C|p_6 \Rightarrow A \land A$$

14

Why do you reassign identification numbers to occurrences of \( p' \)'s in \( G_{H;G^*}^{\vec{G}} \)? It makes different occurrences of \( p' \)'s to be assigned different identification numbers in two nodes \( G_{H;G^*}^{\vec{G}} \) and \( G_{H;G^*}^{\vec{G}} \) of the proof of \( G' \).

By applying \( \langle EC \rangle \) to \( G' \), we get \( \vec{GUL}_\tau G_1^0 \), where

\[
G_1^0 \iff p_2, B|B ⇒ p_4, ¬A ⊗ ¬A|p_1 ⇒ C|C, p_2 ⇒ A ⊗ A| ⇒ p_1, B
\]

\[
p_3 ⇒ C|C, p_4 ⇒ A ⊗ A|B ⇒ p_3, ¬A ⊗ ¬A.
\]

A great change happens here! We have eliminated all sequents which are copies of some sequents in \( G' \) and convert \( G(G') \) into \( G_1^0 \) in which each sequent is some copy of a sequent in \( G_0 \).

Then \( \vec{GUL} \mathcal{D}(G_1^0) \) by Lemma 5.6, where \( \mathcal{D}(G_1^0) = H_0 = \Rightarrow C, B|C ⇒ B, A ⊗ A|B ⇒ C, ¬A ⊗ ¬A|C, B ⇒ A ⊗ A, ¬A ⊗ ¬A. \)

So we have built up one-one correspondences between the proof of \( G_1^0 \) and that of \( H_0 \), i.e., the proof of \( H_0 \) can be constructed by applying \( \langle D \rangle \) to the proof of \( G_1^0 \). The major steps of constructing \( G_1^0 \) are shown in the following figure, where \( \mathcal{D}(G(G')) = H_1, \mathcal{D}(G_{H;G^*}^\vec{G}) = H_2, \mathcal{D}(G_{H;G^*}^\vec{G}) = H_3 \) and \( \mathcal{D}(G_1^0) = H_0 \).

In the above example, \( \mathcal{D}(G_1^0) = \mathcal{D}(G_0) \). But it is not always the case. In general, we can prove that \( \vec{GUL} \mathcal{D}(G_0) \) if \( \vec{GUL} \mathcal{D}(G_1^0) \), which is shown in the proof of Main theorem in Page 46. This example shows that the proof of Main theorem essentially presents an algorithm to construct a proof of \( \mathcal{D}(G_0) \) from \( \tau \).

4. Preprocessing of Proof Tree

Let \( \tau \) be a cut-free proof of \( G_0 \) in Main theorem in \( GL \) by Lemma 2.15. Starting with \( \tau \), we will construct a proof \( \tau^* \) which contains no application of \( (EC) \) and has some other properties in this section.
Lemma 4.1. (i) If $\vdash_{GL} \Gamma_1 \Rightarrow A, \Delta_1$ and $\vdash_{GL} \Gamma_2 \Rightarrow B, \Delta_2$

then $\vdash_{GL} \Gamma_1 \Rightarrow A \land B, \Delta_1 | \Gamma_2 \Rightarrow A \land B, \Delta_2$;

(ii) If $\vdash_{GL} \Gamma_1, A \Rightarrow \Delta_1$ and $\vdash_{GL} \Gamma_2, B \Rightarrow \Delta_2$

then $\vdash_{GL} \Gamma_1, A \lor B \Rightarrow \Delta_1 | \Gamma_2, A \lor B \Rightarrow \Delta_2$.

Proof. (i)

($\vdash_{GL} \Gamma_1 \Rightarrow A, \Delta_1$)

$\vdash_{GL} \Gamma_1 \Rightarrow B, \Delta_2$

$\vdash_{GL} \Gamma_2 \Rightarrow B, \Delta_2$

$\vdash_{GL} \Gamma_1 \Rightarrow A, \Delta_1$

$\vdash_{GL} \Gamma_1 \Rightarrow B, \Delta_2$

$\vdash_{GL} \Gamma_2 \Rightarrow B, \Delta_2$

($\vdash_{GL} \Gamma_1 \Rightarrow A \land B, \Delta_1 | \Gamma_2 \Rightarrow A \land B, \Delta_2$) (CUT)

($\vdash_{GL} \Gamma_1 \Rightarrow A \land B, \Delta_1 | \Gamma_2 \Rightarrow A \land B, \Delta_2$) (CUT)

(ii) is proved by a procedure similar to that of (i) and omitted. \qed

We introduce two new rules by Lemma 4.1.

Definition 4.2. $G_1[\Gamma_1 \Rightarrow A, \Delta_1$ $G_2[\Gamma_2 \Rightarrow B, \Delta_2$ ($\land_{rw}$)

and $G_1[\Gamma_1, A \Rightarrow \Delta_1$ $G_2[\Gamma_2, B \Rightarrow \Delta_2$ ($\lor_{lw}$) are called the generalized ($\land_\gamma$) and ($\lor_\gamma$) rules, respectively.

Now, we begin to process $\tau$ as follows.

Step 1 A proof $\tau^1$ is constructed by replacing inductively all applications of

$\vdash_{GL} \Gamma \Rightarrow A, \Delta$ $\vdash_{GL} \Gamma \Rightarrow B, \Delta$

in $\tau$ with

$\vdash_{GL} \Gamma \Rightarrow A \land B, \Delta$

$\vdash_{GL} \Gamma \Rightarrow A \lor B, \Delta$

The replacements in Step 1 are local and the root of $\tau^1$ is also labeled by $G_0$.

Definition 4.3. We sometimes may regard $G'$ as a structural rule of $GL$ and denote it by ($ID_{\gamma_1}$)

for convenience. The focus sequent for ($ID_{\gamma_2}$) is undefined.

Lemma 4.4. Let $G'[S^m / G] \in \tau^1$, $Th_{C^*}(G'[S]) = (H_0, H_1, \ldots, H_n)$, where $H_0 = G'[S]$ and $H_n = G_0$. A tree $\tau'$ is constructed by replacing each $H_k$ in $\tau^1$ with $H_k[S^{m-1}]$ for all $0 \leq k \leq n$. Then $\tau'$ is a proof of $G_0[S^{m-1}]$. 

16
The proof is by induction on $n$. Since $\tau^1(G'[S']^m)$ is a proof and $\frac{G'[S']^m}{H_0[S']^{m-1}}(ID_2)$ is valid in GL, then $\tau'(H_0[S']^{m-1})$ is a proof. Suppose that $\tau'(H_{n-1}[S']^{m-1})$ is a proof. Since $\frac{H_{n-1}}{H_n}[G'']^n$ (II) or $\frac{H_{n-1}-1}{H_n}(I)$ in $\tau'$, then $\frac{H_{n-1}}{H_n}[S']^{m-1}$ (II) or $\frac{H_{n-1}}{H_n}[S']^{m-1}$ is an application of the same rule (II) (or (I)). Thus $\tau'(H_n[S']^{m-1})$ is a proof. 

**Definition 4.5.** The manipulation described in Lemma 4.4 is called sequent-inserting operation.

Clearly, the number of $(EC^*)$-applications in $\tau'$ is less than $\tau'$. Next, we continue to process $\tau$.

**Step 2** Let $\frac{G''_{T}}{(S'_{\alpha})_{m}^{\delta_{\alpha}}}, \ldots, \frac{G''_{T}}{(S'_{\gamma})_{m}^{\delta_{\gamma}}}$ be all applications of $(EC^*)$ in $\tau'$ and

$G'_{T} := (S'_{\alpha})_{m}^{\delta_{\alpha}} \ldots (S'_{\gamma})_{m}^{\delta_{\gamma}}$. By repeatedly applying sequent-inserting operations, we construct a proof of $G_0[G'_{T}]$ in GL without applications of $(EC^*)$ and denote it by $\tau'$. 

**Remark 4.6.** (i) $\tau'$ is constructed by converting $(EC)$ into $(ID_2)$; (ii) Each node of $\tau'$ has the form $H_0[H'_{T}]$, where $H_0 \in \tau'$ and $H'_{T}$ is a (possibly empty) subset of $G'_{T}$.

We need the following construction to eliminate applications of $(EW)$ in $\tau'$.

**Construction 4.7.** Let $H \in \tau'$, $H' \subseteq H$ and $TH_{2}(H) = (H_0, \ldots, H_n)$, where $H_0 = H$, $H_n = G_0[G'_{T}]$. Hypersequents $(H_k)^{H_{2}}$, and trees $\tau_{H_{2}}^{H_{2}}((H_k)^{H_{2}})$ for all $0 \leq k \leq n$ are constructed inductively as follows.

(i) $(H_0)^{H_{2}} := H'$ and $\tau_{H_{2}}^{H_{2}}((H_0)^{H_{2}})$ consists of a single node $H'$;

(ii) Let $\frac{G'[S']^m}{H_0[S']^{m-1}}(II)$ or $\frac{G'[S']^m}{H_0[S']^{m-1}}(I)$ be in $\tau'$, $H_k = G'[S']$ and $H_{k+1} = G'[S']^m$ (accordingly $H_{k+1} = G'[S']^m$ for (I)) so $0 \leq k \leq n - 1$.

If $S' \in (H_k)^{H_{2}}$,

\[
(H_k)^{H_{2}} := (H_k)^{H_{2}} \setminus \{S'\} | G'[S']^m
\]

(accordingly $(H_{k+1})^{H_{2}} := (H_{k+1})^{H_{2}} \setminus \{S'\} | G'[S']^m$ for (I)) and $\tau_{H_{2}}^{H_{2}}((H_{k+1})^{H_{2}})$ is constructed by combining the trees

\[
\tau_{H_{2}}^{H_{2}}((H_k)^{H_{2}})^{H_{2}}, \tau_{H_{2}}^{H_{2}}((H_k)^{H_{2}})^{H_{2}}(\frac{(H_k)^{H_{2}}}{(H_{k+1})^{H_{2}}})^{(I)}
\]

(accordingly $\tau_{H_{2}}^{H_{2}}((H_k)^{H_{2}})^{H_{2}}$ with $\frac{(H_k)^{H_{2}}}{(H_{k+1})^{H_{2}}}(I)$ for (I)) otherwise $(H_{k+1})^{H_{2}} := (H_{k+1})^{H_{2}}$ and $\tau_{H_{2}}^{H_{2}}((H_{k+1})^{H_{2}})$ is constructed by combining

\[
\tau_{H_{2}}^{H_{2}}((H_{k+1})^{H_{2}})^{H_{2}} = \frac{(H_{k+1})^{H_{2}}}{(H_{k+1})^{H_{2}}}(ID_\alpha).
\]

(iii) Let $G'[S']^{m} \in \tau'$, $H_k = G'$ and $H_{k+1} = G'[S']$ then $H_{k+1}^{H_{2}} := (H_{k+1})^{H_{2}}$ and

$\tau_{H_{2}}^{H_{2}}((H_{k+1})^{H_{2}})$ is constructed by combining $\tau_{H_{2}}^{H_{2}}((H_k)^{H_{2}})^{H_{2}}$ with $\frac{(H_k)^{H_{2}}}{(H_{k+1})^{H_{2}}}(ID_\alpha)$.
Lemma 4.8. (i) $\langle H_k \rangle_{H^k} \subseteq H_k$ for all $0 \leq k \leq n$;
(ii) $\tau^2_{H^k}(\langle H_k \rangle_{H^k})$ is a derivation of $\langle H_k \rangle_{H^k}$ from $H^k$ without (EC).

Proof. The proof is by induction on $k$. For the base step, $\langle H_0 \rangle_{H^0} = H^0$ and $\tau^2_{H^0}(\langle H_0 \rangle_{H^0})$ consists of a single node $H^0$. Then $\langle H_0 \rangle_{H^0} \subseteq H_0 = H$, $\tau^2_{H^0}(\langle H_0 \rangle_{H^0})$ is a derivation of $\langle H_0 \rangle_{H^0}$ from $H^0$ without (EC).

For the induction step, suppose that $\langle H_k \rangle_{H^k}$ and $\tau^2_{H^k}(\langle H_k \rangle_{H^k})$ be constructed such that (i) and (ii) hold for some $0 \leq k \leq n - 1$. There are two cases to be considered.

Case 1 Let $G[S']^k(I) \in \tau^2$, $H_k = G'[S']$ and $H_{k+1} = G'[S'']$. If $S' \notin \langle H_k \rangle_{H^k}$ then $\langle H_k \rangle_{H^k} \setminus \{S'\} \subseteq G'$ by $\langle H_k \rangle_{H^k} \subseteq H^k = G'[S']$. Thus $\langle H_{k+1} \rangle_{H^k} = ((\langle H_k \rangle_{H^k} \setminus \{S'\}))|S'' \subseteq G'[S''] = H_{k+1}$. Otherwise $S' \notin \langle H_k \rangle_{H^k}$ then $\langle H_k \rangle_{H^k} \subseteq G'$ by $\langle H_k \rangle_{H^k} \subseteq H_k = G'[S']$. Thus $\langle H_{k+1} \rangle_{H^k} \subseteq H_{k+1} \subseteq (\langle H_k \rangle_{H^k} \setminus \{S'\})|S''(\langle H_{k+1} \rangle_{H^k})$ is a derivation of $\langle H_{k+1} \rangle_{H^k}$ from $H^k$ without (EC) since $\tau^2_{H^k}(\langle H_k \rangle_{H^k})$ is such one and $\tau^2_{H^k}(\langle H_k \rangle_{H^k})$ is a valid instance of a rule (I) of GL. The case of applications of two-premise rule is proved by a similar procedure and omitted.

Case 2 Let $G[S']^k(EW) \in \tau^2$, $H_k = G'$ and $H_{k+1} = G'[S']$. Then $\langle H_{k+1} \rangle_{H^k} \subseteq H_{k+1}$ by $\langle H_{k+1} \rangle_{H^k} = \langle H_k \rangle_{H^k} \subseteq H_k \subseteq H_{k+1}$. $\tau^2_{H^k}(\langle H_k \rangle_{H^k})$ is a derivation of $\langle H_{k+1} \rangle_{H^k}$ from $H^k$ without (EC) since $\tau^2_{H^k}(\langle H_k \rangle_{H^k})$ is such one and $\tau^2_{H^k}(\langle H_k \rangle_{H^k})$ is valid by Definition 4.3.

Definition 4.9. The manipulation described in Construction 4.7 is called derivation-pruning operation.

Notation 4.10. We denote $\langle H_n \rangle_{H^k}$ by $G[S']^k \langle \langle H_n \rangle_{H^k} \rangle$ by $\tau^2_{H^k}(\langle H_n \rangle_{H^k})$ and say that $H^k$ is transformed into $G[S']^k$ in $\tau^2$.

Then Lemma 4.8 shows that $\frac{H^k}{G[S']^k} \langle \tau^2_{H^k}(\langle H_n \rangle_{H^k}) \rangle \subseteq G_0|G_1^*$. Now, we continue to process $\tau$ as follows.

Step 3 Let $G[S']^k(EW) \in \tau^2$ then $\tau^2_{G[S']^k}((\langle H_0 \rangle_{G[S']^k}))$ is a derivation of $\langle H_0 \rangle_{G[S']^k}$ from $G'$ thus a proof of $\langle H_0 \rangle_{G[S']^k}$ is constructed by combining $\tau^2(G')$ and $\tau^2_{G[S']^k}((\langle H_0 \rangle_{G[S']^k}))$ with $G'[ID_{\Delta}]$. By repeatedly applying the procedure above, we construct a proof $\tau^3$ of $G[G]^\* \subseteq G_0|G_1^*$ without (EW) in GL, where $G_1 \subseteq G_0, G_1^* \subseteq G_0^*$ by Lemma 4.8 (i).

Step 4 Let $\Gamma, p, \perp \Rightarrow \Delta \in \tau^1$ (or $\Gamma, p \Rightarrow \tau, \Delta \in \frac{G[\Gamma] \Rightarrow \Delta}{G[\Gamma], p \Rightarrow \Delta} (WL)$) then there exists $\Gamma' \Rightarrow \Delta' \in H$ such that $p \in \tau^1$ for all $H \in TH_\tau(\Gamma, p, \perp \Rightarrow \Delta)$ (accordingly $H \in TH_\tau(\Gamma, p \Rightarrow \tau, \Delta)$, $H \in TH_\tau(\Gamma \Rightarrow \perp, \Delta)$) thus a proof is constructed by replacing top-down $p$ in each $\Gamma'$ with $\perp$.

Let $\Gamma, \perp \Rightarrow p, \Delta$ (or $\Gamma \Rightarrow \tau, p, \Delta, \frac{G[\Gamma] \Rightarrow \Delta}{G[\Gamma], p \Rightarrow \Delta} (WR)$) is a leaf of $\tau^3$ then there exists $\Gamma' \Rightarrow \Delta' \in H$ such that $p \in \Delta'$ for all $H \in TH_\tau(\Gamma, \perp \Rightarrow p, \Delta)$ (accordingly $H \in TH_\tau(\Gamma \Rightarrow \tau, p, \Delta)$ or $H \in TH_\tau(\Gamma \Rightarrow p, \Delta)$ thus a proof is constructed by replacing top-down $p$ in each $\Gamma'$ with $\perp$. 18
Repeatedly applying the procedure above, we construct a proof $\tau^*$ of $G_2|G_2^*$ in GL such that there doesn’t exist occurrence of $p$ in $\Gamma$ or $\Delta$ at each leaf labeled by $\Gamma, \perp \Rightarrow \Delta$ or $\Gamma \Rightarrow \perp, \Delta$, or $p$ is not the weakening formula $A$ in $G[\Gamma \Rightarrow \Delta]$ or $G[\Gamma \Rightarrow \Delta]''$ when (WR) or (WL) is available. Define two operations $\sigma_p$ and $\sigma_e$ on sequents by $\sigma_p(\Gamma, p \Rightarrow \Delta) := \Gamma, \perp \Rightarrow \Delta$ and $\sigma_e(\Gamma \Rightarrow p, \Delta) := \Gamma \Rightarrow \perp, \Delta$. Then $G_2|G_2^*$ is obtained by applying $\sigma_p$ and $\sigma_e$ to some designated sequents in $G_1|G_1^*$.

**Definition 4.11.** The manipulation described in Step 4 is called eigenvariable-replacing operation.

**Step 5** A proof $\tau^*$ is constructed from $\tau^*$ by assigning inductively one unique identification number to each occurrence of $p$ in $\tau^*$ as follows.

One unique identification number, which is a positive integer, is assigned to each leaf of the form $p \Rightarrow p$ in $\tau^*$ which corresponds to $p_k \Rightarrow p_k$ in $\tau^*$. Other nodes of $\tau^*$ are processed as follows.

- Let $G_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ and $G_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]''$ are assigned identification numbers and have the form $G_1[\Gamma, p_{i_1}, \ldots, p_{i_k} \Rightarrow p_{j_1}, \ldots, p_{j_k}, \Delta]$ in $\tau^*$, which we often write as $G_1[\Gamma, \{p_{i_1}\}_{k=1}^j \Rightarrow \{p_{j_1}\}_{k=1}^j, \Delta]$. Then $G_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ has the form $G_1[\Gamma, \{p_{i_1}\}_{k=1}^j \Rightarrow \{p_{j_1}\}_{k=1}^j, \Delta]$.

- Let $G'_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ and $G''_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ are assigned identification numbers and have the form $G'_1[\Gamma, p_{i_1}, \ldots, p_{i_k} \Rightarrow p_{j_1}, \ldots, p_{j_k}, \Delta]$ in $\tau^*$, respectively. Then $G''_1$ has the form $G'_1[\Gamma, \{p_{i_1}\}_{k=1}^j \Rightarrow \{p_{j_1}\}_{k=1}^j, \Delta]$.

- Let $G'_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ and $G''_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ are assigned identification numbers and have the form $G'_1[\Gamma, p_{i_1}, \ldots, p_{i_k} \Rightarrow p_{j_1}, \ldots, p_{j_k}, \Delta]$ in $\tau^*$, respectively. Then $G''_1$ has the form $G'_1[\Gamma, \{p_{i_1}\}_{k=1}^j \Rightarrow \{p_{j_1}\}_{k=1}^j, \Delta]$.

- Let $G'_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ and $G''_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ are assigned identification numbers and have the form $G'_1[\Gamma, p_{i_1}, \ldots, p_{i_k} \Rightarrow p_{j_1}, \ldots, p_{j_k}, \Delta]$ in $\tau^*$, respectively. Then $G''_1$ has the form $G'_1[\Gamma, \{p_{i_1}\}_{k=1}^j \Rightarrow \{p_{j_1}\}_{k=1}^j, \Delta]$.

- Let $G'_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ and $G''_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ are assigned identification numbers and have the form $G'_1[\Gamma, p_{i_1}, \ldots, p_{i_k} \Rightarrow p_{j_1}, \ldots, p_{j_k}, \Delta]$ in $\tau^*$, respectively. Then $G''_1$ has the form $G'_1[\Gamma, \{p_{i_1}\}_{k=1}^j \Rightarrow \{p_{j_1}\}_{k=1}^j, \Delta]$.

- Let $G'_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ and $G''_1[\Gamma, \lambda_p \Rightarrow \mu_p, \Delta]$ are assigned identification numbers and have the form $G'_1[\Gamma, p_{i_1}, \ldots, p_{i_k} \Rightarrow p_{j_1}, \ldots, p_{j_k}, \Delta]$ in $\tau^*$, respectively. Then $G''_1$ has the form $G'_1[\Gamma, \{p_{i_1}\}_{k=1}^j \Rightarrow \{p_{j_1}\}_{k=1}^j, \Delta]$.

**Definition 4.12.** The manipulation described in Step 5 is called eigenvariable-labeling operation.
Notation 4.13. Let $G_2$ and $G_2'$ be converted to $G$ and $G^*$ in $\tau^*$, respectively. Then $\tau^*$ is a proof of $G|G^*$.

In the preprocessing of $\tau$, each $\frac{G''_i}{G''_i(|S'_i|^{\omega_i})}(\text{EC}^*)$, is converted into $\frac{G''_i}{G''_i(|S'_i|^{\omega_i})(ID_\Omega)}$, in Step 2, where $G''_i \subseteq G''_i$ by Lemma 4.4. $G'/G''_i(\text{EW}) \in \tau^2$ is converted into $\frac{G'}{G''_i,ID_\Omega}$ in Step 3, where $G''_i \subseteq G'$ by Lemma 4.8. Some $G'|\Gamma^i,p \Rightarrow \Delta^i \in \tau^3$ (or $G'|\Gamma^i \Rightarrow p,\Delta^i$) is revised as $G'|\Gamma^i,\top \Rightarrow \Delta^i$ (or $G'|\Gamma^i \Rightarrow 1,\Delta^i$) in Step 4. Each occurrence of $p$ in $\tau^4$ is assigned the unique identification number in Step 5. The whole preprocessing above is depicted by Figure 13.

$$
\begin{array}{c}
\text{Step 1: } \tau^1 \\
\tau \Rightarrow G_0 \\
\frac{\forall w,\forall u}{\text{Step 2: } \tau^2} \\
\frac{G_0}{EC} \Rightarrow G_0[\text{ID numbers]} \\
\frac{\text{Step 3: } \tau^3}{\text{Step 4: } \tau^4} \\
\frac{G_0}{\text{Step 5: } \tau^*} \\
\frac{G_2}{G^*} \Rightarrow G_i[|G^*|] \\
\end{array}
$$

FIGURE 13 Preprocessing of $\tau$.

Notation 4.14. Let $\frac{G''_i}{G''_i(|S'_i|^{\omega_i})}(\text{EC}^*)$, $1 \leq i \leq N$ be all ($\text{EC}^*$)-nodes of $\tau^4$ and $G''_i(|S'_i|^{\omega_i})$ be converted to $G''_i(|S'_i|^{\omega_i})$ in $\tau^*$. Note that there are no identification numbers for occurrences of variable $p$ in $S'_i \in G''_i(|S'_i|^{\omega_i})$ meanwhile they are assigned to $p$ in $S'_i \in G''_i(|S'_i|^{\omega_i})$. But we use the same notations for $S'_i \in G''_i(|S'_i|^{\omega_i})$ and $S'_i \in G''_i(|S'_i|^{\omega_i})$ for simplicity.

In the whole paper, let $H_i^e = G''_i(|S'_i|^{\omega_i})$ denote the unique node of $\tau^*$ such that $H_i^e \subseteq G''_i(|S'_i|^{\omega_i})$, and $S'_i$ is the focus sequent of $H_i^e$ in $\tau^*$, in which case we denote the focus one $S'_i$ and others $S'_i[\cdot \cdot \cdot |S'_i]$ among $\{S'_i\}^m$ for simplicity. We sometimes denote $H_i^e$ also by $G''_i(|S'_i|^{\omega_i})$ or $G''_i(|S'_i|^{\omega_i})$. We then write $G^*$ as $\{S'_i\}^{\omega_i=2\cdots m}$. We call $H_i^e, S'_i$ the $i$-th pseudo-($\text{EC}$) node of $\tau^*$ and pseudo-($\text{EC}$) sequent, respectively. We abbreviate pseudo-EC as pEC. Let $H \in \tau^*$, by $S'_i \in H$ we mean that $S'_i \in H$ for some $1 \leq u \leq m_i$.

It is possible that there doesn’t exist $H_i^e \in G''_i(|S'_i|^{\omega_i})$ such that $S'_i$ is the focus sequent of $H_i^e$ in $\tau^*$, in which case $\{S'_i\}^{\omega_i} \subseteq G^*$, then it hasn’t any effect on our argument to treat all such $S'_i$ as members of $G$. So we assume that all $H_i^e$ are always defined for all $G''_i(|S'_i|^{\omega_i})$ in $\tau^*$, i.e., $H_i^e > G|G^*$.

Proposition 4.15. (i) $\{S'_i\}^{\omega_i=2\cdots m}$ is $\forall H \leq H_i^e$; (ii) $G^* = \{S'_i\}^{\omega_i=2\cdots m}$.

Now, we replace locally each $\frac{G'}{G''_i,ID_\Omega}$ in $\tau^*$ with $G'$ and denote the resulting proof also by $\tau^*$, which has no essential difference with the original one but could simplify subsequent arguments. We introduce the system $GL_{\Omega}$ as follows.

Definition 4.16. GL_{\Omega} is a restricted subsystem of GL such that

(i) $p$ is designated as the unique eigenvariable by which we mean that it is not used to built up any formula containing logical connectives and only used as a sequent-formula.

(ii) Each occurrence of $p$ on each side of every component of a hypersequent in GL is assigned one unique identification number $i$ and written as $p_i$ in $GL_{\Omega}$. Initial sequent $\Rightarrow p$ of GL has the form $p_1 \Rightarrow p_i$ in $GL_{\Omega}$.
(iii) Each sequent \( S \) of \( GL \) in the form \( \Gamma, \lambda p \Rightarrow \mu p, \Delta \) has the form
\[
\Gamma, \{p_i\}_{k=1}^l \Rightarrow \{p_j\}_{k=1}^m, \Delta
\]
in \( GL_\Omega \), where \( p \) does not occur in \( \Gamma \) or \( \Delta \). \( i_k \neq i_l \) for all \( 1 \leq k < l \leq \lambda \), \( j_k \neq j_l \) for all \( 1 \leq k < l \leq \mu \). Define \( v_\lambda(S) = \{i_1, \ldots, i_\lambda\} \) and \( v_\mu(S) = \{j_1, \ldots, j_\mu\} \). Let \( G \) be a hypersequent of \( GL_\Omega \) in the form \( S \mid \cdots \mid S_n \) then \( v_\lambda(S) \cap v_\mu(S) = \emptyset \) and \( v_\lambda(S) \cap v_\mu(S) = \emptyset \) for all \( 1 \leq k < l \leq n \). Define \( v_r(G) = \bigcup_{k=1}^n v_r(S_k) \). Here, \( l \) and \( r \) in \( v_\lambda \) and \( v_\mu \) indicate the left and right side of a sequent, respectively.

(iv) A hypersequent \( G \) of \( GL_\Omega \) is called closed if \( v_\lambda(G) = v_\mu(G) \). Two hypersequents \( G' \) and \( G'' \) of \( GL_\Omega \) are called disjoint if \( v_\lambda(G') \cap v_\lambda(G'') = \emptyset \), \( v_\mu(G') \cap v_\mu(G'') = \emptyset \) and \( v_\lambda(G') \cap v_\mu(G'') = \emptyset \). \( G'' \) is a copy of \( G' \) if they are disjoint and there exist two bijections \( \sigma_1 : v_\lambda(G') \rightarrow v_\lambda(G'') \) and \( \sigma_2 : v_\mu(G') \rightarrow v_\mu(G'') \) such that \( G'' \) can be obtained by applying \( \sigma_1 \) to antecedents of sequents in \( G' \) and \( \sigma_2 \) to succedents of sequents in \( G' \), i.e., \( G'' = \sigma_1 \circ \sigma_2(G') \).

(v) A closed hypersequent \( G''|G'''|G'''' \) can be contracted as \( G''|G'''' \) in \( GL_\Omega \) under the condition that \( G'' \) and \( G''' \) are closed and \( G'''' \) is a copy of \( G'''' \).

Furthermore, if there doesn’t exist two closed hypersequents \( H', H'' \subseteq G''|G'''' \) such that \( H'' \) is a copy of \( H' \), then we call it the fully constraint contraction rule and denote by
\[
\frac{G''|G''''}{G''|G'''}(EC_\Omega).
\]

(vi) \((EW)\) and \((CUT)\) of \( GL \) are forbidden. \((EC), (\wedge), \) and \((\vee)\) of \( GL \) are replaced with \((EC_\Omega), (\wedge_\Omega), \) and \((\vee_\Omega)\) in \( GL_\Omega \), respectively.

(vii) \( G_1[S_1] \) and \( G_2[S_2] \) are closed and disjoint for each two-premise rule
\[
\frac{G_1[S_1]}{G_2[S_2]}(H) \text{ of } GL_\Omega \text{ and, } G[S'] \text{ is closed for each one-premise rule } \frac{G[S']}{G}(I).
\]

(viii) \( p \) doesn’t occur in \( \Gamma \) or \( \Delta \) for each initial sequent \( \Gamma, \Delta \Rightarrow \Delta \) or \( \Gamma \Rightarrow \Gamma \) or \( \Delta \) and, \( p \) doesn’t act as the weakening formula \( A \) in \( G[\Gamma \Rightarrow \Delta] \) or \( G[\Gamma] \Rightarrow A \) or \( G[\Delta] \Rightarrow A \) (\( WR \)) or \( G[\Gamma, A] \Rightarrow \Delta \) (\( WL \)) when \( WR \) or \( WL \) is available.

**Lemma 4.17.** Let \( \tau \) be a cut-free proof of \( G_0 \) in \( L \) and \( \tau^* \) be the tree resulting from preprocessing of \( \tau \).

(i) If \( \frac{G[S]}{G[S']} (I) \in \tau^* \) then \( v_\lambda(G[S']) = v_\lambda(G[S]) = v_\mu(G[S']) = v_\mu(G[S]) \);

(ii) If \( \frac{G[S']} {G[G|G''|H]} (II) \in \tau^* \) then \( v_\lambda(G[G''|H]) = v_\lambda(G[S']) \cup v_\lambda(G[S'']) = v_\mu(G[G|G''|H]) = v_\mu(G[S']) \cup v_\mu(G[S'']) \);

(iii) If \( H \in \tau^* \) and \( k \notin v_\lambda(H) \) then \( k \in v_\lambda(H) \);

(iv) If \( H \in \tau^* \) and \( k \notin v_\lambda(H) \) then \( k \notin \mu \Rightarrow p_k \);

(v) \( \tau^* \) is a proof of \( G|G" \) in \( GL_\Omega \) without \( (EC_\Omega) \);

(vi) If \( H', H'' \in \tau^* \) and \( H' \parallel H'' \) then \( v_\lambda(H') \cap v_\lambda(H'') = \emptyset, v_\mu(H') \cap v_\mu(H'') = \emptyset \).

**Proof.** Claims from (i) to (iv) are immediately from Step 5 in preprocessing of \( \tau \) and Definition 4.16. (v) is from Notation 4.13 and Definition 4.16. Only (vi) is proved as follows.
Suppose that \( k \in v_l(D') \cap v_l(D'') \). Then \( H' \leq p_k \Rightarrow p_k, H'' \leq p_k \Rightarrow p_k \) by Claim (iv). Thus \( H' \leq H'' \) or \( H'' \leq H' \), a contradiction with \( H' \cap H'' \) hence \( v_l(D') \cap v_l(D'') = \emptyset \).

\[ v_l(D') \cap v_l(D'') = \emptyset \] is proved by a similar procedure and omitted.

5. The generalized density rule (D) for GL\( G \)

In this section, we define the generalized density rule (D) for GL\( G \) and prove that it is admissible in GL\( G \).

**Definition 5.1.** Let \( G \) be a closed hypersequent of GL\( G \) and \( S \in G \). Define \( [S]_G = \cap \{ H : S \in H \subseteq G, v_l(H) = v_l(H') \} \), i.e., \( [S]_G \) is the minimal closed unit of \( G \) containing \( S \). In general, for \( G' \subseteq G \), define \( [G']_G = \cap \{ H : G' \subseteq H \subseteq G, v_l(H) = v_l(H') \} \).

Clearly, \( [S]_G = S \) if \( v_l(S) = v_l(S) \) or \( p \) does not occur in \( S \). The following construction procedure gives a construction to prove \( [S]_G \) for any given \( S \in G \).

**Construction 5.2.** Let \( G \) and \( S \) be as above. A sequence \( G_1, G_2, \ldots, G_n \) of hypersequents is constructed recursively as follows. (i) \( G_1 = \{ S \} \); (ii) Suppose that \( G_k \) is constructed for \( k \geq 1 \). If \( v_l(G_k) \neq v_l(G_k) \) then there exists \( i_k \in v_l(G_k) \setminus v_l(G_k) \) (or \( i_k \in v_l(G_k) \setminus v_l(G_k) \)) thus there exists the unique \( S_{k+1} \in G_k \) such that \( i_k \in v_l(S_{k+1}) \setminus v_l(S_{k+1}) \) (or \( i_k \in v_l(S_{k+1}) \setminus v_l(S_{k+1}) \)) by \( v_l(G) = v_l(G) \) and Definition 4.16 then let \( G_{k+1} = G(S_{k+1}) \) otherwise the procedure terminates and \( n := k \).

**Lemma 5.3.** (i) \( G_k = \{ S \} \).
(ii) \( [S]_G \).
(iii) \( [S]_G \).
(iv) \( [S]_G \).
(v) \( [S]_G \).

**Proof.** (i) Since \( G_1 \subseteq G_{k+1} \) for \( 1 \leq k \leq n - 1 \) and \( S \in G_1 \) then \( S \in G_k \subseteq G_k \) if \( 1 \leq k \leq n \) by induction on \( k \) in the following. Clearly, \( G_1 \subseteq [S]_G \). Suppose that \( G_k \subseteq [S]_G \) for some \( 1 \leq k \leq n - 1 \). Since \( i_k \in v_l(G_k) \setminus v_l(G_k) \) (or \( i_k \in v_l(G_k) \setminus v_l(G_k) \)) and \( i_k \in v_l(S_{k+1}) \) then \( S_{k+1} \subseteq [S]_G \) by \( G_k \subseteq [S]_G \) and \( [S]_G \). Let \( [S]_G \).

(ii) By (i), \( [S]_G \). For some \( 1 \leq k \leq n \) hence \( S \subseteq S \). Then \( S' = S \). For some \( 1 \leq k \leq n \) thus \( i_k \in v_l(S_k) \). Hence \( S \subseteq S \). For some \( 1 \leq k \leq n \) thus \( i_k \in v_l(S_k) \). Hence \( S \subseteq S \).

(iii) It holds immediately from Construction 5.2 (i) and (iv).

(iv) The proof is by induction on \( k \). For the base step, let \( k = 1 \) then \( |G_k| = 1 \) thus \( |v_l(G_k)| = 1 \). For the induction step, suppose that \( |v_l(G_k)| = 1 \). Then \( |v_l(G_{k+1})| \leq |v_l(G_k)| + 1 \) by \( i_k \in v_l(G_{k+1}) \) and \( v_l(G_k) \subseteq v_l(G_{k+1}) \). Then \( |v_l(G_{k+1})| = 1 \) by \( |G_{k+1}| = |G_k| + 1 = k + 1 \).

(v) It holds by (iv) and \( v_l(S) = v_l(S) \).

Definition 5.4. Let $G = S_1 \cdots |S_r$ and $S_i$ be in the form $\Gamma_i, \{p_{k_i}^{\mu_i}\}_{k_i=1}^{r_i} \Rightarrow \{p_{k_i}^{\mu_i}\}_{k_i=1}^{r_i}, \Delta_i$ for $1 \leq i \leq r$.

(i) If $S \in G$ and $[S]_G$ be $S_k \cdots |S_r$ then $D_G(S)$ is defined as

$\Gamma_k, \cdots \Gamma_r \Rightarrow \{v_i([S]_G)\} - [S]_G + 1, \Delta_k$,

(ii) Let $\bigcup_{k=1}^{v} [S]_G = G$ and $[S_k]_G \cap [S_v]_G = \emptyset$ for all $1 \leq k < v \leq v$ then $D(G)$ is defined as

$D_G(S_v) : \cdots |D_G(S_k)$.

(iii) We call $(D)$ the generalized density rule of $GL_{\Omega}$, whose conclusion $D(G)$ is defined by

(ii) if its premise is $G$.

Clearly, $D(p_k \Rightarrow p_k)$ is $\Rightarrow t$ and $D(S) = S$ if $p$ does not occur in $S$.

Lemma 5.5. Let $G' \equiv G |S$ and $G'' \equiv G |S_2$ be closed and $[S_1]_{G'} \cap [S_2]_{G''} = \emptyset$, where $S_1 = \Gamma_1, \{p_{k_1}^{\mu_1}\}_{k_1=1}^{l_1}, \Delta_1; S_2 = \Gamma_2, \{p_{k_2}^{\mu_2}\}_{k_2=1}^{l_2}, \Delta_2$.

$S = \Gamma_1, \Gamma_2, \{p_{k_1}^{\mu_1}\}_{k_1=1}^{l_1}, \{p_{k_2}^{\mu_2}\}_{k_2=1}^{l_2} \Rightarrow \{p_{k_1}^{\mu_1}\}_{k_1=1}^{l_1}, \{p_{k_2}^{\mu_2}\}_{k_2=1}^{l_2}, \Delta_1, \Delta_2; D_G'(S_1) = \Gamma_1, \Sigma_1 \Rightarrow \Pi_1, \Delta_1$ and

$D_G''(S_2) = \Gamma_2, \Sigma_2 \Rightarrow \Pi_2, \Delta_2$. Then $D_G(S) = \Gamma_1, \Sigma_1, \Gamma_2, \Sigma_2 \Rightarrow \Pi_1, \Delta_1, \Pi_2, \Delta_2$.

Proof. Since $[S_1]_{G'} \cap [S_2]_{G''} = \emptyset$ then $[S]_{G'} = [S_1]_{G'} \cup [S_2]_{G''}$ and $[S]_G = [S_1]_{G'} \cup [S_2]_{G''} \cup \{S\}$ by $v_i(S) = v_i(S_1S_2)$ and $v_i(S) = v_i(S_1S_2)$ and Lemma 5.3 (iii). Thus

$D_G(S) = \Gamma_1, \Sigma_1, \Gamma_2, \Sigma_2 \Rightarrow \Pi_1, \Delta_1, \Pi_2, \Delta_2$.

Therefore $D_G(S) = \Gamma_1, \Sigma_1, \Gamma_2, \Sigma_2 \Rightarrow \Pi_1, \Delta_1, \Pi_2, \Delta_2$ by

$\Pi_1 = \{v_i([S_1]_{G'}) - [S_1]_{G'} + 1, \Pi_1 \\Pi_i \{v_i([S_1]_{G'}) - [S_1]_{G'} + 1\} \Pi_i$

$\Pi_2 = \{v_i([S_2]_{G''}) - [S_2]_{G''} + 1, \Pi_2 \\Pi_i \{v_i([S_2]_{G''}) - [S_2]_{G''} + 1\} \Pi_i$

$D_G(S) = \Gamma_1, \Sigma_1, \Gamma_2, \Sigma_2 \Rightarrow \{v_i([S]_G) - [S]_G + 1\} t, \Pi_1 \Pi_i \{v_i([S]_G) - [S]_G + 1\} \Pi_i$

$\Pi_2 \{v_i([S_2]_{G''}) - [S_2]_{G''} + 1\} t, \Pi_2 \Pi_i \{v_i([S_2]_{G''}) - [S_2]_{G''} + 1\} \Pi_i$

$\Delta t = \{t, \cdots, t\}.$

Lemma 5.6. ([A.5.1]) If there exists a proof $\tau$ of $G$ in $GL_\Omega$ then there exists a proof of $D(G)$ in $GL_\Omega$, i.e., $(D)$ is admissible in $GL_\Omega$.

Proof. We proceed by induction on the height of $\tau$. For the base step, if $G$ has $p_k \Rightarrow p_k$ then $D(G)$ is $\Rightarrow t$ otherwise $D(G)$ is $G$ then $\Rightarrow GL_\Omega D(G)$ holds. For the induction step, the following cases are considered.

- Let

$$G |S' \Rightarrow \{p_{k_i}^{\mu_i}\}_{k_i=1}^{r_i}, \Delta, B$$

and $S' \equiv \Delta, \Gamma, \{p_{k_i}^{\mu_i}\}_{k_i=1}^{r_i}, \Delta, A \Rightarrow B$.

Then $G |S' \Rightarrow \{S\} \Rightarrow \{S\} \Rightarrow \{S\} \Rightarrow \{S\} \Rightarrow \{S\} \Rightarrow \{S\} \Rightarrow \{S\}$.

Let $D_G(S') = \Delta, \Delta, A \Rightarrow B$ and $D_G(S) = \Delta, \Delta, A \Rightarrow B$.

Thus a proof of $D(G)$ is $\Rightarrow GL_\Omega D(G)$ holds. For the induction step, the following cases are considered.
\( \mathcal{D}(G'|S'') \) is constructed by combining the proof of \( \mathcal{D}(G'|S') \) and \( \frac{\mathcal{D}_{G'|S'}(S')}{{\mathcal{D}_{G'|S''}(S'')}}(\rightarrow_r) \). Other rules of type (I) are processed by a procedure similar to above.

- Let

\[
\frac{G_1|S_1 \quad G_2|S_2}{G_1|G_2|S_3} (\land_r) \in \tau
\]

where

\[
S_1 \equiv \Gamma_1, \{p_G^1\}_{k=1}^{l_1} \Rightarrow \{p_G^1\}_{k=1}^{l_1}, A, \Delta_1
\]

\[
S_2 \equiv \Gamma_2, \{p_G^2\}_{k=1}^{l_2} \Rightarrow \{p_G^2\}_{k=1}^{l_2}, B, \Delta_2
\]

\[
S_3 \equiv \Gamma_1, \Gamma_2, \{p_G^3\}_{k=1}^{l_3}, \{p_G^3\}_{k=1}^{l_3}, \{p_G^3\}_{k=1}^{l_3} \Rightarrow \{p_G^3\}_{k=1}^{l_3}, A \lor B, \Delta_1, \Delta_2.
\]

Let

\[
\mathcal{D}_{G_1|S_1}(S_1) = \Gamma_1, \Gamma_1 \Rightarrow \Delta_1, (v([S_1]_{\mathcal{G}_1|S_1}) - [S_1]_{\mathcal{G}_1|S_1} + 1)t, A, \Delta_1,
\]

\[
\mathcal{D}_{G_2|S_2}(S_2) = \Gamma_2, \Gamma_2 \Rightarrow \Delta_2, (v([S_2]_{\mathcal{G}_2|S_2}) - [S_2]_{\mathcal{G}_2|S_2} + 1)t, B, \Delta_2.
\]

Then \( \mathcal{D}_{G_1|G_2|S_3}(S_3) \) is

\[
\Gamma_1, \Gamma_2, \Gamma_1 \Rightarrow \Delta_1, \Delta_2, A \lor B, \Delta_1, \Delta_2,
\]

\[
(v([S_1]_{\mathcal{G}_1|S_1}) + v([S_2]_{\mathcal{G}_2|S_2}) - [S_1]_{\mathcal{G}_1|S_1} - [S_2]_{\mathcal{G}_2|S_2} + 2)t
\]

by \( [S_3]_{\mathcal{G}_1|G_2|S_3} = ([S_1]_{\mathcal{G}_1|S_1} \setminus \{S_1\}) \cup ([S_2]_{\mathcal{G}_2|S_2} \setminus \{S_2\}) \cup \{S_3\} \). Then the proof of \( \mathcal{D}(G_1|G_2|S_3) \) is constructed by combining \( \triangledown \mathcal{D}(G_1|S_1) \) and \( \triangledown \mathcal{D}(G_2|S_2) \) with \( \frac{\mathcal{D}_{G_1|G_2|S_3}(S_3)}{\mathcal{D}_{G_1|G_2|S_3}(S_3)}(\lor_r) \). All applications of \( (\rightarrow_r) \) are processed by a procedure similar to that of \( \land_r \) and omitted.

- Let

\[
\frac{G' \quad G''}{G'''}(\land_w) \in \tau
\]

where

\[
G' \equiv G_1|S_1, \quad G'' \equiv G_2|S_2, \quad G''' \equiv G_1|G_2|S_1'|S_2',
\]

\[
S_w \equiv \Gamma_w, \{p_G^w\}_{k=1}^{l_w} \Rightarrow \{p_G^w\}_{k=1}^{l_w}, A_w, \Delta_w,
\]

\[
S_w' \equiv \Gamma_w, \{p_G^w\}_{k=1}^{l_w} \Rightarrow \{p_G^w\}_{k=1}^{l_w}, A_1 \land A_2, \Delta_w
\]

for \( w = 1, 2 \). Then \( [S_1]_{G'''} = [S_1]_{G'} \setminus [S_1]', [S_1']_{G'''} = [S_2]_{G'''} \setminus [S_1]' \) by Lemma 5.3 (iii).

Let

\[
\mathcal{D}_{G_1|S_1}(S_1) = \Gamma_w, \Gamma_w \Rightarrow \Delta_w, (v([S_w]_{\mathcal{G}_1|S_1}) - [S_w]_{\mathcal{G}_1|S_1} + 1)t, A_w, \Delta_1
\]

for \( w = 1, 2 \). Then

\[
\mathcal{D}_{G'''}(S_w'') = \Gamma_w, \Gamma_w \Rightarrow \Delta_w, (v([S_w]_{\mathcal{G}_1|S_1}) - [S_w]_{\mathcal{G}_1|S_1} + 1)t, A_1 \land A_2, \Delta_w
\]

for \( w = 1, 2 \). Then the proof of \( \mathcal{D}(G''') \) is constructed by combining \( \triangledown \mathcal{D}(G') \) and \( \triangledown \mathcal{D}(G'') \) with \( \frac{\mathcal{D}_{G'''}(S_1')}{{\mathcal{D}_{G'''}(S_2')}}(\land_w) \). All applications of \( (\lor_w) \) are processed by a procedure similar to that of \( (\land_w) \) and omitted.
Let
\[ G' \equiv G_1[S_1], \quad G'' \equiv G_2[S_2], \quad G''' \equiv G_1[G_2[S_3]S_4] \]
where
\[ \{ p_{l_1}^{l_1} \}_{k=1}^{l_1} \Rightarrow \{ p_{j_1}^{j_1} \}_{k=1}^{j_1}, \Sigma_1, \Delta_1, \]
\[ \{ p_{l_2}^{l_2} \}_{k=1}^{l_2} \Rightarrow \{ p_{j_2}^{j_2} \}_{k=1}^{j_2}, \Sigma_2, \Delta_2, \]
\[ \{ p_{l_3}^{l_3} \}_{k=1}^{l_3} \Rightarrow \{ p_{j_3}^{j_3} \}_{k=1}^{j_3}, \Delta_1, \Delta_2, \]
\[ \{ p_{l_4}^{l_4} \}_{k=1}^{l_4}, \{ p_{l_5}^{l_5} \}_{k=1}^{l_5}, \{ p_{l_6}^{l_6} \}_{k=1}^{l_6}, \{ p_{j_1}^{j_1} \}_{k=1}^{j_1}, \{ p_{j_2}^{j_2} \}_{k=1}^{j_2}, \{ p_{j_3}^{j_3} \}_{k=1}^{j_3}, \{ p_{j_4}^{j_4} \}_{k=1}^{j_4}, \{ p_{j_5}^{j_5} \}_{k=1}^{j_5}, \{ p_{j_6}^{j_6} \}_{k=1}^{j_6} \]

**Case 1** \( S_3 \in [S_3]_{G''} \). Then \([S_3]_{G''} = [S_2]_{G''} \) by Lemma 5.3 (ii) and \([S_3]_{G'''} = ([S_1]_{G'} - [S_2]_{G''})[S_3] \) by Lemma 5.3 (iii). Then
\[ |v((S_3)_{G''})| = |(S_3)_{G''}| + 1 = |v(S_1)_{G'}| + |v(S_2)_{G''}| - |(S_1)_{G'}| - |(S_2)_{G''}| + 1 \geq 0. \]
Thus \( |v(S_1)_{G'}| - |(S_2)_{G''}| + 1 \geq 1 \) or \( |v(S_2)_{G''}| + 1 \geq 1 \). Hence we assume that, without loss of generality,
\[ D_{G'}(S_1) = \Gamma_1, \Pi_1, \Gamma' \Rightarrow \Gamma', \Sigma_1, \Delta_1, \]
\[ D_{G''}(S_2) = \Gamma_2, \Pi_2, \Gamma'' \Rightarrow \Delta', \Sigma_2, \Delta_2. \]

Then
\[ D_{G''}(S_3|S_4) = \Gamma_1, \Pi_1, \Gamma', \Pi_2, \Gamma'' \Rightarrow \Delta', \Sigma_1, \Delta_1, \Delta', \Sigma_2, \Delta_2. \]

Thus the proof of \( D_{G''}(S_3|S_4) \) is constructed by
\[ \Gamma_1, \Pi_1, \Gamma' \Rightarrow \Delta', \Pi_1, \Gamma'' \Rightarrow \Delta', \Sigma_1, \Delta_1, \Delta', \Sigma_2, \Delta_2 \]

**Case 2** \( S_3 \notin [S_3]_{G''} \). Then \([S_3]_{G''} \cap [S_4]_{G''} = \emptyset \) by Lemma 5.3 (ii). Let
\[ S_{3w} \equiv \Gamma_3, \Pi_3, \{ p_{l_3}^{l_3} \}_{k=1}^{l_3} \Rightarrow \{ p_{j_3}^{j_3} \}_{k=1}^{j_3}, \Delta_w, \]
\[ S_{4w} \equiv \Pi_4, \{ p_{l_4}^{l_4} \}_{k=1}^{l_4}, \{ p_{l_5}^{l_5} \}_{k=1}^{l_5} \Rightarrow \{ p_{j_4}^{j_4} \}_{k=1}^{j_4}, \Sigma_w, \]
for \( w = 1, 2 \).

\[ [S_3]_{G''} = [S_3]_{G}|G_1[S_3] | S_4 \setminus \{S_3\} \cup \{S_3\} \cup \{S_4\}, \]
\[ [S_4]_{G''} = [S_4]_{G}|G_1[S_3] | S_4 \setminus \{S_4\} \cup \{S_4\} \cup \{S_4\}, \]
by \( v(S_3) = v(S_3|S_3), v(S_1) = v(S_3|S_3), v(S_2) = v(S_3|S_4) \) and \( v(S_4) = v(S_4|S_4) \). Let
\[ D_{G_1}[S_3w]_{G_2} (S_{3w}) = \Gamma_3, \Pi_3, \{ p_{j_3}^{j_3} \}_{k=1}^{j_3}, \Delta_w. \]
\[ D_{G, \mathcal{S}_m \mid \mathcal{S}_n}(S_{4w}) = \Pi_{w}, X_{4w} \Rightarrow \Psi_{4w}, \Sigma_{w} \]

for \( w = 1, 2 \). Then

\[
\begin{align*}
D_G(S_1) &= \Gamma_1, \Pi_1, X_{31}, X_{41} \Rightarrow \Psi_{31}, \Psi_{41}, \Sigma_1, \Delta_1, \\
D_G(S_2) &= \Gamma_2, \Pi_2, X_{32}, X_{42} \Rightarrow \Psi_{32}, \Psi_{42}, \Sigma_2, \Delta_2, \\
D_G(S_3) &= \Gamma_1, X_{31}, \Gamma_2, X_{32} \Rightarrow \Psi_{31}, \Delta_1, \Psi_{32}, \Delta_2, \\
D_G(S_4) &= \Pi_1, X_{41}, \Pi_2, X_{42} \Rightarrow \Psi_{41}, \Sigma_1, \Psi_{42}, \Sigma_2
\end{align*}
\]

by Lemma 5.5, \([S_3]_{G_{\mathcal{S}_m}} \cap [S_4]_{G_{\mathcal{S}_n}} = \emptyset\), \([S_3]_{G_{\mathcal{S}_m}} \cap [S_4]_{G_{\mathcal{S}_n}} = \emptyset\), \([S_3]_{G_{\mathcal{S}_m}} \cap [S_4]_{G_{\mathcal{S}_n}} = \emptyset\). Then the proof of \( D_{G_m}(S_3) \) is constructed by combining the proofs of \( D_G(S_1) \) and \( D_G(S_2) \) with \( \frac{D_G(S_1)}{D_G(S_3)}, \frac{D_G(S_2)}{D_G(S_4)} \) (COM).

- \( G\vdash G''|G'''(EC_G) \in \tau \). Then \( G', G'' \) and \( G''' \) are closed and \( G'''' \) is a copy of \( G'' \) thus \( D_{G,G''}|G'''(G') = D_{G,G''}|G'''(G'') \) hence a proof of \( D(G|G'') \) is constructed by combining the proof of \( D(G'|G'') \) and \( \frac{D(G'|G'')}{D(G''|G''')} \) (EC*).

The following two lemmas are corollaries of Lemma 5.6.

**Lemma 5.7.** If there exists a derivation of \( G_0 \) from \( G_1, \ldots, G_i \) in \( \text{GL}_{\mathcal{S}_m} \) then there exists a derivation of \( D(G_0) \) from \( D(G_1), \ldots, D(G_i) \) in \( \text{GL} \).

**Lemma 5.8.** Let \( \tau \) be a cut-free proof of \( G_0 \) in \( \text{GL} \) and \( \tau^* \) be the proof of \( G|G' \) in \( \text{GL}_{\mathcal{S}_m} \) resulting from preprocessing of \( \tau \). Then \( \forall_G \mathcal{D}(G|G') \).

### 6. Extraction of Elimination Rules

In this section, we will investigate Construction 4.7 further to extract more derivations from \( \tau^* \).

Any two sequents in a hypersequent seem independent of one another in the sense that they can only be contracted into one by \((EC)\) when it is applicable. Note that one-premise logical rules just modify one sequent of a hypersequent and two-premise rules associate a sequent in a hypersequent with one in a different hypersequent.

\( \tau^* \) (or any proof without \((EC_G)\) in \( \text{GL}_{\mathcal{S}_m} \)) has an essential property, which we call the distinguishability of \( \tau^* \), i.e., any variables, formulas, sequents or hypersequents which occur at the node \( H \) of \( \tau^* \) occur inevitably at \( H' < H \) in some forms.

Let \( H \equiv G|S'|S'' \in \tau^* \). If \( S' \) is equal to \( S'' \) as two sequents then the case that \( \tau_{H,S'} \) is equal to \( \tau_{H,S''} \) as two derivations could possibly happen. This means that both \( S' \) and \( S'' \) are the focus sequent of one node in \( \tau^* \) when \( G^*_{H'} \neq S' \) and \( G^*_{H,H'} \neq S'' \), which contradicts that each node has the unique focus sequent in any derivation. Thus we need differentiate \( S' \) from \( S'' \) for all \( G^*|S'|S'' \in \tau^* \).

Define \( S^\tau \in \tau^* \) such that \( G^*|S'|S'' \in S^\tau, S' \in S^\tau \) and \( S' \) is the principal sequent of \( S^\tau \). If \( S^\tau \) has the unique principal sequent, \( N_{S'} := 0 \), otherwise \( N_{S'} := 1 \) (or \( N_{S'} := 2 \) for another) of such an application as \((COM), (\wedge_m) \) or \((\lor_m)\). Then we may regard \( S' \) as \( (S'; P(S^\tau), N_{S'}) \). Thus \( S' \) is always different from \( S'' \) by \( P(S^\tau) \neq P(S^\beta) \) or, \( P(S^\tau) = P(S^\beta) \) and \( N_{S'} \neq N_{S''} \). We formulate it by the following construction.
Construction 6.1. ([A.5.2]) A labeled tree $\tau^{**}$, which has the same tree structure as $\tau^*$, is constructed as follows.

(i) If $S$ is a leaf $\tau^*$, define $S = S$, $N_S = 0$ and the node $\mathcal{P}(S)$ of $\tau^{**}$ is labeled by $(S; \mathcal{P}(S), N_S)$.

(ii) If $H \equiv G'|S'| \ni (I) \in \tau^*$ and $\mathcal{P}(G'|S')$ be labeled by $G'|((S'; \mathcal{P}(S'), N_S'))$ in $\tau^{**}$. Then define $\overline{S'} = H$, $N_S'' = 0$ and the node $\mathcal{P}(H)$ of $\tau^{**}$ is labeled by $G'|((S''; \mathcal{P}(\overline{S'}), N_S''))$.

(iii) If $H \equiv G'_1|H''$ $(H) \in \tau^*$, $\mathcal{P}(G'_1|S')$ and $\mathcal{P}(G''|S'')$ be labeled by $G'_1|((S'_1; \mathcal{P}(S'_1), N_S'_1))$ and $G''|((S''; \mathcal{P}(\overline{S''}), N_S''))$ in $\tau^{**}$, respectively. If $H'' = S'_1|S_2$ then define $\overline{S'_1} = \overline{S_2} = H$, $N_S'_1 = 1$, $N_S_2 = 2$ and the node $\mathcal{P}(H)$ of $\tau^{**}$ is labeled by $G'_1|(S'_1; \mathcal{P}(\overline{S'_1}), N_S'_1)$(S'_2; \mathcal{P}(\overline{S'_2}), N_S'_2). If $H'' = S_1$ then define $\overline{S'_1} = H, N_S'_1 = 0$ and $\mathcal{P}(H)$ is labeled by $G'_1|(S'_1; \mathcal{P}(\overline{S'_1}), N_S'_1)$.

In the whole paper, we treat $\tau^*$ as $\tau^{**}$ without mention of $\tau^{**}$. Note that in preprocessing $\tau$, some logical applications could also be converted to $(I\mathcal{D}_G)$ in Step 3 and we need to fix the focus sequent at each node $H$ and subsequently assign valid identification numbers to each $H' < H$ by eigenvariable-labeling operation.

Proposition 6.2. (i) $G'|S'|S'' \in \tau^*$ implies $\{S'\} \cap \{S''\} = \emptyset$; (ii) $H \in \tau^*$ and $H'\cap H'' \subseteq H$ imply $H'\cap H'' \leq H$.

Proof. (i) Let $S'_1 \in H$ then $S'_1 = S'_u$ for some $1 \leq u \leq m$, by Notation 14.1. Thus $S'_1 \in H'_1$ also by Notation 14.14. Hence $H \leq S'_1$ and $H'_1 \leq H$ by Construction 6.1. Therefore $H \leq H'_1$ or $H'_1 \leq H$.

Lemma 6.3. Let $H \in \tau^*$ and $\mathcal{P}(H) = (H_0, \cdots, H_n)$, where $H_0 = H, H_n = G|G^*, G_k \subseteq H$ for $1 \leq k \leq 3$.

(i) If $G_1 = G_1|G_2$ then $(H_0|G_1) = (H_0|G_1) \land (H_0|G_2)$ for all $0 \leq i \leq n$;

(ii) If $G_3 = G_1|G_2$ then $(H_0|G_3) = (H_0|G_1) \lor (H_0|G_2)$ for all $0 \leq i \leq n$.

Proof. The proof is by induction on $i$ for $0 \leq i \leq n$. Only (i) is proved as follows and (ii) by a similar procedure and omitted.

For the base step, $(H_0|G_1) = (H_0|G_1) \cap (H_0|G_2)$ holds by $(H_0|G_1) = G_1, (H_0|G_2) = G_2, (H_0|G_3) = G_1$ and $(H_0|G_2) = G_1|G_2$.

For the induction step, suppose that $(H_0|G_1) = (H_0|G_1) \cap (H_0|G_2)$ for some $0 \leq i < n$. Only is the case of one-premise rule given in the following and other cases are omitted.

Let $G'_1|S' \in \tau^*$, $H_i = G'_1|S'$ and $H_{i+1} = G'_1|S''$.

Let $S' \in (H_i|G_i)$, Then $(H_{i+1}|G_{i+1} = ((H_i|G_i) \backslash \{S'\})|S''$, $(H_{i+1}|G_{i+1} = ((H_i|G_i) \backslash \{S'\})|S''$ by $S' \in (H_i|G_i)$. and

$(H_{i+1}|G_{i+1} = (H_i|G_i) \backslash \{S'\})|S''$ by $S' \in (H_i|G_i)$. Thus

$(H_{i+1}|G_{i+1} = (H_{i+1}|G_{i+1}) \cap (H_{i+1}|G_{i+1})$, by $(H_{i+1}|G_{i+1} = (H_i|G_i) \cap (H_i|G_i)$. Let $S' \notin (H_i|G_i)$ and $S' \notin (H_i|G_i)$ and $(H_{i+1}|G_{i+1} = (H_{i+1}|G_{i+1})$, by $(H_{i+1}|G_{i+1} = (H_i|G_i) \cap (H_i|G_i)$. Let $S' \notin (H_i|G_i)$, $(H_{i+1}|G_{i+1} = (H_{i+1}|G_{i+1})$, by $(H_{i+1}|G_{i+1} = (H_i|G_i) \cap (H_i|G_i)$. Let $S' \notin (H_i|G_i)$, $(H_{i+1}|G_{i+1} = (H_{i+1}|G_{i+1})$, by $(H_{i+1}|G_{i+1} = (H_i|G_i) \cap (H_i|G_i)$. Let $S' \notin (H_i|G_i)$, $(H_{i+1}|G_{i+1} = (H_{i+1}|G_{i+1})$, by $(H_{i+1}|G_{i+1} = (H_i|G_i) \cap (H_i|G_i)$. Let $S' \notin (H_i|G_i)$, $(H_{i+1}|G_{i+1} = (H_{i+1}|G_{i+1})$, by $(H_{i+1}|G_{i+1} = (H_i|G_i) \cap (H_i|G_i)$. Let $S' \notin (H_i|G_i)$, $(H_{i+1}|G_{i+1} = (H_{i+1}|G_{i+1})$, by $(H_{i+1}|G_{i+1} = (H_i|G_i) \cap (H_i|G_i)$. Let $S' \notin (H_i|G_i)$, $(H_{i+1}|G_{i+1} = (H_{i+1}|G_{i+1})$, by $(H_{i+1}|G_{i+1} = (H_i|G_i) \cap (H_i|G_i)$.
Lemma 6.4. (i) Let $G[S'] \in \tau^*$ then $G^*_{G[S']} \cap G^*_{G[S']G^*} = \emptyset$, $G^*_{G[S']G^*} = G[G^*]$; (ii) $H \in \tau^*$, $H'H'' \subseteq H$ then $G^*_{H'H''} = G^*_{H'H''}$. 

Proof. (i) and (ii) are immediately from Lemma 6.3.

Note 6.5. We write $\tau^*_{H_i}G_{i}^{S_{i}^{c}}$, $\tau^*_{S_{i}^{c}}G_{i}^{S_{i}^{c}}$, respectively, for the sake of simplicity.

Lemma 6.6. (i) $G[S_i^c] \subseteq GG^*$; 

(ii) $\tau^*_{S_i^c}$ is a derivation of $G[S_i^c]$ from $S_i^c$, which we denote by $S_i^c \xrightarrow{G[S_i^c]} \{\tau^*_{S_i^c}\}$; 

(iii) $G[S_i^c] = S_i^c$ and $\tau^*_{S_i^c}$ consists of a single node $S_i^c$ for all $2 \leq u \leq m$; 

(iv) $v_i(G[S_i^c]) = v_i(G[S_i^c])\{S_i^c\}$; 

(v) $\langle H \rangle_{S_i^c} \in \tau^*_{S_i^c}$ implies $H \subseteq H_f$. Note that $\langle H \rangle_{S_i^c}$ is undefined for any $H > H_i$ or $H_i$. 

(vi) $S_j^c \in G[S_i^c]$ implies $H_f \notin H_i$. 

Proof. Claims from (i) to (v) are immediately from Construction 4.7 and Lemma 4.8.

(vi) Since $S_j^c \subseteq G[S_i^c] \subseteq G[H]^*$ then $S_j^c$ has the form $S_j^c$ for some $2 \geq 2$ by Notation 4.14. Then $G[S_j^c] = S_j^c$ by (iii). Suppose that $H_f^c \notin H_f$. Then $S_j^c$ is transferred from $H_f^c$ downward to $H_f^c$ and in side-hydersequent of $H_f^c$ by Notation 4.14 and $G[H]^* < H_f^c \subseteq H_f^c$. Thus $\{S_j^c\} \cap \{S_j^c\} = \emptyset$ at $H_f^c$ since $S_i^c$ is the unique focus sequent of $H_f^c$. Hence $S_j^c \notin G[S_i^c]$ by Lemma 6.3 and (iii), a contradiction therefore $H_f^c \notin H_f^c$.

Lemma 6.7. Let $G[S'] \xrightarrow{G^*[S]} H \subseteq G[G^*[S]] \xrightarrow{(II)} \tau^*$. (i) If $S_j^c \in G[H^c]$, then $H_f^c \subseteq H$ or $H_f^c \subseteq H$; (ii) If $S_j^c \in G[H^c]$, then $H_f^c \subseteq H$ or $H_f^c \subseteq G[S']$. 

Proof. (i) We impose a restriction on (II) such that each sequent in $H'$ is different from $S'$ or $S''$ otherwise we treat it as an (EW)-application. Since $S_j^c \subseteq G[H^c] \subseteq G[H^c] \subseteq S_j^c$ has the form $S_j^c$ for some $u \geq 2$ by Notation 4.14. Thus $G[S_j^c] = S_j^c$. Suppose that $H_f^c > H$. Then $S_j^c$ is transferred from $H_f^c$ downward to $H$. Thus $S_j^c \subseteq H'$ by $G[S_j^c] = S_j^c \subseteq G[H^c]$ and Lemma 6.3. Hence $S_j^c = S_j$ or $S_j^c = S''$, a contradiction with the restriction above. Therefore $H_f^c \subseteq H$ or $H_f^c \subseteq H$. 

(ii) Let $S_j^c \subseteq G[H^c]$. If $H_f^c > H$ then $S_j^c \subseteq H$ by Proposition 4.15(i) and thus $S_j^c \subseteq G[H]$. 

By Lemma 6.3, hence $H_f^c \subseteq G[S']$ by $H_f^c \subseteq G[S'] \subseteq G[S'] \subseteq G[S']$. If $H_f^c \subseteq H$ then $H_f^c \subseteq G[S']$. 

Thus $H_f^c \subseteq H$ or $H_f^c \subseteq G[S']$. 

Definition 6.8. (i) By $H_f^c \sim H_f^c$ we mean that $S_j^c \subseteq G[S_i^c]$ for some $2 \leq u \leq m_j$; (ii) By $H_f^c \leftrightarrow H_f^c$ we mean that $H_f^c \sim H_f^c$ and $H_f^c \sim H_f^c$; (iii) $H_f^c \rightarrow H_f^c$ means that $S_j^c \subseteq G[S_i^c]$ for all $2 \leq u \leq m_j$.

Then Lemma 6.6 (vi) shows that $H_f^c \rightarrow H_f^c$ implies $H_f^c \subseteq H_f^c$.

Lemma 6.9. Let $H_f^c \in H_f^c \rightarrow H_f^c \xrightarrow{G[S'] \cap G[S')} \tau^*$ such that $G[S'] \subseteq H_f^c$, $G[S'] \subseteq H_f^c$. 

Then $S' \notin \langle G[S'] \rangle_{S_i^c}$. 

Proof. Suppose that $S' \notin \langle G[S'] \rangle_{S_i^c}$. Then $\langle G[S'] \rangle_{S_i^c} \subseteq G[S'] \subseteq \langle G[S'] \rangle_{S_i^c} \subseteq G[S'] \subseteq \langle G[S'] \rangle_{S_i^c} = \langle G[S'] \rangle_{S_i^c}$ by Construction 4.7. Thus $\langle G[S'] \rangle_{S_i^c} \subseteq G[S']$. Hence $G[S'] \subseteq H_f^c \cap \{G[S'] \rangle_{S_i^c} = \emptyset$ by Proposition 6.2 (ii). Therefore $S_j^c \notin G[S_i^c]$ for all $2 \leq u \leq m_j$ by Lemma 6.3, i.e., $H_f^c \sim H_f^c$, a contradiction and hence $S' \notin \langle G[S'] \rangle_{S_i^c}$. 

28
Lemma 6.6 (ii) shows that $\tau_{S_0}^*$ is a derivation of $G*_{S_0}$ from one premise $S_{0i}$. We generalize it by introducing derivations from multiple premises in the following. In the remainder of this section, let $I = \{H^0_{i}, \ldots, H^m_{i}\} \subseteq \{H^0_{j}, \ldots, H^m_{j}\}$, $H^k_{i} \leftrightarrow H^k_{j}$ for all $1 \leq k < q \leq m$. Then $H^q_{i} \not\leftrightarrow H^q_{j}$ and $H^q_{i} \not\leftrightarrow H^q_{j}$ by Lemma 6.6 (vi) thus $H^q_{i} \mid H^q_{j}$ for all $1 \leq k < q \leq m$.

**Notation 6.10.** $H^q_j$ denotes the intersection node of $H^c_{i_1}, \ldots, H^c_{i_q}$. We sometimes write the intersection node of $H^q_i$ and $H^q_j$ as $H^q_{ij}$. If $I = \{H^q_i\}$, $H^q_i \Rightarrow H^q_j$, i.e., the intersection node of a single node is itself.

Let \( G^i|S'^i \mid G'^i|S'^i \mid (II) \in \tau^* \) such that $G^i|G'^i|H^j = H^j_i$. Then $I$ is divided into two subsets $I_1 = \{H^0_{i_1}, \ldots, H^m_{i_1}\}$ and $I_2 = \{H^0_{i_2}, \ldots, H^m_{i_2}\}$, which occur in the left subtree $\tau^*(G^i|S'^i)$ and right subtree $\tau^*(G'^i|S'^i)$, respectively.

Let $I = \{S^c_{i_1}, \ldots, S^c_{i_q}\}$, $I_1 = \{S^c_{i_1}, \ldots, S^c_{i_{q_1}}\}$, $I_2 = \{S^c_{i_{q_1+1}}, \ldots, S^c_{i_{q_1+q_2}}\}$ such that $I = I_1 \cup I_2$. A derivation $\tau^*_2$ of $\langle G|G^i \rangle_{I_2}$ from $S^c_{i_1}, \ldots, S^c_{i_{q_1+q_2}}$ is constructed by induction on $|I|$. The base case of $|I| = 1$ has been done by Construction 4.7. For the induction case, suppose that derivations $\tau^*_2$ of $\langle G|G^i \rangle_{I_2}$ from $S^c_{i_1}, \ldots, S^c_{i_{q_1+1}}$ and $\tau^*_2$ of $\langle G|G^i \rangle_{I_2}$ from $S^c_{i_{q_1+1}}, \ldots, S^c_{i_{q_1+q_2}}$ are constructed. Then $\tau^*_2$ of $\langle G|G^i \rangle_{I_2}$ from $S^c_{i_1}, \ldots, S^c_{i_{q_1+q_2}}$ is constructed as follows.

**Construction 6.11.** ([A.5.2]) (i) 
\[
\langle H \rangle_{I_1} := \langle H \rangle_{I_2} \quad \text{for all } G^i|S' \mid H \mid H^j_i \text{ for some } H^j_i \in I_1, \\
\langle H \rangle_{I_2} := \langle H \rangle_{I_2} \quad \text{for all } G'^i|S'' \mid H \mid H^j_i \text{ for some } H^j_i \in I_2, \\
\tau^*_2(\langle G^i|S'^i \rangle_{I_1}) := \tau^*_2(\langle G^i|S'^i \rangle_{I_2}), \\
\tau^*_2(\langle G'^i|S''^i \rangle_{I_2}) := \tau^*_2(\langle G'^i|S''^i \rangle_{I_1});
\]
and 
\[
\frac{\langle G^i|G'^i \rangle_{I_2}}{\langle G^i|G'^i \rangle_{I_2}} (II) \in \tau^*_2;
\]
(iii) Other nodes of $\tau^*_2$ are built up by Construction 4.7 (ii).

The following lemma is a generalization of Lemma 6.6.

**Lemma 6.12.** Let $\text{Th}(H^c_{i_0}) = \{H^c_{i_0}, \ldots, H^c_{i_{n_q}}\}$, where $1 \leq k \leq m$, $H^c_{i_0} = H^c_i$ and $H^c_{i_{n_q}} = G|G^*$. Then, for all $0 \leq u \leq n_q$, 
(i) 
\[
\langle H^c_{i_0} \rangle_{I} \cap \{\langle H^c_{i_u} \rangle_{S^c_j} : H^j_j \in I, H^c_{i_u} \leq H^j_j \};
\]
(ii) 
\[
\langle S^c_j \rangle_{I} \cap \{\langle H^c_{i_u} \rangle_{S^c_j} : H^j_j \in I, H^c_{i_u} \leq H^j_j \};
\]
(iii) 
\[
\nu_r(\langle H^c_{i_0} \rangle_{I}) \cup \nu_r(\langle S^c_j \rangle_{I}) : H^j_j \in I, H^c_{i_u} \leq H^j_j \};
\]
(iv) $\langle H \rangle_{I} \in \tau^*_2$ if and only if $H \leq H^j_j$ for some $H^j_j \in I$. Note that $\langle H \rangle_{I}$ is undefined if $H > H^j_j$ or $H \parallel H^j_j$ for all $H^j_j \in I$. 

29
We will use all hypothesis then comes from \( v \) and \( \varepsilon \).

**Remark**

**Definition 6.14.**

\( \varepsilon \) holds by a procedure similar to above then

\[
\langle G\mid G''\mid H' \rangle_x = \langle G''\mid H' \rangle_x \cap \langle G\mid H' \rangle_x
\]

by \( \langle G\rangle_x \subseteq G' \) and \( \langle G''\rangle_x \subseteq G'' \). Other claims hold immediately from Construction 6.11.

**Lemma 6.13.** (i) Let \( G_z \) denote \( \langle G\rangle_x \) then \( G_z = \cap_{H_I \in H} G_{S_{g_j}} \); (ii) \( S_{x_{i_1}} \cdots S_{x_{m_k}} (\tau^2_x) \); (iii) \( v(\varepsilon_z) \cup v(S_{j_1}) = v(\varepsilon) \cup v(S_{j_1}) \); (iv) \( S^\prime_j \in G_z^\prime \) implies \( H_I \notin H' \) for all \( H'_I \in I \).

**Proof.** (i), (ii) and (iii) are immediately from Lemma 6.12. (iv) holds by (i) and Lemma 6.6 (vi).

Lemma 6.13 (iv) shows that there exists no copy of \( S^\prime_j \) in \( G_z \) for any \( 1 \leq k \leq m \). Then we may regard them to be eliminated in \( \tau^2_x \). We then call \( \tau^2_x \) an elimination derivation.

Let \( I' = \langle G_{b_1} \rangle_{S_{i_{1_k}}} \cdots G_{b_m} \rangle_{S_{l_{m_k}}} \) be another set of sequents to \( I \) such that \( G' = S_{i_{1_k}} \cdots S_{l_{m_k}} \) is a copy of \( G'' = S_{c_{i_{1_k}}} \cdots S_{c_{l_{m_k}}} \). Then \( G' \) and \( G'' \) are disjoint and there exist two bijections \( \sigma_l : v(\varepsilon) \rightarrow v(\varepsilon') \) and \( \sigma_r : v(\varepsilon') \rightarrow v(\varepsilon) \) such that \( \sigma_l \circ \sigma_r (G) = G'' \). By applying \( \sigma_r \circ \sigma_l \) to \( \tau^2_x \), we construct a derivation from \( S_{i_{1_k}} \cdots S_{l_{m_k}} \) and denote it by \( \tau^2_x \) and its root by \( G^*_z \).

Let \( I' = \langle G_{b_1} \rangle_{S_{i_{1_k}}} \cdots G_{b_m} \rangle_{S_{l_{m_k}}} \) be a set of hypersequents to \( I \), where \( G_{b_1} \rangle_{S_{i_{1_k}}} \) is closed for all \( 1 \leq k \leq m \). By applying \( \tau^2_x \) to \( S_{i_{1_k}} \cdots S_{l_{m_k}} \) in \( G_{b_1} \rangle_{S_{i_{1_k}}} \cdots G_{b_m} \rangle_{S_{l_{m_k}}} \), we construct a derivation from \( G_{b_1} \rangle_{S_{i_{1_k}}} \cdots G_{b_m} \rangle_{S_{l_{m_k}}} \) and denote it by \( \tau^2_x \) and its root by \( G^*_z \). Then \( G^*_z = \langle G_{b_1} \rangle_{S_{i_{1_k}}} \cdots G_{b_m} \rangle_{S_{l_{m_k}}} \).

**Definition 6.14.** We will use all \( \tau^2_x \) as rules of \( GL_\Omega \) and call them elimination rules. Further, we call \( S_{i_{1_k}} \cdots S_{l_{m_k}} \) focus sequents and, all sequents in \( G_z^* \), principal sequents and, \( G_{b_1} \rangle_{S_{i_{1_k}}} \cdots G_{b_m} \rangle_{S_{l_{m_k}}} \) side-hypersequents of \( \tau^2_x \).

**Remark 6.15.** We regard Construction 4.7 as a procedure \( F \), whose inputs are \( \tau^2, H, H' \) and output \( \tau^2_{H,H'} \). With such a viewpoint, we write \( \tau^2_{H,H'} \) as \( F(\tau^2, H, H') \). Then \( \tau^2_x \) can be constructed by iteratively applying \( F \) to \( \tau^* \), i.e., \( \tau^2_x = F(\varepsilon_x, \varepsilon'_{x}) \cdots F(\varepsilon_x, \varepsilon'_{x}) \tau^* \).

We replace locally each \( \frac{G'}{G''} (ID_\Omega) \) in \( \tau^2_x \) with \( G' \) and denote the resulting derivation also by \( \tau^2_x \). Then each non-root node in \( \tau^2_x \) has the focus sequent.

Let \( H \in \tau^2_x \). Then there exists a unique node in \( \tau^* \), which we denote by \( O(H) \) such that \( H \) comes from \( O(H) \) by Construction 4.7 and 6.11. Then the focus sequent of \( O(H) \) in \( \tau^* \) is the focus of \( H \) in \( \tau^2_x \) if \( H \) is a non-root node and, \( O(H) = H \) or \( H \leq O(H) \) as two hypersequents.
Since the relative position of any two nodes in $\tau^*$ keep unchanged in constructing $\tau^*_Z$, $H_1 \leq_{\tau^*_Z} H_2$ if and only if $O(H_1) \subseteq_{\tau^*_Z} O(H_2)$ for any $H_1, H_2 \in \tau^*_Z$. Especially, $O(S_{i_A}^{\leq}) = H_{i_A}^c$ for $S_{i_A}^{\leq} \in \tau^*_Z$.

Let $H \in \tau^*_Z$. Then $H' \equiv \sigma_r o \sigma_l(H) \in \tau^*_Z$ and $H'' \equiv \{G_{i_{h_k}} : H \leq_{\tau^*_Z} S_{i_{h_k}}^{\leq} \text{and} 1 \leq k \leq m \} \mid H' \in \tau^*_Z$. Define $O(H'') = O(H') = O(H)$. Then $O(G_{i_{h_k}}^c) = G_{i_{h_k}}^c G^*$ and $O(G_{i_{h_k}}|S_{i_{m_k}}^{\leq}) = H_{i_{m_k}}^c$ for all $G_{i_{h_k}}|S_{i_{m_k}}^{\leq} \in \tau^*_Z$.

Since $G_{i_{h_k}}^c = (G|G^*)_{i_{h_k}} \subseteq G^*$, then each $(pEC)$-sequent in $G_{i_{h_k}}^c$ has the form $S_{j_{v_k}}^{\leq}$ for some $1 \leq j \leq N, 2 \leq v \leq m_j$ by Proposition 4.15(ii). Then we introduce the following definition.

**Definition 6.16.** (i) By $S_{j_{v_k}}^{\leq} \in G_{i_{h_k}}^c$ we means that there exists $H \in \tau^*_Z$ such that $S_{j_{v_k}}^{\leq} \in H, O(H) = H_j^c$.

(ii) Let $S_{j_{v_k}}^{\leq} \in G_{i_{h_k}}^c$. By $H_j^c \leq_{\tau^*_Z} H_{i_{h_k}}^c$ we means that there exist $H, H' \in \tau^*_Z$ such that $S_{j_{v_k}}^{\leq} \in H$, $O(H) = H_j^c, O(H') = H_j^c$ and $H_j^c \leq_{\tau^*_Z} H_{i_{h_k}}^c$. We usually write $\leq_{\tau^*_Z}$ as $\leq$.

7. Separation of one branch

In the remainder of this paper, we assume that $p$ occur at most one time for each sequent in $G_0$ as the one in Main theorem, $\tau$ be a cut-free proof of $G_0$ in GL and $\tau^*$ the proof of $G^*$ in $G_{i_{h_k}}^c$ resulting from preprocessing of $\tau$. Then $|v_i(S)| + |v_i(S)| \leq 1$ for all $S \in G$, which plays a key role in discussing the separation of branches.

**Definition 7.1.** By $S' \subseteq G'$ we mean that there exists some copy of $S'$ in $G'$. $G' \subseteq_{c} G''$ if $S' \subseteq G'$ for all $S' \subseteq G'$. $G' =_{c} G''$ if $G' \subseteq_{c} G''$ and $G'' \subseteq_{c} G'$. Let $G_1, \ldots, G_m$ be $m$ copies of $G_1$ then we denote $G[G_1, \ldots, G_m]$ by $G_1^{m}$. We will denote $S_1, \ldots, S_m$ for all $1 \leq k \leq m$. $\{S_{j_{0}}^{\leq}\}$ is called a branch of $H_{i_{0}}^c$ to $I$ if it is a closed hypersequent such that

(i) $\{S_{j_{0}}^{\leq}\} \subseteq_{c} G^*$.

(ii) $S_{j_{0}}^{\leq} \in \{S_{j_{0}}^{\leq}\}$.

(iii) $S_{j_{0}}^{\leq} \in \{S_{j_{0}}^{\leq}\}$ implies $H_{j_{0}}^c \leq H_{i_{0}}^c$ or $H_{j_{0}}^c \leq H_{i_{0}}^c$ for all $H_{i_{0}}^c \in I$.

Then (i) $S_{j_{0}}^{\leq} \in \{S_{j_{0}}^{\leq}\}$ for all $1 \leq k, l \leq m, k \neq l$; (ii) $S_{j_{0}}^{\leq} \in \{S_{j_{0}}^{\leq}\}$ and $H_{j_{0}}^c \leq H_{i_{0}}^c$ imply $H_{j_{0}}^c \in I$.

In this section, let $I = \{H_j^c\}, I = \{S_{j_{0}}^{\leq}\}$, we will give an algorithm to eliminate all $S_{j_{0}}^{\leq} \in \{S_{j_{0}}^{\leq}\}$ satisfying $H_{j_{0}}^c \leq H_{i_{0}}^c$.

**Construction 7.3.** ([A.3]) A sequence of hypersequents $G_{1}^{(q)}$ and their derivations $\tau_{1}^{(q)}$ from $\{S_{j_{0}}^{\leq}\}$ for all $q \geq 0$ are constructed inductively as follows.

For the base case, define $G_{1}^{(0)}$ to be $\{S_{j_{0}}^{\leq}\}$ and, $\tau_{1}^{(0)}$ be $\overline{G_{1}^{(0)}}$. For the induction case, suppose that $\tau_{1}^{(q)}$ and $G_{1}^{(q)}$ are constructed for some $0 \leq q$. If there exists no $S_{j_{0}}^{\leq} \in G_{1}^{(q)}$ such that $H_j^c \leq H_{i_{0}}^c$, then the procedure terminates and define $J_{1}$ to be $q$, otherwise define $H_{i_{0}}^c$ such that $S_{j_{0}}^{\leq} \in G_{1}^{(q)}, H_{j_{0}}^c \leq H_{i_{0}}^c$ and $H_{j_{0}}^c \leq H_{i_{0}}^c$ for all $S_{j_{0}}^{\leq} \in G_{1}^{(q)}, H_j^c \leq H_{i_{0}}^c$. Let $S_{j_{0}1}^{\leq}, \ldots, S_{j_{0}m_{q}}^{\leq}$ be all copies of $S_{j_{0}}^{\leq}$ in $G_{1}^{(q)}$ then define $G_{1}^{(q+1)} = G_{1}^{(q)} \setminus \{S_{j_{u}1}^{\leq}, \ldots, S_{j_{u}m_{q}}^{\leq}\}$ and its derivation $\tau_{1}^{(q+1)}$ is constructed by sequentially applying $\tau_{j_{u}1}^{(q)}, \ldots, \tau_{j_{u}m_{q}}^{(q)}$ to $S_{j_{u}1}^{\leq}, \ldots, S_{j_{u}m_{q}}^{\leq}$ in $G_{1}^{(q)}$, respectively. Notice that we assign new identification numbers to new occurrences of $p$ in $\tau_{j_{u}1}^{(q)}$ for all $0 \leq q < J_{1} - 1, 1 \leq u \leq m_{q}$.
Lemma 7.4. (i) $H_q^c = H_q^f$ and $H_{q+1}^c < H_q^c$ for all $0 \leq q < J_1 - 2$;

(ii) $G_{1}^{(q)}(c) \subseteq G_{1}^{(q)}(G^*)$ is closed for all $0 \leq q < J_1$;

(iii) $\frac{[S^c]}{G_{1}^{(q)}(G^*)} = \frac{[S^c]}{G_{1}^{(q)}(G^*)}$ for all $0 \leq q < J_1$, especially, $\frac{[S^c]}{G_{1}^{(q)}(G^*)} = \frac{[S^c]}{G_{1}^{(q)}(G^*)}$;

(iv) $S^c_i \in G_{1}^{(q)}(c)$ implies $H_q^c \parallel H_q^c$ and $S^c_i \in G_{1}^{(q)}(c)$ for some $\tau_{G_1(S^c_i)}^* \in \tau_{G_1(S^c_i)}^*$ or $S^c_i \in \{S^c_i\}$.

Proof. (i) Since $S^c_i \in G_{1}^{(q)}(c)$ by $S^c_i \in \{S^c_i\}$, then $H_q^c < H_q^c$ for all $S^c_i \in G_{1}^{(q)}(c)$.

(ii) Suppose that $v_i(G_{1}^{(q)}(c)) = v_i(G_{1}^{(q)}) \subseteq c G^*$. Then $\tau_{G_1(S^c_i)}^* \in \tau_{G_1(S^c_i)}^*$ and $\tau_{G_1(S^c_i)}^* \subseteq c G^*$.

(iii) $\tau_{G_1(S^c_i)}^* \subseteq \tau_{G_1(S^c_i)}^*$ by Lemma 6.6 (iv) then $\tau_{G_1(S^c_i)}^* < \tau_{G_1(S^c_i)}^*$ by $\tau_{G_1(S^c_i)}^* \subseteq \tau_{G_1(S^c_i)}^*$. Then $\tau_{G_1(S^c_i)}^* < \tau_{G_1(S^c_i)}^*$ by $G_{1}^{(q+1)} = G_{1}^{(q)} \cap \{S^c_i \mid \tau_{G_1(S^c_i)}^* \subseteq \tau_{G_1(S^c_i)}^* \}$.

(iv) Let $S^c_i \in G_{1}^{(q)}(c)$. Then $H_q^c \parallel H_q^c$ by the definition of $J_1$. If $S^c_i \in \{S^c_i\}$, then $H_q^c \parallel H_q^c$ by $H_q^c \parallel H_q^c$ and the definition of $\{S^c_i\}$. Otherwise, by Construction 7.3, there exists some $\tau_{G_1(S^c_i)}^*$. 


in $\tau_1^{\cal A}$ such that $S_j^c \in G_{\cal I_{\cal E}^*}$. Then $H^c_j \not\subset H_j$ by Lemma 6.6 (vi). Thus $H^c_j \not\subset H_j$ by $H_{\cal I_{\cal E}^*} \not\subset H_{\cal I}$. Hence $H_j^c \not\subset H_j^c$.

Lemma 7.4 shows that Construction 7.3 presents a derivation $\tau_1^{\cal A}$ of $G_{\cal I_{\cal E}^*}$ from $\mathcal{S}_{1,\cal I}^*$, such that there doesn’t exist $S_j^c \in G_{\cal I_{\cal E}^*}$ satisfying $H^c_j \not\subset H_j$, i.e., all $S_j^c \in \mathcal{S}_{1,\cal I}^*$ satisfying $H^c_j \not\subset H_j$ are eliminated by Construction 7.3. We generalize this procedure as follows.

**Construction 7.5.** Let $H \in \tau^*$, $H_1 \leq H$ and $H_2 \leq G|G^*$. Then $G_{\cal I_{\cal E}^*}$ and its derivation $\tau_{H:H_2}^{\cal A}$ for $l = 1, 2$ are constructed by procedures similar to that of Construction 7.3 such that $H^c_j \not\subset H$ for all $S_j^c \in G_{\cal I_{\cal E}^*}$, where $G_{\cal I_{\cal E}^*} = G_{\cal I_{\cal E}^*}$, $\tau_{H:H_2}^{\cal A}$, which are defined by Construction 4.7.

We sometimes write $I_{\cal I_{\cal E}^*}$, $J_{\cal I_{\cal E}^*}$ as $J$ for simplicity. Then the following lemma holds clearly.

**Lemma 7.6.** (i) $H_j \not\subset H$ for all $S_j^c \in G_{\cal I_{\cal E}^*}$.

(ii) If $S_j^c \in H$ and $H_{\cal I_{\cal E}^*} \not\subset H$ then $G_{\cal I_{\cal E}^*} = S_j^c$.

(iii) If $S \in G$ or $S \in G^*$ is a copy of $S_{\cal I_{\cal E}^*}$ and $H_j \not\subset H$ then $G_{\cal I_{\cal E}^*} = S$.

(iv) Let $H|H^* \not\subset H$. Then $G_{\cal I_{\cal E}^*} = G_{\cal I_{\cal E}^*}$ by suitable assignments of identification numbers to new occurrences of $\cal P$ in constructing $\tau_1^{\cal A}$, $\tau_{H:H_2}^{\cal A}$ and $\tau_{H:H_2}^{\cal A}$.

(v) $G_{\cal I_{\cal E}^*} = \bigcup\{G_{\cal I_{\cal E}^*} \mid S_j^c \in \mathcal{S}_{j,\cal I}^*, H_j \not\subset H_j\} \bigcup\{S_j^c \in \mathcal{S}_{j,\cal I}^*, H_j \not\subset H_j\} \bigcup\{S \in \mathcal{S}_{j,\cal I}^*, H_j \not\subset H_j\}$.

*Proof.* (i) is proved by a procedure similar to that of Lemma 7.4 (iii), (iv) and omitted.

(ii) Since $S_{\cal I_{\cal E}^*}$ is the focus sequent of $H_j$ then it is revised by some rule at the node lower than $H_j$. Thus $S_j^c \in H$ is some copy of $S_{\cal I_{\cal E}^*}$ by $H_j \not\subset H$. Hence $S_j^c$ has the form $S_{\cal I_{\cal E}^*}$ for some $\alpha \geq 2$. Therefore it is transferred downward to $G|G^*$, i.e., $S_j^c \in G|G^*$. Then $G_{\cal I_{\cal E}^*} = G_{\cal I_{\cal E}^*} = S_j^c$. Since there exists no $S_j^c \in G_{\cal I_{\cal E}^*}$, $H_j \not\subset H$ then $J = 0$. Thus $G_{\cal I_{\cal E}^*} = S_j^c$.

(iii) is proved by a procedure similar to that of (ii) and omitted.

(iv) Since $H_j^*|H^* \not\subset H$, then $H_j^* \not\subset H^*$. Hence $H_j^* \not\subset H^* \not\subset H$. Then $G_{\cal I_{\cal E}^*} = G_{\cal I_{\cal E}^*}$ by Proposition 6.2. Thus $G_{\cal I_{\cal E}^*} = G_{\cal I_{\cal E}^*}$ and $G_{\cal I_{\cal E}^*} = G_{\cal I_{\cal E}^*}$. Suppose that $G_{\cal I_{\cal E}^*} = G_{\cal I_{\cal E}^*}$ for some $\alpha \geq 0$. Then all copies $\{S_{\alpha}^c \}_{\alpha=1}^{m_\alpha}$ in $G_{\cal I_{\cal E}^*}$ are divided into two subsets $\{S_{\alpha}^c \}_{\alpha=1}^{m_\alpha} \cap G_{\cal I_{\cal E}^*}$ and $\{S_{\alpha}^c \}_{\alpha=1}^{m_\alpha} \cap G_{\cal I_{\cal E}^*}$. Thus we can construct $G_{\cal I_{\cal E}^*}$, $G_{\cal I_{\cal E}^*}$ and $G_{\cal I_{\cal E}^*}$ simultaneously and assign the same identification numbers to new occurrences of $\cal P$ in $G_{\cal I_{\cal E}^*}$ and $G_{\cal I_{\cal E}^*}$ as the corresponding one in $G_{\cal I_{\cal E}^*}$. Hence $G_{\cal I_{\cal E}^*} = G_{\cal I_{\cal E}^*}$. Then $G_{\cal I_{\cal E}^*} = G_{\cal I_{\cal E}^*}$ and $G_{\cal I_{\cal E}^*}$.

Note that the requirement is imposed only on one derivation that distinct occurrence of $p$ has distinct identification number. We permit $G_{\cal I_{\cal E}^*} = G_{\cal I_{\cal E}^*}$ or $G_{\cal I_{\cal E}^*} = G_{\cal I_{\cal E}^*}$ in the proof above, which has no essential effect on the proof of the claim.

(v) is immediately from (iv).
Lemma 7.6(v) shows that \( G_1^{\ast(j)} \) could be constructed by applying \( \tau_{H}^{\ast(j)} \) sequentially to each \( S_{\ast}^j \in [S_{\ast}]_l \) satisfying \( H_{\ast}^j \leq H_{\ast}^i \). Thus the requirement \( H_{\ast+1}^j < H_{\ast+1}^i \) in Construction 7.3 is not necessary, but which make the termination of the procedure obvious.

**Construction 7.7.** Apply \( (EC_\Omega^1) \) to \( G_1^{\ast(j)} \) and denote the resulting hypersequent by \( G_1^{\ast(j)} \) and its derivation by \( \tau_1^{\ast(j)} \). It is possible that \( (EC_\Omega^1) \) is not applicable to \( G_1^{\ast(j)} \) in which case we apply \( (ID_\Omega) \) to it for the regularity of the derivation.

**Lemma 7.8.**

(i) \( \frac{[S_{\ast}^j]}{G_1^{\ast(j)}}(\tau_1^{\ast(j)}), G_1^{\ast(j)} \) is closed and \( H_{\ast}^j \leq H_{\ast}^i \) for all \( S_{\ast}^j \in G_1^{\ast(j)} \);

(ii) \( \tau_1^{\ast(j)} \) is constructed by applying elimination rules, say, \( \frac{G_b[S_{\ast}^j]}{G_1[G_{\ast-1}]}(\tau_1^{\ast(j)} G_b[S_{\ast}^j]), \) and the fully constrained contraction rules, say, \( \frac{G_1(G_{\ast-1})}{G_{\ast-1}}(EC_\Omega^1), \) where \( H_{\ast}^j \leq H_{\ast}^i, G_{\ast}S_{\ast}^j \) is closed for \( 0 \leq q \leq J - 1, 1 \leq u \leq m_{\ast} \).

**Proof.** Immediately from Lemma 7.4.

**Definition 7.9.** Let \( G' \in G_1^{\ast(j)}(H' \subseteq G' \land S' \in H') \). (i) For any sequent-formula \( A \) of \( S' \), define \( \Lambda \) to be the sequent \( S' \in S_1 \) such that \( A \) is a sequent-formula of \( S_1 \) or subformula of a sequent-formula of \( S_1 \). (ii) Let \( S' \) be in the form \( A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m \). Define \( \tilde{S}' \) to be the hypersequent which consists of all distinct sequents among \( \tilde{A}_1, \ldots, \tilde{A}_n, \tilde{B}_1, \ldots, \tilde{B}_m \); (iii) Let \( H' \) be in the form \( S_{\ast} \mid S_{\ast} \), define \( \tilde{H}' \) to be \( S_{\ast} \mid S_{\ast} \), \( \ast \) We call \( H' \) to be separable if \( \tilde{H}' \subseteq_c G \) and, call it to be separated into \( \tilde{H}' \) and \( \ast \) Note that \( \tau_1^{\ast(j)} \) is a derivation without \( (EC_\Omega) \) in \( GL_\Omega \). Then we can extract elimination derivations from it by Construction 4.7.

**Notation 7.10.** Let \( H' \subseteq G' \in \tau_1^{\ast(j)} \). \( \tau_1^{\ast(j)}(H') \) denotes the derivation from \( H' \), which extracts from \( \tau_1^{\ast(j)}(H') \) by Construction 4.7, and denote its root by \( G_1^{\ast(j)}(H') \).

The following two lemmas show that Construction 7.3 and 7.5 force some sequents in \( [S_{\ast}]_l \) or \( H' \) to be separable.

**Lemma 7.11.** Let \( G[S_{\ast}^j] \subseteq G'[S_{\ast}^j] \subseteq H' \subseteq \tau^* \). Then (i) \( H' \) is separable in \( \tau_{H}^{\ast(j)} \): \( \frac{G[H][G''][H]}{G_b[H][G''][H]}(\tau_{H}^{\ast(j)} G_b[H][G''][H]) \). (ii) If \( \frac{G[H] [H] [H]}{G_b[H][G''][H]}(\tau_{H}^{\ast(j)} G_b[H][G''][H]) \), then \( H' \) is separable in \( \tau_1^{\ast(j)}(H') \) and there is a unique copy of \( \tilde{S}'(G_1^{\ast(j)}(H';G'')) \) in \( G_1^{\ast(j)} \).

**Proof.** (i) We write \( \leq_{H;H'} \) and \( \leq_{\ast} \) respectively as \( \leq_{\ast} \) and \( \leq_{\ast} \) for simplicity. Since \( G_1^{\ast(j)}(H';G'') \leq_{\ast} G[G^*] \), we divide it into two hypersequents \( G_1^{\ast(j)}(H') \) and \( G_1^{\ast(j)}(H') \) such that \( G_1^{\ast(j)}(H';G'') \leq_{\ast} G, G_1^{\ast(j)}(H';G'') \leq_{\ast} G^* \).

34
Let $S_j^c \in G_{H, H'}^{\ast (J)}$ then $H_j^c \not\in H$ by Construction 7.5. We prove that $H_j^c || H'$ in $\tau_{H, H'}^{\ast (J)}$ as follows. If $S_j^c \in G_{H, H'}^{\ast (J)}$ then $H_j^c || H'$ by Lemma 6.7(i), $\tau_{H, H'}^{\ast (J)} || H_j^c$ and $H_j^c \not\in H$. Thus we assume that $S_j^c \not\in G_{H, H'}$. In the following.

Then, by Lemma 7.4(iv), there exists some $\tau_{G_n, H, H'}^{\ast (J)}$ in $\tau_{H, H'}^{\ast (J)}$ such that $H_j^c \not\in H$, $S_j^c \in G_{H, H'}^{\ast (J)}$. Then $H_j^c \not\in H_j^c$ by Lemma 6.6(vi). $H_j^c \not\in H_j^c$ in $H$, $H_j^c \not\in H$. Thus $H_j^c || H_j^c$. Let $G_n|H_1 = G_n|H_2 \in \tau^*$, where $G_n|G_n|H_1 = H_j^c$, $G_n|H_1 \not\in H_j^c$, $G_n|H_2 \not\in H_j^c$. Then $S_j \in (G_n|S_j^c)$, $G_n|H_1 \not\in H_j^c$. $G_n|S_j^c \not\in H_j^c$ by Lemma 6.6(vi). $S_j^c \not\in H_j^c$. Thus $H_j^c || H_j^c$ by $G_n|S_j^c \not\in H_j^c$, $G_n|S_j^c \not\in H_j^c$. $G_n|S_j^c \not\in H_j^c$, $G_n|S_j^c \not\in H_j^c$. Thus $H_j^c || H_j^c$. Therefore $G_{H, H'}^\ast \cap \bar{H} = \emptyset$. Then $\bar{H} \subseteq G_{H, H'}^\ast \subseteq G$, i.e., $H'$ is separable in $\tau_{H, H'}^{\ast (J)}$.

(ii) Clearly, $G_{H, H'}^{\ast (J)}$ is a copy of $G_{H, H'}^{\ast (J)}$ and, $\tau_{G_n, H, H'}^{\ast (J)}$ has no difference with $\tau_{H, H'}^{\ast (J)}$ except some applications of $\{ID_{\alpha}\}$ and identification numbers of some $\rho$'s. Then $H'$ is separated into $\bar{H}$ in $G_{H, H'}^{\ast (J)}$ by the same reason as that of (i). Then $S_j^c, S_j^c$ are separated into $\bar{S}_j^c$ and $\bar{S}_j^c$ in $\tau_{H, H'}^{\ast (J)}$, respectively. Then $\bar{S}_j^c|G_{H, H'}^{\ast (J)} \subseteq G_{H, H'}^{\ast (J)}$ is closed since $G_{H, H'}^{\ast (J)}$ is closed. Thus all copies of $\bar{S}_j^c|G_{H, H'}^{\ast (J)}$ in $\tau_{H, H'}^{\ast (J)}$ are contracted into one by (EC) in $G_{H, H'}^{\ast (J)}$.

**Lemma 7.12.** (i) All copies of $S_j^c$ in $[S_j^c]$ are separable in $\tau_{H, H'}^{\ast (J)}$.

(ii) Let $H \in \tau^*$, $H' \subseteq H$, $H_j^c \subseteq H$ or $H_j^c H_j^c$ for all $S_j^c \in G_{H, H'}^{\ast (J)}$. Then $H'$ is separable in $\tau_{H, H'}^{\ast (J)}$.

**Proof.** (i) and (ii) are proved by a procedure similar to that of Lemma 7.11 and omitted.

**Definition 7.13.** The skeleton of $\tau_{H, H'}^{\ast (J)}$, which we denote by $\bar{\tau}_{H, H'}^{\ast (J)}$, is constructed by replacing all $G_{H, H'}^{\ast (J)}$ with $G_{H, H'}^{\ast (J)}$, i.e., $G_{H, H'}^{\ast (J)}$ is the parent node of $G_{H, H'}^{\ast (J)}$ in $\tau_{H, H'}^{\ast (J)}$.

**Lemma 7.14.** $\bar{\tau}_{H, H'}^{\ast (J)}$ is a linear structure with the lowest node $G_{H, H'}^{\ast (J)}$ and the highest $[S_j^c]_l$.

**Proof.** It holds by all $\tau_{H, H'}^{\ast (J)}$ and (EC) in $\tau_{H, H'}^{\ast (J)}$ being one-premise rules.

**Definition 7.15.** We call Construction 7.3 together with 7.7 the separation algorithm of one branch and, Construction 7.5 the separation algorithm along $H$.

8. Separation algorithm of multiple branches

In this section, let $I = \{H_1^c, \ldots, H_m^c\} \subseteq \{H_1^c, \ldots, H_m^c\}$ such that $H_k^c || H_k^c$ for all $1 \leq k < l \leq m$. We will generalize the separation algorithm of one branch to that of multiple branches. Roughly speaking, we give an algorithm to eliminate all $S_j^c \in G|G^*$ satisfying $H_k^c \not\in H_k^c$ for some $H_k^c \in I$.

**Definition 8.1.** $I := \{H_k^c : H_k^c \not\in H_k^c$ for some $H_k^c \in I\}$. 


Theorem 8.2. ([A.4, A.5.4]) Let $I = \{[S'_1], \ldots, [S'_m]\}$. Then there exist one closed hypersequent $G^*_1 \subseteq G^*$ and its derivation $\tau^*_1$ from $[S'_1], \ldots, [S'_m]$ in $GL_\Omega$ such that

(i) $\tau^*_1$ is constructed by applying elimination rules, say,

$$
G_b|S'_h \vdash G_b|S'_h \ldots \vdash G_b|S'_h \{\tau^*_1\},
$$

and the fully constraint contraction rules, say $\frac{G_2}{G_1}(\text{EC}^*_\Omega)$, where $1 \leq w \leq m$, $H'_h \Rightarrow H'_h$, for all $1 \leq k < l \leq m$, $I_k = \{H'_h, \ldots, H'_h\} \subseteq \mathcal{T}$, $I_\mathcal{T} = \{S'_1, \ldots, S'_l\}$, $I_b = \{G_b|S'_h, \ldots, G_b|S'_h\}$ and $G_b|S'_h$ is closed for all $1 \leq k \leq m$. Then $H'_h \not\models H'_h$ for all $S'_j \in G^*_1$ and $H'_h \not\models I$.

(ii) For all $H \in \mathcal{T}_1^m$,

$$
\partial_{\mathcal{T}_1^m}(H) := \begin{cases} 
G(G^*) & H \text{ is the root of } \mathcal{T}_1^m \text{ or } G_2 \in \frac{G_2}{G_1}(\text{EC}^*_\Omega \text{ or } \text{ID}_\Omega) \in \mathcal{T}_1^m, \\
H'_h & H \text{ is } G_b|S'_h \text{ in } \tau^*_1 \text{ for some } 1 \leq k \leq m,
\end{cases}
$$

where, $\mathcal{T}_1^m$ is the skeleton of $\tau^*_1$ which is defined as Definition 7.13. Then $\partial_{\mathcal{T}_1^m}(G^*_1) \subseteq \partial_{\mathcal{T}_1^m}(G_b|S'_h)$ for some $1 \leq k \leq m$ in $\tau^*_1$.

(iii) Let $H \in \mathcal{T}_1^m$, $G(G^*) < \partial_{\mathcal{T}_1^m}(H) \subseteq H'_h$, then $G_{H'_h}^0(I)$ $\in \tau^*_1$ and it is constructed by applying the separation algorithm along $H'_h$ to $H$ and is, an upper hypersequent of either $(\text{EC}^*_\Omega)$ if it is applicable, or $(\text{ID}_\Omega)$ otherwise.

(iv) $S'_j \in G^*_1$ implies $H'_h \not\models H'_h$ for all $H'_h \in I$ and, $S'_j \in G^*_1$ for some $\tau^*_1 \in \mathcal{T}_1^m$ or $S'_j \in [S'_h]$ for some $H'_h \in I$ satisfying $H'_h \not\models H'_h$.

Note that in Claim (i), bold $j$ in $I_k, I_T$ or $I_b$ indicates the $w$-tuple $(j_1, \ldots, j_w)$ in $S'_1, \ldots, S'_w$.

Claim (iv) shows the final aim of Theorem 8.2, i.e., there exists no $S'_j \in G^*_1$ such that $H'_h \not\models H'_h$ for some $H'_h \in I$. It is almost impossible to construct $\tau^*_1$ in a non-recursive way. Thus we use Claims (i), (ii) and (iii) in Theorem 8.2 to characterize the structure of $\tau^*_1$ in order to construct it recursively.

Proof. $\tau^*_1$ is constructed by induction on $|I|$. For the base case, let $|I| = 1$. Then $\tau^*_1$ is constructed by Construction 7.3 and 7.7. Here, Claim (i) holds by Lemma 7.8(ii), Lemma 7.4(i) and Lemma 6.6 (vi), Claim (ii) by Lemma 7.4(i), (iii) is clear and (iv) by Lemma 7.4(iv).

For the induction case, let $|I| \geq 2$. Let $G^*|S' \subseteq G^*|S'' \subseteq (I) \in \tau^*$, where $G^*|G^*|H' = H'_h$. Then $\{H'_h, \ldots, H'_h\}$ is divided into two subsets $I_r = \{H'_h, \ldots, H'_h\}$, $I_r = \{H'_h, \ldots, H'_h\}$, which occur in the left subtree $\tau^*(G^*|S')$ and right subtree $\tau^*(G^*|S'')$ of $\tau^*(H')$, respectively. Then $m(l) + m(r) = m$. Let $I_r = \{[S'_1], \ldots, [S'_c]\}$, $I_r = \{[S'_1], \ldots, [S'_c]\}$. Suppose that derivations $\tau^*_h$ of $G^*_h$ and $\tau^*_h$ of $G^*_h$ are constructed such that Claims from (i) to (iv) hold. There are three cases to be considered in the following.

Case 1: $S' \not\subseteq (G^*|S')$ for all $\tau^*_h \in \tau^*_1$. Then $\tau^*_1 := \tau^*_1$ and $G^*_1 := G^*_1$.

• For Claim (i), let $\tau^*_h \in \tau^*_1$ and $S'_j \in G^*_1$. By the induction hypothesis, $H'_h \not\models H'_h$ for all $H'_h \in I_r$. Since $S' \not\subseteq (G^*|S')$, then $G^*|H' \cap (G^*|H') \not\models \emptyset$. Thus $G^*|H'; G^*|H' \cap G^*_1 = \emptyset$ by
Lemma 6.3 and 6.4. Then $S_j^* \notin G_{H_j^0,G''}$. Thus $G''|S'' \notin H_j^0$ by Proposition 4.15(i). Hence, for all $H_j^0 \in I_j$ and, $S_j^* \in G_{H_j^0}$. for some $\tau_j^0 \in \tau_k^0$ or $S_j^* \notin [S_j^*]_k$ for some $H_j^0 \in I_j$. Then $H_j^0 \notin H_j^0$ by the definition of branches to $I$. Thus we assume that $S_j^* \in G_{H_j^0}$ for some $\tau_j^0 \in \tau_k^0$ in the following. If $G''|S'' \notin H_j^0$ then $H_j^0 \notin H_j^0$ for all $H_j^0 \in I_j$, Thus $H_j^0 \notin H_j^0$ for all $H_j^0 \in I_j$. Thus let $G''|S'' \notin H_j^0$ in the following. By the proof of Claim (i) above, $G''|S'' \notin H_j^0$. Then $H_j^0 \notin H_j^0$ by $G''|S'' \notin H_j^0$ and $G''|S'' \notin H_j^0$. Thus $H_j^0 \notin H_j^0$. Hence $H_j^0 \notin H_j^0$ for all $H_j^0 \in I_j$.

**Case 2** $S'' \notin (G''|S'')_{\tau_k^0}^j$ for all $\tau_j^0 \in \tau_k^0$. Then $\tau_j^0 := \tau_j^0$ and $G_{\tau_j^0} := G_{\tau_j^0}$. This case is proved by a procedure similar to that of Case 1 and omitted.

**Case 3** $S'' \notin (G''|S'')_{\tau_k^0}^j$ and $S'' \notin (G''|S'')_{\tau_k^0}^j$ for some $\tau_j^0 \in \tau_k^0$.

Given $G_{\tau_k^0} := \{G_{\tau_k^0}: \tau_k^0 \in \tau_k^0\}$.

such that $S'' \notin (G''|S'')_{\tau_k^0}^j$, and $H_j^0 \notin H_j^0$ for all $1 \leq k \leq v$, where, $1 \leq v \leq m(r)$, $G_{\tau_k^0}|S_j^0$ is closed for all $1 \leq k \leq v$, $I_{\tau_k^0} = \{H_j^0, H_j^0, \ldots, H_j^0\} \leq T_j^0$, $I_{\tau_k^0} = \{S_j^0, S_j^0, \ldots, S_j^0\}$, $I_{\tau_k^0} = \{G_{\tau_k^0}|S_j^0, \ldots, G_{\tau_k^0}|S_j^0\}$. Then $H_j^0 \notin G''|S''$ by $I_{\tau_k^0} \leq T_j^0$ and $H_j^0 \notin H_j^0$ for all $1 \leq k \leq v$. Thus $H_j^0 \notin H_j^0$ for all $H_j^0 \in I_j$, and $H_j^0 \notin I_j$ by $S'' \notin (G''|S'')_{\tau_k^0}^j$ and Construction 6.11.

For each $\tau_j^0 \in \tau_k^0$, above, we construct a derivation $\lambda_j^0(\tau_j^0)$ in which you may regard $\tau_j^0$ as a subroutine, and $\tau_j^0$ as a subroutine in the following stage 1. Then a derivation $\lambda_j^0(\tau_j^0)$ is constructed by calling $\lambda_j^0(\tau_j^0)$ in Stage 2, in which you may regard $\lambda_j^0(\tau_j^0)$ as a routine and $\tau_j^0$ as its subroutine.

Firstly, we present some properties of $\tau_j^0$ which are derived from Claims (i) – (iv) and applicable to $\tau_j^0$ or $\tau_j^0$ under the induction hypothesis.

**Notation 8.3.** Let $G_+ := \{G^{0|0}(I) : G^{0|0}(I) \downarrow (S'\{S''\})\}$ and $G_+ := \{G_{\tau_k^0}: \tau_k^0 \in \tau_k^0\}$ be two close hypersequents, $G_+ \subseteq H$ for some $H \in \tau_k^0$, and $G_+ \notin \tau_k^0 \subseteq H$ for some $H \in \tau_k^0$.

Generally, $\tau_k^0 \subseteq G_+$ and $G_+ \notin G_+$, i.e., eigenvariables in $\tau_k^0 \subseteq G_+$ have different identification numbers with those in $\tau_k^0 \subseteq G_+$, so are $H', G''', S'$.

**Lemma 8.4.** $S_j^* \in G_+$ implies $H_j^* \parallel G|S'$.

**Proof.** Let $S_j^* \in G_+ \subseteq G^{0|0}(I)$. Then $H_j^* \notin H_j^*$ by Lemma 7.6. Thus $H_j^* \notin H_j^*$ or $H_j^* \parallel H_j^*$. If $H_j^* \parallel H_j^*$ then $H_j^* \parallel G|S'$ by $H_j^* \parallel G|S'$ and Proposition 2.12(ii). If $H_j^* \parallel H_j^*$ then $S_j^* \in H_j^*$ by Proposition 4.15(i). Thus $S_j^* \in G_+$ by Lemma 6.3, Lemma 6.7(i). Hence $H_j^* \parallel G|S'$ by $H_j^* \parallel G|S'$, $G''|S''|G''|S''$.\[\Box\]
The proof is by induction on $I$. Let $H \in T^0_1$ then $\partial_{\Gamma_1}(H) \leq H^c_{i_k}$ for some $1 \leq k \leq m$; (3) Let $H \in T^0_1$ then $H^V_1 \parallel \partial_{\Gamma_1}(H)$; (4) Let $w > 1$ in $T^0_1$ then $H^V_w < H^c_{j_k}$ for all $1 \leq k \leq w$. (5) Let $T^0_{1w} \subseteq T^0_{1} \parallel \partial_{\Gamma_1}(G ś_{i_k}) \leq H^V_w$ for some $1 \leq k \leq w$. Then $w = 1$.

**Proof.** (1) is immediately from Claim (i). (2) holds by $G \leq H^c_{i_k}$ and $H^c_{i_k} \leq H^c_{i_k}$ for some $H^c_{i_k} \in I$ by $I \subseteq T$. (3) holds by Proposition 2.12(iii), (2) and $H^V_w \leq H^c_{i_k}$. (4) Let $w > 1$. Then $H^V_w < H^c_{i_k}$ for each $2 \leq k \leq w$, $H^V_w < H^c_{i_k}$ and $H^c_{i_k} < H^c_{i_k}$ for some $H^c_{i_k}, H^c_{i_k} \in I$ by (2). Thus $H^V_w \parallel H^V_w$ and $H^V_w \parallel H^c_{i_k}$ by Proposition 2.12(ii). Hence $H^V_w \leq H^c_{i_k}$ by $H^V_w < H^c_{i_k}$ and $H^V_w < H^c_{i_k}$. Thus $H^V_w < H^c_{i_k}$ and $H^V_w < H^c_{i_k}$ by (3), $H^V_w < H^c_{i_k}$. Hence $H^V_w < H^c_{i_k}$ for all $1 \leq k \leq w$. (5) is from (4).

**Lemma 8.6.** Let $\frac{H_{1,i_1} \cdots H_{1,i_n}}{H_{1,i_1}} \left\{\tau^\varphi_{r,l}(H)\right\} \in \tau^\varphi_{i_k}$ for all $1 \leq i \leq n$ such that $\partial_{\Gamma_1}(H_{0,1}) = G \parallel G^*$ and $\partial_{\Gamma_1}(H_{n,1}) \leq H^V_w$. Then $\partial_{\Gamma_1}(H_{1,1}) \in H^V_w$ and $w_i = 1$ for all $1 \leq i \leq n$.

**Proof.** The proof is by induction on $n$. Let $n = 1$ then $w_1 = 1$ by Lemma 8.5(5) and $\partial_{\Gamma_1}(H_{1,1}) \leq H^V_w$. For the induction step, let $\partial_{\Gamma_1}(H_{1,1}) \leq H^V_w$ for some $1 \leq i \leq n$ then $w_i = 1$ by Lemma 8.5(5).

Since $\frac{H_{1,i_1} \cdots H_{1,i_n}}{H_{1,i_1}} \left\{\tau^\varphi_{r,l}(H)\right\} \in \tau^\varphi_{i_k}$ then $\partial_{\Gamma_1}(H_{1,i_1}) \leq \partial_{\Gamma_1}(H_{1,i_k})$ for some $1 \leq k \leq w_i$ by Claim (ii). Then $\partial_{\Gamma_1}(H_{1,i_1}) \leq \partial_{\Gamma_1}(H_{1,i_k}) \in H^V_w$ by $w_i = 1$. Thus $w_{i-1} = 1$ by Lemma 8.5(5).

**Definition 8.7.** Let $\frac{G_3}{G_1} \left\{EC^*_G\right\} \in \tau^\varphi_{i_k}$ in $\mathbf{GL}_G$. The module of $\tau^\varphi_{i_k}$ at $G_2$, which we denote by $\tau^\varphi_{i_k}$, is defined as follows: (1) $G_2 \in \tau^\varphi_{i_k}$, (2) $\frac{H_{1,i_k}}{H_0} \left\{\tau^\varphi_{i_k}\right\} \in \tau^\varphi_{i_k}$ if $G_0 \in \tau^\varphi_{i_k}$; (3) $H_1 \notin \tau^\varphi_{i_k}$ if $\frac{H_1}{H_0} \left\{EC^*_G\right\} \in \tau^\varphi_{i_k}$, $H_0 \in \tau^\varphi_{i_k}$.

Each node of $\tau^\varphi_{i_k}$ is determined bottom-up, starting with $G_2$, whose root is $G_2$ and leaves may be branches, leaves of $\tau^*$ or lower hypersequents of $\left\{EC^*_G\right\}$-applications. While each node of $\tau^\varphi_{i_k}$ is determined top-down, starting with $H^c$, whose root is a subset of $G \parallel G^*$ and leaves contain $H^c$ and some leaves of $\tau^*$.

**Lemma 8.8.** (1) $\tau^\varphi_{i_k}$ is a derivation without $\left\{EC^*_G\right\}$ in $\mathbf{GL}_G$. (2) Let $H' \in \tau^\varphi_{i_k}$ and $\partial_{\Gamma_1}(H') \leq H^V_w$. Then $\partial_{\Gamma_1}(H) \leq H^V_w$ for all $H \in \tau^\varphi_{i_k}$ and $H \geq H'$.

**Proof.** (1) is clear and (2) immediately from Lemma 8.6.

**Stage 1 Construction of Subroutine** $\tau^\varphi_{i_k}(\tau^\varphi_{i_k})$. Roughly speaking, $\tau^\varphi_{i_k}(\tau^\varphi_{i_k})$ is constructed by replacing some nodes $\tau^\varphi_{i_k} \in \tau^\varphi_{i_k}$ with $\tau^\varphi_{i_k} \cup \tau^\varphi_{i_k}$ in post-order. However, the ordinal postorder-traversal algorithm cannot be used to construct $\tau^\varphi_{i_k}(\tau^\varphi_{i_k})$ because the tree structure of $\tau^\varphi_{i_k}(\tau^\varphi_{i_k})$ is
generally different from that of $\tau^\circ_l$ at some nodes $H \in \tau^\circ_l$ satisfying $\partial_{\tau^\circ_l}(H) < H'_l$. Thus we construct a sequence $\tau^\circ_{l}(q)$ of trees for all $q > 0$ inductively as follows.

For the base case, we mark all $\langle EC^*_l \rangle$-applications in $\tau^\circ_l$ as unprocessed and define such marked derivation to be $\tau^\circ_l(0)$. For the induction case, let $\tau^\circ_{l}(q)$ be constructed. If all applications of $\langle EC^*_l \rangle$ in $\tau^\circ_{l}(q)$ are marked as processed, we firstly delete the root of the tree resulting from the procedure and then, apply $\langle EC^*_l \rangle$ to the root of the resulting derivation if it is applicable otherwise add an $(ID_{l_j})$-application to it and finally, terminate the procedure. Otherwise we select one of the outermost unprocessed $\langle EC^*_l \rangle$-applications in $\tau^\circ_{l}(q)$, say, $\frac{G^\circ_{q+1}}{G^\circ_{q+1}}(\langle EC^*_l \rangle)^{q+1}$, and perform the following steps to construct $\tau^\circ_{l}(q+1)$ in which $\frac{G^\circ_{q+1}}{G^\circ_{q+1}}(\langle EC^*_l \rangle)^{q+1}$ be revised as $\frac{G^\circ_{q+1}}{G^\circ_{q+1}}(\langle EC^*_l \rangle)^{q+1}$ such that

(a) $\tau^\circ_{l}(q+1)$ is constructed by locally revising $\tau^\circ_{l}(q)$ and leaving other nodes of $\tau^\circ_{l}(q)$ unchanged, particularly including $G^\circ_{q+1}$;

(b) $\tau^\circ_{l}(q+1)(G^\circ_{q+1})$ is a derivation in $GL_{l_i}$;

(c) $G^\circ_{q+1} = G^\circ_{q+1}$ if $S' \notin (G'|S')_{I_{l_i}}$ for all $\tau^*_{l} \in \tau^\circ_{l}(G^\circ_{q+1})$ otherwise $G^\circ_{q+1} = G^\circ_{q+1}|G^\circ_{q+1}|G^\circ_{q+1}$ for some $m_{q+1} \geq 1$.

Remark 8.9. By two superscripts $\circ$ and $\cdot$ in $\langle EC^*_l \rangle$ and $\langle EC^*_l \rangle$, we indicate the unprocessed state and processed state, respectively. This procedure determines an ordering for all $\langle EC^*_l \rangle$-applications in $\tau^\circ_{l}$ and the subscript $q$ + 1 indicates that it is the $q + 1$-th application of $\langle EC^*_l \rangle$ in a post-order transversal of $\tau^\circ_{l}$. $G^\circ_{q+1}$ and $G^\circ_{q+1}$ in $G^\circ_{q+1}$ and $G^\circ_{q+1}$ are the premise and conclusion of $\langle EC^*_l \rangle$.

Step 1 (Delete). Take the module $\tau^\circ_{l}(q)$ out of $\tau^\circ_{l}(q)$. Since $\langle EC^*_l \rangle$ is the unique unprocessed $\langle EC^*_l \rangle$-applications in $\tau^\circ_{l}(q)$ $(G^\circ_{q+1})$ by its choice criteria, $\tau^\circ_{l}(G^\circ_{q+1})$ is the same as $\tau^\circ_{l}(G^\circ_{q+1})$ by Claim (a). Thus it is a derivation. If $\partial_{\tau^\circ_{l}}(H) \leq H'_l$ for all $H \in \tau^\circ_{l}(q)$, delete all internal nodes of $\tau^\circ_{l}(q)$. Otherwise there exists

$$\frac{G_{b_{k+l}}|S'_{b_{k+l}}|G_{b_{k+l}}|S'_{b_{k+l}} \ldots G_{b_{k+l}}|S'_{b_{k+l}}}{G_r \equiv \propto G^\circ_{b_{k+l}}|G^\circ_{b_{k+l}}}$$

such that $\partial_{\tau^\circ_{l}}(G_{b_{k+l}}|S'_{b_{k+l}}) > H'_l$ for all $1 \leq k \leq u'$ and $\partial_{\tau^\circ_{l}}(G_r) \leq H'_l$ by Lemma 8.8(2) and $\partial_{\tau^\circ_{l}}(G^\circ_{q+1}) = G|G| \leq H'_l$, then delete all $H \in \tau^\circ_{l}(q)$, $G^\circ_{q+1} \leq H < G_r$. We denote the structure resulting from the deletion operation above by $\tau^\circ_{l}(q)_{l_i}^{G^\circ_{q+1}(1)}$. Since $\partial_{\tau^\circ_{l}}(G_r) \leq H'_l$ then $\tau^\circ_{l}(q)_{l_i}^{G^\circ_{q+1}(1)}$ is a tree by Lemma 8.6. Thus it is also a derivation.

Step 2 (Update). For each $G^\circ_{q+1} \in \tau^\circ_{l}(q)_{l_i}^{G^\circ_{q+1}(1)}$ which satisfies $\frac{G^\circ_{q+1}}{G^\circ_{q+}}(\langle EC^*_l \rangle)^{q+1} \in \tau^\circ_{l}(q)$ and $S' \in (G'|S')_{I_{l_i}}$ for some $\tau^*_{l} \in \tau^\circ_{l}(G^\circ_{q+1})$, we replace $H$ with $H|G|G_l$ for each $H \in \tau^\circ_{l}(q)$, $G_l \leq
$H \leq G'_{q'}$. 

Since $\frac{G'_{q'}}{G'_q} \langle EC^*_\Omega \rangle_{q'} \in \tau^{(q)}_{H_{k'}}(G'_{q+1})$ and $(EC^*_\Omega)^\circ_{q+1}$ is the outermost unprocessed $(EC^*_\Omega)$-application in $\tau^{(q)}_{H_{k'}}$ then $q' \leq q$ and $(EC^*_\Omega)^\circ_{q'}$ has been processed. Thus Claims (b) and (c) hold for $\tau^{(q)}_{H_{k'}}(G'_{q'})$ by the induction hypothesis. Then $\frac{G'_{q'}}{G'_q}$ is a valid $(EC^*_\Omega)$-application since $\frac{G'_{q'}}{G'_q}, G'_l, G_1, G_2$ are valid, where $G'_{q'} = G'_{q'} \cdot G_1^{m'} | G_2^{m''}, G'_{q'} = G'_{q'} \cdot G_3$. 

**Lemma 8.10.** Let $G_{q'} < H \leq G'_q$. Then $\partial_{q'}(H) \geq G'|S'$. 

**Proof.** Since $G_{q'} < H$ then $G_{b(r)}|S'_{j_{k}} \leq H$ for some $1 \leq k \leq u'$. If $\partial_{q'}(H) \geq H'_{k'}$ then $\partial_{q'}(H) \geq G'|S'$. Otherwise all applications between $G_{q'}$ and $H$ are one-premise rules by Lemma 8.6. Then $H'_{k_{a}} \leq \partial_{q'}(H)$ by Claim (ii). Thus $\partial_{q'}(H) \geq G'|S'$ by $H'_{q'} < H'_{k_{a}}, \partial_{q'}(H) \leq H'_{k_{a}}$ for some $1 \leq k' \leq m(l)$ by Claim (i). 

Since $\partial_{q'}(H) \geq G'|S'$ by Lemma 8.10 and $H'_{q'} \geq G'|S'$ for each $S_{j_{k}} \in G_1$ by Lemma 8.4, then $G_{l} \leq H$ as side-hypersequent of $H$. Thus this step updates the revision of $G_{q'}$ downward to $G_{q'}$. 

Let $m'$ be the number of $G'_{q'}$ satisfying the above conditions, $\tau^{(q)}_{b', G'_{q',1}(1)}(G_{q'}, G_{b(r)}|S'_{j_{k}})$ for all $1 \leq k \leq u'$ are updated as $\tau^{(q)}_{b', G'_{q',1}(2)}(G_{q'}, G_{b(r)}|S'_{j_{k}})$, respectively. Then $\tau^{(q)}_{b', G'_{q',1}(2)}(G_{q'}, G_{b(r)}|S'_{j_{k}})$ is a derivation and $G_{m'} = G_{r'} \cdot G_{m''}$. 

**Step 3** (Replace). All $\tau^{(q)}_{b'}(G_{q'})_{k} \in \tau^{(q)}_{b', G'_{q',1}(2)}(G_{q'}, G_{b(r)}|S'_{j_{k}})$ are processed in post-order. If $H'_{k'} \sim H'_{k}$ for all $H'_{k'} \in \theta_{b'}$ and $H'_{k} \in \theta_{b'}$, it proceeds by the following procedure otherwise it remains unchanged. Let $\tau^{(q)}_{b'}$ be in the form 

$$G_{b(r)}|S'_{j_{k}} \cdot G_{b(r)}|S'_{j_{k}} \ldots \cdot G_{b(r)}|S'_{j_{k}}$$

Then $H'_{k_{a}} \geq G'|S'$ for all $1 \leq k \leq u$ by Lemma 8.10, $G_{b(r)}|S'_{j_{k}} > G_{m'}$. Firstly, replace $\tau^{(q)}_{b'}$ with $\tau^{(q)}_{b', G'_{q',1}(2)}$. We may rewrite the roots of $\tau^{(q)}_{b'}$ and $\tau^{(q)}_{b', G'_{q',1}(2)}$ as 

$$G_{l} = \langle G_{b(r)} \rangle_{k=1}^{k_{a}}|G_{m'}^{H_{r'}} | G_{m''}^{H_{r'}}$$ and 

$$G_{l} = \langle G_{b(r)} \rangle_{k=1}^{k_{a}}|G_{m'}^{H_{r'}} | \langle G_{b(r)} \rangle_{k=1}^{k_{a}}|G_{m''}^{H_{r'}}$$,

respectively. 

Let $G_{m'} < H' \leq G_{l}$. By Lemma 8.10, $\partial_{q'}(H) \geq G'|S'$. By Lemma 6.7, $H'_{k'} \leq H'_{k} < G'|S'$ or $H'_{k} | G'|S'$ for all $S'_{j_{k}} \in G_{m'}^{H_{r'}} \cdot G_{m''}^{H_{r'}}$. Thus $G_{m'}^{H_{r'}} \cdot G_{m''}^{H_{r'}} \leq H$. Secondly, we replace $H$ with $H_{k'} | G_{m'}^{H_{r'}} \cdot G_{m''}^{H_{r'}} | G_{b(r)} \langle G_{m'}^{H_{r'}} \rangle_{k=1}^{k_{a}} | G_{m''}^{H_{r'}} \langle G_{m''}^{H_{r'}} \rangle_{k=1}^{k_{a}} | H''$, for all $G_{q'} \leq H \leq G_{l}$. Let $m''$ be the number of $\tau^{(q)}_{b'} \in \tau^{(q)}_{b', G'_{q',1}(2)}$ satisfying the replacement conditions above, $\tau^{(q)}_{b', G'_{q',1}(2)}(G_{q'}, G_{b(r)}|S'_{j_{k}})$ for all $1 \leq k \leq u'$ be
updated as $\tau_{1(G)}^{(q)}$, $G^m$, $G_{1}\langle S_{r} \rangle_{\omega}$, respectively. Then $\tau_{1(G)}^{(q)}$ is a derivation of $G^m$ and 
$G_1 = G^m \backslash \left\{ \langle G_{b_{1}} \rangle_{k+1} | G_{H_{1}}^{(\omega)} \backslash \{ G_{b_{1}} \} \right\}.$

**Step 4 (Separation along $H_{1}^{\prime}$).** Apply the separation algorithm along $H_{1}^{\prime}$ to $G^m$ and denote 
the resulting derivation by $\tau_{1(G)}^{(q)}$. Then all $G_1$, $G_{\omega}^{(\omega)}$, $G_{b_{1}}$, and $G_{b_{1}}$ keep unchanged in $G_1$.

Since $G^m = G_1 \backslash \left\{ \langle G_{b_{1}} \rangle_{k+1} | G_{H_{1}}^{(\omega)} \backslash \{ G_{b_{1}} \} \right\}$, we have $\tau_{1(G)}^{(q)}$.

$H^\prime$, $S^\prime$, and $S''$ are separable in $\tau_{1(G)}^{(q)}$, by a procedure similar to that of Lemma 7.11. Let $S^\prime$ 
and $S''$ be separated into $\bar{S}^\prime$ and $\bar{S}''$, respectively. By Claim (iii), $G_1^{(\omega)} = G_{\omega}^{(\omega)}$, 
$G^{(\omega)} = G_1^{(\omega)} \backslash \left\{ \langle G_{b_{1}} \rangle_{k+1} | G_{H_{1}}^{(\omega)} \backslash \{ G_{b_{1}} \} \right\}$ by Lemma 7.6(iv),

$G_{1}^{(\omega)} = G_{\omega}^{(\omega)}$.

where $m_1 := m + m''$.

**Step 5 (Put back).** Replace $\tau_{1(G)}^{(q)}$ in $\tau_{1(G)}^{(q)}$ with $\tau_{1(G)}^{(q)}$ and mark $G_{1}^{(\omega)} = G_{\omega}^{(\omega)}$ 
as processed, i.e., revise $\{ EC_{1}\}^{\langle q \rangle}$ as $\{ EC_{1}\}^{\langle q \rangle}$, Among leaves of $\tau_{1(G)}^{(q)}$, all $G_{\omega}^{(\omega)}$ are updated as 
$G_{\omega}^{(\omega)}$, and others keep unchanged in $\tau_{1(G)}^{(q)}$. Then this replacement is feasible, especially, $G_{1}^{(\omega)}$ 
be replaced with $G_{1}^{(\omega)}$. Define the tree resulting from Step 5 to be $\tau_{1}^{(q)}$. Then Claims (a), (b) 
and (c) hold for $q + 1$ by the above construction.

Finally, we construct a derivation of $G_1^{(\omega)} \backslash \langle G_r \rangle_{\omega}$ from $\langle [S_{r}^\prime], \ldots, [S_{r}^\prime]_{\omega} \rangle$, $G_{b_{1}}|S_{r}^\prime$, $\ldots$, $G_{b_{n}}|S_{r}^\prime$ in $GL_{\omega}$, which we denote by $\tau_{1}^{(q)}$.

**Remark 8.11.** All elimination rules used in constructing $\tau_{1}^{(q)}$ are extracted from $\tau^{*}$. Since $\tau_{1}^{(q)}$ 
is a derivation in $GL_{\omega}$ without $\langle EC_{1}\rangle$, we may extract elimination rules from $\tau_{1}^{(q)}$ which 
may use to construct $\tau_{1}^{(q)}$ by a procedure similar to that of constructing $\tau_{1}^{(q)}$ with minor revision.
at every node $H$ that $\partial_{\bar{\nu}}(H) \leq H'$. Note that updates and replacements in Steps 2 and 3 are essentially inductive operations but we neglect it for simplicity.

We may also think of constructing $\tau^\oplus_L(\tau^\lambda_L)$ as grafting $\tau^\lambda_L$ in $\tau^\oplus_L$ by adding $\tau^\lambda_L$ to some $\tau^\lambda_L \in \tau^\oplus_L$. Since the rootstock $\tau^\oplus_L$ of the grafting process is invariant in Stage 2, we encapsulate $\tau^\lambda_L(\tau^\lambda_L)$ as an rule in $G_{L,\Omega}$ whose premises are $G_{b_i}|S_{j_1}^c, G_{b_2}|S_{j_2}^c, \ldots, G_{b_n}|S_{j_n}^c$ and conclusion is $\overline{S}'|\{G_{b_1}|S_{j_1}^c \cup \cdots \cup G_{b_n}|S_{j_n}^c\}$ for all $q \geq 0$.

**Step 1 (Delete).** $\tau^\oplus_{L, G_{b_i}^q}(q)$ and $\tau^\lambda_{L, G_{b_i}^q}(q)$ are defined as before.

$$\begin{align*}
G_{b_i}|S_{j_1}^c, G_{b_2}|S_{j_2}^c, \ldots, G_{b_n}|S_{j_n}^c & \vdash \{\tau^\lambda_L(\tau^\lambda_L)\} \\
& \vdash \{\tau^\lambda_L(\tau^\lambda_L)\} \\
& \vdash \{\tau^\lambda_L(\tau^\lambda_L)\}
\end{align*}$$

satisfies $\partial_{\bar{\nu}}(G_{b_i}|S_{j_i}^c) > H'_{\nu}$ for all $1 \leq k \in \nu'$ and $\partial_{\bar{\nu}}(G_{\nu}) \leq H'_{\nu}$.

**Step 2 (Update).** For all $G_{b_i}' \in \tau^\lambda_L(\tau^\lambda_L)$ which satisfy $G_{b_i}' \vdash \{\tau^\lambda_L(\tau^\lambda_L)\}$ and $S_{j_i}^c \vdash \{\tau^\lambda_L(\tau^\lambda_L)\}$, we replace $H$ with $H'_{\nu} = \{\tau^\lambda_L(\tau^\lambda_L)\}$ for all $H \in \tau^\lambda_L(\tau^\lambda_L)$, $G_{b_i}' \vdash \{\tau^\lambda_L(\tau^\lambda_L)\}$ if and only if $G_{b_i}' \vdash \{\tau^\lambda_L(\tau^\lambda_L)\}$.

Then Claims (a) and (b) are proved by a procedure as before. Let $m'$ be the number of $G_{b_i}'$ satisfying the above conditions. $G_{b_i}'$ is revised as

$$\begin{align*}
G_{b_i}' & \vdash \{\tau^\lambda_L(\tau^\lambda_L)\} \\
& \vdash \{\tau^\lambda_L(\tau^\lambda_L)\} \\
& \vdash \{\tau^\lambda_L(\tau^\lambda_L)\}
\end{align*}$$

$$\begin{align*}
G_{b_i}' & \vdash \{\tau^\lambda_L(\tau^\lambda_L)\} \\
& \vdash \{\tau^\lambda_L(\tau^\lambda_L)\} \\
& \vdash \{\tau^\lambda_L(\tau^\lambda_L)\}
\end{align*}$$

$$\begin{align*}
G_{b_i}' & \vdash \{\tau^\lambda_L(\tau^\lambda_L)\} \\
& \vdash \{\tau^\lambda_L(\tau^\lambda_L)\} \\
& \vdash \{\tau^\lambda_L(\tau^\lambda_L)\}
\end{align*}$$

satisfies $\partial_{\bar{\nu}}(G_{b_i}|S_{j_i}^c) > H'_{\nu}$ for all $1 \leq k \in \nu'$ and $\partial_{\bar{\nu}}(G_{\nu}) \leq H'_{\nu}$.

**Step 3 (Replace).** All $\tau^\lambda_L \in \tau^\lambda_{L, G_{b_i}^q}(q)$ are processed in post-order. If $H'_{\nu} \sim H'_{\nu}$ for all $H'_{\nu} \in \tau^\lambda_L$ and $H'_{\nu} \in \tau^\lambda_L$ it proceeds by the following procedure otherwise it remains unchanged. Let $\tau^\lambda_L$ be in
the form

$$G_{b_1}\{S_{j_1}^c, G_{b_2}\{S_{j_2}^c, \ldots, G_{b_n}\{S_{j_n}^c, G_r \} \} \}$$

Then there exists the unique $1 \leq k' \leq v'$ such that $G_{v''} < G_{b_{j_k'}}|S_{j_{k'}}^c \leq G_r$.

Firstly, we replace $\tau_{b_k}$ with $\tau_{b_k}^\phi$ ($\tau_{b_k}^\phi$). We may rewrite the roots of $\tau_{b_k}^\phi$ ($\tau_{b_k}^\phi$) as $G_r = \{G_{b_{k+1}}\{G_{b_{k+2}}\{S_{j_k}^c, G_{b_{k+3}}\{S_{j_{k+1}}^c, \ldots, G_{b_n}\{S_{j_n}^c, G_r \} \} \} \}$. We may rewrite the roots of $\tau_{b_k}^\phi$ ($\tau_{b_k}^\phi$) as $G_r = \{G_{b_{k+1}}\{G_{b_{k+2}}\{S_{j_k}^c, G_{b_{k+3}}\{S_{j_{k+1}}^c, \ldots, G_{b_n}\{S_{j_n}^c, G_r \} \} \} \}$.

Let $G_{v''} < H \leq G_r$. Then $\partial(H) \supseteq S''$ by Lemma 8.10. Thus $G_{H;H'}^{\phi(0)} \subseteq H, \{S_j^c : S_j^c \in G_{H;H'}^{\phi(0)}\} \supseteq \{S_j^c : S_j^c \in G_{H;H'}^{\phi(0)}\}$. Define $G_{H;H'}^{\phi} = \{S_j^c : S_j^c \in G_{H;H'}^{\phi(0)}\}$.

Then we replace $H$ with $\tau_{b_k}^\phi$ in $\tau_{b_k}^\phi$.

Let $G_r^\phi$ be the focus sequent of some $H' \in \tau_{L,G_{v''}(3)}^\phi, H \prec H' \prec G_r$.

Then we replace $H$ with $H\{G_{H;H'}^{\phi}(S_{j_k}^c)\} \cup G_{H;H'}^{\phi}(S_{j_k}^c) \{S_k^c\} \cup \{G_{H;H'}^{\phi}(S_{j_k}^c)\} \cup G_{H;H'}^{\phi}(S_{j_k}^c) \{S_k^c\} \cup \{G_{H;H'}^{\phi}(S_{j_k}^c)\}$ for all $G_{b_{j_k'}}|S_{j_k'}^c \leq H \leq G_r$.

Let $m'\phi$ be the number of $\tau_{b_k}^\phi$ in $\tau_{L,G_{v''}(3)}^\phi$ satisfying the replacement conditions as above, $\tau_{L,G_{v''}(3)}^\phi, G_{v''}$ and $G_{b_{j_k'}}|S_{j_k'}^c$ for all $1 \leq k' \leq v'$ be updated as $\tau_{L,G_{v''}(3)}^\phi, G_{v''}, G_{b_{j_k'}}|S_{j_k'}^c$, respectively. Then $\tau_{L,G_{v''}(3)}^\phi$ is a derivation and $G_{v''} = H \cup H_{m''}^\phi \{H_{m'}^\phi\}$, where

$$H_0 = G_{H;H'}^{\phi}(S_{j_k}^c) \cup G_{H;H'}^{\phi}(S_{j_k}^c) \{S_k^c\} \cup \{G_{H;H'}^{\phi}(S_{j_k}^c)\} \cup G_{H;H'}^{\phi}(S_{j_k}^c) \{S_k^c\} \cup \{G_{H;H'}^{\phi}(S_{j_k}^c)\}$$

**Step 4 (Separation along $H_{v}(3)$).** Apply the separation algorithm along $H_{v}(3)$ to $G_{v''}$ and denote the resulting derivation by $\tau_{L,G_{v''}(4)}^\phi$ whose root is labeled by $G_{v+1}$.

By Claim (iii), $G_{H;G_{v'''}}^{\phi(3)} = G_{H;G_{v'''}}^{\phi(1)}$.

$$G_{H;G_{v'''}}^{\phi(3)} = G_{H;G_{v'''}}^{\phi(1)} \cup \{G_{H;G_{v'''}}^{\phi(3)}\} \cup \{G_{H;G_{v'''}}^{\phi(1)}\}$$

Then

$$G_{H;G_{v'''}}^{\phi(3)} = G_{H;G_{v'''}}^{\phi(1)} \cup \{G_{H;G_{v'''}}^{\phi(3)}\} \cup \{G_{H;G_{v'''}}^{\phi(1)}\}$$

43
Then

\[ G_{q+1}^* = G_{H_j^{c_0}}^{\varphi(J)} = G_{q+1}^{\varphi(J)} \backslash \{ \hat{S} | G_H^{\varphi(J)} \}^{m_{q+1}} \} \{ G_{H_j^{c_0}}^{\varphi(J)} \}^{m_{q+1}} \]

where \( m_{q+1} := m' + m'' \).

**Step 5 (Put back).** Replace \( \tau_L^{\varphi(q)} \) in \( \tau_L^{\varphi(q)} \) with \( \tau_L^{\varphi(q)} \) and revise \( \frac{G_{q+1}^*}{G_{q+1}^*} \langle EC_{\varphi(J)} \rangle_{q+1} \) as \( \frac{G_{q+1}^*}{G_{q+1}^*} \langle EC_{\varphi(J)} \rangle_{q+1} \). Define the resulting tree from Step 5 to be \( \tau_L^{\varphi(q+1)} \) then Claims (a), (b) and (c) hold for \( q + 1 \) by the above construction.

Finally, we construct a derivation of \( G_{H_j^{c_0}}^{\varphi(J)} \in \mathcal{GL}_M \) from \( \{ S_1', \ldots, S_j' \} \) in \( \mathcal{GL}_M \).

Since the major operation of Stage 2 is to replace \( \tau_L^{\varphi(q)} \) with \( \tau_L^{\varphi(q+1)} \) for all \( \tau_L^{\varphi(q)} \) satisfying \( S'' \in (G''|S')_J \), then we denote the resulting derivation from Stage 2 by \( \tau_L^{\varphi(q)} \).

In the following, we prove that the claims from (i) to (iv) hold if \( \tau_L^{\varphi} := \tau_L^{\varphi(q)} \) and \( G_L^{\varphi} := G_{H_j^{c_0}}^{\varphi(J)} \).

- For Claim (i), (ii): Let \( H_1 \cdots H_w H_0 \) be \( \tau_L^{\varphi} \) and \( S_j' \in G_{Z_j} \). Then \( \partial_\varphi(H_0) \not\subseteq H_j' \) for all \( 1 \leq k \leq w \) by Lemma 6.13(iv).

  If \( \partial_\varphi(H_0) \not\subseteq H_j' \) for some \( 1 \leq k' \leq w \), then \( H_j' \not\subseteq H_j' \) for all \( H_j' \in I \) by \( \partial_\varphi(H_0) \not\subseteq H_j' \). Thus Claim (i) holds by Lemma 8.5(5) and Lemma 7.6(i). Note that Lemma 8.5(5) is independent of Claims from (ii) to (iv).

  Otherwise \( \tau_L^{\varphi(q+1)} \) is built up from \( \tau_L^{\varphi(q)} \) by keeping their focus and principal sequents unchanged and making their side-hypersequents possibly to be modified, but which has no effect on discussing Claim (ii) and then Claim (ii) holds for \( \tau_L^{\varphi} \) by the induction hypothesis on Claim (ii) of \( \tau_L^{\varphi(q+1)} \).

- If \( \tau_L^{\varphi(q)} \) is from \( \tau_L^{\varphi(q+1)} \) at Stage 1. By the induction hypothesis, \( H_j' \not\subseteq H_j' \) for all \( S_j' \in G_{Z_j} \). Hence \( H_j' \not\subseteq H_j' \) for all \( S_j' \in G_{Z_j} \) and \( I \) is proved by a procedure similar to above and omitted.

- Claim (iii) holds by Step 4 at Stage 1 and 2. Note that in the whole of Stage 1, we treat \( \{ G_{b_k} \}_{k=1} \) as a side-hypersequent. But it is possible that there exists \( S_j' \in \{ G_{b_k} \}_{k=1} \) such that \( H_j' \not\subseteq H_j' \). Since we haven’t applied the separation algorithm to \( \{ G_{b_k} \}_{k=1} \) in Step 4 at Stage 1, then it could make Claim (iii) invalid. But it is not difficult to find that we just move the separation of such \( S_j' \) to Step 4 at Stage 2. Of course, we can move it to Step 4 at Stage 1, but which make the discussion complicated.

- For Claim (iv), we prove (1) \( H_j' \not\subseteq H_j' \) for all \( S_j' \in G_{H_j} \) and \( H_j' \in I \), (2) \( H_j' \not\subseteq H_j' \) for all \( S_j' \in G_{H_j} \) \( \{ \hat{S} | G_H^{\varphi(J)} \} \). Only (1) is proved as follows and (2) by a similar procedure and omitted.
Let $S'_j \in G_{k_1}^\Pi$. Then $S'_j \in G_{k_1}^\Pi$ and $S'_j \notin \overline{S'}(G_{H_j';H_j''}^\Pi)|G_{H_j';H_j''}^\Pi(\overline{S'})$. By the definition of $G_{k_1}^\Pi$.

By a procedure similar to that of Claim (iv) in Case 1, we get $H'_j \not\preceq H'_1$ and assume that $S'_j \in G_{k_1}^\Pi$ for some $r'_k \in r'_k$ and let $G'|S'| \not\preceq H'_j$ in the following.

Suppose that $G''|S'' \preceq H'_j$. Then $S'_j \in G_{H_j';G''}^\Pi$ and $S' \in \langle G'|S'| \rangle_{r_k}$ by $S'_j \in G_{k_1}^\Pi$. Hence $S'_j \in G_{H_j';G''}^\Pi$ by $H'_j \not\preceq G''|S'' \not\preceq H'_j$. Therefore $S'_j \notin \overline{S'}(G_{H_j';G''}^\Pi)|G_{H_j';G''}^\Pi(\overline{S'})$, a contradiction thus $G''|S'' \not\preceq H'_j$. Then $H'_j \not\preceq H''_j$ by $G'|S'| \not\preceq H'_j$ and $G''|S'' \not\preceq H'_j$. Thus $H'_j \parallel H''_j$. Hence $H'_j \parallel H''_j$ for all $H''_j \in I$. This completes the proof of Theorem 8.2.

\(\square\)

**Definition 8.12.** The manipulation described in Theorem 8.2 is called derivation-grafting operation.

9. The proof of Main theorem

Recall that in Main theorem $G_0 \equiv G' \langle \Gamma, p \Rightarrow \Delta_j \}_{j=1\ldots m} \langle \Pi_j \Rightarrow p, \Sigma_j \}_{j=1\ldots m}$.

**Lemma 9.1.** (i) If $G_2 = G_0 \langle \Gamma, p \Rightarrow \Delta_j \rangle$ and $\rightarrow_{GL} D_0(G_2)$ then $\rightarrow_{GL} D_0(G_0)$

(ii) If $G_2 = G_0 \langle \Pi_j \Rightarrow p, \Sigma_j \rangle$ and $\rightarrow_{GL} D_0(G_2)$ then $\rightarrow_{GL} D_0(G_0)$

(iii) If $G_2 = G_0 \langle \Gamma, p \Rightarrow \Delta_j \rangle \rightarrow_{GL} D_0(G_2)$ then $\rightarrow_{GL} D_0(G_0)$

Proof. (i) Since $D_0(G_2) = G' \langle \Gamma, p \Rightarrow \Delta_j \rangle (\Gamma_1 \Rightarrow \Delta_j, \Sigma_j)_{j=1\ldots m}$ then $\rightarrow_{GL} D_0(G_2)$ holds. If $n = 1$, we replace all $p$ in $\Pi_j \Rightarrow p, \Sigma_j$ with $\top$. Then $\rightarrow_{GL} D_0(G_0)$ holds by applying (CUT) to $\Gamma_1 \parallel \Rightarrow \Delta_j$ and $G'/\langle \Pi_j \parallel \Rightarrow \Delta_j, \Sigma_j \rangle_{j=1\ldots m}$ then $\rightarrow_{GL} D_0(G_2)$ holds by applying (ECT) to $\rightarrow_{GL} D_0(G_2)$

(ii) Since $D_0(G_2) = G' \langle \Gamma, p \Rightarrow \Delta_j \rangle (\Gamma_1 \Rightarrow \Delta_j, \Sigma_j)_{j=1\ldots m}$ then $\rightarrow_{GL} D_0(G_0)$ holds by applying (ECT) to $\rightarrow_{GL} D_0(G_2)$

(iii) Since $D_0(G_2) = G' \langle \Gamma, \parallel \Rightarrow \Delta_j \rangle (\Gamma_1 \Rightarrow \Delta_j, \Sigma_j)_{j=1\ldots m}$ then $\rightarrow_{GL} G'' \equiv G' \langle \Gamma, \parallel \Rightarrow \Delta_j \rangle (\Gamma_1 \Rightarrow \Delta_j, \Sigma_j)_{j=1\ldots m}$ holds by applying (CUT) to $\Gamma_1 \parallel \Rightarrow \Delta_j$ and $\Pi_j \Rightarrow \top, \Sigma_j$. Thus $\rightarrow_{GL} D_0(G_2)$ holds by applying (ECT) to $G''$.

(1)', (1'') and (1''') are proved by a procedure respectively similar to those of (i), (ii) and (iii) and omitted.

Let $I = \langle H''_1, \ldots, H''_n \rangle \subset \langle H''_1, \ldots, H''_n \rangle$, $G_i$ denote a closed hypersequent such that $G_i \subseteq_c G_i$ and $H''_j \parallel H''_k$ for all $S'_j \in G_i$ and $H''_k$. Then $H''_j \parallel H''_k$.

**Lemma 9.2.** There exists $G_i$ such that $\rightarrow_{GL} G_i$ for all $I \subset \langle H''_1, \ldots, H''_n \rangle$.

Proof. The proof is by induction on $m$. For the base step, let $m = 0$, then $I = \emptyset$ and $G_i ::= G_i$ and $\rightarrow_{GL} G_i$ by Lemma 4.17(v).

For the induction step, suppose that $m \geq 1$ and there exists $G_i$ such that $\rightarrow_{GL} G_i$ for all $\emptyset \subseteq I \subseteq \langle H''_1, \ldots, H''_n \rangle$. Then there exist $G_{i \langle \Pi_k \rangle}$ for all $1 \leq k \leq m$ such that $\rightarrow_{GL} G_{i \langle \Pi_k \rangle}$ and $H''_j \parallel H''_k$ for all $S'_j \in G_{i \langle \Pi_k \rangle}$ and $H''_k \in I \langle H''_k \rangle$

If $H''_j \parallel H''_k$ for all $S'_j \in G_{i \langle \Pi_k \rangle}$ then $G_i ::= G_{i \langle \Pi_k \rangle}$ and the claim holds clearly. Otherwise there exists $S'_j \in G_{i \langle \Pi_k \rangle}$ such that $H''_j \not\preceq H''_k$ or $H''_j \not\preceq H''_k$. Then we rewrite $G_{i \langle \Pi_k \rangle}$ as $\Gamma_j \parallel H''_j$.
where we define $H^c_i$ such that $S^c_1 \in G_{H^c_i}$ and, $S^c_j \in G_{H^c_i}$ implies $H^c_i \leq H^c_j$ or $H^c_j \parallel H^c_i$ for all $H^c_i \in \{H^c_1, \ldots, H^c_N\}$. If we can’t define $G_I$ to be $G_{H^c_i}$ for each $1 \leq k \leq m$, let $I' := \{H^c_1', \ldots, H^c_m'\}$. Then $G_{I'}$ is constructed by applying the separation algorithm of multiple branches (or one branch if $m = 1$) to $[S^c_1']$, $\cdots$, $[S^c_m']$.

The proof of Main theorem: Let $I = \{H^c_1, \ldots, H^c_N\}$ in Lemma 9.2. Then there exists $G_I$ such that $\vdash_{GL_{\alpha}} G_I$, $G_I \subseteq G[G^*]$ and $H^c_i \parallel H^c_j$ for all $S^c_i \in G_I$ and $H^c_i \in I$. Then $\vdash_{GL} D(G_I)$ by Lemma 5.6.

Suppose that $S^c_j \in G_I$. Then $H^c_j \parallel H^c_i$ for all $H^c_i \in I$. Thus $H^c_j \parallel H^c_i$ by $H^c_i \in I$, a contradiction with $H^c_i \leq H^c_j$ and hence there doesn’t exist $S^c_j \in G_I$. Therefore $G_I \subseteq G$ by Lemma 5.6.

By removing the identification number of each occurrence of $p$ in $G$, we obtain the sub-hypersequent $G_2$ of $G_2[G^*]$, which is the root of $\tau^4$ resulting from Step 4 in Section 4. Then $\vdash_{GL} D_0(G_2)$ by $\vdash_{GL} D(G_I)$ and $G_I \subseteq G$. Since $G_2$ is constructed by adding or removing some $\Gamma, p \Rightarrow \Delta_i$ or $\Pi_j \Rightarrow p, \Sigma_j$ from $G_0$, or replacing $\Gamma, p \Rightarrow \Delta_i$ with $\Gamma, \top \Rightarrow \Delta_i$, or $\Pi_j \Rightarrow p, \Sigma_j$ with $\Pi_j \Rightarrow \top, \Sigma_j$, then $\vdash_{GL} D_0(G_0)$ by Lemma 9.1. This completes the proof of Main theorem.

Theorem 9.3. Density elimination holds for all GL in \{GUL, GIUL, GMTL, GMTL\}.

Proof. It follows immediately from Main theorem.

10. Final remarks and open problems

Recently, we have generalized our method described in this paper to the non-commutative substructural logic $GpsUL^*$ in [24]. This result shows that $GpsUL^*$ is the logic of pseudo-uninorms and their residua and answered the question posed by Prof. Metcalfe, Olivetti, Gabbay and Tsinakis in [17, 18].

It has often been the case in the past that metamathematical proofs of the standard completeness of UL and their method had also been extended by Galatos and Horcik [11]. A natural problem is whether there is an algebraic proof corresponding to our proof-theoretic one. It seems difficult to obtain it by using the insights gained from the approach described in this paper because ideas and syntactic manipulations introduced here are complicated and specialized. In addition, Baldi and Terui [3] also mentioned some open problems. Whether our method could be applied to their problems is another research direction.

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References


Appendices

A.1 Why do we adopt Avron-style hypersequent calculi?

A hypersequent calculus is called Pottinger-style if its two-premise rules are in the form of
\[ \frac{G[S'] \ G[S'']}{G[H']} (II) \] and, Avron-style if in the form of
\[ \frac{G[S'] \ G[H']}{G[G[S'']]} (II) \]. In the viewpoint of Avron-style systems, each application of two-premise rules contains implicitly applications of (EC) in Pottinger-style systems, as shown in the following.

\[
\begin{array}{c}
G[S'] \ G[S''] \quad \text{corresponds to} \\
G[H'] (II) \quad \text{in Avron-style system}
\end{array}
\begin{array}{c}
G[S'] \ G[H'] \\
\text{(EC*)}
\end{array}
\]

47
The choice of the underlying system of hypersequent calculus is vital to our purpose and it gives the background or arena. In Pottinger-style system, $G_0$ in Section 3 is proved without application of $(EC)$ as follows. But it seems helpless to prove that $H_0$ is a theorem of IUL.

<table>
<thead>
<tr>
<th>$C \Rightarrow C$</th>
<th>$B \Rightarrow B$</th>
<th>$B \Rightarrow B[p,p \Rightarrow A] \Rightarrow A \Rightarrow A[p \Rightarrow A] \Rightarrow A[p \Rightarrow p \Rightarrow A]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C \Rightarrow C[p,B \Rightarrow p, \sim A \Rightarrow A]$</td>
<td>$B \Rightarrow p, \sim A \Rightarrow A[p,p \Rightarrow A]$</td>
<td></td>
</tr>
</tbody>
</table>

The peculiarity of our method is not only to focus on controlling the role of the external contraction rule in the hypersequent calculus but also introduce other syntactic manipulations. For example, we label occurrences of the eigenvariable $p$ introduced by an application of the density rule in order to be able to trace these occurrences from the leaves (axioms) of the derivation to the root (the derived hypersequent).

### A.2 Why do we need the constrained external contraction rule?

We use the example in Section 3 to answer this question. Firstly, we illustrate Notation 4.14 as follows. In Figure 4, let $S^1_{11} = A \Rightarrow p_2; S^1_{12} = A \Rightarrow p_4; S^2_{22} = A \Rightarrow p_3; S^2_{31} = p_1; p_2 \Rightarrow A \Rightarrow A; S^2_{32} = p_3, p_4 \Rightarrow A \Rightarrow A; G^1 = p_1, p_2 \Rightarrow A \Rightarrow A; G^2 = p_3, p_4 \Rightarrow A \Rightarrow A; G^3 = A \Rightarrow p_2, B \Rightarrow p_4, \sim A \Rightarrow A \Rightarrow p_3$. Then $H^i\vdash G|S^i_{11}|S^i_{12}$ for $i = 1, 2, 3$. $H^i$ are $(pEC)$-nodes and, $S^i_{11}$ and $S^i_{12}$ are $(pEC)$-sequences.

Let $G^i_{H^i\vdash A \Rightarrow p_2} \Rightarrow p_2, B \Rightarrow p_4, \sim A \Rightarrow A \Rightarrow p_3, p_4 \Rightarrow A \Rightarrow A$. We denote the derivation $\tau^i_{H^i\vdash A \Rightarrow p_2}$ of $G^i_{H^i\vdash A \Rightarrow p_2}$ from $A \Rightarrow p_2$ by $\frac{A \Rightarrow p_2}{G^i_{H^i\vdash A \Rightarrow p_2}}(\tau^i_{H^i\vdash A \Rightarrow p_2})$. Since we focus on sequents in $G^*$ in the separation algorithm, we abbreviate $\frac{A \Rightarrow p_2}{G^i_{H^i\vdash A \Rightarrow p_2}}(\tau^i_{H^i\vdash A \Rightarrow p_2})$ to $\frac{1}{2}\frac{S^i_{11}}{S^i_{22}}(\tau^i_{S^i_{11}11})$ and further to $\frac{1}{2}\frac{1}{2}\frac{1}{2}(\tau^i_{S^i_{11}11})$. Then the separation algorithm $\tau^i_{G^i\vdash G^*}$ is abbreviated as

\[
\begin{array}{l}
\frac{1}{2}(\tau^i_1) \rightarrow \frac{1}{2}(\tau^i_2), \tau^i_3)
\end{array}
\]

where $2' \text{ and } 3'$ are abbreviations of $A \Rightarrow p_2$ and $p_3, p_6 \Rightarrow A \Rightarrow A$, respectively. We also write $2'$ and $3'$ respectively as $2$ and $3$ for simplicity. Then the whole separation derivation is given as follows.

\[
\begin{array}{l}
\frac{1}{2}(\tau^i_1) \rightarrow \frac{1}{2}(\tau^i_2), \tau^i_3)
\end{array}
\]
where $\emptyset$ is an abbreviation of $G''$ in Page 14 and means that all sequents in it are copies of sequents in $G_0$. Note that the simplified notations become intractable when we decide whether $(EC_{1\alpha})$ is applicable to resulting hypersequents. If no application of $(EC_{1\alpha})$ is used in it, all resulting hypersequents fall into the set \( \{ \underbrace{[2|3|\cdots|3}, \underbrace{2|3|\cdots|3}, \underbrace{1|3|\cdots|3}: l \geq 0, m \geq 0, n \geq 0 \} \)
and $\emptyset$ is never obtained.

A.3 Why do we need the separation of branches?

In Figure 11, $p_1$ and $p_2$ in the premise of 
\[ \frac{p_1, p_2 \Rightarrow A \otimes A}{p_1 \Rightarrow C, p_2 \Rightarrow A \otimes A} \left( \tau_{S^1_{11}}^{c} \right) \]
could be viewed as being tangled in one sequent $p_1, p_2 \Rightarrow A \otimes A$ but in the conclusion of \( \tau_{S^1_{11}}^{c} \) they are separated into two sequents $p_1 \Rightarrow C$ and $C, p_2 \Rightarrow A \otimes A$. which are copies of sequents in $G_0$. In Figure 5, $p_2$ in $A \Rightarrow p_2$ falls into $= p_2, B$ in the root of $\tau_{P^1_{H,K \Rightarrow P}}^{c}$ and $\Rightarrow p_2, B$ is a copy of a sequent in $G_0$. The same is true for $p_2$ in $A \Rightarrow p_4$ in Figure 8. But it’s not the case.

Lemma 6.6 (vi) shows that in the elimination rule $\frac{G^c_{S^1_{11}} \left( \tau_{S^1_{11}}^{c} \right), S_j^c \in G^c_{S^1_{11}}}{H_j^c < H_j}$ implies $H_j^c < H_j^c$. If there exists no $S_j^c \in G^c_{S^1_{11}}$ such that $H_j^c < H_j^c$, then $S_j^c \in G^c_{S^1_{11}}$ implies $H_j^c < H_j^c$ and thus each occurrence of $p \prime$'s is separated in $S^c_{11}$ is fell into a unique sequent which is a copy of a sequent in $G_0$. Otherwise there exists $S_j^c \in G^c_{S^1_{11}}$ such that $H_j^c < H_j^c$, then we apply $\left( \tau_{S^1_{11}}^{c} \right)$ to $S_j^c$ in $G^c_{S^1_{11}}$ and the whole operations can be written as
\[
\frac{S^c_{11}}{G^c_{S^1_{11}}(0) \equiv G^c_{S^1_{11}}(\{S_j^c\}) | S_j^c} \left( \tau_{S^1_{11}}^{c} \right) \]
\[
\frac{G^c_{S^1_{11}}(0) \equiv G^c_{S^1_{11}}(\{S_j^c\}) | S_j^c}{G^c_{S^1_{11}}(1) \equiv G^c_{S^1_{11}}(\{S_j^c\}) | S_j^c} \left( \tau_{S^1_{11}}^{c} \right). \]

Repeatedly we can get $G^c_{S^1_{11}}(J)$ such that $S_j^c \in G^c_{S^1_{11}}(J)$ implies $H_j^c < H_j^c$. Then each occurrence of $p \prime$'s is fell into a unique sequent in $G^c_{S^1_{11}}(J)$ which is a copy of a sequent in $G_0$. In such case, we call occurrences of $p \prime$'s in $S^c_{11}$ are separated in $G^c_{S^1_{11}}(J)$ and call such a procedure the separation algorithm. It is the starting point of the separation algorithm. We introduce branches in order to tackle the case of multiple-premise separation derivations for which it is necessary to apply $(EC_{1\alpha})$ to the resulting hypersequents.

A.4 Some questions about Theorem 8.2

In Theorem 8.2, $\tau_{I \rightarrow I}$ is constructed by induction on the number $|I|$ of branches. As usual, we take the algorithm of $|I| - 1$ branches as the induction hypothesis. Why do we take $\tau_{I \rightarrow I}$ and $\tau_{I \rightarrow I}$ as the induction hypothesises?

Roughly speaking, it degenerates the case of $|I|$ branches into the case of two branches in the following sense. The subtree $\tau^\ast(G''/S'')$ of $\tau^\ast$ is as a whole contained in $\tau_{I \rightarrow I}$ or not in it. Similarly, $\tau^\ast(G'/S')$ of $\tau^\ast$ is as a whole contained in $\tau_{I \rightarrow I}$ or not in it. It is such a division of $I$ into $I$ and $I$, that makes the whole algorithm possible.
Claim (i) of Theorem 8.2 asserts that $H^*_j \not\subseteq H^*_j$ for all $S^*_j \in G^*_j$ and $H^*_j \in I$. It guarantees that $\tau^*_j$ is not far from the final aim of Theorem 8.2 but roughly close to it if we define some complexity to calculate it. If $H^*_j \subseteq H^*_j$, the complexity of $G^*_i$ is more than or equal to that of $[S^*_i]_i$ under such a definition of complexity and thus such an application of $\tau^*_j$ is redundant at least. Claim (iii) of Theorem 8.2 guarantees the validity of the step 4 of Stage 1 and 2.

The tree structure of the skeleton of $\tau^*_i (\tau^*_h)$ can be obtained by deleting some node $H \in \tau^*_i$ satisfying $\partial_H (H) \subseteq H^*_i$. The same is true for $\tau^*_i (\tau^*_h)$ if $\tau^*_i (\tau^*_h)$ is treated as a rule or a subroutine whose premises are same as one of $\tau^*_h$. However, it is incredibly difficult to imagine or describe the structure of $\tau^*_i$ if you want to expand it as a normal derivation, a binary tree.

All syntactic manipulations in constructing $\tau^*_i$ are performed on the skeletons of $\tau^*_i$ or $\tau^*_h$.

The structure of the proof of Theorem 8.2 is depicted in the following figure.

**A.5 Illustrations of notations and algorithms**

We use the example in Section 3 to illustrate some notations and algorithms in this paper.

**A.5.1 Illustration of two cases of (COM) in the proof of Lemma 5.6**

Let $G' \rightarrow G''$ (COM) be $p_1 \Rightarrow p_1 \ A \Rightarrow A$, $A \Rightarrow p_1 | p_1 \Rightarrow A$ (COM), where $G' = S_1 = p_1 \Rightarrow p_1 \ ; G'' = S_2 = A \Rightarrow A ; S_3 = A \Rightarrow p_1 ; S_4 = p_1 \Rightarrow A$ and $G''' = S_3[S_4]$. Then $S_3[G''' = S_3[S_4] ; D_{G''}(S_3) \Rightarrow t ; D_{G''}(S_2) = A \Rightarrow A ; D_{G''}(S_3[S_4] = A \Rightarrow A$. Thus the proof of \( \frac{D_{G''}(S_3) \ D_{G''}(S_2)}{D_{G''}(S_3[S_4])} \) is constructed by $A \Rightarrow A (t)$.

Let $G' \rightarrow G''$ (COM) be $B \Rightarrow B \left( \begin{array}{c} \Rightarrow p_2, p_4, A \Rightarrow A, p_1 \Rightarrow p_1, p_2 \Rightarrow A \right) \ A \Rightarrow p_1 \Rightarrow p_3, p_4 \Rightarrow A \Rightarrow A \ (COM)$,

where $G' = S_1 = B \Rightarrow B ; G_2 = p_1, p_2 \Rightarrow A \Rightarrow A \Rightarrow A ; G'' = G_3[S_2 ; S_3] = p_2, p_3, p_4 \Rightarrow A \Rightarrow A \Rightarrow A ; S_2 = p_2, p_4, A \Rightarrow A ; G'' = G_3[S_3[S_4]$.

Then $D_{G''}(S_1) = B \Rightarrow B ; D_{G''}(S_2) = A \Rightarrow A \Rightarrow A \Rightarrow A \Rightarrow A ; D_{G''}(S_3) = A \Rightarrow B, A \Rightarrow A ; D_{G''}(S_4) = A \Rightarrow A \Rightarrow A \Rightarrow A ; D_{G''}(S_3[S_4]) = D_{G''}(S_4) | D_{G''}(S_4)$. 

50
Thus the proof of \( \frac{D_G(S_1)}{D_G(S_2)} \) is constructed by

\[
\begin{align*}
B &\Rightarrow B \quad A, A \Rightarrow A \odot A, \neg A \odot \neg A, A \odot A \\
A &\Rightarrow B, A \odot A | A, B \Rightarrow A \odot A, \neg A \odot \neg A (COM).
\end{align*}
\]

A.5.2 Illustration of Construction 6.1

Let \( \tau^* \) be

\[
\begin{align*}
H_8 &\equiv B \Rightarrow B H_9 \equiv A \Rightarrow A (COM) \\
H_4 &\equiv A \Rightarrow B | B \Rightarrow A (COM) \\
H_5 &\equiv A \Rightarrow B | B \Rightarrow A (COM) \\
H_1 &\equiv A \Rightarrow B \Rightarrow B, \neg A | B, B \Rightarrow A \odot A
\end{align*}
\]

By Construction 6.1, \( \tau^{**} \) is then given as follows.

\[
\begin{align*}
(B \Rightarrow B; 8, 0) &\Rightarrow (A \Rightarrow A; 9, 0) (COM) \\
(A \Rightarrow B; 4, 1) &\Rightarrow (B \Rightarrow B; 4, 2) (COM) \\
(A \Rightarrow B; 5, 1) &\Rightarrow (B \Rightarrow A; 5, 2) (COM) \\
(A \Rightarrow B; 4, 1) &\Rightarrow (B \Rightarrow A; 4, 2) (COM)
\end{align*}
\]

As an example, we calculate \( \varphi(H_8) \). Since \( Th(H_8) = (H_8, H_4, H_5, H_1) \), then \( b_3 = 1, b_2 = b_1 = b_0 = 0 \) by Definition 2.13. Thus \( \varphi(H_8) = b_0 2^0 + b_1 2^1 + b_2 2^2 + b_3 2^3 = 8 \).

Note that we can’t distinguish the one from the other for two \( A \Rightarrow B \)'s in \( H_2 \in \tau^* \). If we divide \( H_2 \) into \( H' | H'' \), where \( H' \equiv A \Rightarrow B \) and \( H'' \equiv A \Rightarrow B | B \Rightarrow A \odot A, \neg A \), then \( H' \cap H'' = \emptyset (A \Rightarrow B) \) in the conventional meaning of hypersequents. Thus only in the sense that we treat \( \tau^* \) as \( \tau^{**} \), the assertion that \( H' \cap H'' = \emptyset \) for any \( H' | H'' \subseteq H \) in Proposition 6.2 holds.

A.5.3 Illustration of Notation 6.10 and Construction 6.11

Let \( I = \{ H_1, H_2 \}, J = \{ H_3, H_4 \}, K = \{ H_5 \}, L = \{ S_{11}, S_{12} \}, M = \{ S_{11}, S_{12} \}, N = \{ S_{21} \}, O = \{ S_{21} \}, \)

\[
G[S' G'' | H'] (\ominus_r) \in \tau^*,
\]

where \( G[G' | G'' | H'] = H'_1 \); \( G' \equiv A \Rightarrow p_1 | p_2, p_2 \Rightarrow A \odot A; S' \equiv \Rightarrow p_2, \neg A; \)

\( G'' \equiv A \Rightarrow p_3 | p_4, p_4 \Rightarrow A \odot A; S'' \equiv \Rightarrow p_4, \neg A; H' \equiv \Rightarrow p_2, \neg A \odot \neg A \)

(See Figure 4).

\[
\begin{align*}
\langle G' | S' \rangle_{L_1} &\Rightarrow p_2, \neg A; \langle G' | S' \rangle_{L_2} = \emptyset; \langle G' | G'' | H' \rangle_{L_1} = A \Rightarrow p_3 | p_4, p_4 \neg A \odot \neg A | p_3, \neg A \odot \neg A A | A \Rightarrow p_3, \neg A \odot \neg A A \Rightarrow A \odot A \rangle \\
\langle G' | G'' \rangle_{L_2} = G'_{S_1} = A \Rightarrow p_1 | p_2, p_2 | p_4, \neg A \odot \neg A | p_3, \neg A \odot \neg A | p_1, \Rightarrow A \odot A \rangle (See Figure 5)
\end{align*}
\]

\[
\begin{align*}
\langle G'' | S' \rangle_{L_2} &\Rightarrow p_4, \neg A; \langle G'' | G'' | H' \rangle_{L_2} = A \Rightarrow p_1 | p_2, p_2 | p_4, \neg A \odot \neg A | p_1, \Rightarrow A \odot A; \\
\langle G' | G'' \rangle_{L_1} = G''_{S_1} = A \Rightarrow p_1 | p_2, p_2 | p_4, \neg A \odot \neg A | p_1, \Rightarrow C | C, \Rightarrow A \odot A \rangle (See Figure 8)
\end{align*}
\]

\[
\begin{align*}
\langle G' | G'' | H' \rangle_{L_2} &\Rightarrow p_2, \neg A \odot \neg A; \langle G' | G'' \rangle_{L_2} = G''_{S_1, S_2} = G''_{S_1} \cap G''_{S_2} \Rightarrow p_2, B | B \Rightarrow p_2, \neg A \odot \neg A (See Figure 10).
\end{align*}
\]
A.5.4 Illustration of Theorem 8.2

Note that sequents in [ ] are principal sequents of elimination rules in the following. Let $I, I, I$ be the same as in A.5.3 and, $I = \{\{S\}_{I}\}, f, f = \{\{S\}_{f}\}, f = \{\{S\}_{f}\}$.

$[S]_{I} = G_{H:G}^{\varnothing} = A \Rightarrow \phi, B \Rightarrow \phi, \phi \Rightarrow A \Rightarrow \phi, A \Rightarrow \phi, C, p_0 \Rightarrow A \Rightarrow \phi, A \Rightarrow \phi, C, p_3 \Rightarrow A \Rightarrow \phi, B$.

Note that sequents in $[ ]$ are principal sequents of elimination rules in the following. Let $I, I, I$ be the same as in A.5.3 and, $I = \{\{S\}_{I}\}, f, f = \{\{S\}_{f}\}, f = \{\{S\}_{f}\}$.

$[S]_{I} = G_{H:G}^{\varnothing} = A \Rightarrow \phi, B \Rightarrow \phi, \phi \Rightarrow A \Rightarrow \phi, A \Rightarrow \phi, C, p_0 \Rightarrow A \Rightarrow \phi, A \Rightarrow \phi, C, p_3 \Rightarrow A \Rightarrow \phi, B$.

A.5.4 Illustration of Theorem 8.2

Note that sequents in [ ] are principal sequents of elimination rules in the following. Let $I, I, I$ be the same as in A.5.3 and, $I = \{\{S\}_{I}\}, f, f = \{\{S\}_{f}\}, f = \{\{S\}_{f}\}$.

$[S]_{I} = G_{H:G}^{\varnothing} = A \Rightarrow \phi, B \Rightarrow \phi, \phi \Rightarrow A \Rightarrow \phi, A \Rightarrow \phi, C, p_0 \Rightarrow A \Rightarrow \phi, A \Rightarrow \phi, C, p_3 \Rightarrow A \Rightarrow \phi, B$.

Then $f = 1, S^{\varnothing} = A \Rightarrow \phi, G_{H:G}^{\varnothing} = A \Rightarrow \phi, B \Rightarrow \phi, \phi \Rightarrow A \Rightarrow \phi, A \Rightarrow \phi, C, p_2 \Rightarrow A \Rightarrow \phi, A \Rightarrow \phi, C, p_3 \Rightarrow A \Rightarrow \phi, A$ in $\tau^{\varnothing}$.
$\tau_{L;G_1}^{\phi(0)} = \frac{[S_1^J]}{G_{1;A,p_1}} \left( \tau_{H_r;A,p_1}^{H_2;A,p_1} \right) \left( EC_{G_1}^* \right)_1,$

where $\hat{\partial}_{L;G_1}([S_1^J]) = H_1', \hat{\partial}_{L;G_1}^{\phi(1)} = H_2' < H_1', \hat{\partial}_{L;G_1}^{\phi(2)} = \hat{\partial}_{L;G_1}^{\phi(3)} (G_{L}^{\phi(2)}) = G_1G^\phi, G_1^{\phi(0)} = G_1^{\phi(2)}, G_1^{\phi(0)} = G_1^{\phi(2)}.

Then $u = 1, S_{(A)}^{(0)} = A \Rightarrow p_3; G_{b_1} = p_5, B \Rightarrow p_8, B \Rightarrow p_8; A \Rightarrow p_8$.

$p_5 \Rightarrow C, p_6 \Rightarrow A \Rightarrow p_1, \neg A \Rightarrow p_1$.

$p_3 \Rightarrow C, p_4 \Rightarrow A \Rightarrow p_1, \neg A \Rightarrow p_1$.

$p_5 \Rightarrow C, p_6 \Rightarrow A \Rightarrow p_1, \neg A \Rightarrow p_1$.

$p_3 \Rightarrow C, p_4 \Rightarrow A \Rightarrow p_1, \neg A \Rightarrow p_1$.

$p_5 \Rightarrow C, p_6 \Rightarrow A \Rightarrow p_1, \neg A \Rightarrow p_1$.

$p_3 \Rightarrow C, p_4 \Rightarrow A \Rightarrow p_1, \neg A \Rightarrow p_1$.

Replacing $\tau_{L;G_1}^{\phi(0)}$ in $\tau_{L;G_1}^{\phi(3)}$, then deleting $G_{b_1}$ and after that applying $< EC_{G_1}^* >$ to $G_{l;A}$ and keeping $G_{b_1}$ unchanged, we get

$\tau_{L;G_1}^{\phi(0)} = \frac{[S_1^J]}{G_{1;A,p_1}} \left( \tau_{H_r;A,p_1}^{H_2;A,p_1} \right) \left( EC_{G_1}^* \right)_1,$

where $G_{H_r;A,p_1}^{\phi(0)} = G_{H_r;A,p_1}^{\phi(0)} = \emptyset; S = p_5, B$.

$\tau_{L;G_1}^{\phi(0)} = \frac{[S_1^J]}{G_{1;A,p_1}} \left( \tau_{H_r;A,p_1}^{H_2;A,p_1} \right) \left( EC_{G_1}^* \right)_1,
Stage 2 \( \tau(L,G_{\Omega}^0) = \tau(L,G_{\Omega}^0(1)) = \tau(L,G_{\Omega}^0(2)) = \frac{[S_2^1]}{G_{1}^n} \{ \tau_{H_2^1}^{A \rightarrow p_1} \} \).

Replacing \( \tau(L,G_{\Omega}^0) \) in \( \tau(L,G_{\Omega}^0(3)) \) with \( \tau(L,G_{\Omega}^0(4)) \), then deleting \( G_{1}^n \) and after that applying \( \langle EC \rangle^0 \) to \( \tilde{S}^{|G_{b_1}|G_{H_2^1; (G_{p})}^{S(J)} |G_{H_2^1; (G_{p})}^{S(J)} \setminus (\tilde{S}^{(S^\circ)})|G_{1}^n} \), we get \( \tau(L,G_{\Omega}^0) \).