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Density Elimination for Semilinear Substructural Logics☆

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Abstract

We present a uniform method of density elimination for several semilinear substructural logics. Especially, the density elimination for the involutive uninorm logic **IUL** is proved. Then the standard completeness of **IUL** follows as a lemma by virtue of previous work by Metcalfe and Montagna.

Keywords: Density elimination, Involutive uninorm logic, Standard completeness of IUL, Semilinear substructural logics, Fuzzy logic
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Contents

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Notation

$G_1 \equiv G_2$ ..... The symbol G_1 denotes a complex hypersequent G_2 temporarily for convenience.
$X := Y$ ..... Define X as Y for two hypersequents (sets or derivations) X and Y.
G_0 ..... The upper hypersequent of strong density rule in Theorem 1.1, Page 4
τ ..... A cut-free proof of G_0 in **GL**, in Theorem 1.1, Page 4
$\mathcal{P}(H)$ ..... The position of $H \in \tau$, Def. 2.13, Construction 6.1, Page 7,26
$\langle H_k \rangle_{H:H'}$ and $\tau^2_{H:H'}(\langle H_k \rangle_{H:H'})$ ..... Construction 4.7, Page 17
$G^2_{H:H'}$ and $\tau^2_{H:H'}$ ..... Notation 4.10, Page 18
τ^* ..... The proof of $G|G^*$ in $\mathbf{GL}_\Omega$ resulting from preprocessing of τ, Notation 4.13, Page 20
$G|G^*$ ..... The root of τ^* corresponding to the root G_0 of τ, Notation 4.13, Page 20
H^c_i ... The i-th (pEC)-node in τ^*, the superscript $'c'$ means contraction, Notation 4.14, Page 20
S^c_{i1} ..... The focus sequent of H^c_i, Notation 4.14, Page 20
S^c_i or S^c_{iu} ..... S^c_{i1} or one copy of S^c_{i1}, Notation 4.14, Page 20
$\{H^c_1, \cdots, H^c_N\}$ ..... The set of all (pEC)-nodes in τ^*, Notation 4.14, Page 20
$\mathbf{GL}_\Omega$ ..... A restricted subsystem of **GL**, Definition 4.16, Page 20
$[S]_G, [G']_G$ ... The minimal closed unit of S and G' in G, respectively, Definition 5.1, Page 22
$(\mathcal{D})$ ..... The generalized density rule of $\mathbf{GL}_\Omega$, Definition 5.4, Page 22
$\tau^*_{S^c_{i1}}$ and $G^*_{S^c_{i1}}$ ..... Notation 6.5. , Page 28
$H^c_i \rightsquigarrow H^c_j, H^c_i \leftrightsquigarrow H^c_j$ ..... Definition 6.8, Page 28
$I = \{H^c_{i_1}, \cdots, H^c_{i_m}\}$ ..... A subset of $\{H^c_1, \cdots, H^c_N\}$, Notation 6.10, Page 29
H^V_I, H^V_{ij} ..... The intersection nodes of I and, that of H^c_i and H^c_j, Notation 6.10, Page 29
$\mathcal{I}' = \{S^c_{i_1u_1}, \cdots, S^c_{i_mu_m}\}$ ..... A subset of (pEC)-sequents to I, Definition 6.14, Page 30
$\mathbf{I}' = \{G_{b_1}|S^c_{i_1u_1}, \cdots, G_{b_m}|S^c_{i_mu_m}\}$ ..... A set of closed hypersequents to I, Def. 6.14, Page 30
$\langle H \rangle_{\mathcal{I}}, \tau^*_{\mathcal{I}}$ and $G^*_{\mathcal{I}}$ ..... The elimination derivation, Construction 6.11, Lemma 6.13, Page 29, 30
$\tau^*_{\mathbf{I}'}$ ..... The elimination rule, Definition 6.14, , Page 30
$\left[S^c_{i_k}\right]_I$ ..... A branch of $H^c_{i_k}$ to I, Definition 7.2, Page 31
$G^{\star(q)}_{\mathbf{I}}, G^{\star(J_1)}_{\mathbf{I}}, \tau^{\star(q)}_{\mathbf{I}}$ ..... Construction 7.3, Page 31
$G^{\star(J)}_{H:H_1}, \tau^{\star(J)}_{H:H_1}$ ..... Construction 7.5, Page 33
$G^{\star}_{\mathbf{I}}, \tau^{\star}_{\mathbf{I}}$ ..... Construction 7.7, Theorem 8.2, Page 34, 36
$\overline{\tau}^{\star}_{\mathbf{I}}$ ..... The skeleton of $\tau^{\star}_{\mathbf{I}}$, Definition 7.13, Page 35
$\partial_{\tau^{\star}_{\mathbf{I}}}(H)$ ..... Theorem 8.2 (ii), Page 36
$\tau^{\star}_{\mathbf{I}:G_2}$ ..... The module of $\tau^{\star}_{\mathbf{I}}$ at G_2, Definition 8.7, Page 38

1. Introduction

The problem of the completeness of Łukasiewicz infinite-valued logic (**Ł** for short) was posed by Łukasiewicz and Tarski in the 1930s. It was twenty-eight years later that it was syntactically solved by Rose and Rosser [20]. Chang [4] developed at almost the same time a theory of algebraic systems for **Ł**, which is called **MV**-algebras, with an attempt to make **MV**-algebras correspond to **Ł** as Boolean algebras to the classical two-valued logic. Chang [5] subsequently finished another proof for the completeness of **Ł** by virtue of his **MV**-algebras.

It was Chang who observed that the key role in the structure theory of **MV**-algebras is not locally finite **MV**-algebras but linearly ordered ones. The observation was formalized by Hájek

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[12] who showing the completeness for his basic fuzzy logic (**BL** for short) with respect to linearly ordered **BL**-algebras. Starting with the structure of **BL**-algebras, Hájek [13] reduced the problem of the standard completeness of **BL** to two formulas to be provable in **BL**. Here and thereafter, by the standard completeness we mean that logics are complete with respect to algebras with lattice reduct [0, 1]. Cignoli et al. [6] subsequently proved the standard completeness of **BL**, i.e., **BL** is the logic of continuous t-norms and their residua.

Hajek's approach toward fuzzy logic has been extended by Esteva and Godo in [9], where the authors introduced the logic **MTL** which aims at capturing the tautologies of left-continuous t-norms and their residua. The standard completeness of **MTL** was proved by Jenei and Montagna in [15], where the major step is to embed linearly ordered **MTL**-algebras into the dense ones under the situation that the structure of **MTL**-algebras has been unknown as yet.

Esteva and Godo's work was further promoted by Metcalfe and Montagna [16] who introduced the uninorm logic **UL** and involutive uninorm logic **IUL** which aim at capturing tautologies of left-continuous uninorms and their residua and those of involutive left-continuous ones, respectively. Recently, Cintula and Noguera [8] introduced semilinear substructural logics which are substructural logics complete with respect to linearly ordered models. Almost all well-known families of fuzzy logics such as **Ł**, **BL**, **MTL**, **UL** and **IUL** belong to the class of semilinear substructural logics.

Metcalfe and Montagna's method to prove standard completeness for **UL** and its extensions is of proof theory in nature and consists of two key steps. Firstly, they extended **UL** with the density rule of Takeuti and Titani [21]:

$$\frac{\Gamma \vdash (A \to p) \vee (p \to B) \vee C}{\Gamma \vdash (A \to B) \vee C}(D)$$

where p does not occur in Γ, A, B or C, and then prove the logics with (D) are complete with respect to algebras with lattice reduct [0,1]. Secondly, they give a syntactic elimination of (D) that was formulated as a rule of the corresponding hypersequent calculus.

Hypersequents are a natural generalization of sequents which were introduced independently by Avron [1] and Pottinger [19] and have proved to be particularly suitable for logics with prelinearity [2, 16]. Following the spirit of Gentzen's cut elimination, Metcalfe and Montagna succeeded to eliminate the density rule for **GUL** and several extensions of **GUL** by induction on the height of a derivation of the premise and shifting applications of the rule upwards, but failed for **GIUL** and therefore left it as an open problem.

There are several relevant works about the standard completeness of **IUL** as follows. With an attempt to prove the standard completeness of **IUL**, we generalized Jenei and Montagna's method for **IMTL** in [22], but our effort was only partially successful. It seems that the subtle reason why it does not work for **UL** and **IUL** is the failure of FMP of these systems [23]. Jenei [14] constructed several classes of involutive FL_e-algebras, as he said, in order to gain a better insight into the algebraic semantic of the substructural logic **IUL**, and also to the long-standing open problem about its standard completeness. Ciabattoni and Metcalfe [7] introduced the method of density elimination by substitutions which is applicable to a general classes of (first-order) hypersequent calculi but fails to the case of **GIUL**.

We reconsidered Metcalfe and Montagna's proof-theoretic method to investigate the standard completeness of **IUL**, because they have proved the standard completeness of **UL** by their method and we can't prove such a result by the Jenei and Montagna's model-theoretic method. In order to prove the density elimination for **GUL**, they prove that the following generalized density rule $(\mathcal{D}_1)$:

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$$\frac{G_0 \equiv \{\Gamma_i, \lambda_i p \Rightarrow \Delta_i\}_{i=1\cdots n} | \{\Sigma_k, (\mu_k+1)p \Rightarrow p\}_{k=1\cdots o} | \{\Pi_j \Rightarrow p\}_{j=1\cdots m}}{\mathcal{D}_1(G_0) \equiv \{\Gamma_i, \lambda_i \Pi_j \Rightarrow \Delta_i\}_{i=1\cdots n}^{j=1\cdots m} | \{\Sigma_k, \mu_k \Pi_j \Rightarrow t\}_{k=1\cdots o}^{j=1\cdots m}} (\mathcal{D}_1)$$

is admissible for **GUL**, where they set two constraints to the form of G_0: (i) $n, m \geqslant 1$ and $\lambda_i \geqslant 1$ for some $1 \leqslant i \leqslant n$; (ii) p does not occur in Γ_i, Δ_i, Π_j, Σ_k for $i = 1 \cdots n$, $j = 1 \cdots m$, $k = 1 \cdots o$.

We may regard $(\mathcal{D}_1)$ as a procedure whose input and output are the premise and conclusion of $(\mathcal{D}_1)$, respectively. We denote the conclusion of $(\mathcal{D}_1)$ by $\mathcal{D}_1(G_0)$ when its premise is G_0. Observe that Metcalfe and Montagna had succeeded to define the suitable conclusion for almost arbitrary premise in $(\mathcal{D}_1)$, but it seems impossible for **GIUL** (See Section 3 for an example). We then define the following generalized density rule $(\mathcal{D}_0)$ for

$$\mathbf{GL} \in \{\mathbf{GUL}, \mathbf{GIUL}, \mathbf{GMTL}, \mathbf{GIMTL}\}$$

and prove its admissibility in Section 9.

Theorem 1.1 (Main theorem). *Let $n, m \geqslant 1$, p does not occur in $G', \Gamma_i, \Delta_i, \Pi_j$ or Σ_j for all $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$. Then the strong density rule*

$$\frac{G_0 \equiv G' | \{\Gamma_i, p \Rightarrow \Delta_i\}_{i=1\cdots n} | \{\Pi_j \Rightarrow p, \Sigma_j\}_{j=1\cdots m}}{\mathcal{D}_0(G_0) \equiv G' | \{\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j\}_{i=1\cdots n; j=1\cdots m}} (\mathcal{D}_0)$$

is admissible in **GL**.

In proving the admissibility of $(\mathcal{D}_1)$, Metcalfe and Montagna made some restriction on the proof τ of G_0, i.e., converted τ into an r-proof. The reason why they need an r-proof is that they set the constraint (i) to G_0. We may imagine the restriction on τ and the constraints to G_0 as two pallets of a balance, i.e., one is strong if another is weak and vice versa. Observe that we select the weakest form of G_0 in $(\mathcal{D}_0)$ that guarantees the validity of (D). Then it is natural that we need make the strongest restriction on the proof τ of G_0. But it seems extremely difficult to follow such a way to prove the admissibility of $(\mathcal{D}_0)$.

In order to overcome such a difficulty, we first of all choose Avron-style hypersequent calculi as the underlying systems (See A.1). Let τ be a cut-free proof of G_0 in **GL**. Starting with τ, we construct a proof τ^* of $G|G^*$ in a restricted subsystem $\mathbf{GL}_\Omega$ of **GL** by a systematic novel manipulations in Section 4. Roughly speaking, each sequent of G is a copy of some sequent of G_0, and each sequent of G^* is a copy of some contraction sequent in τ. In Section 5, we define the generalized density rule $(\mathcal{D})$ in $\mathbf{GL}_\Omega$ and prove that it is admissible.

Now, starting with $G|G^*$ and its proof τ^*, we construct a proof $\tau^{\star}$ of $G^{\star}$ in $\mathbf{GL}_\Omega$ such that each sequent of $G^{\star}$ is a copy of some sequent of G. Then $\vdash_{\mathbf{GL}_\Omega} \mathcal{D}(G^{\star})$ by the admissibility of $(\mathcal{D})$. Then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$ by Lemma 9.1. Hence the density elimination theorem holds in **GL**. Especially, the standard completeness of **IUL** follows from Theorem 62 of [16].

$G^{\star}$ is constructed by eliminating (pEC)-sequents in $G|G^*$ one by one. In order to control the process, we introduce the set $I = \{H_{i_1}^c, \cdots, H_{i_m}^c\}$ of (pEC)-nodes of τ^* and the set $\mathbf{I}$ of the branches relative to I and construct $G_{\mathbf{I}}^{\star}$ such that $G_{\mathbf{I}}^{\star}$ doesn't contain (pEC)-sequents lower than any node in I, i.e., $S_j^c \in G_{\mathbf{I}}^{\star}$ implies $H_j^c || H_i^c$ for all $H_i^c \in I$. The procedure is called the separation algorithm of branches in which we introduce another novel manipulation and call it derivation-grafting operation in Section 7, 8.

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2. Preliminaries

In this section, we recall the basic definitions and results involved, which are mainly from [16]. Substructural fuzzy logics are based on a countable propositional language with formulas FOR built inductively as usual from a set of propositional variables VAR, binary connectives $\odot, \to, \wedge, \vee$, and constants $\bot, \top, t, f$ with definable connective $\neg A := A \to f$.

Definition 2.1. ([1, 16]) A sequent is an ordered pair (Γ, Δ) of finite multisets (possibly empty) of formulas, which we denote by $\Gamma \Rightarrow \Delta$. Γ and Δ are called the antecedent and succedents, respectively, of the sequent and each formula in Γ and Δ is called a sequent-formula. A hypersequent G is a finite multiset of the form $\Gamma_1 \Rightarrow \Delta_1|\cdots|\Gamma_n \Rightarrow \Delta_n$, where each $\Gamma_i \Rightarrow \Delta_i$ is a sequent and is called a component of G for each $1 \leqslant i \leqslant n$. If Δ_i contains at most one formula for $i = 1\cdots n$, then the hypersequent is single-conclusion, otherwise it is multiple-conclusion.

Definition 2.2. Let S be a sequent and $G = S_1|\cdots|S_m$ a hypersequent. We say that $S \in G$ if S is one of $S_1, \cdots, S_m$.

Notation 2.3. *Let G_1 and G_2 be two hypersequents. We will assume from now on that all set terminology refers to multisets, adopting the conventions of writing Γ, Δ for the multiset union of Γ and Δ, A for the singleton multiset $\{A\}$, and $\lambda\Gamma$ for the multiset union of λ copies of Γ for $\lambda \in \mathbf{N}$. By $G_1 \subseteq G_2$ we mean that $S \in G_2$ for all $S \in G_1$ and the multiplicity of S in G_1 is not more than that of S in G_2. We will use $G_1 = G_2$, $G_1 \bigcap G_2$, $G_1 \bigcup G_2$, $G_1 \backslash G_2$ by their standard meaning for multisets by default and we will declare when we use them for sets. We sometimes write $S_1|\cdots|S_m$ and $G|\overbrace{S|\cdots|S}^{n\ copies}$ as $\{S_1, \cdots, S_m\}$, $G|S^n$(or $G|\{S\}^n$), respectively.*

Definition 2.4. ([1]) A hypersequent rule is an ordered pair consisting of a sequence of hypersequents $G_1, \cdots, G_n$ called the premises (upper hypersequents) of the rule, and a hypersequent G called the conclusion (lower hypersequent), written by $\dfrac{G_1 \cdots G_n}{G}$. If $n = 0$, then the rule has no premise and is called an initial sequent. The single-conclusion version of a rule adds the restriction that both the premises and conclusion must be single-conclusion; otherwise the rule is multiple-conclusion.

Definition 2.5. ([16]) **GUL** and **GIUL** consist of the single-conclusion and multiple-conclusion versions of the following initial sequents and rules, respectively:
Initial Sequents

$$\frac{}{A \Rightarrow A}(ID) \quad \frac{}{\Gamma \Rightarrow \top, \Delta}(\top_r) \quad \frac{}{\Gamma, \bot \Rightarrow \Delta}(\bot_l) \quad \frac{}{\Rightarrow t}(t_r) \quad \frac{}{f \Rightarrow}(f_l)$$

Structural Rules

$$\frac{G|\Gamma \Rightarrow A|\Gamma \Rightarrow A}{G|\Gamma \Rightarrow A}(EC) \quad \frac{G}{G|\Gamma \Rightarrow A}(EW)$$

$$\frac{G_1|\Gamma_1, \Pi_1 \Rightarrow \Sigma_1, \Delta_1 \quad G_2|\Gamma_2, \Pi_2 \Rightarrow \Sigma_2, \Delta_2}{G_1|G_2|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|\Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}(COM)$$

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Logical Rules

$$\frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma, t, \Rightarrow \Delta}(t_l) \qquad \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow f, \Delta}(f_r)$$

$$\frac{G_1|\Gamma_1 \Rightarrow A, \Delta_1 \quad G_2|\Gamma_2, B \Rightarrow \Delta_2}{G_1|G_2|\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2}(\to_l) \qquad \frac{G|\Gamma, A \Rightarrow B, \Delta}{G|\Gamma \Rightarrow A \to B, \Delta}(\to_r)$$

$$\frac{G|\Gamma, A, B \Rightarrow \Delta}{G|\Gamma, A \odot B \Rightarrow \Delta}(\odot_l) \qquad \frac{G_1|\Gamma_1 \Rightarrow A, \Delta_1 \;\; G_2|\Gamma_2 \Rightarrow B, \Delta_2}{G_1|G_2|\Gamma_1, \Gamma_2 \Rightarrow A \odot B, \Delta_1, \Delta_2}(\odot_r)$$

$$\frac{G|\Gamma, A \Rightarrow \Delta}{G|\Gamma, A \wedge B \Rightarrow \Delta}(\wedge_{lr}) \qquad \frac{G|\Gamma, B \Rightarrow \Delta}{G|\Gamma, A \wedge B \Rightarrow \Delta}(\wedge_{ll})$$

$$\frac{G_1|\Gamma \Rightarrow A, \Delta \quad G_2|\Gamma \Rightarrow B, \Delta}{G_1|G_2|\Gamma \Rightarrow A \wedge B, \Delta}(\wedge_r) \qquad \frac{G|\Gamma \Rightarrow A, \Delta}{G|\Gamma \Rightarrow A \vee B, \Delta}(\vee_{rr})$$

$$\frac{G|\Gamma \Rightarrow B, \Delta}{G|\Gamma \Rightarrow A \vee B, \Delta}(\vee_{rl}) \qquad \frac{G_1|\Gamma, A \Rightarrow \Delta \quad G_2|\Gamma, B \Rightarrow \Delta}{G_1|G_2|\Gamma, A \vee B \Rightarrow \Delta}(\vee_l)$$

Cut Rule

$$\frac{G_1|\Gamma_1, A \Rightarrow \Delta_1 \quad G_2|\Gamma_2 \Rightarrow A, \Delta_2}{G_1|G_2|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}(CUT)$$

Definition 2.6. ([16]) **GMTL** and **GIMTL** are **GUL** and **GIUL** plus the single conclusion and multiple-conclusion versions, respectively, of:

$$\frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma, A \Rightarrow \Delta}(WL), \quad \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow A, \Delta}(WR).$$

Definition 2.7. (i) $(I) \in \{(t_l), (f_r), (\to_r), (\odot_l), (\wedge_{lr}), (\wedge_{ll}), (\vee_{rr}), (\vee_{rl}), (WL), (WR)\}$ and $(II) \in \{(\to_l), (\odot_r), (\wedge_r), (\vee_l), (COM)\}$;

(ii) By $\frac{G'|S' \quad G''|S''}{G'|G''|H'}(II)$ (or $\frac{G'|S'}{G'|H'}(I)$) we denote an instance of a two-premise rule (II) (or one-premise rule (I)) of **GL**, where S' and S'' are its focus sequents and H' is its principle sequent (for $(\to_l)$, $(\odot_r)$, $(\wedge_r)$ and $(\vee_l)$) or hypersequent (for (COM), $(\wedge_{rw})$ and $(\vee_{lw})$, see Definition 4.2).

Definition 2.8. ([16]) **GL**$^{\mathrm{D}}$ is **GL** extended with the following density rule:

$$\frac{G|\Gamma_1, p \Rightarrow \Delta_1|\Gamma_2 \Rightarrow p, \Delta_2}{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}(D)$$

where p does not occur in $G, \Gamma_1, \Gamma_2, \Delta_1$ or Δ_2.

Definition 2.9. ([1]) A derivation τ of a hypersequent G from hypersequents $G_1, \cdots, G_n$ in a hypersequent calculus **GL** is a labeled tree with the root labeled by G, leaves labeled initial sequents or some $G_1, \cdots, G_n$, and for each node labeled G'_0 with parent nodes labeled $G'_1, \cdots, G'_m$ (where possibly $m = 0$), $\frac{G'_1 \cdots G'_m}{G'_0}$ is an instance of a rule of **GL**.

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Notation 2.10. *(i)* $\dfrac{G_1 \cdots G_n}{G_0}\langle\tau\rangle$ *denotes that* τ *is a derivation of* G_0 *from* $G_1, \cdots, G_n$;

(ii) Let H *be a hypersequent.* $H \in \tau$ *denotes that* H *is a node of* τ*. We call* H *a leaf hypersequent if* H *is a leaf of* τ*, the root hypersequent if it is the root of* τ*.* $\dfrac{G'_1 \cdots G'_m}{G'_0} \in \tau$ *denotes that* $G'_0 \in \tau$ *and its parent nodes are* $G'_1, \cdots, G'_m$;

(iii) Let $H \in \tau$ *then* $\tau(H)$ *denotes the subtree of* τ *rooted at* H;

(iv) τ *determines a partial order* $\leqslant_\tau$ *with the root as the least element.* $H_1 \| H_2$ *denotes* $H_1 \nleqslant_\tau H_2$ *and* $H_2 \nleqslant_\tau H_1$ *for any* $H_1, H_2 \in \tau$*. By* $H_1 =_\tau H_2$ *we mean that* H_1 *is the same node as* H_2 *in* τ*. We sometimes write* $\leqslant_\tau$ *as* $\leqslant$;

(v) An inference of the form $\dfrac{G'|S^n}{G'|S} \in \tau$ *is called the full external contraction and denoted by* (EC^*)*, if* $n \geqslant 2$*,* $G'|S^n$ *is not a lower hypersequent of an application of* (EC) *whose contraction sequent is* S*, and* $G'|S$ *not an upper one in* τ*.*

Definition 2.11. Let τ be a derivation of G and $H \in \tau$. The thread $Th_\tau(H)$ of τ at H is a sequence $H_0, \cdots, H_n$ of node hypersequents of τ such that $H_0 =_\tau H$, $H_n =_\tau G$, $\dfrac{H_k}{H_{k+1}} \in \tau$ or there exists $G' \in \tau$ such that $\dfrac{H_k \quad G'}{H_{k+1}}$ or $\dfrac{G' \quad H_k}{H_{k+1}}$ in τ for all $0 \leqslant k \leqslant n-1$.

Proposition 2.12. *Let* $H_1, H_2 \in \tau$*. Then*
(i) $H_1 \leqslant H_2$ *if and only if* $H_1 \in Th_\tau(H_2)$;
(ii) $H_1 \| H_2$ *and* $H_1 \leqslant H_3$ *imply* $H_2 \| H_3$;
(iii) $H_1 \leqslant H_3$ *and* $H_2 \leqslant H_3$ *imply* $H_1 \nparallel H_2$.

We need the following definition to give each node of τ an identification number, which is used in Construction 6.1 to differentiate sequents in a hypersequent in a proof.

Definition 2.13. ([A.5.2]) Let $H \in \tau$ and $Th(H) = (H_0, \cdots, H_n)$. Let $b_n := 1$,

$$b_k := \begin{cases} 1 & \text{if } \dfrac{G' \ H_k}{H_{k+1}} \in \tau, \\ 0 & \text{if } \dfrac{H_k}{H_{k+1}} \in \tau \text{ or } \dfrac{H_k \ G'}{H_{k+1}} \in \tau \end{cases}$$

for all $0 \leqslant k \leqslant n-1$. Then $\mathcal{P}(H) := \sum_{k=0}^{k=n} 2^k b_k$ and call it the position of H in τ.

Definition 2.14. A rule is admissible for a calculus **GL** if whenever its premises are derivable in **GL**, then so is its conclusion.

Lemma 2.15. ([16]) *Cut-elimination holds for* **GL***, i.e., proofs using* (CUT) *can be transformed syntactically into proofs not using* (CUT)*.*

3. Proof of the main theorem: A computational example

In this section, we present an example to illustrate the proof of Main theorem.

Let $G_0 \equiv \Rightarrow p, B|B \Rightarrow p, \neg A \odot \neg A|p \Rightarrow C|C, p \Rightarrow A \odot A$. G_0 is a theorem of **IUL** and a cut-free proof τ of G_0 is shown in Figure 1, where we use an additional rule $\dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}(\neg_r)$ for simplicity.

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Note that we denote three applications of (EC) in τ respectively by $(EC)_1, (EC)_2, (EC)_3$ and three $(\odot_r)$ by $(\odot_r)_1, (\odot_r)_2$ and $(\odot_r)_3$.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p \quad A \Rightarrow A}{A \Rightarrow p|p \Rightarrow A}(COM) \quad \dfrac{p \Rightarrow p \quad A \Rightarrow A}{A \Rightarrow p|p \Rightarrow A}(COM)}{A \Rightarrow p|A \Rightarrow p|p, p \Rightarrow A \odot A}(\odot_r)_1}{A \Rightarrow p|p, p \Rightarrow A \odot A}(EC)_1}{\Rightarrow p, \neg A|p, p \Rightarrow A \odot A}(\neg_r)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p \quad A \Rightarrow A}{A \Rightarrow p|p \Rightarrow A}(COM) \quad \dfrac{p \Rightarrow p \quad A \Rightarrow A}{A \Rightarrow p|p \Rightarrow A}(COM)}{A \Rightarrow p|A \Rightarrow p|p, p \Rightarrow A \odot A}(\odot_r)_2}{A \Rightarrow p|p, p \Rightarrow A \odot A}(EC)_2}{\Rightarrow p, \neg A|p, p \Rightarrow A \odot A}(\neg_r)$$

(continued)

$$\dfrac{\overline{C \Rightarrow C} \quad \dfrac{\overline{B \Rightarrow B} \quad \dfrac{\dfrac{\begin{array}{c}\vdots \\ \Rightarrow p, \neg A|p, p \Rightarrow A \odot A\end{array} \quad \begin{array}{c}\vdots \\ \Rightarrow p, \neg A|p, p \Rightarrow A \odot A\end{array}}{H_{\times\times} \equiv\Rightarrow p, p, \neg A \odot \neg A|p, p \Rightarrow A \odot A|p, p \Rightarrow A \odot A}(\odot_r)_3}{H_{\times} \equiv\Rightarrow p, p, \neg A \odot \neg A|p, p \Rightarrow A \odot A}(EC)_3}{\Rightarrow p, B|B \Rightarrow p, \neg A \odot \neg A|p, p \Rightarrow A \odot A}(COM)}{\Rightarrow p, B|B \Rightarrow p, \neg A \odot \neg A|p \Rightarrow C|C, p \Rightarrow A \odot A}(COM)$$

FIGURE 1 A proof τ of G_0

By applying (D) to free combinations of all sequents in $\Rightarrow p, B|B \Rightarrow p, \neg A \odot \neg A$ and in $p \Rightarrow C|C, p \Rightarrow A \odot A$, we get that $H_0 \equiv\Rightarrow B, C|C \Rightarrow A \odot A, B|B \Rightarrow C, \neg A \odot \neg A|C, B \Rightarrow A \odot A, \neg A \odot \neg A$. H_0 is a theorem of **IUL** and a cut-free proof ρ of H_0 is shown in Figure 2. It supports the validity of the generalized density rule $(\mathcal{D}_0)$ in Section 1, as an instance of $(\mathcal{D}_0)$.

$$\dfrac{\overline{C \Rightarrow C} \quad \dfrac{\overline{B \Rightarrow B} \quad \dfrac{\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A, A \Rightarrow A \odot A}}{A \Rightarrow \neg A, A \odot A} \quad \dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A, A \Rightarrow A \odot A}}{A \Rightarrow \neg A, A \odot A}}{A, A \Rightarrow \neg A \odot \neg A, A \odot A, A \odot A}}{A, B \Rightarrow A \odot A, \neg A \odot \neg A|A \Rightarrow A \odot A, B}}{H_1 \equiv A \Rightarrow C|A, B \Rightarrow A \odot A, \neg A \odot \neg A|C \Rightarrow A \odot A, B}$$

$$\dfrac{\dfrac{\overline{C \Rightarrow C} \quad \dfrac{\overline{C \Rightarrow C} \quad \dfrac{\overline{B \Rightarrow B} \quad \dfrac{\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A, A \Rightarrow A \odot A}}{A \Rightarrow \neg A, A \odot A} \quad \dfrac{\begin{array}{c}\vdots \\ H_1 = A \Rightarrow C|A, B \Rightarrow A \odot A, \neg A \odot \neg A|C \Rightarrow A \odot A, B\end{array}}{\Rightarrow \neg A, C|A, B \Rightarrow A \odot A, \neg A \odot \neg A|C \Rightarrow A \odot A, B}}{A \Rightarrow \neg A \odot \neg A, A \odot A, C|A, B \Rightarrow A \odot A, \neg A \odot \neg A|C \Rightarrow A \odot A, B}}{\Rightarrow B, C|A, B \Rightarrow \neg A \odot \neg A, A \odot A|A, B \Rightarrow A \odot A, \neg A \odot \neg A|C \Rightarrow A \odot A, B}}{A \Rightarrow C| \Rightarrow B, C|C, B \Rightarrow \neg A \odot \neg A, A \odot A|A, B \Rightarrow A \odot A, \neg A \odot \neg A|C \Rightarrow A \odot A, B}}{A \Rightarrow C|A \Rightarrow C| \Rightarrow B, C|C, B \Rightarrow \neg A \odot \neg A, A \odot A|C, B \Rightarrow A \odot A, \neg A \odot \neg A|C \Rightarrow A \odot A, B}}{H_2 \equiv A \Rightarrow C| \Rightarrow B, C|C, B \Rightarrow A \odot A, \neg A \odot \neg A|C \Rightarrow A \odot A, B}$$

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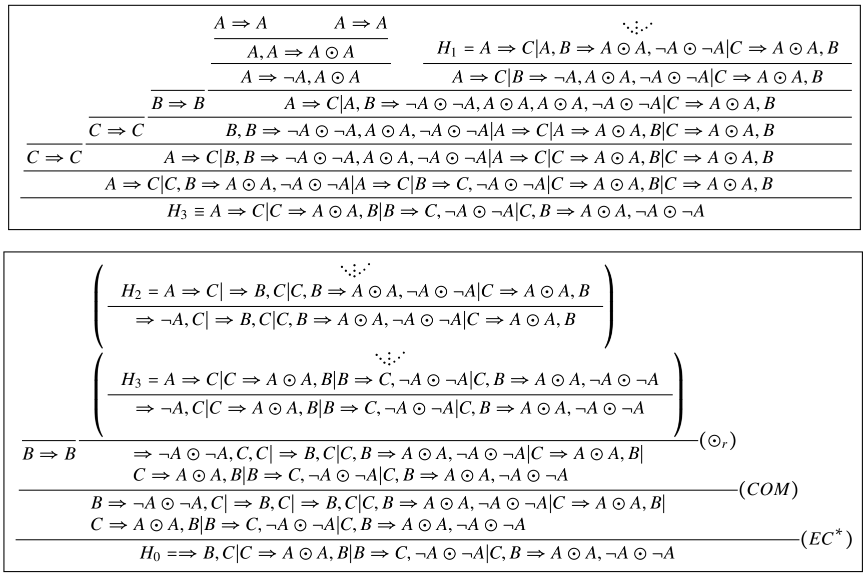

Figure 2 a proof ρ of H_0

Our task is to construct ρ, starting from τ. The tree structure of ρ is more complicated than that of τ. Compared with **UL**, **MTL** and **IMTL**, there is no one-to-one correspondence between nodes in τ and ρ.

Following the method given by G. Metcalfe and F. Montagna, we need to define a generalized density rule for **IUL**. We denote such an expected unknown rule by $(\mathcal{D}_x)$ for convenience. Then $\mathcal{D}_x(H)$ must be definable for all $H \in \tau$. Naturally, $\mathcal{D}_x(p \Rightarrow p) \Longrightarrow t; \mathcal{D}_x(A \Rightarrow p|p \Rightarrow A) = A \Rightarrow A; \mathcal{D}_x(\Rightarrow p, \neg A|p, p \Rightarrow A \odot A) \Longrightarrow \neg A, \neg A, A \odot A; \mathcal{D}_x(\Rightarrow p, B|B \Rightarrow p, \neg A \odot \neg A|p, p \Rightarrow A \odot A) \Longrightarrow B, B, A \odot A|B, B \Rightarrow A \odot A, \neg A \odot \neg A, \neg A \odot \neg A|B \Rightarrow A \odot A, B, \neg A \odot \neg A; \mathcal{D}_x(G_0) = \mathcal{D}_0(G_0) = H_0$.

However, we couldn't find a suitable way to define $\mathcal{D}_x(H_{\times\times})$ and $\mathcal{D}_x(H_\times)$ for $H_{\times\times}$ and $H_\times$ in τ, see Figure 1. This is the biggest difficulty we encounter in the case of **IUL** such that it is hard to prove density elimination for **IUL**. A possible way is to define $\mathcal{D}_x(\Rightarrow p, p, \neg A \odot \neg A|p, p \Rightarrow A \odot A)$ as $\Rightarrow t, A \odot A, \neg A \odot \neg A$. Unfortunately, it is not a theorem of **IUL**.

Notice that two upper hypersequents $\Rightarrow p, \neg A|p, p \Rightarrow A \odot A$ of $(\odot_r)_3$ are permissible inputs of $(\mathcal{D}_x)$. Why is $H_{\times\times}$ an invalid input? One reason is that, two applications $(EC)_1$ and $(EC)_2$ cut off two sequents $A \Rightarrow p$ such that two $p's$ disappear in all nodes lower than upper hypersequent of $(EC)_1$ or $(EC)_2$, including $H_{\times\times}$. These make occurrences of $p's$ to be incomplete in $H_{\times\times}$. We then perform the following operation in order to get complete occurrences of $p's$ in $H_{\times\times}$.

Step 1 (Preprocessing of τ). Firstly, we replace H with $H|S'$ for all $\dfrac{G'|S'|S'}{G'|S'}(EC)_k \in \tau$, $H \leqslant G'|S'$ then replace $\dfrac{G'|S'|S'}{G'|S'|S'}(EC)_k$ with $G'|S'|S'$ for all $k = 1, 2, 3$. Then we construct a proof without (EC), which we denote by τ_1, as shown in Figure 3. We call such manipulations

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sequent-inserting operations, which eliminate applications of (EC) in τ.

$$
\begin{array}{c}
p \Rightarrow p \quad A \Rightarrow A \qquad p \Rightarrow p \quad A \Rightarrow A \qquad\qquad p \Rightarrow p \quad A \Rightarrow A \qquad p \Rightarrow p \quad A \Rightarrow A \\
\hline
A \Rightarrow p|p \Rightarrow A \qquad A \Rightarrow p|p \Rightarrow A \qquad\qquad A \Rightarrow p|p \Rightarrow A \qquad A \Rightarrow p|p \Rightarrow A \\
\hline
A \Rightarrow p|A \Rightarrow p|p, p \Rightarrow A \odot A \qquad\qquad A \Rightarrow p|A \Rightarrow p|p, p \Rightarrow A \odot A \\
\hline
B \Rightarrow B \quad A \Rightarrow p| \Rightarrow p, \neg A|p, p \Rightarrow A \odot A \qquad A \Rightarrow p| \Rightarrow p, \neg A|p, p \Rightarrow A \odot A \\
\hline
C \Rightarrow C \quad H'_{\times\times} \equiv A \Rightarrow p| \Rightarrow p, p, \neg A \odot \neg A|p, p \Rightarrow A \odot A|A \Rightarrow p|p, p \Rightarrow A \odot A \\
\hline
A \Rightarrow p| \Rightarrow p, B|B \Rightarrow p, \neg A \odot \neg A|p, p \Rightarrow A \odot A|A \Rightarrow p|p, p \Rightarrow A \odot A \\
\hline
A \Rightarrow p| \Rightarrow p, B|B \Rightarrow p, \neg A \odot \neg A|p \Rightarrow C|C, p \Rightarrow A \odot A|A \Rightarrow p|p, p \Rightarrow A \odot A
\end{array}
$$

FIGURE 3 A proof τ_1

However, we also can't define $\mathcal{D}_x(H'_{\times\times})$ for $H'_{\times\times} \in \tau_1$ in that $\Rightarrow p, p, \neg A \odot \neg A|p, p \Rightarrow A \odot A \subseteq H'_{\times\times}$. The reason is that the origins of $p's$ in $H'_{\times\times}$ are indistinguishable if we regard all leaves in the form $p \Rightarrow p$ as the origins of $p's$ which occur in the inner node. For example, we don't know which p comes from the left subtree of $\tau_1(H'_{\times\times})$ and which from the right subtree in two occurrences of $p's$ in $\Rightarrow p, p, \neg A \odot \neg A \in H'_{\times\times}$. We then perform the following operation in order to make all occurrences of $p's$ in $H'_{\times\times}$ distinguishable.

We assign the unique identification number to each leaf in the form $p \Rightarrow p \in \tau_1$ and transfer these identification numbers from leaves to the root, as shown in Figure 4. We denote the proof of $G|G^*$ resulting from this step by τ^*, where $G \equiv \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A$ in which each sequent is a copy of some sequent in G_0 and $G^* \equiv A \Rightarrow p_1|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A$ in which each sequent is a copy of some external contraction sequent in (EC)-node of τ. We call such manipulations eigenvariable-labeling operations, which make us to trace eigenvariables in τ.

$$
\begin{array}{c}
p_1 \Rightarrow p_1 \quad A \Rightarrow A \qquad p_2 \Rightarrow p_2 \quad A \Rightarrow A \qquad\qquad p_3 \Rightarrow p_3 \quad A \Rightarrow A \quad p_4 \Rightarrow p_4 \quad A \Rightarrow A \\
\hline
A \Rightarrow p_1|p_1 \Rightarrow A \qquad A \Rightarrow p_2|p_2 \Rightarrow A \qquad\qquad A \Rightarrow p_3|p_3 \Rightarrow A \quad A \Rightarrow p_4|p_4 \Rightarrow A \\
\hline
H_1^c \equiv A \Rightarrow p_1|A \Rightarrow p_2|p_1, p_2 \Rightarrow A \odot A \quad H_2^c \equiv A \Rightarrow p_3|A \Rightarrow p_4|p_3, p_4 \Rightarrow A \odot A \\
\hline
B \Rightarrow B \quad A \Rightarrow p_1| \Rightarrow p_2, \neg A|p_1, p_2 \Rightarrow A \odot A \qquad A \Rightarrow p_3| \Rightarrow p_4, \neg A|p_3, p_4 \Rightarrow A \odot A \\
\hline
C \Rightarrow C \quad A \Rightarrow p_1| \Rightarrow p_2, p_4, \neg A \odot \neg A|p_1, p_2 \Rightarrow A \odot A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A \\
\hline
H_3^c \equiv A \Rightarrow p_1| \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1, p_2 \Rightarrow A \odot A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A \\
\hline
A \Rightarrow p_1| \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A
\end{array}
$$

FIGURE 4 A proof τ^* of $G|G^*$

Then all occurrences of p in τ^* are distinguishable and we regard them as distinct eigenvariables (See Definition 4.16 (i)). Firstly, by selecting p_1 as the eigenvariable and applying (D) to $G|G^*$, we get

$$G' \equiv A \Rightarrow C| \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|C, p_2 \Rightarrow A \odot A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A.$$

Secondly, by selecting p_2 and applying (D) to G', we get

$$G'' \equiv A \Rightarrow C|B \Rightarrow p_4, \neg A \odot \neg A|C \Rightarrow B, A \odot A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A.$$

Repeatedly, we get

$$G'''' \equiv A \Rightarrow C|A, B \Rightarrow A \odot A, \neg A \odot \neg A|C \Rightarrow A \odot A, B.$$

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We define such iterative applications of (D) as $\mathcal{D}$-rule (See Definition 5.4). Lemma 5.8 shows that $\vdash_{\mathbf{GIUL}} \mathcal{D}(G|G^*)$ if $\vdash_{\mathbf{GIUL}} G|G^*$. Then we obtain $\vdash_{\mathbf{GIUL}} \mathcal{D}(G|G^*)$, i.e., $\vdash_{\mathbf{GIUL}} G''''$.

A miracle happens here! The difficulty that we encountered in **GIUL** is overcome by converting $H'_{\times\times} = A \Rightarrow p| \Rightarrow p, p, \neg A \odot \neg A|p, p \Rightarrow A \odot A|A \Rightarrow p|p, p \Rightarrow A \odot A$ into $A \Rightarrow p_1| \Rightarrow p_2, p_4, \neg A \odot \neg A|p_1, p_2 \Rightarrow A \odot A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A$ and using $(\mathcal{D})$ to replace $(\mathcal{D}_x)$.

Why do we assign the unique identification number to each $p \Rightarrow p \in \tau_1$? We would return back to the same situation as that of τ_1 if we assign the same indices to all $p \Rightarrow p \in \tau_1$ or, replace $p_3 \Rightarrow p_3$ and $p_4 \Rightarrow p_4$ by $p_2 \Rightarrow p_2$ in τ^*.

Note that $\mathcal{D}(G|G^*) = H_1$. So we have built up a one-one correspondence between the proof τ^* of $G|G^*$ and that of H_1. Observe that each sequent in G^* is not a copy of any sequent in G_0. In the following steps, we work on eliminating these sequents in G^*.

Step 2 (Extraction of Elimination Rules). We select $A \Rightarrow p_2$ as the focus sequent in H_1^c in τ^* and keep $A \Rightarrow p_1$ unchanged from H_1^c downward to $G|G^*$ (See Figure 4). So we extract a derivation from $A \Rightarrow p_2$ by pruning some sequents (or hypersequents) in τ^*, which we denote by $\tau^*_{H_1^c:A\Rightarrow p_2}$, as shown in Figure 5.

$$\dfrac{B \Rightarrow B \quad \dfrac{\dfrac{A \Rightarrow p_2}{\Rightarrow p_2, \neg A} \quad \dfrac{\dfrac{\dfrac{p_3 \Rightarrow p_3 \quad A \Rightarrow A}{A \Rightarrow p_3|p_3 \Rightarrow A} \quad \dfrac{p_4 \Rightarrow p_4 \quad A \Rightarrow A}{A \Rightarrow p_4|p_4 \Rightarrow A}}{A \Rightarrow p_3|A \Rightarrow p_4|p_3, p_4 \Rightarrow A \odot A}}{A \Rightarrow p_3| \Rightarrow p_4, \neg A|p_3, p_4 \Rightarrow A \odot A}}{\Rightarrow p_2, p_4, \neg A \odot \neg A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A}}{\Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A}$$

FIGURE 5 A derivation $\tau^*_{H_1^c:A\Rightarrow p_2}$ from $A \Rightarrow p_2$

A derivation $\tau^*_{H_1^c:A\Rightarrow p_1}$ from $A \Rightarrow p_1$ is constructed by replacing p_2 with p_1, p_3 with p_5 and p_4 with p_6 in $\tau^*_{H_1^c:A\Rightarrow p_2}$, as shown in Figure 6. Notice that we assign new identification numbers to new occurrences of p in $\tau^*_{H_1^c:A\Rightarrow p_1}$.

$$\dfrac{B \Rightarrow B \quad \dfrac{\dfrac{A \Rightarrow p_1}{\Rightarrow p_1, \neg A} \quad \dfrac{\dfrac{\dfrac{p_5 \Rightarrow p_5 \quad A \Rightarrow A}{A \Rightarrow p_5|p_5 \Rightarrow A} \quad \dfrac{p_6 \Rightarrow p_6 \quad A \Rightarrow A}{A \Rightarrow p_6|p_6 \Rightarrow A}}{A \Rightarrow p_5|A \Rightarrow p_6|p_5, p_6 \Rightarrow A \odot A}}{A \Rightarrow p_5| \Rightarrow p_6, \neg A|p_5, p_6 \Rightarrow A \odot A}}{\Rightarrow p_1, p_6, \neg A \odot \neg A|A \Rightarrow p_5|p_5, p_6 \Rightarrow A \odot A}}{\Rightarrow p_1, B|B \Rightarrow p_6, \neg A \odot \neg A|A \Rightarrow p_5|p_5, p_6 \Rightarrow A \odot A}$$

FIGURE 6 A derivation $\tau^*_{H_1^c:A\Rightarrow p_1}$ from $A \Rightarrow p_1$

Next, we apply $\tau^*_{H_1^c:A\Rightarrow p_1}$ to $A \Rightarrow p_1$ in $G|G^*$. Then we construct a proof $\tau^{\star(1)}_{H_1^c:G|G^*}$, as shown in Figure 7, where $G' \equiv G|G^*\backslash\{A \Rightarrow p_1\}$.

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$$\dfrac{B \Rightarrow B \quad \dfrac{\dfrac{G'|A \Rightarrow p_1}{G'| \Rightarrow p_1, \neg A} \quad \dfrac{\dfrac{\dfrac{p_5 \Rightarrow p_5 \quad A \Rightarrow A}{A \Rightarrow p_5|p_5 \Rightarrow A} \quad \dfrac{p_6 \Rightarrow p_6 \quad A \Rightarrow A}{A \Rightarrow p_6|p_6 \Rightarrow A}}{A \Rightarrow p_5|A \Rightarrow p_6|p_5, p_6 \Rightarrow A \odot A}}{A \Rightarrow p_5| \Rightarrow p_6, \neg A|p_5, p_6 \Rightarrow A \odot A}}{G'| \Rightarrow p_1, p_6, \neg A \odot \neg A|A \Rightarrow p_5|p_5, p_6 \Rightarrow A \odot A}}{G^{\star(1)}_{H_1^c:G|G^*} \equiv G'| \Rightarrow p_1, B|B \Rightarrow p_6, \neg A \odot \neg A|A \Rightarrow p_5|p_5, p_6 \Rightarrow A \odot A}$$

FIGURE 7 A proof $\tau^{\star(1)}_{H_1^c:G|G^*}$ of $G^{\star(1)}_{H_1^c:G|G^*}$

However, $G^{\star(1)}_{H_1^c:G|G^*} \Longrightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A| \Rightarrow p_1, B|B \Rightarrow p_6, \neg A \odot \neg A|A \Rightarrow p_5|p_5, p_6 \Rightarrow A \odot A$ contains more copies of sequents from G^* and seems more complex than $G|G^*$. We will present a unified method to tackle with it in the following steps. Other derivations are shown in Figures 8, 9, 10, 11.

$$\dfrac{C \Rightarrow C \quad \dfrac{B \Rightarrow B \quad \dfrac{A \Rightarrow p_1| \Rightarrow p_2, \neg A|p_1, p_2 \Rightarrow A \odot A \quad \dfrac{A \Rightarrow p_4}{\Rightarrow p_4, \neg A}}{A \Rightarrow p_1| \Rightarrow p_2, p_4, \neg A \odot \neg A|p_1, p_2 \Rightarrow A \odot A}}{A \Rightarrow p_1| \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1, p_2 \Rightarrow A \odot A}}{A \Rightarrow p_1| \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A}$$

FIGURE 8 A derivation $\tau^*_{H_2^c:A \Rightarrow p_4}$ from $A \Rightarrow p_4$

$$\dfrac{C \Rightarrow C \quad \dfrac{B \Rightarrow B \quad \dfrac{A \Rightarrow p_5| \Rightarrow p_6, \neg A|p_5, p_6 \Rightarrow A \odot A \quad \dfrac{A \Rightarrow p_3}{\Rightarrow p_3, \neg A}}{A \Rightarrow p_5| \Rightarrow p_6, p_3, \neg A \odot \neg A|p_5, p_6 \Rightarrow A \odot A}}{A \Rightarrow p_5| \Rightarrow p_6, B|B \Rightarrow p_3, \neg A \odot \neg A|p_5, p_6 \Rightarrow A \odot A}}{A \Rightarrow p_5| \Rightarrow p_6, B|B \Rightarrow p_3, \neg A \odot \neg A|p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A}$$

FIGURE 9 A derivation $\tau^*_{H_2^c:A \Rightarrow p_3}$ from $A \Rightarrow p_3$

$$\dfrac{B \Rightarrow B \quad \dfrac{\dfrac{A \Rightarrow p_2}{\Rightarrow p_2, \neg A} \quad \dfrac{A \Rightarrow p_4}{\Rightarrow p_4, \neg A}}{\Rightarrow p_2, p_4, \neg A \odot \neg A}}{\Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A} \qquad \dfrac{B \Rightarrow B \quad \dfrac{\dfrac{A \Rightarrow p_5}{\Rightarrow p_5, \neg A} \quad \dfrac{A \Rightarrow p_3}{\Rightarrow p_3, \neg A}}{\Rightarrow p_5, p_3, \neg A \odot \neg A}}{\Rightarrow p_5, B|B \Rightarrow p_3, \neg A \odot \neg A}$$

FIGURE 10 $\tau^*_{\{H_1^c:A \Rightarrow p_2, H_2^c:A \Rightarrow p_4\}}$ and $\tau^*_{\{H_1^c:A \Rightarrow p_5, H_2^c:A \Rightarrow p_3\}}$

$$\dfrac{C \Rightarrow C \quad p_1, p_2 \Rightarrow A \odot A}{p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A} \qquad \dfrac{C \Rightarrow C \quad p_3, p_4 \Rightarrow A \odot A}{p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A}$$

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$$\frac{C \Rightarrow C \quad p_5, p_6 \Rightarrow A \odot A}{p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A}$$

FIGURE 11 $\tau^*_{H_3^c:p_1,p_2\Rightarrow A\odot A}$, $\tau^*_{H_3^c:p_3,p_4\Rightarrow A\odot A}$ and $\tau^*_{H_3^c:p_5,p_6\Rightarrow A\odot A}$

Step 3 (Separation of one branch). A proof $\tau^{\star(2)}_{H_1^c:G|G^*}$ is constructed by applying sequentially

$$\tau^*_{H_3^c:p_3,p_4\Rightarrow A\odot A}, \tau^*_{H_3^c:p_5,p_6\Rightarrow A\odot A}$$

to $p_3, p_4 \Rightarrow A \odot A$ and $p_5, p_6 \Rightarrow A \odot A$ in $G^{\star(1)}_{H_1^c:G|G^*}$, as shown in Figure 12, where $G'' \equiv G^{\star(1)}_{H_1^c:G|G^*}\backslash\{p_3, p_4 \Rightarrow A \odot A, p_5, p_6 \Rightarrow A \odot A\}$

$$\frac{C \Rightarrow C \quad \dfrac{C \Rightarrow C \quad G''|p_3, p_4 \Rightarrow A \odot A|p_5, p_6 \Rightarrow A \odot A}{G''|p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A|p_5, p_6 \Rightarrow A \odot A}}{G^{\star(2)}_{H_1^c:G|G^*} \equiv G''|p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A|p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A}$$

FIGURE 12 A proof $\tau^{\star(2)}_{H_1^c:G|G^*}$ of $G^{\star(2)}_{H_1^c:G|G^*}$

$$G^{\star(2)}_{H_1^c:G|G^*} \Longrightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A|A \Rightarrow p_3| \Rightarrow p_1, B|$$
$$B \Rightarrow p_6, \neg A \odot \neg A|A \Rightarrow p_5|p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A|p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A.$$

Notice that

$$\mathcal{D}(B \Rightarrow p_4, \neg A \odot \neg A|A \Rightarrow p_3|p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A)$$
$$= \mathcal{D}(B \Rightarrow p_6, \neg A \odot \neg A|A \Rightarrow p_5|p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A)$$
$$= A \Rightarrow C|C, B \Rightarrow A \odot A, \neg A \odot \neg A.$$

Then it is permissible to cut off the part

$$B \Rightarrow p_6, \neg A \odot \neg A|A \Rightarrow p_5|p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A$$

of $G^{\star(2)}_{H_1^c:G|G^*}$, which corresponds to applying (EC) to $\mathcal{D}(G^{\star(2)}_{H_1^c:G|G^*})$. We regard such a manipulation as a constrained contraction rule applied to $G^{\star(2)}_{H_1^c:G|G^*}$ and denote it by (EC_Ω). Define $G^{\star}_{H_1^c:G|G^*}$ to be

$$\Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A|$$
$$A \Rightarrow p_3| \Rightarrow p_1, B|p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A.$$

Then we construct a proof of $G^{\star}_{H_1^c:G|G^*}$ by $\dfrac{G^{\star(2)}_{H_1^c:G|G^*}}{G^{\star}_{H_1^c:G|G^*}}(EC_\Omega)$, which guarantees the validity of

$$\vdash_{\mathbf{GIUL}} \mathcal{D}(G^{\star}_{H_1^c:G|G^*})$$

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under the condition

$$\vdash_{\mathbf{GIUL}} \mathcal{D}(G^{\star(2)}_{H_1^c:G|G^*}).$$

A change happens here! There is only one sequent which is a copy of a sequent in G^* in $G^{\star}_{H_1^c:G|G^*}$. It is simpler than $G|G^*$. So we are moving forward. The above procedure is called the separation of $G|G^*$ as a branch of H_1^c and reformulated as follows (See Section 7 for details).

$$\cfrac{\cfrac{\cfrac{G|G^*}{G^{\star(1)}_{H_1^c:G|G^*}}\left\langle\tau^*_{H_1^c:A\Rightarrow p_1}\right\rangle}{G^{\star(2)}_{H_1^c:G|G^*}}\left\langle\tau^*_{H_3^c:p_3,p_4\Rightarrow A\odot A},\tau^*_{H_3^c:p_5,p_6\Rightarrow A\odot A}\right\rangle}{G^{\star}_{H_1^c:G|G^*}}\langle EC_\Omega\rangle$$

The separation of $G|G^*$ as a branch of H_2^c is constructed by a similar procedure as follows.

$$\cfrac{\cfrac{\cfrac{G|G^*}{G^{\star(1)}_{H_2^c:G|G^*}}\left\langle\tau^*_{H_2^c:A\Rightarrow p_3}\right\rangle}{G^{\star(2)}_{H_2^c:G|G^*}}\left\langle\tau^*_{H_3^c:p_3,p_4\Rightarrow A\odot A}\right\rangle}{G^{\star}_{H_2^c:G|G^*}}\langle EC_\Omega\rangle$$

Note that $\mathcal{D}(G^{\star}_{H_1^c:G|G^*}) = H_2$ and $\mathcal{D}(G^{\star}_{H_2^c:G|G^*}) = H_3$. So we have built up one-one correspondences between proofs of $G^{\star}_{H_1^c:G|G^*}, G^{\star}_{H_1^c:G|G^*}$ and those of H_2, H_3.

Step 3 (Separation algorithm of multiple branches). We will prove $\vdash_{\mathbf{GIUL}} \mathcal{D}_0(G_0)$ in a direct way, i.e., only the major step of Theorem 8.2 is presented in the following. (See A.5.4 for a complete illustration.) Recall that

$$G^{\star}_{H_1^c:G|G^*} \equiv\Rightarrow p_2, B|B \Rightarrow p_4, \neg A\odot\neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A\odot A|$$

$$A \Rightarrow p_3| \Rightarrow p_1, B|p_3 \Rightarrow C|C, p_4 \Rightarrow A\odot A,$$

$$G^{\star}_{H_2^c:G|G^*} = A \Rightarrow p_1| \Rightarrow p_2, B|B \Rightarrow p_4, \neg A\odot\neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A\odot A|$$

$$B \Rightarrow p_3, \neg A\odot\neg A|p_3 \Rightarrow C|C, p_4 \Rightarrow A\odot A.$$

By reassigning identification numbers to occurrences of $p's$ in $G^{\star}_{H_2^c:G|G^*}$,

$$G^{\star}_{H_2^c:G|G^*} = A \Rightarrow p_5| \Rightarrow p_6, B|B \Rightarrow p_8, \neg A\odot\neg A|p_5 \Rightarrow C|C, p_6 \Rightarrow A\odot A|$$

$$B \Rightarrow p_7, \neg A\odot\neg A|p_7 \Rightarrow C|C, p_8 \Rightarrow A\odot A.$$

By applying $\tau^*_{\{H_1^c:A\Rightarrow p_5, H_2^c:A\Rightarrow p_3\}}$ to $A \Rightarrow p_3$ in $G^{\star}_{H_1^c:G|G^*}$ and $A \Rightarrow p_5$ in $G^{\star}_{H_2^c:G|G^*}$, we get $\vdash_{\mathbf{GIUL}} G'$, where

$$G' \equiv\Rightarrow p_2, B|B \Rightarrow p_4, \neg A\odot\neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A\odot A| \Rightarrow p_1, B|$$

$$p_3 \Rightarrow C|C, p_4 \Rightarrow A\odot A| \Rightarrow p_6, B|B \Rightarrow p_8, \neg A\odot\neg A|p_5 \Rightarrow C|C, p_6 \Rightarrow A\odot A|$$

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$$B \Rightarrow p_7, \neg A \odot \neg A | p_7 \Rightarrow C | C, p_8 \Rightarrow A \odot A | \Rightarrow p_5, B | B \Rightarrow p_3, \neg A \odot \neg A.$$

Why do you reassign identification numbers to occurrences of $p's$ in $G^{\star}_{H^c_2:G|G^*}$? It makes different occurrences of $p's$ to be assigned different identification numbers in two nodes $G^{\star}_{H^c_1:G|G^*}$ and $G^{\star}_{H^c_2:G|G^*}$ of the proof of G'.

By applying $\langle EC^*_{\Omega} \rangle$ to G', we get $\vdash_{\mathbf{GIUL}_{\Omega}} G^{\star}_{\mathbf{I}}$, where

$$G^{\star}_{\mathbf{I}} \equiv \Rightarrow p_2, B | B \Rightarrow p_4, \neg A \odot \neg A | p_1 \Rightarrow C | C, p_2 \Rightarrow A \odot A | \Rightarrow p_1, B |$$

$$p_3 \Rightarrow C | C, p_4 \Rightarrow A \odot A | B \Rightarrow p_3, \neg A \odot \neg A.$$

A great change happens here! We have eliminated all sequents which are copies of some sequents in G^* and convert $G|G^*$ into $G^{\star}_{\mathbf{I}}$ in which each sequent is some copy of a sequent in G_0.

Then $\vdash_{\mathbf{GIUL}} \mathcal{D}(G^{\star}_{\mathbf{I}})$ by Lemma 5.6, where $\mathcal{D}(G^{\star}_{\mathbf{I}}) = H_0 =$

$$\Rightarrow C, B | C \Rightarrow B, A \odot A | B \Rightarrow C, \neg A \odot \neg A | C, B \Rightarrow A \odot A, \neg A \odot \neg A.$$

So we have built up one-one correspondences between the proof of $G^{\star}_{\mathbf{I}}$ and that of H_0, i.e., the proof of H_0 can be constructed by applying $(\mathcal{D})$ to the proof of $G^{\star}_{\mathbf{I}}$. The major steps of constructing $G^{\star}_{\mathbf{I}}$ are shown in the following figure, where $\mathcal{D}(G|G^*) = H_1$, $\mathcal{D}(G^{\star}_{H^c_1:G|G^*}) = H_2$, $\mathcal{D}(G^{\star}_{H^c_2:G|G^*}) = H_3$ and $\mathcal{D}(G^{\star}_{\mathbf{I}}) = H_0$.

$$G|G^* \equiv \Rightarrow p_2, B | B \Rightarrow p_4, \neg A \odot \neg A | p_1 \Rightarrow C | C, p_2 \Rightarrow A \odot A | A \Rightarrow p_1 | A \Rightarrow p_3 | p_3, p_4 \Rightarrow A \odot A$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$G^{\star}_{H^c_2:G|G^*} = A \Rightarrow p_1 | \Rightarrow p_2, B | B \Rightarrow p_4, \neg A \odot \neg A | p_1 \Rightarrow C | C, p_2 \Rightarrow A \odot A | p_3 \Rightarrow C | B \Rightarrow p_3, \neg A \odot \neg A | C, p_4 \Rightarrow A \odot A$$

$$G^{\star}_{H^c_1:G|G^*} \equiv \Rightarrow p_2, B | B \Rightarrow p_4, \neg A \odot \neg A | p_1 \Rightarrow C | C, p_2 \Rightarrow A \odot A | A \Rightarrow p_3 | A \Rightarrow p_1, B | p_3 \Rightarrow C | C, p_4 \Rightarrow A \odot A$$

$$\vdots$$

$$G^{\star}_{\mathbf{I}} \equiv \Rightarrow p_2, B | B \Rightarrow p_4, \neg A \odot \neg A | p_1 \Rightarrow C | C, p_2 \Rightarrow A \odot A | B \Rightarrow \neg A \odot \neg A, p_3 | \Rightarrow p_1, B | p_3 \Rightarrow C | C, p_4 \Rightarrow A \odot A$$

In the above example, $\mathcal{D}(G^{\star}_{\mathbf{I}}) = \mathcal{D}_0(G_0)$. But it is not always the case. In general, we can prove that $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$ if $\vdash_{\mathbf{GL}} \mathcal{D}(G^{\star}_{\mathbf{I}})$, which is shown in the proof of Main theorem in Page 46. This example shows that the proof of Main theorem essentially presents an algorithm to construct a proof of $\mathcal{D}_0(G_0)$ from τ.

4. Preprocessing of Proof Tree

Let τ be a cut-free proof of G_0 in Main theorem in **GL** by Lemma 2.15. Starting with τ, we will construct a proof τ^* which contains no application of (EC) and has some other properties in this section.

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Lemma 4.1. *(i) If* $\vdash_{\mathbf{GL}} \Gamma_1 \Rightarrow A, \Delta_1$ *and* $\vdash_{\mathbf{GL}} \Gamma_2 \Rightarrow B, \Delta_2$

$$\textit{then} \quad \vdash_{\mathbf{GL}} \Gamma_1 \Rightarrow A \wedge B, \Delta_1 | \Gamma_2 \Rightarrow A \wedge B, \Delta_2;$$

(ii) If $\vdash_{\mathbf{GL}} \Gamma_1, A \Rightarrow \Delta_1$ *and* $\vdash_{\mathbf{GL}} \Gamma_2, B \Rightarrow \Delta_2$

$$\textit{then} \quad \vdash_{\mathbf{GL}} \Gamma_1, A \vee B \Rightarrow \Delta_1 | \Gamma_2, A \vee B \Rightarrow \Delta_2.$$

Proof. (i)

$$\dfrac{\Gamma_2 \Rightarrow B, \Delta_2 \quad \dfrac{\Gamma_1 \Rightarrow A, \Delta_1 \quad \dfrac{B \Rightarrow B \quad \dfrac{A \Rightarrow A \quad \dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \Rightarrow B | B \Rightarrow A}(COM)}{A \Rightarrow A \wedge B | B \Rightarrow A}(\wedge_r)}{A \Rightarrow A \wedge B | B \Rightarrow A \wedge B}(\wedge_r)}{\Gamma_1 \Rightarrow A \wedge B, \Delta_1 | B \Rightarrow A \wedge B}(CUT)}{\Gamma_1 \Rightarrow A \wedge B, \Delta_1 | \Gamma_2 \Rightarrow A \wedge B, \Delta_2}(CUT)$$

(ii) is proved by a procedure similar to that of (i) and omitted. □

We introduce two new rules by Lemma 4.1.

Definition 4.2. $\dfrac{G_1|\Gamma_1 \Rightarrow A, \Delta_1 \quad G_2|\Gamma_2 \Rightarrow B, \Delta_2}{G_1|G_2|\Gamma_1 \Rightarrow A \wedge B, \Delta_1 | \Gamma_2 \Rightarrow A \wedge B, \Delta_2}(\wedge_{rw})$ and $\dfrac{G_1|\Gamma_1, A \Rightarrow \Delta_1 \quad G_2|\Gamma_2, B \Rightarrow \Delta_2}{G_1|G_2|\Gamma_1, A \vee B \Rightarrow \Delta_1 | \Gamma_2, A \vee B \Rightarrow \Delta_2}(\vee_{lw})$ are called the generalized $(\wedge_r)$ and $(\vee_l)$ rules, respectively.

Now, we begin to process τ as follows.
Step 1 A proof τ^1 is constructed by replacing inductively all applications of

$$\dfrac{G_1|\Gamma \Rightarrow A, \Delta \quad G_2|\Gamma \Rightarrow B, \Delta}{G_1|G_2|\Gamma \Rightarrow A \wedge B, \Delta}(\wedge_r) \ \left(\text{or } \dfrac{G_1|\Gamma, A \Rightarrow \Delta \quad G_2|\Gamma, B \Rightarrow \Delta}{G_1|G_2|\Gamma, A \vee B \Rightarrow \Delta}(\vee_l)\right)$$

in τ with

$$\dfrac{\dfrac{G_1|\Gamma \Rightarrow A, \Delta \quad G_2|\Gamma \Rightarrow B, \Delta}{G_1|G_2|\Gamma \Rightarrow A \wedge B, \Delta | \Gamma \Rightarrow A \wedge B, \Delta}(\wedge_{rw})}{G_1|G_2|\Gamma \Rightarrow A \wedge B, \Delta}(EC)$$

$$\left(\text{accordingly } \dfrac{\dfrac{G_1|\Gamma, A \Rightarrow \Delta \quad G_2|\Gamma, B \Rightarrow \Delta}{G_1|G_2|\Gamma, A \vee B \Rightarrow \Delta | \Gamma, A \vee B \Rightarrow \Delta}(\vee_{lw})}{G_1|G_2|\Gamma, A \vee B \Rightarrow \Delta}(EC) \text{ for } (\vee_l)\right).$$

The replacements in Step 1 are local and the root of τ^1 is also labeled by G_0.

Definition 4.3. We sometimes may regard $\dfrac{G'}{G'}$ as a structural rule of **GL** and denote it by (ID_Ω) for convenience. The focus sequent for (ID_Ω) is undefined.

Lemma 4.4. *Let* $\dfrac{G'|S^m}{G'|S}(EC^*) \in \tau^1$, $Th_{\tau^1}(G'|S) = (H_0, H_1, \cdots, H_n)$, *where* $H_0 = G'|S$ *and* $H_n = G_0$. *A tree* τ' *is constructed by replacing each* H_k *in* τ^1 *with* $H_k|S^{m-1}$ *for all* $0 \leqslant k \leqslant n$. *Then* τ' *is a proof of* $G_0|S^{m-1}$.

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Proof. The proof is by induction on n . Since $\tau^1(G'|S^m)$ is a proof and $\frac{G'|S^m}{H_0|S^{m-1}}(ID_\Omega)$ is valid in **GL**, then $\tau'(H_0|S^{m-1})$ is a proof. Suppose that $\tau'(H_{n-1}|S^{m-1})$ is a proof. Since $\frac{H_{n-1} \quad G''}{H_n}(II)$ (or $\frac{H_{n-1}}{H_n}(I)$) in τ^1, then $\frac{H_{n-1}|S^{m-1} \quad G''}{H_n|S^{m-1}}$ (or $\frac{H_{n-1}|S^{m-1}}{H_n|S^{m-1}}$) is an application of the same rule (II) (or (I)). Thus $\tau'(H_n|S^{m-1})$ is a proof. □

Definition 4.5. The manipulation described in Lemma 4.4 is called sequent-inserting operation.

Clearly, the number of (EC^*)-applications in τ' is less than τ^1. Next, we continue to process τ.

Step 2 Let $\frac{G_1'''|\{S_1^c\}^{m_1'}}{G_1'|S_1^c}(EC^*), \cdots, \frac{G_N'''|\{S_N^c\}^{m_N'}}{G_N'|S_N^c}(EC^*)$ be all applications of (EC^*) in τ^1 and $G_0^* := \{S_1^c\}^{m_1'-1}|\cdots|\{S_N^c\}^{m_N'-1}$. By repeatedly applying sequent-inserting operations, we construct a proof of $G_0|G_0^*$ in **GL** without applications of (EC^*) and denote it by τ^2.

Remark 4.6. (i) τ^2 is constructed by converting (EC) into (ID_Ω); (ii) Each node of τ^2 has the form $H_0|H_0^*$, where $H_0 \in \tau^1$ and H_0^* is a (possibly empty) subset of G_0^*.

We need the following construction to eliminate applications of (EW) in τ^2.

Construction 4.7. *Let $H \in \tau^2$, $H' \subseteq H$ and $Th_{\tau^2}(H) = (H_0, \cdots, H_n)$, where $H_0 = H$, $H_n = G_0|G_0^*$. Hypersequents $\langle H_k \rangle_{H:H'}$ and trees $\tau^2_{H:H'}(\langle H_k \rangle_{H:H'})$ for all $0 \leqslant k \leqslant n$ are constructed inductively as follows.*

(i) $\langle H_0 \rangle_{H:H'} := H'$ and $\tau^2_{H:H'}(\langle H_0 \rangle_{H:H'})$ consists of a single node H';

(ii) Let $\frac{G'|S' \quad G''|S''}{G'|G''|H''}(II)$ (or $\frac{G'|S'}{G'|S''}(I)$) be in τ^2, $H_k = G'|S'$ and $H_{k+1} = G'|G''|H''$ (accordingly $H_{k+1} = G'|S''$ for (I)) for some $0 \leqslant k \leqslant n-1$.

If $S' \in \langle H_k \rangle_{H:H'}$

$$\langle H_{k+1} \rangle_{H:H'} := \langle H_k \rangle_{H:H'} \backslash \{S'\}|G''|H''$$

$$(\text{accordingly } \langle H_{k+1} \rangle_{H:H'} := \langle H_k \rangle_{H:H'} \backslash \{S'\}|S'' \text{ for } (I))$$

and $\tau^2_{H:H'}(\langle H_{k+1} \rangle_{H:H'})$ is constructed by combining trees

$$\tau^2_{H:H'}(\langle H_k \rangle_{H:H'}), \tau^2(G''|S'') \text{ with } \frac{\langle H_k \rangle_{H:H'} \quad G''|S''}{\langle H_{k+1} \rangle_{H:H'}}(II)$$

$$(\text{accordingly } \tau^2_{H:H'}(\langle H_k \rangle_{H:H'}) \text{ with } \frac{\langle H_k \rangle_{H:H'}}{\langle H_{k+1} \rangle_{H:H'}}(I) \text{ for } (I))$$

otherwise $\langle H_{k+1} \rangle_{H:H'} := \langle H_k \rangle_{H:H'}$ and $\tau^2_{H:H'}(\langle H_{k+1} \rangle_{H:H'})$ is constructed by combining

$$\tau^2_{H:H'}(\langle H_k \rangle_{H:H'}) \text{ with } \frac{\langle H_k \rangle_{H:H'}}{\langle H_{k+1} \rangle_{H:H'}}(ID_\Omega).$$

(iii) Let $\frac{G'}{G'|S'}(EW) \in \tau^2$, $H_k = G'$ and $H_{k+1} = G'|S'$ then $\langle H_{k+1} \rangle_{H:H'} := \langle H_k \rangle_{H:H'}$ and $\tau^2_{H:H'}(\langle H_{k+1} \rangle_{H:H'})$ is constructed by combining $\tau^2_{H:H'}(\langle H_k \rangle_{H:H'})$ with $\frac{\langle H_k \rangle_{H:H'}}{\langle H_{k+1} \rangle_{H:H'}}(ID_\Omega)$.

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Lemma 4.8. *(i)* $\langle H_k\rangle_{H:H'} \subseteq H_k$ *for all* $0 \leqslant k \leqslant n$*;*
(ii) $\tau^2_{H:H'}(\langle H_k\rangle_{H:H'})$ *is a derivation of* $\langle H_k\rangle_{H:H'}$ *from* H' *without* (EC)*.*

Proof. The proof is by induction on k. For the base step, $\langle H_0\rangle_{H:H'} = H'$ and $\tau^2_{H:H'}(\langle H_0\rangle_{H:H'})$ consists of a single node H'. Then $\langle H_0\rangle_{H:H'} \subseteq H_0 = H$, $\tau^2_{H:H'}(\langle H_0\rangle_{H:H'})$ is a derivation of $\langle H_0\rangle_{H:H'}$ from H' without (EC).

For the induction step, suppose that $\langle H_k\rangle_{H:H'}$ and $\tau^2_{H:H'}(\langle H_k\rangle_{H:H'})$ be constructed such that (i) and (ii) hold for some $0 \leqslant k \leqslant n-1$. There are two cases to be considered.

Case 1 Let $\dfrac{G'|S'}{G'|S''}(I) \in \tau^2$, $H_k = G'|S'$ and $H_{k+1} = G'|S''$. If $S' \in \langle H_k\rangle_{H:H'}$ then $\langle H_k\rangle_{H:H'}\backslash\{S'\} \subseteq G'$ by $\langle H_k\rangle_{H:H'} \subseteq H_k = G'|S'$. Thus $\langle H_{k+1}\rangle_{H:H'} = (\langle H_k\rangle_{H:H'}\backslash\{S'\})|S'' \subseteq G'|S'' = H_{k+1}$. Otherwise $S' \notin \langle H_k\rangle_{H:H'}$ then $\langle H_k\rangle_{H:H'} \subseteq G'$ by $\langle H_k\rangle_{H:H'} \subseteq H_k = G'|S'$. Thus $\langle H_{k+1}\rangle_{H:H'} \subseteq H_{k+1}$ by $\langle H_{k+1}\rangle_{H:H'} = \langle H_k\rangle_{H:H'} \subseteq G' \subseteq H_{k+1}$. $\tau^2_{H:H'}(\langle H_{k+1}\rangle_{H:H'})$ is a derivation of $\langle H_{k+1}\rangle_{H:H'}$ from H' without (EC) since $\tau^2_{H:H'}(\langle H_k\rangle_{H:H'})$ is such one and $\dfrac{\langle H_k\rangle_{H:H'}}{\langle H_{k+1}\rangle_{H:H'}}(I)$ is a valid instance of a rule (I) of **GL**. The case of applications of two-premise rule is proved by a similar procedure and omitted.

Case 2 Let $\dfrac{G'}{G'|S'}(EW) \in \tau^2$, $H_k = G'$ and $H_{k+1} = G'|S'$. Then $\langle H_{k+1}\rangle_{H:H'} \subseteq H_{k+1}$ by $\langle H_{k+1}\rangle_{H:H'} = \langle H_k\rangle_{H:H'} \subseteq H_k \subseteq H_{k+1}$. $\tau^2_{H:H'}(\langle H_{k+1}\rangle_{H:H'})$ is a derivation of $\langle H_{k+1}\rangle_{H:H'}$ from H' without (EC) since $\tau^2_{H:H'}(\langle H_k\rangle_{H:H'})$ is such one and $\dfrac{\langle H_k\rangle_{H:H'}}{\langle H_{k+1}\rangle_{H:H'}}(ID_\Omega)$ is valid by Definition 4.3. □

Definition 4.9. The manipulation described in Construction 4.7 is called derivation-pruning operation.

Notation 4.10. *We denote* $\langle H_n\rangle_{H:H'}$ *by* $G^2_{H:H'}$*,* $\tau^2_{H:H'}(\langle H_n\rangle_{H:H'})$ *by* $\tau^2_{H:H'}$ *and say that* H' *is transformed into* $G^2_{H:H'}$ *in* τ^2*.*

Then Lemma 4.8 shows that $\dfrac{H'}{G^2_{H:H'}}\langle\tau^2_{H:H'}\rangle$, $G^2_{H:H'} \subseteq G_0|G_0^*$. Now, we continue to process τ as follows.

Step 3 Let $\dfrac{G'}{G'|S'}(EW) \in \tau^2$ then $\tau^2_{G'|S':G'}(\langle H_n\rangle_{G'|S':G'})$ is a derivation of $\langle H_n\rangle_{G'|S':G'}$ from G' thus a proof of $\langle H_n\rangle_{G'|S':G'}$ is constructed by combining $\tau^2(G')$ and $\tau^2_{G'|S':G'}(\langle H_n\rangle_{G'|S':G'})$ with $\dfrac{G'}{G'}(ID_\Omega)$. By repeatedly applying the procedure above, we construct a proof τ^3 of $G_1|G_1^*$ without (EW) in **GL**, where $G_1 \subseteq G_0, G_1^* \subseteq G_0^*$ by Lemma 4.8 (i).

Step 4 Let $\Gamma, p, \bot \Rightarrow \Delta \in \tau^3$ (or $\Gamma, p \Rightarrow \top, \Delta$, $\dfrac{G|\Gamma \Rightarrow \Delta}{G|\Gamma, p \Rightarrow \Delta}(WL)$) then there exists $\Gamma' \Rightarrow \Delta' \in H$ such that $p \in \Gamma'$ for all $H \in Th_{\tau^3}(\Gamma, p, \bot \Rightarrow \Delta)$ (accordingly $H \in Th_{\tau^3}(\Gamma, p \Rightarrow \top, \Delta)$, $H \in Th_{\tau^3}(\Gamma, p \Rightarrow \Delta)$) thus a proof is constructed by replacing top-down p in each Γ' with $\top$.

Let $\Gamma, \bot \Rightarrow p, \Delta$ (or $\Gamma \Rightarrow \top, p, \Delta$, $\dfrac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow p, \Delta}(WR)$) is a leaf of τ^3 then there exists $\Gamma' \Rightarrow \Delta' \in H$ such that $p \in \Delta'$ for all $H \in Th_{\tau^3}(\Gamma, \bot \Rightarrow p, \Delta)$ (accordingly $H \in Th_{\tau^3}(\Gamma \Rightarrow \top, p, \Delta)$ or $H \in Th_{\tau^3}(\Gamma \Rightarrow p, \Delta)$) thus a proof is constructed by replacing top-down p in each Γ' with $\bot$.

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Repeatedly applying the procedure above, we construct a proof τ^4 of $G_2|G_2^*$ in **GL** such that there doesn't exist occurrence of p in Γ or Δ at each leaf labeled by $\Gamma, \bot \Rightarrow \Delta$ or $\Gamma \Rightarrow \top, \Delta$, or p is not the weakening formula A in $\dfrac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow A, \Delta}(WR)$ or $\dfrac{G|\Gamma \Rightarrow \Delta}{G|\Gamma, A \Rightarrow \Delta}(WL)$ when (WR) or (WL) is available. Define two operations σ_l and σ_r on sequents by $\sigma_l(\Gamma, p \Rightarrow \Delta) := \Gamma, \top \Rightarrow \Delta$ and $\sigma_r(\Gamma \Rightarrow p, \Delta) := \Gamma \Rightarrow \bot, \Delta$. Then $G_2|G_2^*$ is obtained by applying σ_l and σ_r to some designated sequents in $G_1|G_1^*$.

Definition 4.11. The manipulation described in Step 4 is called eigenvariable-replacing operation.

Step 5 A proof τ^* is constructed from τ^4 by assigning inductively one unique identification number to each occurrence of p in τ^4 as follows.

One unique identification number, which is a positive integer, is assigned to each leaf of the form $p \Rightarrow p$ in τ^4 which corresponds to $p_k \Rightarrow p_k$ in τ^*. Other nodes of τ^4 are processed as follows.

- Let $\dfrac{G_1|\Gamma, \lambda p \Rightarrow \mu p, \Delta}{G_1|\Gamma', \lambda p \Rightarrow \mu p, \Delta'}(I) \in \tau^4$. Suppose that all occurrences of p in $G_1|\Gamma, \lambda p \Rightarrow \mu p, \Delta$ are assigned identification numbers and have the form $G_1'|\Gamma, p_{i_1}, \cdots, p_{i_\lambda} \Rightarrow p_{j_1}, \cdots, p_{j_\mu}, \Delta$ in τ^*, which we often write as $G_1'|\Gamma, \{p_{i_k}\}_{k=1}^{\lambda} \Rightarrow \{p_{j_k}\}_{k=1}^{\mu}, \Delta$. Then $G_1|\Gamma', \lambda p \Rightarrow \mu p, \Delta'$ has the form $G_1'|\Gamma', \{p_{i_k}\}_{k=1}^{\lambda} \Rightarrow \{p_{j_k}\}_{k=1}^{\mu}, \Delta'$.
- Let $\dfrac{G' \quad G''}{G'''}(\wedge_{rw}) \in \tau^4$, where $G' \equiv G_1|\Gamma, \lambda p \Rightarrow \mu p, A, \Delta$, $G'' \equiv G_2|\Gamma, \lambda p \Rightarrow \mu p, B, \Delta$, $G''' \equiv G_1|G_2|\Gamma, \lambda p \Rightarrow \mu p, A \wedge B, \Delta|\Gamma, \lambda p \Rightarrow \mu p, A \wedge B, \Delta$. Suppose that G' and G'' have the forms $G_1'|\Gamma, \{p_{i_{1k}}\}_{k=1}^{\lambda} \Rightarrow \{p_{j_{1k}}\}_{k=1}^{\mu}, A, \Delta$ and $G_2'|\Gamma, \{p_{i_{2k}}\}_{k=1}^{\lambda} \Rightarrow \{p_{j_{2k}}\}_{k=1}^{\mu}, B, \Delta$ in τ^*, respectively. Then G''' has the form $G_1'|G_2'|\Gamma, \{p_{i_{1k}}\}_{k=1}^{\lambda} \Rightarrow \{p_{j_{1k}}\}_{k=1}^{\mu}, A \wedge B, \Delta|\Gamma, \{p_{i_{2k}}\}_{k=1}^{\lambda} \Rightarrow \{p_{j_{2k}}\}_{k=1}^{\mu}, A \wedge B, \Delta$. All applications of $(\vee_{lw})$ are processed by the procedure similar to that of $(\wedge_{rw})$.
- Let $\dfrac{G' \quad G''}{G'''}(\odot_r) \in \tau^4$, where $G' \equiv G_1|\Gamma_1, \lambda_1 p \Rightarrow \mu_1 p, A, \Delta_1$, $G'' \equiv G_2|\Gamma_2, \lambda_2 p \Rightarrow \mu_2 p, B, \Delta_2$, $G''' \equiv G_1|G_2|\Gamma_1, \Gamma_2, (\lambda_1 + \lambda_2)p \Rightarrow (\mu_1 + \mu_2)p, A \odot B, \Delta_1, \Delta_2$. Suppose that G' and G'' have the forms $G_1'|\Gamma_1, \{p_{i_{1k}}\}_{k=1}^{\lambda_1} \Rightarrow \{p_{j_{1k}}\}_{k=1}^{\mu_1}, A, \Delta_1$ and $G_2'|\Gamma_2, \{p_{i_{2k}}\}_{k=1}^{\lambda_2} \Rightarrow \{p_{j_{2k}}\}_{k=1}^{\mu_2}, B, \Delta_2$ in τ^*, respectively. Then G''' has the form $G_1'|G_2'|\Gamma_1, \Gamma_2, \{p_{i_{1k}}\}_{k=1}^{\lambda_1}, \{p_{i_{2k}}\}_{k=1}^{\lambda_2} \Rightarrow \{p_{j_{1k}}\}_{k=1}^{\mu_1}, \{p_{j_{2k}}\}_{k=1}^{\mu_2}, A \odot B, \Delta_1, \Delta_2$. All applications of $(\rightarrow_l)$ are processed by the procedure similar to that of $(\odot_r)$.
- Let $\dfrac{G' \quad G''}{G'''}(COM) \in \tau^4$, where $G' \equiv G_1|\Gamma_1, \Pi_1, \lambda_1 p \Rightarrow \mu_1 p, \Sigma_1, \Delta_1$, $G'' \equiv G_2|\Gamma_2, \Pi_2, \lambda_2 p \Rightarrow \mu_2 p, \Sigma_2, \Delta_2$, $G''' \equiv G_1|G_2|\Gamma_1, \Gamma_2, (\lambda_{11} + \lambda_{21})p \Rightarrow (\mu_{11} + \mu_{21})p, \Delta_1, \Delta_2|$ $\Pi_1, \Pi_2, (\lambda_{12} + \lambda_{22})p \Rightarrow (\mu_{12} + \mu_{22})p, \Sigma_1, \Sigma_2$, where $\lambda_{11} + \lambda_{12} = \lambda_1, \lambda_{21} + \lambda_{22} = \lambda_2$, $\mu_{11} + \mu_{12} = \mu_1, \mu_{21} + \mu_{22} = \mu_2$.

  Suppose that G' and G'' have the forms $G_1'|\Gamma_1, \Pi_1, \{p_{i_k^1}\}_{k=1}^{\lambda_1} \Rightarrow \{p_{j_k^1}\}_{k=1}^{\mu_1}, \Sigma_1, \Delta_1$ and $G_2'|\Gamma_2, \Pi_2, \{p_{i_k^2}\}_{k=1}^{\lambda_2} \Rightarrow \{p_{j_k^2}\}_{k=1}^{\mu_2}, \Sigma_2, \Delta_2$ in τ^*, respectively. Then G''' has the form $G_1'|G_2'|\Gamma_1, \Gamma_2, \{p_{i_{1k}^1}\}_{k=1}^{\lambda_{11}}, \{p_{i_{1k}^2}\}_{k=1}^{\lambda_{21}} \Rightarrow \{p_{j_{1k}^1}\}_{k=1}^{\mu_{11}}, \{p_{j_{1k}^2}\}_{k=1}^{\mu_{21}}, \Delta_1, \Delta_2|$ $\Pi_1, \Pi_2, \{p_{i_{2k}^1}\}_{k=1}^{\lambda_{12}}, \{p_{i_{2k}^2}\}_{k=1}^{\lambda_{22}} \Rightarrow \{p_{j_{2k}^1}\}_{k=1}^{\mu_{12}}, \{p_{j_{2k}^2}\}_{k=1}^{\mu_{22}}, \Sigma_1, \Sigma_2$, where $\{p_{i_k^w}\}_{k=1}^{\lambda_w} = \{p_{i_{1k}^w}\}_{k=1}^{\lambda_{w1}} \bigcup \{p_{i_{2k}^w}\}_{k=1}^{\lambda_{w2}}, \{p_{j_k^w}\}_{k=1}^{\mu_w} = \{p_{j_{1k}^w}\}_{k=1}^{\mu_{w1}} \bigcup \{p_{j_{2k}^w}\}_{k=1}^{\mu_{w2}}$ for $w = 1, 2$.

Definition 4.12. The manipulation described in Step 5 is called eigenvariable-labeling operation.

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Notation 4.13. *Let G_2 and G_2^* be converted to G and G^* in τ^*, respectively. Then τ^* is a proof of $G|G^*$.*

In the preprocessing of τ, each $\dfrac{G_i'''|\{S_i^c\}^{m_i'}}{G_i'''|S_i^c}(EC^*)_i$ is converted into $\dfrac{G_i''|\{S_i^c\}^{m_i'}}{G_i''|\{S_i^c\}^{m_i'}}(ID_\Omega)_i$ in Step 2, where $G_i''' \subseteq G_i''$ by Lemma 4.4. $\dfrac{G'}{G'|S'}(EW) \in \tau^2$ is converted into $\dfrac{G''}{G''}(ID_\Omega)$ in Step 3, where $G'' \subseteq G'$ by Lemma 4.8(i). Some $G'|\Gamma', p \Rightarrow \Delta' \in \tau^3$ (or $G'|\Gamma' \Rightarrow p, \Delta'$) is revised as $G'|\Gamma', \top \Rightarrow \Delta'$ (or $G'|\Gamma' \Rightarrow \bot, \Delta'$) in Step 4. Each occurrence of p in τ^4 is assigned the unique identification number in Step 5. The whole preprocessing above is depicted by Figure 13.

$$\xrightarrow{\tau} G_0 \xrightarrow[\wedge_{rw},\vee_{lw}]{Step\ 1:\ \tau^1} G_0 \xrightarrow[EC]{Step\ 2:\ \tau^2} G_0|G_0^* \xrightarrow[EW]{Step\ 3:\ \tau^3} G_1|G_1^*$$

$$\xrightarrow[\top,\bot,W]{Step\ 4:\ \tau^4} G_2|G_2^* \xrightarrow[ID\ numbers]{Step\ 5:\ \tau^*} G|G^*$$

FIGURE 13 Preprocessing of τ

Notation 4.14. *Let $\dfrac{G_i'''|\{S_i^c\}^{m_i'}}{G_i'''|S_i^c}(EC^*)_i, 1 \leqslant i \leqslant N$ be all (EC^*)-nodes of τ^1 and $G_i'''|\{S_i^c\}^{m_i'}$ be converted to $G_i''|\{S_i^c\}^{m_i}$ in τ^*. Note that there are no identification numbers for occurrences of variable p in $S_i^c \in G_i'''|\{S_i^c\}^{m_i'}$ meanwhile they are assigned to p in $S_i^c \in G_i''|\{S_i^c\}^{m_i}$. But we use the same notations for $S_i^c \in G_i'''|\{S_i^c\}^{m_i'}$ and $S_i^c \in G_i''|\{S_i^c\}^{m_i}$ for simplicity.*

In the whole paper, let $H_i^c = G_i'|\{S_i^c\}^{m_i}$ denote the unique node of τ^* such that $H_i^c \leqslant G_i''|\{S_i^c\}^{m_i}$ and S_i^c is the focus sequent of H_i^c in τ^*, in which case we denote the focus one S_{i1}^c and others $S_{i2}^c|\cdots|S_{im_i}^c$ among $\{S_i^c\}^{m_i}$. We sometimes denote H_i^c also by $G_i'|\{S_{iv}^c\}_{v=1}^{m_i}$ or $G_i'|S_{i1}^c|\{S_{iv}^c\}_{v=2}^{m_i}$. We then write G^* as $\{S_{iv}^c\}_{i=1\cdots N}^{v=2\cdots m_i}$.

We call H_i^c, S_{iu}^c the i-th pseudo-(EC) node of τ^* and pseudo-(EC) sequent, respectively. We abbreviate pseudo-EC as pEC. Let $H \in \tau^*$, by $S_i^c \in H$ we mean that $S_{iu}^c \in H$ for some $1 \leqslant u \leqslant m_i$.

It is possible that there doesn't exist $H_i^c \leqslant G_i''|\{S_i^c\}^{m_i}$ such that S_i^c is the focus sequent of H_i^c in τ^*, in which case $\{S_i^c\}^{m_i} \subseteq G|G^*$, then it hasn't any effect on our argument to treat all such S_i^c as members of G. So we assume that all H_i^c are always defined for all $G_i''|\{S_i^c\}^{m_i}$ in τ^*, i.e., $H_i^c > G|G^*$.

Proposition 4.15. *(i) $\{S_{iv}^c\}_{v=2\cdots m_i} \subseteq H$ for all $H \leqslant H_i^c$; (ii) $G^* = \{S_{iv}^c\}_{i=1\cdots N}^{v=2\cdots m_i}$.*

Now, we replace locally each $\dfrac{G'}{G'}(ID_\Omega)$ in τ^* with G' and denote the resulting proof also by τ^*, which has no essential difference with the original one but could simplify subsequent arguments. We introduce the system $\mathbf{GL}_\Omega$ as follows.

Definition 4.16. $\mathbf{GL}_\Omega$ is a restricted subsystem of **GL** such that

(i) p is designated as the unique eigenvariable by which we mean that it is not used to built up any formula containing logical connectives and only used as a sequent-formula.

(ii) Each occurrence of p on each side of every component of a hypersequent in **GL** is assigned one unique identification number i and written as p_i in $\mathbf{GL}_\Omega$. Initial sequent $p \Rightarrow p$ of **GL** has the form $p_i \Rightarrow p_i$ in $\mathbf{GL}_\Omega$.

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(iii) Each sequent S of **GL** in the form $\Gamma, \lambda p \Rightarrow \mu p, \Delta$ has the form

$$\Gamma, \{p_{i_k}\}_{k=1}^{\lambda} \Rightarrow \{p_{j_k}\}_{k=1}^{\mu}, \Delta$$

in $\mathbf{GL}_\Omega$, where p does not occur in Γ or Δ, $i_k \neq i_l$ for all $1 \leqslant k < l \leqslant \lambda$, $j_k \neq j_l$ for all $1 \leqslant k < l \leqslant \mu$. Define $v_l(S) = \{i_1, \cdots, i_\lambda\}$ and $v_r(S) = \{j_1, \cdots, j_\mu\}$. Let G be a hypersequent of $\mathbf{GL}_\Omega$ in the form $S_1|\cdots|S_n$ then $v_l(S_k) \bigcap v_l(S_l) = \varnothing$ and $v_r(S_k) \bigcap v_r(S_l) = \varnothing$ for all $1 \leqslant k < l \leqslant n$. Define $v_l(G) = \bigcup_{k=1}^{n} v_l(S_k)$, $v_r(G) = \bigcup_{k=1}^{n} v_r(S_k)$. Here, l and r in v_l and v_r indicate the left side and right side of a sequent, respectively.

(iv) A hypersequent G of $\mathbf{GL}_\Omega$ is called closed if $v_l(G) = v_r(G)$. Two hypersequents G' and G'' of $\mathbf{GL}_\Omega$ are called disjoint if $v_l(G') \bigcap v_l(G'') = \varnothing$, $v_l(G') \bigcap v_r(G'') = \varnothing$, $v_r(G') \bigcap v_l(G'') = \varnothing$ and $v_r(G') \bigcap v_r(G'') = \varnothing$. G'' is a copy of G' if they are disjoint and there exist two bijections $\sigma_l : v_l(G') \to v_l(G'')$ and $\sigma_r : v_r(G') \to v_r(G'')$ such that G'' can be obtained by applying σ_l to antecedents of sequents in G' and σ_r to succedents of sequents in G', i.e., $G'' = \sigma_r \circ \sigma_l(G')$.

(v) A closed hypersequent $G'|G''|G'''$ can be contracted as $G'|G''$ in $\mathbf{GL}_\Omega$ under the condition that G'' and G''' are closed and G''' is a copy of G''. We call it the constraint external contraction rule and denote by

$$\frac{G'|G''|G'''}{G'|G''}(EC_\Omega).$$

Furthermore, if there doesn't exist two closed hypersequents $H', H'' \subseteq G'|G''$ such that H'' is a copy of H' then we call it the fully constraint contraction rule and denote by $\dfrac{G'|G''|G'''}{G'|G''}\langle EC^*_\Omega \rangle$.

(vi) (EW) and (CUT) of **GL** are forbidden. (EC), $(\wedge_r)$ and $(\vee_l)$ of **GL** are replaced with (EC_Ω), $(\wedge_{rw})$ and $(\vee_{lw})$ in $\mathbf{GL}_\Omega$, respectively.

(vii) $G_1|S_1$ and $G_2|S_2$ are closed and disjoint for each two-premise rule $\dfrac{G_1|S_1 \quad G_2|S_2}{G_1|G_2|H'}(II)$ of $\mathbf{GL}_\Omega$ and, $G'|S'$ is closed for each one-premise rule $\dfrac{G'|S'}{G'|S''}(I)$.

(viii) p doesn't occur in Γ or Δ for each initial sequent $\Gamma, \bot \Rightarrow \Delta$ or $\Gamma \Rightarrow \top, \Delta$ and, p doesn't act as the weakening formula A in $\dfrac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow A, \Delta}(WR)$ or $\dfrac{G|\Gamma \Rightarrow \Delta}{G|\Gamma, A \Rightarrow \Delta}(WL)$ when (WR) or (WL) is available.

Lemma 4.17. *Let τ be a cut-free proof of G_0 in* **L** *and τ^* be the tree resulting from preprocessing of τ.*

(i) If $\dfrac{G'|S'}{G'|S''}(I) \in \tau^*$ then $v_l(G'|S'') = v_r(G'|S'') = v_r(G'|S') = v_l(G'|S')$;

(ii) If $\dfrac{G'|S' \quad G''|S''}{G'|G''|H'}(II) \in \tau^*$ then $v_l(G'|G''|H') = v_l(G'|S') \bigcup v_l(G''|S'') = v_r(G'|G''|H') = v_r(G'|S') \bigcup v_r(G''|S'')$;

(iii) If $H \in \tau^*$ and $k \in v_l(H)$ then $k \in v_r(H)$;

(iv) If $H \in \tau^*$ and $k \in v_l(H)$ (or $k \in v_r(H)$) then $H \leqslant p_k \Rightarrow p_k$;

(v) τ^* is a proof of $G|G^*$ in $\mathbf{GL}_\Omega$ without (EC_Ω);

(vi) If $H', H'' \in \tau^*$ and $H' \| H''$ then $v_l(H') \bigcap v_l(H'') = \varnothing$, $v_r(H') \bigcap v_r(H'') = \varnothing$.

Proof. Claims from (i) to (iv) are immediately from Step 5 in preprocessing of τ and Definition 4.16. (v) is from Notation 4.13 and Definition 4.16. Only (vi) is proved as follows.

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Suppose that $k \in v_l(H') \bigcap v_l(H'')$. Then $H' \leqslant p_k \Rightarrow p_k$, $H'' \leqslant p_k \Rightarrow p_k$ by Claim (iv). Thus $H' \leqslant H''$ or $H'' \leqslant H'$, a contradiction with $H' \| H''$ hence $v_l(H') \bigcap v_l(H'') = \varnothing$.
$v_r(H') \bigcap v_r(H'') = \varnothing$ is proved by a similar procedure and omitted. □

5. The generalized density rule $(\mathcal{D})$ for $\mathbf{GL}_\Omega$

In this section, we define the generalized density rule $(\mathcal{D})$ for $\mathbf{GL}_\Omega$ and prove that it is admissible in $\mathbf{GL}_\Omega$.

Definition 5.1. Let G be a closed hypersequent of $\mathbf{GL}_\Omega$ and $S \in G$. Define $[S]_G = \bigcap\{H : S \in H \subseteq G, v_l(H) = v_r(H)\}$, i.e., $[S]_G$ is the minimal closed unit of G containing S. In general, for $G' \subseteq G$, define $[G']_G = \bigcap\{H : G' \subseteq H \subseteq G, v_l(H) = v_r(H)\}$.

Clearly, $[S]_G = S$ if $v_l(S) = v_r(S)$ or p does not occur in S. The following construction gives a procedure to construct $[S]_G$ for any given $S \in G$.

Construction 5.2. *Let G and S be as above. A sequence $G_1, G_2, \cdots, G_n$ of hypersequents is constructed recursively as follows. (i) $G_1 = \{S\}$; (ii) Suppose that G_k is constructed for $k \geqslant 1$. If $v_l(G_k) \neq v_r(G_k)$ then there exists $i_{k+1} \in v_l(G_k) \backslash v_r(G_k)$ (or $i_{k+1} \in v_r(G_k) \backslash v_l(G_k)$) thus there exists the unique $S_{k+1} \in G \backslash G_k$ such that $i_{k+1} \in v_r(S_{k+1}) \backslash v_l(S_{k+1})$ (or $i_{k+1} \in v_l(S_{k+1}) \backslash v_r(S_{k+1})$) by $v_l(G) = v_r(G)$ and Definition 4.16 then let $G_{k+1} = G_k | S_{k+1}$ otherwise the procedure terminates and $n := k$.*

Lemma 5.3. *(i) $G_n = [S]_G$;*
(ii)Let $S' \in [S]_G$ then $[S']_G = [S]_G$;
(iii)Let $G' \equiv G|H'$, $G'' \equiv G|H''$,$v_l(G') = v_r(G')$, $v_l(G'') = v_r(G'')$, $v_l(H') \ominus v_r(H') = v_l(H'') \ominus v_r(H'')$ then $[H']_{G'} \backslash H' = [H'']_{G''} \backslash H''$, where $A \ominus B$ is the symmetric difference of two multisets A, B;
(iv)Let $v_{lr}(G_k) = v_l(G_k) \bigcap v_r(G_k)$ then $|v_{lr}(G_k)| + 1 \geqslant |G_k|$ for all $1 \leqslant k \leqslant n$;
(v) $\left|v_l([S]_G)\right| + 1 \geqslant \left|[S]_G\right|$.

Proof. (i) Since $G_k \subseteq G_{k+1}$ for $1 \leqslant k \leqslant n-1$ and $S \in G_1$ then $S \in G_n \subseteq G$ thus $[S]_G \subseteq G_n$ by $v_l(G_n) = v_r(G_n)$. We prove $G_k \subseteq [S]_G$ for $1 \leqslant k \leqslant n$ by induction on k in the following. Clearly, $G_1 \subseteq [S]_G$. Suppose that $G_k \subseteq [S]_G$ for some $1 \leqslant k \leqslant n-1$. Since $i_{k+1} \in v_l(G_k) \backslash v_r(G_k)$ (or $i_{k+1} \in v_r(G_k) \backslash v_l(G_k)$) and $i_{k+1} \in v_r(S_{k+1})$ (or $i_{k+1} \in v_l(S_{k+1})$) then $S_{k+1} \in [S]_G$ by $G_k \subseteq [S]_G$ and $v_l([S]_G) = v_r([S]_G)$ thus $G_{k+1} \subseteq [S]_G$. Then $G_n \subseteq [S]_G$ thus $G_n = [S]_G$.

(ii) By (i), $[S]_G = S_1 | S_2 | \cdots | S_n$, where $S_1 = S$. Then $S' = S_k$ for some $1 \leqslant k \leqslant n$ thus $i_k \in v_r(S_k) \backslash v_l(S_k)$ (or $i_k \in v_l(S_k) \backslash v_r(S_k)$) hence there exists the unique $k' < k$ such that $i_k \in v_l(S_{k'}) \backslash v_r(S_{k'})$ (or $i_k \in v_r(S_{k'}) \backslash v_l(S_{k'})$) if $k \geq 2$ hence $S_{k'} \in [S_k]_G$. Repeatedly, $S_1 \in [S_k]_G$, i.e., $S \in [S']_G$ then $[S]_G \subseteq [S']_G$. $[S']_G \subseteq [S]_G$ by $S' \in [S]_G$ then $[S']_G = [S]_G$.

(iii) It holds immediately from Construction 5.2 and (i).

(iv) The proof is by induction on k. For the base step, let $k = 1$ then $|G_k| = 1$ thus $|v_{lr}(G_k)| + 1 \geqslant |G_k|$ by $|v_{lr}(G_k)| \geqslant 0$. For the induction step, suppose that $|v_{lr}(G_k)| + 1 \geqslant |G_k|$ for some $1 \leqslant k < n$. Then $|v_{lr}(G_{k+1})| \geqslant |v_{lr}(G_k)| + 1$ by $i_{k+1} \in v_{lr}(G_{k+1}) \backslash v_{lr}(G_k)$ and $v_{lr}(G_k) \subseteq v_{lr}(G_{k+1})$. Then $|v_{lr}(G_{k+1})| + 1 \geqslant |G_{k+1}|$ by $|G_{k+1}| = |G_k| + 1 = k + 1$.

(v) It holds by (iv) and $v_{lr}([S]_G) = v_l([S]_G)$. □

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Definition 5.4. Let $G = S_1|\cdots|S_r$ and S_l be in the form $\Gamma_l, \{p_{i_k^l}\}_{k=1}^{\lambda_l} \Rightarrow \{p_{j_k^l}\}_{k=1}^{\mu_l}, \Delta_l$ for $1 \leqslant l \leqslant r$.

(i) If $S \in G$ and $[S]_G$ be $S_{k_1}|\cdots|S_{k_u}$ then $\mathcal{D}_G(S)$ is defined as $\Gamma_{k_1}, \cdots, \Gamma_{k_u} \Rightarrow (|v_l([S]_G)| - |[S]_G| + 1)t, \Delta_{k_1}, \cdots, \Delta_{k_u}$;

(ii) Let $\bigcup_{k=1}^{v}[S_{q_k}]_G = G$ and $[S_{q_k}]_G \bigcap [S_{q_l}]_G = \varnothing$ for all $1 \leqslant k < l \leqslant v$ then $\mathcal{D}(G)$ is defined as $\mathcal{D}_G(S_{q_1})|\cdots|\mathcal{D}_G(S_{q_v})$.

(iii) We call $(\mathcal{D})$ the generalized density rule of $\mathbf{GL}_\Omega$, whose conclusion $\mathcal{D}(G)$ is defined by (ii) if its premise is G.

Clearly, $\mathcal{D}(p_k \Rightarrow p_k)$ is $\Rightarrow t$ and $\mathcal{D}(S) = S$ if p does not occur in S.

Lemma 5.5. *Let $G' \equiv G|S$ and $G'' \equiv G|S_1|S_2$ be closed and $[S_1]_{G''} \bigcap [S_2]_{G''} = \varnothing$, where $S_1 = \Gamma_1, \{p_{i_k^1}\}_{k=1}^{\lambda_1} \Rightarrow \{p_{j_k^1}\}_{k=1}^{\mu_1}, \Delta_1$; $S_2 = \Gamma_2, \{p_{i_k^2}\}_{k=1}^{\lambda_2} \Rightarrow \{p_{j_k^2}\}_{k=1}^{\mu_2}, \Delta_2$; $S = \Gamma_1, \Gamma_2, \{p_{i_k^1}\}_{k=1}^{\lambda_1}, \{p_{i_k^2}\}_{k=1}^{\lambda_2} \Rightarrow \{p_{j_k^1}\}_{k=1}^{\mu_1}, \{p_{j_k^2}\}_{k=1}^{\mu_2}, \Delta_1, \Delta_2$; $\mathcal{D}_{G''}(S_1) = \Gamma_1, \Sigma_1 \Rightarrow \Pi_1, \Delta_1$ and $\mathcal{D}_{G''}(S_2) = \Gamma_2, \Sigma_2 \Rightarrow \Pi_2, \Delta_2$. Then $\mathcal{D}_{G'}(S) = \Gamma_1, \Sigma_1, \Gamma_2, \Sigma_2 \Rightarrow \Pi_1, \Delta_1, \Pi_2, \Delta_2$.*

Proof. Since $[S_1]_{G''} \bigcap [S_2]_{G''} = \varnothing$ then $[S]_{G'} = [S_1]_{G''} \backslash \{S_1\} \bigcup [S_2]_{G''} \backslash \{S_2\} \bigcup \{S\}$ by $v_l(S) = v_l(S_1|S_2)$, $v_r(S) = v_r(S_1|S_2)$ and Lemma 5.3 (iii). Thus $|v_l([S]_{G'})| = |v_l([S_1]_{G''})| + |v_l([S_2]_{G''})|$, $|[S]_{G'}| = |[S_1]_{G''}| + |[S_2]_{G''}| - 1$. Hence

$$|v_l([S]_{G'})| - |[S]_{G'}| + 1 = |v_l([S_1]_{G''})| - |[S_1]_{G''}| + 1 + |v_l([S_2]_{G''})| - |[S_2]_{G''}| + 1.$$

Therefore $\mathcal{D}_{G'}(S) = \Gamma_1, \Sigma_1, \Gamma_2, \Sigma_2 \Rightarrow \Pi_1, \Delta_1, \Pi_2, \Delta_2$ by

$$\Pi_1 = (|v_l([S_1]_{G''})| - |[S_1]_{G''}| + 1)t, \Pi_1 \backslash (|v_l([S_1]_{G''})| - |[S_1]_{G''}| + 1)t$$

$$\Pi_2 = (|v_l([S_2]_{G''})| - |[S_2]_{G''}| + 1)t, \Pi_2 \backslash (|v_l([S_2]_{G''})| - |[S_2]_{G''}| + 1)t$$

$$\mathcal{D}_{G'}(S) = \Gamma_1, \Sigma_1, \Gamma_2, \Sigma_2 \Rightarrow (|v_l([S]_{G'})| - |[S]_{G'}| + 1)t,$$
$$\Pi_1 \backslash (|v_l([S_1]_{G''})| - |[S_1]_{G''}| + 1)t, \Delta_1, \Pi_2 \backslash (|v_l([S_2]_{G''})| - |[S_2]_{G''}| + 1)t, \Delta_2$$

where $\lambda t = \{\underbrace{t, \cdots, t}_{\lambda}\}$. □

Lemma 5.6. ([A.5.1]) *If there exists a proof τ of G in $\mathbf{GL}_\Omega$ then there exists a proof of $\mathcal{D}(G)$ in $\mathbf{GL}$, i.e., $(\mathcal{D})$ is admissible in $\mathbf{GL}_\Omega$.*

Proof. We proceed by induction on the height of τ. For the base step, if G is $p_k \Rightarrow p_k$ then $\mathcal{D}(G)$ is $\Rightarrow t$ otherwise $\mathcal{D}(G)$ is G then $\vdash_{\mathbf{GL}} \mathcal{D}(G)$ holds. For the induction step, the following cases are considered.

- Let

$$\frac{G'|S'}{G'|S''}(\rightarrow_r) \in \tau$$

where

$$S' \equiv A, \Gamma, \{p_{i_k}\}_{k=1}^{\lambda} \Rightarrow \{p_{j_k}\}_{k=1}^{\mu}, \Delta, B,$$

$$S'' \equiv \Gamma, \{p_{i_k}\}_{k=1}^{\lambda} \Rightarrow \{p_{j_k}\}_{k=1}^{\mu}, \Delta, A \rightarrow B.$$

Then $[S'']_{G'|S''} = [S']_{G'|S'} \backslash \{S'\}|S''$ by $v_l(S') = v_l(S'')$, $v_r(S') = v_r(S'')$ and Lemma 5.3 (iii). Let $\mathcal{D}_{G'|S'}(S') = A, \Gamma, \Gamma'' \Rightarrow \Delta'', \Delta, B$ then $\mathcal{D}_{G'|S''}(S'') = \Gamma, \Gamma'' \Rightarrow \Delta'', \Delta, A \rightarrow B$ thus a proof of

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$\mathcal{D}(G'|S'')$ is constructed by combining the proof of $\mathcal{D}(G'|S')$ and $\dfrac{\mathcal{D}_{G'|S'}(S')}{\mathcal{D}_{G'|S''}(S'')}(\to_r)$. Other rules of type (I) are processed by a procedure similar to above.

• Let

$$\frac{G_1|S_1 \quad G_2|S_2}{G_1|G_2|S_3}(\odot_r) \in \tau$$

where

$$S_1 \equiv \Gamma_1, \{p_{i_k^1}\}_{k=1}^{\lambda_1} \Rightarrow \{p_{j_k^1}\}_{k=1}^{\mu_1}, A, \Delta_1$$

$$S_2 \equiv \Gamma_2, \{p_{i_k^2}\}_{k=1}^{\lambda_2} \Rightarrow \{p_{j_k^2}\}_{k=1}^{\mu_2}, B, \Delta_2$$

$$S_3 \equiv \Gamma_1, \Gamma_2, \{p_{i_k^1}\}_{k=1}^{\lambda_1}, \{p_{i_k^2}\}_{k=1}^{\lambda_2} \Rightarrow \{p_{j_k^2}\}_{k=1}^{\mu_2}, \{p_{j_k^1}\}_{k=1}^{\mu_1}, A \odot B, \Delta_1, \Delta_2.$$

Let

$$\mathcal{D}_{G_1|S_1}(S_1) = \Gamma_1, \Gamma_{11} \Rightarrow \Delta_{11}, (\left|v_l([S_1]_{G_1|S_1})\right| - \left|[S_1]_{G_1|S_1}\right| + 1)t, A, \Delta_1,$$

$$\mathcal{D}_{G_2|S_2}(S_2) = \Gamma_2, \Gamma_{21} \Rightarrow \Delta_{21}, (\left|v_l([S_2]_{G_2|S_2})\right| - \left|[S_2]_{G_2|S_2}\right| + 1)t, B, \Delta_2.$$

Then $\mathcal{D}_{G_1|G_2|S_3}(S_3)$ is

$$\Gamma_1, \Gamma_2, \Gamma_{11}, \Gamma_{21} \Rightarrow \Delta_{11}, \Delta_{21}, A \odot B, \Delta_1, \Delta_2,$$

$$(\left|v_l([S_1]_{G_1|S_1})\right| + \left|v_l([S_2]_{G_2|S_2})\right| - \left|[S_1]_{G_1|S_1}\right| - \left|[S_2]_{G_2|S_2}\right| + 2)t$$

by $[S_3]_{G_1|G_2|S_3} = ([S_1]_{G_1|S_1} \backslash \{S_1\}) \cup ([S_2]_{G_2|S_2} \backslash \{S_2\}) \cup \{S_3\}$. Then the proof of $\mathcal{D}(G_1|G_2|S_3)$ is constructed by combining $\vdash_{\mathbf{GL}} \mathcal{D}(G_1|S_1)$ and $\vdash_{\mathbf{GL}} \mathcal{D}(G_2|S_2)$ with $\dfrac{\mathcal{D}_{G_1|S_1}(S_1) \quad \mathcal{D}_{G_2|S_2}(S_2)}{\mathcal{D}_{G_1|G_2|S_3}(S_3)}(\odot_r)$. All applications of $(\to_l)$ are processed by a procedure similar to that of $\odot_r$ and omitted.

• Let

$$\frac{G' \quad G''}{G'''}(\wedge_{rw}) \in \tau$$

where

$$G' \equiv G_1|S_1, \quad G'' \equiv G_2|S_2, \quad G''' \equiv G_1|G_2|S'_1|S'_2,$$

$$S_w \equiv \Gamma_w, \{p_{i_k^w}\}_{k=1}^{\lambda_w} \Rightarrow \{p_{j_k^w}\}_{k=1}^{\mu_w}, A_w, \Delta_w,$$

$$S'_w \equiv \Gamma_w, \{p_{i_k^w}\}_{k=1}^{\lambda_w} \Rightarrow \{p_{j_k^w}\}_{k=1}^{\mu_w}, A_1 \wedge A_2, \Delta_w$$

for $w = 1, 2$. Then $[S'_1]_{G'''} = [S_1]_{G'} \backslash \{S_1\} | S'_1$, $[S'_2]_{G'''} = [S_2]_{G''} \backslash \{S_2\} | S'_2$ by Lemma 5.3 (iii). Let

$$\mathcal{D}_{G_w|S_w}(S_w) = \Gamma_w, \Gamma_{w1} \Rightarrow \Delta_{w1}, (\left|v_l([S_w]_{G_w|S_w})\right| - \left|[S_w]_{G_w|S_w}\right| + 1)t, A_w, \Delta_1$$

for $w = 1, 2$. Then

$$\mathcal{D}_{G'''}(S'_w) = \Gamma_w, \Gamma_{w1} \Rightarrow \Delta_{w1}, (\left|v_l([S_w]_{G_w|S_w})\right| - \left|[S_w]_{G_w|S_w}\right| + 1)t, A_1 \wedge A_2, \Delta_w$$

for $w = 1, 2$. Then the proof of $\mathcal{D}(G''')$ is constructed by combining $\vdash_{\mathbf{GL}} \mathcal{D}(G')$ and $\vdash_{\mathbf{GL}} \mathcal{D}(G'')$ with $\dfrac{\mathcal{D}_{G'}(S_1) \quad \mathcal{D}_{G''}(S_2)}{\mathcal{D}_{G'''}(S'_1|S'_2)}(\wedge_{rw})$. All applications of $(\vee_{lw})$ are processed by a procedure similar to that of $(\wedge_{rw})$ and omitted.

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- Let

$$\frac{G' \quad G''}{G'''}(COM) \in \tau$$

where

$$G' \equiv G_1|S_1, \quad G'' \equiv G_2|S_2, \quad G''' \equiv G_1|G_2|S_3|S_4$$

$$S_1 \equiv \Gamma_1, \Pi_1, \{p_{i_k^1}\}_{k=1}^{\lambda_1} \Rightarrow \{p_{j_k^1}\}_{k=1}^{\mu_1}, \Sigma_1, \Delta_1,$$

$$S_2 \equiv \Gamma_2, \Pi_2, \{p_{i_k^2}\}_{k=1}^{\lambda_2} \Rightarrow \{p_{j_k^2}\}_{k=1}^{\mu_2}, \Sigma_2, \Delta_2,$$

$$S_3 \equiv \Gamma_1, \Gamma_2, \{p_{i_{1k}^1}\}_{k=1}^{\lambda_{11}}, \{p_{i_{1k}^2}\}_{k=1}^{\lambda_{21}} \Rightarrow \{p_{j_{1k}^1}\}_{k=1}^{\mu_{11}}, \{p_{j_{1k}^2}\}_{k=1}^{\mu_{21}}, \Delta_1, \Delta_2,$$

$$S_4 \equiv \Pi_1, \Pi_2, \{p_{i_{2k}^1}\}_{k=1}^{\lambda_{12}}, \{p_{i_{2k}^2}\}_{k=1}^{\lambda_{22}} \Rightarrow \{p_{j_{2k}^1}\}_{k=1}^{\mu_{12}}, \{p_{j_{2k}^2}\}_{k=1}^{\mu_{22}}, \Sigma_1, \Sigma_2$$

where $\{p_{i_k^w}\}_{k=1}^{\lambda_w} = \{p_{i_{1k}^w}\}_{k=1}^{\lambda_{w1}} \bigcup \{p_{i_{2k}^w}\}_{k=1}^{\lambda_{w2}}$, $\{p_{j_k^w}\}_{k=1}^{\mu_w} = \{p_{j_{1k}^w}\}_{k=1}^{\mu_{w1}} \bigcup \{p_{j_{2k}^w}\}_{k=1}^{\mu_{w2}}$ for $w = 1, 2$.

Case 1 $S_3 \in [S_4]_{G'''}$. Then $[S_3]_{G'''} = [S_4]_{G'''}$ by Lemma 5.3 (ii) and $[S_3]_{G'''} = [S_1]_{G'} \,|\, [S_2]_{G''} \,|S_3|S_4 \backslash \{S_1, S_2\}$ by Lemma 5.3 (iii). Then

$$\left|v_l([S_3]_{G'''})\right| - \left|[S_3]_{G'''}\right| + 1 = \left|v_l([S_1]_{G'})\right| + \left|v_l([S_2]_{G''})\right| - \left|[S_1]_{G'}\right| - \left|[S_2]_{G''}\right| + 1 \geqslant 0.$$

Thus $\left|v_l([S_1]_{G'})\right| - \left|[S_1]_{G'}\right| + 1 \geqslant 1$ or $\left|v_l([S_2]_{G''})\right| - \left|[S_2]_{G''}\right| + 1 \geqslant 1$. Hence we assume that, without loss of generality,

$$\mathcal{D}_{G'}(S_1) = \Gamma_1, \Pi_1, \Gamma' \Rightarrow \Delta', t, \Sigma_1, \Delta_1,$$

$$\mathcal{D}_{G''}(S_2) = \Gamma_2, \Pi_2, \Gamma'' \Rightarrow \Delta'', \Sigma_2, \Delta_2.$$

Then

$$\mathcal{D}_{G'''}(S_3|S_4) = \Gamma_1, \Pi_1, \Gamma', \Gamma_2, \Pi_2, \Gamma'' \Rightarrow \Delta', \Sigma_1, \Delta_1, \Delta'', \Sigma_2, \Delta_2.$$

Thus the proof of $\dfrac{\mathcal{D}_{G'}(S_1) \quad \mathcal{D}_{G''}(S_2)}{\mathcal{D}_{G'''}(S_3|S_4)}$ is constructed by

$$\dfrac{\Gamma_1, \Pi_1, \Gamma' \Rightarrow \Delta', t, \Sigma_1, \Delta_1 \qquad \dfrac{\Gamma_2, \Pi_2, \Gamma'' \Rightarrow \Delta'', \Sigma_2, \Delta_2}{\Gamma_2, \Pi_2, \Gamma'', t \Rightarrow \Delta'', \Sigma_2, \Delta_2}(t_l)}{\Gamma_1, \Pi_1, \Gamma', \Gamma_2, \Pi_2, \Gamma'' \Rightarrow \Delta', \Sigma_1, \Delta_1, \Delta'', \Sigma_2, \Delta_2}(CUT).$$

Case 2 $S_3 \notin [S_4]_{G'''}$. Then $[S_3]_{G'''} \bigcap [S_4]_{G'''} = \varnothing$ by Lemma 5.3 (ii). Let

$$S_{3w} \equiv \Gamma_w, \{p_{i_{1k}^w}\}_{k=1}^{\lambda_{w1}} \Rightarrow \{p_{j_{1k}^w}\}_{k=1}^{\mu_{w1}}, \Delta_w,$$

$$S_{4w} \equiv \Pi_w, \{p_{i_{2k}^w}\}_{k=1}^{\lambda_{w2}} \Rightarrow \{p_{j_{2k}^w}\}_{k=1}^{\mu_{w2}}, \Sigma_w,$$

for $w = 1, 2$. Then

$$[S_3]_{G'''} = [S_{31}]_{G_1|S_{31}|S_{41}} \backslash \{S_{31}\} \bigcup [S_{32}]_{G_2|S_{32}|S_{42}} \backslash \{S_{32}\} \bigcup \{S_3\},$$

$$[S_4]_{G'''} = [S_{41}]_{G_1|S_{31}|S_{41}} \backslash \{S_{41}\} \bigcup [S_{42}]_{G_2|S_{32}|S_{42}} \backslash \{S_{42}\} \bigcup \{S_4\}$$

by $v_l(S_3) = v_l(S_{31}|S_{32})$, $v_l(S_1) = v_l(S_{31}|S_{41})$, $v_l(S_2) = v_l(S_{32}|S_{42})$ and $v_l(S_4) = v_l(S_{41}|S_{42})$. Let

$$\mathcal{D}_{G_w|S_{3w}|S_{4w}}(S_{3w}) = \Gamma_w, \mathrm{X}_{3w} \Rightarrow \Psi_{3w}, \Delta_w,$$

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$$\mathcal{D}_{G_w|S_{3w}|S_{4w}}(S_{4w}) = \Pi_w, X_{4w} \Rightarrow \Psi_{4w}, \Sigma_w$$

for $w = 1, 2$. Then

$$\mathcal{D}_{G'}(S_1) = \Gamma_1, \Pi_1, X_{31}, X_{41} \Rightarrow \Psi_{31}, \Psi_{41}, \Sigma_1, \Delta_1,$$
$$\mathcal{D}_{G''}(S_2) = \Gamma_2, \Pi_2, X_{32}, X_{42} \Rightarrow \Psi_{32}, \Psi_{42}, \Sigma_2, \Delta_2,$$
$$\mathcal{D}_{G'''}(S_3) = \Gamma_1, X_{31}, \Gamma_2, X_{32} \Rightarrow \Psi_{31}, \Delta_1, \Psi_{32}, \Delta_2,$$
$$\mathcal{D}_{G'''}(S_4) = \Pi_1, X_{41}, \Pi_2, X_{42} \Rightarrow \Psi_{41}, \Sigma_1, \Psi_{42}, \Sigma_2$$

by Lemma 5.5, $[S_3]_{G'''} \cap [S_4]_{G'''} = \emptyset, [S_{31}]_{G_1|S_{31}|S_{41}} \cap [S_{41}]_{G_1|S_{31}|S_{41}} = \emptyset$, $[S_{32}]_{G_2|S_{32}|S_{42}} \cap [S_{42}]_{G_2|S_{32}|S_{42}} = \emptyset$. Then the proof of $\mathcal{D}_{G'''}(S_3|S_4)$ is constructed by combing the proofs of $\mathcal{D}_{G'}(S_1)$ and $\mathcal{D}_{G''}(S_2)$ with $\dfrac{\mathcal{D}_{G'}(S_1) \quad \mathcal{D}_{G''}(S_2)}{\mathcal{D}_{G'''}(S_3|S_4)}(COM)$.

- $\dfrac{G'|G''|G'''}{G'|G''}(EC_\Omega) \in \tau$. Then G', G'' and G''' are closed and G''' is a copy of G'' thus $\mathcal{D}_{G'|G''|G'''}(G'') = \mathcal{D}_{G'|G''|G'''}(G''')$ hence a proof of $\mathcal{D}(G'|G'')$ is constructed by combining the proof of $\mathcal{D}(G'|G''|G''')$ and $\dfrac{\mathcal{D}(G'|G''|G''')}{\mathcal{D}(G'|G'')}(EC^*)$. □

The following two lemmas are corollaries of Lemma 5.6.

Lemma 5.7. *If there exists a derivation of G_0 from $G_1, \cdots, G_r$ in* $\mathbf{GL}_\Omega$ *then there exists a derivation of $\mathcal{D}(G_0)$ from $\mathcal{D}(G_1), \cdots, \mathcal{D}(G_r)$ in* **GL**.

Lemma 5.8. *Let τ be a cut-free proof of G_0 in* **GL** *and τ^* be the proof of $G|G^*$ in* $\mathbf{GL}_\Omega$ *resulting from preprocessing of τ. Then $\vdash_{\mathbf{GL}} \mathcal{D}(G|G^*)$.*

6. Extraction of Elimination Rules

In this section, we will investigate Construction 4.7 further to extract more derivations from τ^*.

Any two sequents in a hypersequent seem independent of one another in the sense that they can only be contracted into one by (EC) when it is applicable. Note that one-premise logical rules just modify one sequent of a hypersequent and two-premise rules associate a sequent in a hypersequent with one in a different hypersequent.

τ^* (or any proof without (EC_Ω) in $\mathbf{GL}_\Omega$) has an essential property, which we call the distinguishability of τ^*, i.e., any variables, formulas, sequents or hypersequents which occur at the node H of τ^* occur inevitably at $H' < H$ in some forms.

Let $H \equiv G'|S'|S'' \in \tau^*$. If S' is equal to S'' as two sequents then the case that $\tau^*_{H:S'}$ is equal to $\tau^*_{H:S''}$ as two derivations could possibly happen. This means that both S' and S'' are the focus sequent of one node in τ^* when $G^*_{H:S'} \neq S'$ and $G^*_{H:S''} \neq S''$, which contradicts that each node has the unique focus sequent in any derivation. Thus we need differentiate S' from S'' for all $G'|S'|S'' \in \tau^*$.

Define $\overline{S'} \in \tau^*$ such that $G'|S'|S'' \leqslant \overline{S'}$, $S' \in \overline{S'}$ and S' is the principal sequent of $\overline{S'}$. If $\overline{S'}$ has the unique principal sequent, $N_{S'} := 0$, otherwise $N_{S'} := 1$ (or $N_{S'} = 2$) to indicate that S' is one designated principal sequent (or accordingly $N_{S'} = 2$ for another) of such an application as $(COM), (\wedge_{rw})$ or $(\vee_{lw})$. Then we may regard S' as $(S'; \mathcal{P}(\overline{S'}), N_{S'})$. Thus S' is always different from S'' by $\mathcal{P}(\overline{S'}) \neq \mathcal{P}(\overline{S''})$ or, $\mathcal{P}(\overline{S'}) = \mathcal{P}(\overline{S''})$ and $N_{S'} \neq N_{S''}$. We formulate it by the following construction.

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Construction 6.1. ([A.5.2]) *A labeled tree τ^{**}, which has the same tree structure as τ^*, is constructed as follows.*

(i) If S is a leaf τ^*, define $\overline{S} = S$, $N_S = 0$ and the node $\mathcal{P}(S)$ of τ^{**} is labeled by $(S; \mathcal{P}(\overline{S}), N_S)$;

(ii) If $\dfrac{G'|S'}{H \equiv G'|S''}(I) \in \tau^*$ and $\mathcal{P}(G'|S')$ be labeled by $\mathcal{G}'|(S'; \mathcal{P}(\overline{S'}), N_{S'})$ in τ^{**}. Then define $\overline{S''} = H$, $N_{S''} = 0$ and the node $\mathcal{P}(H)$ of τ^{**} is labeled by $\mathcal{G}'|(S''; \mathcal{P}(\overline{S''}), N_{S''})$;

(iii) If $\dfrac{G'|S' \quad G''|S''}{H \equiv G'|G''|H'}(II) \in \tau^*$, $\mathcal{P}(G'|S')$ and $\mathcal{P}(G''|S'')$ be labeled by $\mathcal{G}'|(S'; \mathcal{P}(\overline{S'}), N_{S'})$ and $\mathcal{G}''|(S''; \mathcal{P}(\overline{S''}), N_{S''})$ in τ^{**}, respectively. If $H' = S_1|S_2$ then define $\overline{S_1} = \overline{S_2} = H$, $N_{S_1} = 1$, $N_{S_2} = 2$ and the node $\mathcal{P}(H)$ of τ^{**} is labeled by $\mathcal{G}'|\mathcal{G}''|(S_1; \mathcal{P}(\overline{S_1}), N_{S_1})|(S_2; \mathcal{P}(\overline{S_2}), N_{S_2})$. If $H' = S_1$ then define $\overline{S_1} = H$, $N_{S_1} = 0$ and $\mathcal{P}(H)$ is labeled by $\mathcal{G}'|\mathcal{G}''|(S_1; \mathcal{P}(\overline{S_1}), N_{S_1})$.

In the whole paper, we treat τ^* as τ^{**} without mention of τ^{**}. Note that in preprocessing of τ, some logical applications could also be converted to (ID_Ω) in Step 3 and we need fix the focus sequent at each node H and subsequently assign valid identification numbers to each $H' < H$ by eigenvariable-labeling operation.

Proposition 6.2. *(i) $G'|S'|S'' \in \tau^*$ implies $\{S'\} \bigcap \{S''\} = \emptyset$; (ii) $H \in \tau^*$ and $H'|H'' \subseteq H$ imply $H' \bigcap H'' = \emptyset$; (iii) Let $H \in \tau^*$ and $S_i^c \in H$ then $H \leqslant H_i^c$ or $H_i^c \leqslant H$.*

Proof. (iii) Let $S_i^c \in H$ then $S_i^c = S_{iu}^c$ for some $1 \leqslant u \leqslant m_i$ by Notation 4.14. Thus $S_i^c \in H_i^c$ also by Notation 4.14. Hence $H \leqslant \overline{S_i^c}$ and $H_i^c \leqslant \overline{S_i^c}$ by Construction 6.1. Therefore $H \leqslant H_i^c$ or $H_i^c \leqslant H$. □

Lemma 6.3. *Let $H \in \tau^*$ and $Th(H) = (H_0, \cdots, H_n)$, where $H_0 = H$, $H_n = G|G^*$, $G_k \subseteq H$ for $1 \leqslant k \leqslant 3$.*

(i) If $G_3 = G_1 \bigcap G_2$ then $\langle H_i \rangle_{H:G_3} = \langle H_i \rangle_{H:G_1} \bigcap \langle H_i \rangle_{H:G_2}$ for all $0 \leqslant i \leqslant n$;

(ii) If $G_3 = G_1|G_2$ then $\langle H_i \rangle_{H:G_3} = \langle H_i \rangle_{H:G_1} | \langle H_i \rangle_{H:G_2}$ for all $0 \leqslant i \leqslant n$.

Proof. The proof is by induction on i for $0 \leqslant i < n$. Only (i) is proved as follows and (ii) by a similar procedure and omitted.

For the base step, $\langle H_0 \rangle_{H:G_3} = \langle H_0 \rangle_{H:G_1} \bigcap \langle H_0 \rangle_{H:G_2}$ holds by $\langle H_0 \rangle_{H:G_1} = G_1$, $\langle H_0 \rangle_{H:G_2} = G_2$, $\langle H_0 \rangle_{H:G_3} = G_3$ and $G_3 = G_1 \bigcap G_2$.

For the induction step, suppose that $\langle H_i \rangle_{H:G_3} = \langle H_i \rangle_{H:G_1} \bigcap \langle H_i \rangle_{H:G_2}$ for some $0 \leqslant i < n$. Only is the case of one-premise rule given in the following and other cases are omitted.

Let $\dfrac{G'|S'}{G'|S''}(I) \in \tau^*$, $H_i = G'|S'$ and $H_{i+1} = G'|S''$.

Let $S' \in \langle H_i \rangle_{H:G_3}$. Then $\langle H_{i+1} \rangle_{H:G_3} = (\langle H_i \rangle_{H:G_3} \backslash \{S'\})|S''$, $\langle H_{i+1} \rangle_{H:G_1} = (\langle H_i \rangle_{H:G_1} \backslash \{S'\})|S''$ by $S' \in \langle H_i \rangle_{H:G_1}$ and $\langle H_{i+1} \rangle_{H:G_2} = (\langle H_i \rangle_{H:G_2} \backslash \{S'\})|S''$ by $S' \in \langle H_i \rangle_{H:G_2}$. Thus $\langle H_{i+1} \rangle_{H:G_3} = \langle H_{i+1} \rangle_{H:G_1} \bigcap \langle H_{i+1} \rangle_{H:G_2}$ by $\langle H_i \rangle_{H:G_3} = \langle H_i \rangle_{H:G_1} \bigcap \langle H_i \rangle_{H:G_2}$.

Let $S' \notin \langle H_i \rangle_{H:G_1}$ and $S' \notin \langle H_i \rangle_{H:G_2}$. Then $\langle H_{i+1} \rangle_{H:G_1} = \langle H_i \rangle_{H:G_1}$, $\langle H_{i+1} \rangle_{H:G_2} = \langle H_i \rangle_{H:G_2}$ and $\langle H_{i+1} \rangle_{H:G_3} = \langle H_i \rangle_{H:G_3}$. Thus $\langle H_{i+1} \rangle_{H:G_3} = \langle H_{i+1} \rangle_{H:G_1} \bigcap \langle H_{i+1} \rangle_{H:G_2}$ by $\langle H_i \rangle_{H:G_3} = \langle H_i \rangle_{H:G_1} \bigcap \langle H_i \rangle_{H:G_2}$.

Let $S' \notin \langle H_i \rangle_{H:G_1}$, $S' \in \langle H_i \rangle_{H:G_2}$. Then $\langle H_{i+1} \rangle_{H:G_1} = \langle H_i \rangle_{H:G_1}$, $\langle H_{i+1} \rangle_{H:G_3} = \langle H_i \rangle_{H:G_3}$ and $\langle H_{i+1} \rangle_{H:G_2} = (\langle H_i \rangle_{H:G_2} \backslash \{S'\})|S''$. Thus $\langle H_{i+1} \rangle_{H:G_3} = \langle H_{i+1} \rangle_{H:G_1} \bigcap \langle H_{i+1} \rangle_{H:G_2}$ by $\langle H_i \rangle_{H:G_3} = \langle H_i \rangle_{H:G_1} \bigcap \langle H_i \rangle_{H:G_2}$, $S'' \notin \langle H_{i+1} \rangle_{H:G_1}$.

The case of $S' \notin \langle H_i \rangle_{H:G_2}$, $S' \in \langle H_i \rangle_{H:G_1}$ is proved by a similar procedure and omitted. □

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Lemma 6.4. *(i) Let $G'|S' \in \tau^*$ then $G^*_{G'|S':S'} \cap G^*_{G'|S':G'} = \varnothing, G^*_{G'|S':G'}|G^*_{G'|S':S'} = G|G^*$;*

(ii) $H \in \tau^*$, $H'|H'' \subseteq H$ then $G^*_{H:H'|H''} = G^*_{H:H'}|G^*_{H:H''}$.

Proof. (i) and (ii) are immediately from Lemma 6.3. □

Notation 6.5. *We write $\tau^*_{H_i^c:S_{i1}^c}$, $G^*_{H_i^c:S_{i1}^c}$ as $\tau^*_{S_{i1}^c}$, $G^*_{S_{i1}^c}$, respectively, for the sake of simplicity.*

Lemma 6.6. *(i) $G^*_{S_{i1}^c} \subseteq G|G^*$;*

(ii) $\tau^*_{S_{i1}^c}$ is a derivation of $G^*_{S_{i1}^c}$ from S_{i1}^c, which we denote by $\dfrac{S_{i1}^c}{G^*_{S_{i1}^c}}\left\langle \tau^*_{S_{i1}^c}\right\rangle$;

(iii) $G^*_{S_{iu}^c} = S_{iu}^c$ and $\tau^*_{S_{iu}^c}$ consists of a single node S_{iu}^c for all $2 \leqslant u \leqslant m_i$;

(iv) $v_l(G^*_{S_{i1}^c})\backslash v_l(S_{i1}^c) = v_r(G^*_{S_{i1}^c})\backslash v_r(S_{i1}^c)$;

(v) $\langle H\rangle_{S_{i1}^c} \in \tau^*_{S_{i1}^c}$ implies $H \leqslant H_i^c$. Note that $\langle H\rangle_{S_{i1}^c}$ is undefined for any $H > H_i^c$ or $H\|H_i^c$.

(vi) $S_j^c \in G^*_{S_{i1}^c}$ implies $H_i^c \nleqslant H_j^c$.

Proof. Claims from (i) to (v) are immediately from Construction 4.7 and Lemma 4.8.

(vi) Since $S_j^c \in G^*_{S_{i1}^c} \subseteq G|G^*$ then S_j^c has the form S_{ju}^c for some $u \geq 2$ by Notation 4.14. Then $G^*_{S_j^c} = S_j^c$ by (iii). Suppose that $H_i^c \leqslant H_j^c$. Then S_j^c is transferred from H_j^c downward to H_i^c and in side-hypersequent of H_i^c by Notation 4.14 and $G|G^* < H_i^c \leqslant H_j^c$. Thus $\{S_{i1}^c\}\cap\{S_j^c\} = \varnothing$ at H_i^c since S_{i1}^c is the unique focus sequent of H_i^c. Hence $S_j^c \notin G^*_{S_{i1}^c}$ by Lemma 6.3 and (iii), a contradiction therefore $H_i^c \nleqslant H_j^c$. □

Lemma 6.7. *Let $\dfrac{G'|S' \quad G''|S''}{H \equiv G'|G''|H'}(II) \in \tau^*$. (i) If $S_j^c \in G^*_{H:H'}$ then $H_j^c \leqslant H$ or $H_j^c\|H$; (ii) If $S_j^c \in G^*_{H:G''}$ then $H_j^c \leqslant H$ or $H_j^c\|G'|S'$.*

Proof. (i) We impose a restriction on (II) such that each sequent in H' is different from S' or S'' otherwise we treat it as an (EW)-application. Since $S_j^c \in G^*_{H:H'} \subseteq G|G^*$ then S_j^c has the form S_{ju}^c for some $u \geq 2$ by Notation 4.14. Thus $G^*_{S_j^c} = S_j^c$. Suppose that $H_j^c > H$. Then S_j^c is transferred from H_j^c downward to H. Thus $S_j^c \in H'$ by $G^*_{S_j^c} = S_j^c \in G^*_{H:H'}$ and Lemma 6.3. Hence $S_j^c = S'$ or $S_j^c = S''$, a contradiction with the restriction above. Therefore $H_j^c \leqslant H$ or $H_j^c\|H$.

(ii) Let $S_j^c \in G^*_{H:G''}$. If $H_j^c > H$ then $S_j^c \in H$ by Proposition 4.15(i) and thus $S_j^c \in G''$ by Lemma 6.3 and, hence $H_j^c \parallel G'|S'$ by $H_j^c \geqslant G''|S''$, $G'|S'\|G''|S''$. If $H_j^c\|H$ then $H_j^c \parallel G'|S'$ by $H \leqslant G'|S'$. Thus $H_j^c \leqslant H$ or $H_j^c\|G'|S'$. □

Definition 6.8. (i) By $H_i^c \rightsquigarrow H_j^c$ we mean that $S_{ju}^c \in G^*_{S_{i1}^c}$ for some $2 \leqslant u \leqslant m_j$; (ii) By $H_i^c \leftrightsquigarrow H_j^c$ we mean that $H_i^c \rightsquigarrow H_j^c$ and $H_j^c \rightsquigarrow H_i^c$; (iii) $H_i^c \not\rightsquigarrow H_j^c$ means that $S_{ju}^c \notin G^*_{S_{i1}^c}$ for all $2 \leqslant u \leqslant m_j$.

Then Lemma 6.6 (vi) shows that $H_i^c \rightsquigarrow H_j^c$ implies $H_i^c \nleqslant H_j^c$.

Lemma 6.9. *Let $H_i^c\|H_j^c$, $H_i^c \rightsquigarrow H_j^c$, $\dfrac{G'|S' \quad G''|S''}{G'|G''|H'}(II) \in \tau^*$ such that $G'|S' \leqslant H_i^c$, $G''|S'' \leqslant H_j^c$. Then $S' \in \langle G'|S'\rangle_{S_{i1}^c}$.*

Proof. Suppose that $S' \notin \langle G'|S'\rangle_{S_{i1}^c}$. Then $\langle G'|S'\rangle_{S_{i1}^c} \subseteq G'$ by $\langle G'|S'\rangle_{S_{i1}^c} \subseteq G'|S'$, $\langle G'|G''|H'\rangle_{S_{i1}^c} = \langle G'|S'\rangle_{S_{i1}^c}$ by Construction 4.7. Thus $\langle G'|G''|H'\rangle_{S_{i1}^c} \subseteq G'$. Hence $G''|H' \cap \langle G'|G''|H'\rangle_{S_{i1}^c} = \varnothing$ by Proposition 6.2 (ii). Therefore $S_{ju}^c \notin G^*_{S_{i1}^c}$ for all $1 \leqslant u \leqslant m_j$ by Lemma 6.3, i.e., $H_i^c \not\rightsquigarrow H_j^c$, a contradiction and hence $S' \in \langle G'|S'\rangle_{S_{i1}^c}$. □

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Lemma 6.6 (ii) shows that $\tau^*_{S^c_{i1}}$ is a derivation of $G^*_{S^c_{i1}}$ from one premise S^c_{i1}. We generalize it by introducing derivations from multiple premises in the following. In the remainder of this section, let $I = \{H^c_{i_1}, \cdots, H^c_{i_m}\} \subseteq \{H^c_1, \cdots, H^c_N\}$, $H^c_{i_k} \nleftrightarrow H^c_{i_q}$ for all $1 \leqslant k < q \leqslant m$. Then $H^c_{i_k} \nleqslant H^c_{i_q}$ and $H^c_{i_q} \nleqslant H^c_{i_k}$ by Lemma 6.6 (vi) thus $H^c_{i_k} \| H^c_{i_q}$ for all $1 \leqslant k < q \leqslant m$.

Notation 6.10. *H^V_I denotes the intersection node of $H^c_{i_1}, \cdots, H^c_{i_m}$. We sometimes write the intersection node of H^c_i and H^c_j as H^V_{ij}. If $I = \{H^c_i\}$, $H^V_I := H^c_i$, i.e., the intersection node of a single node is itself.*

Let $\dfrac{G'|S' \quad G''|S''}{G'|G''|H'}(II) \in \tau^*$ such that $G'|G''|H' = H^V_I$. Then I is divided into two subsets $I_l = \{H^c_{l_1}, \cdots, H^c_{l_{m(l)}}\}$ and $I_r = \{H^c_{r_1}, \cdots, H^c_{r_{m(r)}}\}$, which occur in the left subtree $\tau^*(G'|S')$ and right subtree $\tau^*(G''|S'')$ of $\tau^*(G'|G''|H')$, respectively.

Let $\mathcal{I} = \{S^c_{i_11}, \cdots, S^c_{i_m1}\}$, $\mathcal{I}_l = \{S^c_{l_11}, \cdots, S^c_{l_{m(l)}1}\}$, $\mathcal{I}_r = \{S^c_{r_11}, \cdots, S^c_{r_{m(r)}1}\}$ such that $\mathcal{I} = \mathcal{I}_l \bigcup \mathcal{I}_r$. A derivation $\tau^*_{\mathcal{I}}$ of $\langle G|G^*\rangle_{\mathcal{I}}$ from $S^c_{i_11}, \cdots, S^c_{i_m1}$ is constructed by induction on $|I|$. The base case of $|I| = 1$ has been done by Construction 4.7. For the induction case, suppose that derivations $\tau^*_{\mathcal{I}_l}$ of $\langle G|G^*\rangle_{\mathcal{I}_l}$ from $S^c_{l_11}, \cdots, S^c_{l_{m(l)}1}$ and $\tau^*_{\mathcal{I}_r}$ of $\langle G|G^*\rangle_{\mathcal{I}_r}$ from $S^c_{r_11}, \cdots, S^c_{r_{m(r)}1}$ are constructed. Then $\tau^*_{\mathcal{I}}$ of $\langle G|G^*\rangle_{\mathcal{I}}$ from $S^c_{i_11}, \cdots, S^c_{i_m1}$ is constructed as follows.

Construction 6.11. ([A.5.2]) *(i)*

$$\langle H\rangle_{\mathcal{I}} := \langle H\rangle_{\mathcal{I}_l} \text{ for all } G'|S' \leqslant H \leqslant H^c_i \text{ for some } H^c_i \in I_l,$$
$$\langle H\rangle_{\mathcal{I}} := \langle H\rangle_{\mathcal{I}_r} \text{ for all } G''|S'' \leqslant H \leqslant H^c_i \text{ for some } H^c_i \in I_r,$$
$$\tau^*_{\mathcal{I}}(\langle G'|S'\rangle_{\mathcal{I}}) := \tau^*_{\mathcal{I}_l}(\langle G'|S'\rangle_{\mathcal{I}_l}),\ \ \tau^*_{\mathcal{I}}(\langle G''|S''\rangle_{\mathcal{I}}) := \tau^*_{\mathcal{I}_r}(\langle G''|S''\rangle_{\mathcal{I}_r});$$

(ii)

$$\langle G'|G''|H'\rangle_{\mathcal{I}} := \langle G'\rangle_{\mathcal{I}_l}|\langle G''\rangle_{\mathcal{I}_r}|H'$$

and

$$\frac{\langle G'|S'\rangle_{\mathcal{I}_l} \quad \langle G''|S''\rangle_{\mathcal{I}_r}}{\langle G'|G''|H'\rangle_{\mathcal{I}}}(II) \in \tau^*_{\mathcal{I}};$$

(iii) Other nodes of $\tau^*_{\mathcal{I}}$ are built up by Construction 4.7 (ii).

The following lemma is a generalization of Lemma 6.6.

Lemma 6.12. *Let $Th(H^c_{i_k}) = (H^c_{i_k0}, \cdots, H^c_{i_kn_{i_k}})$, where $1 \leqslant k \leqslant m, H^c_{i_k0} = H^c_{i_k}$ and $H^c_{i_kn_{i_k}} = G|G^*$. Then, for all $0 \leqslant u \leqslant n_{i_k}$,*

(i)

$$\langle H^c_{i_ku}\rangle_{\mathcal{I}} = \bigcap\{\langle H^c_{i_ku}\rangle_{S^c_{j1}} : H^c_j \in I, H^c_{i_ku} \leqslant H^c_j\};$$

(ii)

$$\frac{\{S^c_{j1} : H^c_j \in I, H^c_{i_ku} \leqslant H^c_j\}}{\langle H^c_{i_ku}\rangle_{\mathcal{I}}}\langle\tau^*_{\mathcal{I}}(\langle H^c_{i_ku}\rangle_{\mathcal{I}})\rangle;$$

(iii)

$$v_l(\langle H^c_{i_ku}\rangle_{\mathcal{I}})\backslash\bigcup\{v_l(S^c_{j1}) : H^c_j \in I, H^c_{i_ku} \leqslant H^c_j\} =$$
$$v_r(\langle H^c_{i_ku}\rangle_{\mathcal{I}})\backslash\bigcup\{v_r(S^c_{j1}) : H^c_j \in I, H^c_{i_ku} \leqslant H^c_j\};$$

(iv) $\langle H\rangle_{\mathcal{I}} \in \tau^*_{\mathcal{I}}$ if and only if $H \leqslant H^c_i$ for some $H^c_i \in I$. Note that $\langle H\rangle_{\mathcal{I}}$ is undefined if $H > H^c_i$ or $H\|H^c_i$ for all $H^c_i \in I$.

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Proof: (i) is proved by induction on $|I|$. For the base step, let $|I| = 1$ then the claim holds clearly. For the induction step, let $|I| \geqslant 2$ then $|I_l| \geqslant 1$ and $|I_r| \geqslant 1$. Then $S' \in \langle G'|S'\rangle_{S^c_{i1}}$ for all $H^c_i \in I_l$ by Lemma 6.9 and $H^c_i \rightsquigarrow H^c_j$ for all $H^c_j \in I_r$. $\langle G'|S'\rangle_{\mathcal{I}_l} = \bigcap_{H^c_i \in I_l} \langle G'|S'\rangle_{S^c_{i1}}$ by the induction hypothesis then $S' \in \langle G'|S'\rangle_{\mathcal{I}_l}$ thus $\langle G'|G''|H'\rangle_{\mathcal{I}_l} = \langle G'\rangle_{\mathcal{I}_l}|G''|H'$ by $G'|S' \leqslant H^V_{I_l}$.

$\langle G'|G''|H'\rangle_{\mathcal{I}_r} = \langle G''\rangle_{\mathcal{I}_r}|G'|H'$ holds by a procedure similar to above then

$$\begin{aligned}\langle G'|G''|H'\rangle_{\mathcal{I}} &= \langle G'\rangle_{\mathcal{I}_l} | \langle G''\rangle_{\mathcal{I}_r}|H' \\ &= (\langle G'\rangle_{\mathcal{I}_l}|G''|H') \bigcap (\langle G''\rangle_{\mathcal{I}_r}|G'|H') \\ &= \langle G'|G''|H'\rangle_{\mathcal{I}_l} \bigcap \langle G'|G''|H'\rangle_{\mathcal{I}_r}\end{aligned}$$

by $\langle G'\rangle_{\mathcal{I}_l} \subseteq G'$ and $\langle G''\rangle_{\mathcal{I}_r} \subseteq G''$. Other claims hold immediately from Construction 6.11.

Lemma 6.13. *(i) Let $G^*_{\mathcal{I}}$ denote $\langle G|G^*\rangle_{\mathcal{I}}$ then $G^*_{\mathcal{I}} = \bigcap_{H^c_i \in I} G^*_{S^c_{i1}}$;*

(ii) $\dfrac{S^c_{i_1 1} \ \cdots \ S^c_{i_m 1}}{G^*_{\mathcal{I}}} \langle \tau^*_{\mathcal{I}} \rangle$;

(iii) $v_l(G^*_{\mathcal{I}}) \backslash \bigcup_{H^c_j \in I} v_l(S^c_{j1}) = v_r(G^*_{\mathcal{I}}) \backslash \bigcup_{H^c_j \in I} v_r(S^c_{j1})$;

(iv) $S^c_j \in G^*_{\mathcal{I}}$ implies $H^c_i \nleqslant H^c_j$ for all $H^c_i \in I$.

Proof. (i), (ii) and (iii) are immediately from Lemma 6.12. (iv) holds by (i) and Lemma 6.6 (vi). □

Lemma 6.13 (iv) shows that there exists no copy of $S^c_{i_k}$ in $G^*_{\mathcal{I}}$ for any $1 \leqslant k \leqslant m$. Then we may regard them to be eliminated in $\tau^*_{\mathcal{I}}$. We then call $\tau^*_{\mathcal{I}}$ an elimination derivation.

Let $\mathcal{I}' = \{S^c_{i_1 u_1}, \cdots, S^c_{i_m u_m}\}$ be another set of sequents to I such that $G' \equiv S^c_{i_1 u_1}|\cdots|S^c_{i_m u_m}$ is a copy of $G'' \equiv S^c_{i_1 1}|\cdots|S^c_{i_m 1}$. Then G' and G'' are disjoint and there exist two bijections $\sigma_l : v_l(G') \to v_l(G'')$ and $\sigma_r : v_r(G') \to v_r(G'')$ such that $\sigma_r \circ \sigma_l(G') = G''$. By applying $\sigma_r \circ \sigma_l$ to $\tau^*_{\mathcal{I}}$, we construct a derivation from $S^c_{i_1 u_1}, \cdots, S^c_{i_m u_m}$ and denote it by $\tau^*_{\mathcal{I}'}$ and its root by $G^*_{\mathcal{I}'}$.

Let $\mathbf{I}' = \{G_{b_1}|S^c_{i_1 u_1}, \cdots, G_{b_m}|S^c_{i_m u_m}\}$ be a set of hypersequents to I, where $G_{b_k}|S^c_{i_k u_k}$ be closed for all $1 \leqslant k \leqslant m$. By applying $\tau^*_{\mathcal{I}'}$ to $S^c_{i_1 u_1}, \cdots, S^c_{i_m u_m}$ in $G_{b_1}|S^c_{i_1 u_1}, \cdots, G_{b_m}|S^c_{i_m u_m}$, we construct a derivation from

$$G_{b_1}|S^c_{i_1 u_1}, \cdots, G_{b_m}|S^c_{i_m u_m}$$

and denote it by $\tau^*_{\mathbf{I}'}$ and its root by $G^*_{\mathbf{I}'}$. Then $G^*_{\mathbf{I}'} = \{G_{b_k}\}^m_{k=1}|G^*_{\mathcal{I}'}$.

Definition 6.14. We will use all $\tau^*_{\mathbf{I}'}$ as rules of $\mathbf{GL_\Omega}$ and call them elimination rules. Further, we call $S^c_{i_1 u_1}, \cdots, S^c_{i_m u_m}$ focus sequents and, all sequents in $G^*_{\mathcal{I}'}$ principal sequents and, $G_{b_1}, \cdots, G_{b_m}$ side-hypersequents of $\tau^*_{\mathbf{I}'}$.

Remark 6.15. We regard Construction 4.7 as a procedure $\mathcal{F}$, whose inputs are τ^2, H, H' and output $\tau^2_{H:H'}$. With such a viewpoint, we write $\tau^2_{H:H'}$ as $\mathcal{F}_{H:H'}(\tau^2)$. Then $\tau^*_{\mathcal{I}}$ can be constructed by iteratively applying $\mathcal{F}$ to τ^*, i.e., $\tau^*_{\mathcal{I}} = \mathcal{F}_{H^c_{i_m}:S^c_{i_m 1}}(\cdots \mathcal{F}_{H^c_{i_1}:S^c_{i_1 1}}(\tau^*)\cdots)$.

We replace locally each $\dfrac{G'}{G'}(ID_\Omega)$ in $\tau^*_{\mathcal{I}}$ with G' and denote the resulting derivation also by $\tau^*_{\mathcal{I}}$. Then each non-root node in $\tau^*_{\mathcal{I}}$ has the focus sequent.

Let $H \in \tau^*_{\mathcal{I}}$. Then there exists a unique node in τ^*, which we denote by $\mathcal{O}(H)$ such that H comes from $\mathcal{O}(H)$ by Construction 4.7 and 6.11. Then the focus sequent of $\mathcal{O}(H)$ in τ^* is the focus of H in $\tau^*_{\mathcal{I}}$ if H is a non-root node and, $\mathcal{O}(H) = H$ or $H \subseteq \mathcal{O}(H)$ as two hypersequents.

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Since the relative position of any two nodes in τ^* keep unchanged in constructing $\tau^*_{\mathcal{I}}$, $H_1 \leqslant_{\tau^*_{\mathcal{I}}} H_2$ if and only if $\mathcal{O}(H_1) \leqslant_{\tau^*} \mathcal{O}(H_2)$ for any $H_1, H_2 \in \tau^*_{\mathcal{I}}$. Especially, $\mathcal{O}(S^c_{i_k 1}) = H^c_{i_k}$ for $S^c_{i_k 1} \in \tau^*_{\mathcal{I}}$.

Let $H \in \tau^*_{\mathcal{I}}$. Then $H' \equiv \sigma_r \circ \sigma_l(H) \in \tau^*_{\mathcal{I}'}$ and $H'' \equiv \{G_{b_k} : H \leqslant_{\tau^*_{\mathcal{I}}} S^c_{i_k 1} \text{and} 1 \leqslant k \leqslant m\} \mid H' \in \tau^*_{\mathbf{I}'}$. Define $\mathcal{O}(H') = \mathcal{O}(H'') = \mathcal{O}(H)$. Then $\mathcal{O}(G^*_{\mathbf{I}'}) = G|G^*$ and $\mathcal{O}(G_{b_k}|S^c_{i_k u_k}) = H^c_{i_k}$ for all $G_{b_k}|S^c_{i_k u_k} \in \tau^*_{\mathbf{I}'}$.

Since $G^*_{\mathcal{I}} = \langle G|G^* \rangle_{\mathcal{I}} \subseteq G|G^*$, then each (pEC)-sequent in $G^*_{\mathcal{I}}$ has the form S^c_{jv} for some $1 \leqslant j \leqslant N$, $2 \leqslant v \leqslant m_j$ by Proposition 4.15(ii). Then we introduce the following definition.

Definition 6.16. (i) By $S^c_j \in G^*_{\mathcal{I}}$ we means that there exists $H \in \tau^*_{\mathcal{I}}$ such that $S^c_j \in H$, $\mathcal{O}(H) = H^c_j$. So is $S^c_j \in G^*_{\mathcal{I}'}$.

(ii) Let $S^c_j \in G^*_{\mathcal{I}}$. By $H^c_j \leqslant_{\tau^*_{\mathcal{I}}} H^c_i$ we means that there exist $H, H' \in \tau^*_{\mathcal{I}}$ such that $S^c_j \in H$, $\mathcal{O}(H) = H^c_j$, $\mathcal{O}(H') = H^c_i$ and $H^c_j \leqslant_{\tau^*} H^c_i$. We usually write $\leqslant_{\tau^*_{\mathcal{I}}}$ as $\leqslant$.

7. Separation of one branch

In the remainder of this paper, we assume that p occur at most one time for each sequent in G_0 as the one in Main theorem, τ be a cut-free proof of G_0 in **GL** and τ^* the proof of $G|G^*$ in $\mathbf{GL}_\Omega$ resulting from preprocessing of τ. Then $|v_l(S)| + |v_r(S)| \leq 1$ for all $S \in G$, which plays a key role in discussing the separation of branches.

Definition 7.1. By $S' \in_c G'$ we mean that there exists some copy of S' in G'. $G' \subseteq_c G''$ if $S' \in_c G''$ for all $S' \in G'$. $G' =_c G''$ if $G' \subseteq_c G''$ and $G'' \subseteq_c G'$. Let $G_{11}, \cdots, G_{1m}$ be m copies of G_1 then we denote $G'|G_{11}|\cdots|G_{1m}$ by $G'|\{G_{1u}\}^m_{u=1}$ or $G'|\{G_1\}^m$.

Definition 7.2. Let $I = \{H^c_{i_1}, \cdots, H^c_{i_m}\} \subseteq \{H^c_1, \cdots, H^c_N\}$, $H^c_{i_k} \| H^c_{i_l}$ for all $1 \leqslant k < l \leqslant m$. $\lceil S^c_{i_k} \rceil_I$ is called a branch of $H^c_{i_k}$ to I if it is a closed hypersequent such that

(i) $\lceil S^c_{i_k} \rceil_I \subseteq_c G|G^*$,
(ii) $S^c_{i_k} \in \lceil S^c_{i_k} \rceil_I$,
(iii) $S^c_j \in \lceil S^c_{i_k} \rceil_I$ implies $H^c_j \leqslant H^c_{i_k}$ or $H^c_j \| H^c_i$ for all $H^c_i \in I$.

Then (i) $S^c_{i_l} \notin_c \lceil S^c_{i_k} \rceil_I$ for all $1 \leqslant k, l \leqslant m$, $k \neq l$; (ii) $S^c_j \in \lceil S^c_{i_k} \rceil_I$ and $H^c_j \nleqslant H^c_{i_k}$ imply $H^c_j \notin I$.

In this section, let $I = \{H^c_i\}$, $\mathbf{I} = \{\lceil S^c_i \rceil_I\}$, we will give an algorithm to eliminate all $S^c_j \in \lceil S^c_i \rceil_I$ satisfying $H^c_j \leqslant H^c_i$.

Construction 7.3. ([A.3]) *A sequence of hypersequents $G^{\star(q)}_{\mathbf{I}}$ and their derivations $\tau^{\star(q)}_{\mathbf{I}}$ from $\lceil S^c_i \rceil_I$ for all $q \geqslant 0$ are constructed inductively as follows.*

For the base case, define $G^{\star(0)}_{\mathbf{I}}$ to be $\lceil S^c_i \rceil_I$ and, $\tau^{\star(0)}_{\mathbf{I}}$ be $\dfrac{}{G^{\star(0)}_{\mathbf{I}}}$. For the induction case, suppose that $\tau^{\star(q)}_{\mathbf{I}}$ and $G^{\star(q)}_{\mathbf{I}}$ are constructed for some $0 \leqslant q$. If there exists no $S^c_j \in G^{\star(q)}_{\mathbf{I}}$ such that $H^c_j \leqslant H^c_i$, then the procedure terminates and define $J_{\mathbf{I}}$ to be q; otherwise define $H^c_{i_q}$ such that $S^c_{i_q} \in G^{\star(q)}_{\mathbf{I}}$, $H^c_{i_q} \leqslant H^c_i$ and $H^c_j \leqslant H^c_{i_q}$ for all $S^c_j \in G^{\star(q)}_{\mathbf{I}}, H^c_j \leqslant H^c_i$. Let $S^c_{i_q 1}, \cdots, S^c_{i_q m_q}$ be all copies of $S^c_{i_q}$ in $G^{\star(q)}_{\mathbf{I}}$ then define $G^{\star(q+1)}_{\mathbf{I}} = G^{\star(q)}_{\mathbf{I}} \backslash \{S^c_{i_q u}\}^{m_q}_{u=1} | \{G^*_{S^c_{i_q u}}\}^{m_q}_{u=1}$ and its derivation $\tau^{\star(q+1)}_{\mathbf{I}}$ is constructed by sequentially applying $\tau^*_{S^c_{i_q 1}}, \cdots, \tau^*_{S^c_{i_q m_q}}$ to $S^c_{i_q 1}, \cdots, S^c_{i_q m_q}$ in $G^{\star(q)}_{\mathbf{I}}$, respectively. Notice that we assign new identification numbers to new occurrences of p in $\tau^*_{S^c_{i_q u}}$ for all $0 \leqslant q \leqslant J_{\mathbf{I}} - 1$, $1 \leqslant u \leqslant m_q$.

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Lemma 7.4. *(i)* $H^c_{i_0} = H^c_i$ *and* $H^c_{i_{q+1}} < H^c_{i_q}$ *for all* $0 \leqslant q \leqslant J_{\mathbf{I}} - 2$*;*

(ii) $G_{\mathbf{I}}^{\star(q)} \subseteq_c G|G^*$ *is closed for all* $0 \leqslant q \leqslant J_{\mathbf{I}}$*;*

(iii) $\dfrac{\lceil S^c_i \rceil_I}{G_{\mathbf{I}}^{\star(q)}} \left\langle \tau_{\mathbf{I}}^{\star(q)} \right\rangle$ *for all* $0 \leqslant q \leqslant J_{\mathbf{I}}$*, especially,* $\dfrac{\lceil S^c_i \rceil_I}{G_{\mathbf{I}}^{\star(J_{\mathbf{I}})}} \left\langle \tau_{\mathbf{I}}^{\star(J_{\mathbf{I}})} \right\rangle$*;*

(iv) $S^c_j \in G_{\mathbf{I}}^{\star(J_{\mathbf{I}})}$ *implies* $H^c_j \| H^c_i$ *and,* $S^c_j \in G^*_{S^c_{i_q u}}$ *for some* $\tau^*_{G_b|S^c_{i_q u}} \in \tau_{\mathbf{I}}^{\star(J_{\mathbf{I}})}$ *or* $S^c_j \in \lceil S^c_i \rceil_I$*,* $H^c_j \nleqslant H^c_i$*, where* $G_b = G_I^{\star(q)} \backslash \{S^c_{i_q v}\}_{v=1}^{u} | \{G^*_{S^c_{i_q v}}\}_{v=1}^{u-1}$*,* $G_b|S^c_{i_q u}$ *is closed and* $0 \leqslant q \leqslant J_{\mathbf{I}} - 1$*,* $1 \leqslant u \leqslant m_q$*.*

Proof. (i) Since $S^c_i \in G_{\mathbf{I}}^{\star(0)}$ by $S^c_i \in \lceil S^c_i \rceil_I = G_{\mathbf{I}}^{\star(0)}$ and, $H^c_j \leqslant H^c_i$ for all $S^c_j \in G_{\mathbf{I}}^{\star(0)}$, $H^c_j \leqslant H^c_i$ then $H^c_{i_0} = H^c_i$. If $S^c_{i_{q+1}} \in G_{\mathbf{I}}^{\star(q)} \backslash \{S^c_{i_q u}\}_{u=1}^{m_q}$ then $H^c_{i_{q+1}} \leqslant H^c_{i_q}$ by $S^c_{i_{q+1}} \in G_{\mathbf{I}}^{\star(q)}$, $H^c_{i_{q+1}} \leqslant H^c_i$ thus $H^c_{i_{q+1}} < H^c_{i_q}$ by all copies of $S^c_{i_q}$ in $G_{\mathbf{I}}^{\star(q)}$ being collected in $\{S^c_{i_q u}\}_{u=1}^{m_q}$. If $S^c_{i_{q+1}} \in \{G^*_{S^c_{i_q u}}\}_{u=1}^{m_q}$ then $H^c_{i_q} \nleqslant H^c_{i_{q+1}}$ by Lemma 6.6 (vi) thus $H^c_{i_{q+1}} < H^c_{i_q}$ by $H^c_{i_q} \leqslant H^c_i$, $H^c_{i_{q+1}} \leqslant H^c_i$. Then $H^c_{i_{q+1}} < H^c_{i_q}$ by $G_{\mathbf{I}}^{\star(q+1)} = G_{\mathbf{I}}^{\star(q)} \backslash \{S^c_{i_q u}\}_{u=1}^{m_q} | \{G^*_{S^c_{i_q u}}\}_{u=1}^{m_q}$. Note that $H^c_{i_{J_{\mathbf{I}}}}$ is undefined in Construction 7.3.

(ii) $v_l(G_{\mathbf{I}}^{\star(0)}) = v_r(G_{\mathbf{I}}^{\star(0)})$, $G_{\mathbf{I}}^{\star(0)} \subseteq_c G|G^*$ by $G_{\mathbf{I}}^{\star(0)} = \lceil S^c_i \rceil_I$. Suppose that $v_l(G_{\mathbf{I}}^{\star(q)}) = v_r(G_{\mathbf{I}}^{\star(q)})$, $G_{\mathbf{I}}^{\star(q)} \subseteq_c G|G^*$ then $v_l(G_{\mathbf{I}}^{\star(q+1)}) = v_r(G_{\mathbf{I}}^{\star(q+1)})$, $G_{\mathbf{I}}^{\star(q+1)} \subseteq_c G|G^*$ by $G_{\mathbf{I}}^{\star(q+1)} = G_{\mathbf{I}}^{\star(q)} \backslash \{S^c_{i_q u}\}_{u=1}^{m_q} | \{G^*_{S^c_{i_q u}}\}_{u=1}^{m_q}$, $v_l(G^*_{S^c_{i_q u}} \backslash \{S^c_{i_q u}\}) = v_r(G^*_{S^c_{i_q u}} \backslash \{S^c_{i_q u}\})$ and $G^*_{S^c_{i_q u}} \subseteq_c G|G^*$ for all $1 \leqslant u \leqslant m_q$.

(iii) $\tau_{\mathbf{I}}^{\star(0)}$ is $\dfrac{}{G_{\mathbf{I}}^{\star(0)}} \left\langle \tau_{\mathbf{I}}^{\star(0)} \right\rangle$. Given $\dfrac{\lceil S^c_i \rceil_I}{G_{\mathbf{I}}^{\star(q)}} \left\langle \tau_{\mathbf{I}}^{\star(q)} \right\rangle$ then $\dfrac{\lceil S^c_i \rceil_I}{G_{\mathbf{I}}^{\star(q+1)}} \left\langle \tau_{\mathbf{I}}^{\star(q+1)} \right\rangle$ is constructed by linking up the conclusion of previous derivation to the premise of its successor in the sequence of derivations

$$\frac{\lceil S^c_i \rceil_I}{G_{\mathbf{I}}^{\star(q)}} \left\langle \tau_{\mathbf{I}}^{\star(q)} \right\rangle, \frac{G_{\mathbf{I}}^{\star(q)} \backslash \{S^c_{i_q 1}\} | S^c_{i_q 1}}{G_{\mathbf{I}}^{\star(q)} \backslash \{S^c_{i_q 1}\} | G^*_{S^c_{i_q 1}}} \left\langle \tau^*_{S^c_{i_q 1}} \right\rangle, \cdots, \frac{G_{\mathbf{I}}^{\star(q)} \backslash \{S^c_{i_q u}\}_{u=1}^{m_q - 1} | S^c_{i_q m_q} | \{G^*_{S^c_{i_q u}}\}_{u=1}^{m_q - 1}}{G_{\mathbf{I}}^{\star(q+1)} = G_{\mathbf{I}}^{\star(q)} \backslash \{S^c_{i_q u}\}_{u=1}^{m_q} | \{G^*_{S^c_{i_q u}}\}_{u=1}^{m_q}} \left\langle \tau^*_{S^c_{i_q m_q}} \right\rangle,$$

as shown in the following figure.

$$\cfrac{\cfrac{\cfrac{\cfrac{\lceil S^c_i \rceil_I}{G_{\mathbf{I}}^{\star(q)} = G_{\mathbf{I}}^{\star(q)} \backslash \{S^c_{i_q u}\}_{u=1}^{m_q} | \{S^c_{i_q u}\}_{u=2}^{m_q} | S^c_{i_q 1}} \left\langle \tau_{\mathbf{I}}^{\star(q)} \right\rangle}{G_{\mathbf{I}}^{\star(q)} \backslash \{S^c_{i_q u}\}_{u=1}^{m_q} | \{S^c_{i_q u}\}_{u=3}^{m_q} | S^c_{i_q 2} | G^*_{S^c_{i_q 1}}} \left\langle \tau^*_{S^c_{i_q 1}} \right\rangle}{\vdots} \left\langle \tau^*_{S^c_{i_q 2}} \right\rangle}{G_{\mathbf{I}}^{\star(q)} \backslash \{S^c_{i_q u}\}_{u=1}^{m_q} | S^c_{i_q m_q} | \{G^*_{S^c_{i_q u}}\}_{u=1}^{m_q - 1}}}{G_{\mathbf{I}}^{\star(q+1)} = G_{\mathbf{I}}^{\star(q)} \backslash \{S^c_{i_q u}\}_{u=1}^{m_q} | \{G^*_{S^c_{i_q u}}\}_{u=1}^{m_q}} \left\langle \tau^*_{S^c_{i_q m_q}} \right\rangle$$

A derivation of $G_{\mathbf{I}}^{\star(q+1)}$ from $G_{\mathbf{I}}^{\star(q)}$

(iv) Let $S^c_j \in G_{\mathbf{I}}^{\star(J_{\mathbf{I}})}$. Then $H^c_j \nleqslant H^c_i$ by the definition of $J_{\mathbf{I}}$. If $S^c_j \in \lceil S^c_i \rceil_I$, then $H^c_j \| H^c_i$ by $H^c_j \nleqslant H^c_i$ and the definition of $\lceil S^c_i \rceil_I$. Otherwise, by Construction 7.3, there exists some $\tau^*_{G_b|S^c_{i_q u}}$

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in $\tau_{\mathbf{I}}^{\star(J_1)}$ such that $S_j^c \in G^*_{S^c_{i_q u}}$. Then $H^c_{i_q} \nleqslant H^c_j$ by Lemma 6.6 (vi). Thus $H^c_i \nleqslant H^c_j$ by $H^c_{i_q} \leqslant H^c_i$. Hence $H^c_j \| H^c_i$. □

Lemma 7.4 shows that Construction 7.3 presents a derivation $\tau_{\mathbf{I}}^{\star(J_1)}$ of $G_{\mathbf{I}}^{\star(J_1)}$ from $\lceil S_i^c \rceil_I$ such that there doesn't exist $S_j^c \in G_{\mathbf{I}}^{\star(J_1)}$ satisfying $H^c_j \leqslant H^c_i$, i.e., all $S_j^c \in \lceil S_i^c \rceil_I$ satisfying $H^c_j \leqslant H^c_i$ are eliminated by Construction 7.3. We generalize this procedure as follows.

Construction 7.5. *Let $H \in \tau^*$, $H_1 \subseteq H$ and $H_2 \subseteq_c G|G^*$. Then $G_{H:H_l}^{\star(J_{H:H_l})}$ and its derivation $\tau_{H:H_l}^{\star(J_{H:H_l})}$ for $l = 1, 2$ are constructed by procedures similar to that of Construction 7.3 such that $H^c_j \nleqslant H$ for all $S^c_j \in G_{H:H_l}^{\star(J_{H:H_l})}$, where $G_{H:H_1}^{\star(0)} := G^*_{H:H_1}$, $\tau_{H:H_1}^{\star(0)} := \tau^*_{H:H_1}$, which are defined by Construction 4.7.*

We sometimes write $J_{\mathbf{I}}$, $J_{H:H_l}$ as J for simplicity. Then the following lemma holds clearly.

Lemma 7.6. *(i) $\dfrac{H_l}{G_{H:H_l}^{\star(J)}}\left\langle \tau_{H:H_l}^{\star(J)} \right\rangle$, $H^c_j \nleqslant H$ for all $S^c_j \in G_{H:H_l}^{\star(J)}$.*

(ii) If $S^c_i \in H$ and $H^c_i > H$ then $G_{H:S^c_i}^{\star(J)} = S^c_i$.

(iii) If $S \in_c G$ or, $S \in_c G^*$ is a copy of S^c_{i1} and $H^c_i \nleqslant H$ then $G_{H:S}^{\star(J)} = S$.

(iv) Let $H'|H'' \subseteq H \in \tau^*$. Then $G_{H:H'|H''}^{\star(J)} = G_{H:H'}^{\star(J)}|G_{H:H''}^{\star(J)}$ by suitable assignments of identification numbers to new occurrences of p in constructing $\tau_{H:H'|H''}^{\star(J)}$, $\tau_{H:H'}^{\star(J)}$ and $\tau_{H:H''}^{\star(J)}$.

(v) $G_{\mathbf{I}}^{\star(J)} = \bigcup\{G_{H^c_i:S^c_j}^{\star(J)} : S^c_j \in \lceil S^c_i \rceil_I, H^c_j \leqslant H^c_i\} | \bigcup\{S^c_j : S^c_j \in \lceil S^c_i \rceil_I, H^c_j \nleqslant H^c_i\} | \bigcup\{S : S \in \lceil S^c_i \rceil_I, S \in_c G\}$.

Proof. (i) is proved by a procedure similar to that of Lemma 7.4 (iii), (iv) and omitted.

(ii) Since S^c_{i1} is the focus sequent of H^c_i then it is revised by some rule at the node lower than H^c_i. Thus $S^c_i \in H$ is some copy of S^c_{i1} by $H^c_i > H$. Hence S^c_i has the form S^c_{iu} for some $u \geq 2$. Therefore it is transferred downward to $G|G^*$, i.e., $S^c_i \in G|G^*$. Then $G_{H:S^c_i}^{\star(0)} = G^*_{H:S^c_i} = S^c_i$. Since there exists no $S^c_j \in G_{H:S^c_j}^{\star(0)}, H^c_j \leqslant H$ then $J = 0$. Thus $G_{H:S^c_i}^{\star(J)} = S^c_i$.

(iii) is proved by a procedure similar to that of (ii) and omitted.

(iv) Since $H'|H'' \subseteq H \in \tau^*$, then $H' \cap H'' = \varnothing$ by Proposition 6.2. Thus $G_{H:H'|H''}^{\star(0)} = G^*_{H:H'|H''} = G^*_{H:H'}|G^*_{H:H''} = G_{H:H'}^{\star(0)}|G_{H:H''}^{\star(0)}$. Suppose that $G_{H:H'|H''}^{\star(q)} = G_{H:H'}^{\star(q)}|G_{H:H''}^{\star q}$ for some $q \geq 0$. Then all copies $\{S^c_{i_q u}\}_{u=1}^{m_q}$ of $S^c_{i_q}$ in $G_{H:H'|H''}^{\star(q)}$ are divided two subsets $\{S^c_{i_q u}\}_{u=1}^{m_q} \cap G_{H:H'}^{\star(q)}$ and $\{S^c_{i_q u}\}_{u=1}^{m_q} \cap G_{H:H''}^{\star(q)}$. Thus we can construct $G_{H:H'|H''}^{\star(q+1)}, G_{H:H'}^{\star(q+1)}$ and $G_{H:H''}^{\star(q+1)}$ simultaneously and assign the same identification numbers to new occurrences of p in $G_{H:H'}^{\star(q+1)}$ and $G_{H:H''}^{\star(q+1)}$ as the corresponding one in $G_{H:H'|H''}^{\star(q+1)}$. Hence $G_{H:H'|H''}^{\star(q+1)} = G_{H:H'}^{\star(q+1)}|G_{H:H''}^{\star(q+1)}$. Then $G_{H:H'|H''}^{\star(J)} = G_{H:H'}^{\star(J)}|G_{H:H''}^{\star(J)}$.

Note that the requirement is imposed only on one derivation that distinct occurrence of p has distinct identification number. We permit $G_{H:H''}^{\star(q+1)} = G_{H:H''}^{\star(q)}$ or $G_{H:H'}^{\star(q+1)} = G_{H:H'}^{\star(q)}$ in the proof above, which has no essential effect on the proof of the claim.

(v) is immediately from (iv). □

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Lemma 7.6(v) shows that $G_{\mathbf{I}}^{\star(J)}$ could be constructed by applying $\tau_{H_i^c:S_j^c}^{\star(J)}$ sequentially to each $S_j^c \in \lceil S_i^c \rceil_I$ satisfying $H_j^c \leqslant H_i^c$. Thus the requirement $H_{i_{q+1}}^c < H_{i_q}^c$ in Construction 7.3 is not necessary, but which make the termination of the procedure obvious.

Construction 7.7. *Apply (EC_{Ω}^*) to $G_{\mathbf{I}}^{\star(J)}$ and denote the resulting hypersequent by $G_{\mathbf{I}}^{\star}$ and its derivation by $\tau_{\mathbf{I}}^{\star}$. It is possible that (EC_{Ω}^*) is not applicable to $G_{\mathbf{I}}^{\star(J)}$ in which case we apply $\langle ID_{\Omega}\rangle$ to it for the regularity of the derivation.*

Lemma 7.8. *(i) $\dfrac{\lceil S_i^c \rceil_I}{G_{\mathbf{I}}^{\star}}\langle \tau_{\mathbf{I}}^{\star}\rangle$, $G_{\mathbf{I}}^{\star}$ is closed and $H_j^c \| H_i^c$ for all $S_j^c \in G_{\mathbf{I}}^{\star}$;*

(ii) $\tau_{\mathbf{I}}^{\star}$ is constructed by applying elimination rules, say, $\dfrac{G_b|S_{i_{qu}}^c}{G_b|G_{S_{i_{qu}}^c}^*}\left\langle \tau_{G_b|S_{i_{qu}}^c}^*\right\rangle$, and the fully constraint contraction rules, say, $\dfrac{G_2}{G_1}\langle EC_{\Omega}^*\rangle$, where $H_{i_q}^c \leqslant H_i^c$, $G_b|S_{i_{qu}}^c$ is closed for $0 \leqslant q \leqslant J-1$, $1 \leqslant u \leqslant m_q$.

Proof. Immediately from Lemma 7.4. □

Definition 7.9. Let $G' \in G_{\mathbf{I}}^{\star(J)}$, $H' \subseteq G'$ and $S' \in H'$. (i) For any sequent-formula A of S', define $\widehat{A}$ to be the sequent S of $G_{\mathbf{I}}^{\star(J)}$ such that A is a sequent-formula of S or subformula of a sequent-formula of S; (ii) Let S' be in the form $A_1, \cdots, A_n \Rightarrow B_1, \cdots, B_m$, define $\widehat{S'}$ to be the hypersequent which consists of all distinct sequents among $\widehat{A_1}, \cdots, \widehat{A_n}, \widehat{B_1}, \cdots, \widehat{B_m}$; (iii) Let H' be in the form $S_1|\cdots|S_m$, define $\widehat{H'}$ to be $\widehat{S_1}|\cdots|\widehat{S_m}$; (iv) We call H' to be separable if $\widehat{H'} \subseteq_c G$ and, call it to be separated into $\widehat{H'}$.

Note that $\tau_{\mathbf{I}}^{\star(J)}$ is a derivation without (EC_{Ω}) in $\mathbf{GL}_{\Omega}$. Then we can extract elimination derivations from it by Construction 4.7.

Notation 7.10. *Let $H' \subseteq G' \in \tau_{\mathbf{I}}^{\star(J)}$. $\tau_{\mathbf{I}\{\mathrm{G}':\mathrm{H}'\}}^{\star(J)}$ denotes the derivation from H', which extracts from $\tau_{\mathbf{I}}^{\star(J)}$ by Construction 4.7, and denote its root by $G_{\mathbf{I}\{G':H'\}}^{\star(J)}$.*

The following two lemmas show that Construction 7.3 and 7.5 force some sequents in $\lceil S_i^c \rceil_I$ or H' to be separable.

Lemma 7.11. *Let $\dfrac{G'|S' \quad G''|S''}{H \equiv G'|G''|H'}(II) \in \tau^*$. Then (i) H' is separable in $\tau_{\mathrm{H}:\mathrm{H}'}^{\star(J)}$;*

(ii) If $\dfrac{G_b|\langle G'|S'\rangle_{S_{i_{qu}}^c} \quad G''|S''}{H_1 \equiv G_b|\langle G'\rangle_{S_{i_{qu}}^c}|G''|H'}(II) \in \tau_{G_b|S_{i_{qu}}^c}^* \in \tau_{\mathbf{I}}^{\star}$, then H' is separable in $\tau_{\mathbf{I}}^{\star(J)}$ and there is a unique copy of $\widehat{S''}|G_{\mathbf{I}\{H_1:G''\}}^{\star(J)}$ in $G_{\mathbf{I}}^{\star}$.

Proof. (i) We write $\leqslant_{\tau_{H:H'}^{\star(J)}}$ and $\leqslant_{\tau^*}$ respectively as $\leqslant_{\star}$ and $\leqslant$ for simplicity. Since $G_{H:H'}^{\star(J)} \subseteq_c G|G^*$, we divide it into two hypersequents $G_{H:H'}^{0(J)}$ and $G_{H:H'}^{*(J)}$ such that $G_{H:H'}^{\star(J)} = G_{H:H'}^{0(J)}|G_{H:H'}^{*(J)}, G_{H:H'}^{0(J)} \subseteq_c G, G_{H:H'}^{*(J)} \subseteq_c G^*$.

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Let $S_j^c \in G_{H:H'}^{*(J)}$ then $H_j^c \nleqslant H$ by Construction 7.5. We prove that $H_j^c \| H'$ in $\tau_{H:H'}^{\star(J)}$ as follows. If $S_j^c \in G_{H:H'}^{*}$ then $H_j^c \| H'$ in $\tau_{H:H'}^{*}$ by Lemma 6.7(i), $\tau_{H:H'}^{*} \in \tau_{H:H'}^{\star(J)}$ and $H_j^c \nleqslant H$. Thus we assume that $S_j^c \notin G_{H:H'}^{*}$ in the following.

Then, by Lemma 7.4(iv), there exists some $\tau_{G_b|S_i^c}^{*}$ in $\tau_{H:H'}^{\star(J)}$ such that $H_i^c \leqslant H$, $S_j^c \in G_{S_i^c}^{*}$. Then $H_i^c \nleqslant H_j^c$ by Lemma 6.6(vi). $H_j^c \nleqslant H_i^c$ by $H_i^c \leqslant H$, $H_j^c \nleqslant H$. Thus $H_i^c \|_{\tau^*} H_j^c$. Let $\dfrac{G_1|S_1 \quad G_2|S_2}{G_1|G_2|H_2}(II) \in \tau^*$, where $G_1|G_2|H_2 = H_{ij}^V$, $G_1|S_1 \leqslant H_i^c$, $G_2|S_2 \leqslant H_j^c$. Then $S_1 \in \langle G_1|S_1\rangle_{S_i^c}$, $G_b|\langle G_1|S_1\rangle_{S_i^c} \in \tau_{H:H'}^{\star(J)}$, $G_2|S_2 \in \tau_{H:H'}^{\star(J)}$ by $S_j^c \in G_{S_i^c}^{*} \subseteq G_b|G_{S_i^c}^{*} \in \tau_{H:H'}^{\star(J)}$. Thus $H_j^c \|_{\star} H'$ by $G_b|\langle G_1|S_1\rangle_{S_i^c} \leqslant_{\star} G_b|S_i^c \leqslant_{\star} H'$, $G_2|S_2 \leqslant_{\star} H_j^c$ and $G_b|\langle G_1|S_1\rangle_{S_i^c} \|_{\star} G_2|S_2$.

Thus $H_j^c \| H'$ in $\tau_{H:H'}^{\star(J)}$. Therefore $G_{H:H'}^{*(J)} \cap \widehat{H'} = \varnothing$. Then $\widehat{H'} \subseteq G_{H:H'}^{0(J)} \subseteq_c G$, i.e., H' is separable in $\tau_{\mathrm{H:H'}}^{\star(J)}$.

(ii) Clearly, $G_{\mathbf{I}\{\mathrm{H_1:G''|H'}\}}^{\star(J)}$ is a copy of $G_{H:G''|H'}^{\star(J)}$ and, $\tau_{\mathbf{I}\{\mathrm{H_1:G''|H'}\}}^{\star(J)}$ has no difference with $\tau_{H:G''|H'}^{\star(J)}$ except some applications of (ID_Ω) and identification numbers of some $p's$. Then H' is separated into $\widehat{H'}$ in $G_{\mathbf{I}\{H_1:H'\}}^{\star(J)}$ by the same reason as that of (i). Then S', S'' are separated into $\widehat{S'}$ and $\widehat{S''}$ in $\tau_{\mathbf{I}}^{\star(J)}$, respectively. Then $\widehat{S''}|G_{\mathbf{I}\{H_1:G''\}}^{\star(J)} \subseteq G_{\mathbf{I}}^{0(J)}|G_{\mathbf{I}}^{*(J)}$ is closed since $G''|S''$ is closed. Thus all copies of $\widehat{S''}|G_{\mathbf{I}\{H_1:G''\}}^{\star(J)}$ in $\tau_{\mathbf{I}}^{\star(J)}$ are contracted into one by (EC_Ω^*) in $G_{\mathbf{I}}^{\star}$. □

Lemma 7.12. *(i) All copies of S_i^c in $\lceil S_i^c \rceil_I$ are separable in $\tau_{\mathbf{I}}^{\star(J)}$;*

(ii) Let $H \in \tau^*$, $H' \subseteq H$, $H_j^c \leqslant H$ or $H_j^c \| H$ for all $S_j^c \in G_{H:H'}^{*}$. Then H' is separable in $\tau_{H:H'}^{\star(J)}$.

Proof. (i) and (ii) are proved by a procedure similar to that of Lemma 7.11 and omitted. □

Definition 7.13. The skeleton of $\tau_{\mathbf{I}}^{\star}$, which we denote by $\bar{\tau}_{\mathbf{I}}^{\star}$, is constructed by replacing all $\dfrac{G_b|S_{i_qu}^c}{\dfrac{}{G_b|G_{S_{i_qu}^c}^{*}}}\left\langle \tau_{G_b|S_{i_qu}^c}^{*} \right\rangle \in \tau_{\mathbf{I}}^{\star}$ with $\dfrac{G_b|S_{i_qu}^c}{G_b|G_{S_{i_qu}^c}^{*}}(\tau_{G_b|S_{i_qu}^c}^{*})$, i.e, $G_b|S_{i_qu}^c$ is the parent node of $G_b|G_{S_{i_qu}^c}^{*}$ in $\bar{\tau}_{\mathbf{I}}^{\star}$.

Lemma 7.14. *$\bar{\tau}_{\mathbf{I}}^{\star}$ is a linear structure with the lowest node $G_{\mathbf{I}}^{\star}$ and the highest $\lceil S_i^c \rceil_I$.*

Proof. It holds by all $\tau_{G_b|S_{i_qu}^c}^{*}$ and EC_Ω^{*} in $\tau_{\mathbf{I}}^{\star}$ being one-premise rules. □

Definition 7.15. We call Construction 7.3 together with 7.7 the separation algorithm of one branch and, Construction 7.5 the separation algorithm along H.

8. Separation algorithm of multiple branches

In this section, let $I = \{H_{i_1}^c, \cdots, H_{i_m}^c\} \subseteq \{H_1^c, \cdots, H_N^c\}$ such that $H_{i_k}^c \| H_{i_l}^c$ for all $1 \leqslant k < l \leqslant m$. We will generalize the separation algorithm of one branch to that of multiple branches. Roughly speaking, we give an algorithm to eliminate all $S_j^c \in G|G^*$ satisfying $H_j^c \leqslant H_{i_k}^c$ for some $H_{i_k}^c \in I$.

Definition 8.1. $\bar{I} := \{H_j^c : H_j^c \leqslant H_i^c \ \textit{for some} \ H_i^c \in I\}$.

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Theorem 8.2. ([A.4, A.5.4]) *Let* $\mathbf{I} = \{\lceil S^c_{i_1}\rceil_I, \cdots, \lceil S^c_{i_m}\rceil_I\}$. *Then there exist one closed hypersequent* $G^{\star}_{\mathbf{I}} \subseteq_c G|G^*$ *and its derivation* $\tau^{\star}_{\mathbf{I}}$ *from* $\lceil S^c_{i_1}\rceil_I, \ldots, \lceil S^c_{i_m}\rceil_I$ *in* $\mathbf{GL}_{\mathbf{\Omega}}$ *such that*

(i) $\tau^{\star}_{\mathbf{I}}$ *is constructed by applying elimination rules, say,*

$$\dfrac{G_{b_1}|S^c_{j_1} \quad G_{b_2}|S^c_{j_2} \quad \cdots \quad G_{b_w}|S^c_{j_w}}{G^*_{\mathbf{I_j}} = \{G_{b_k}\}^w_{k=1}|G^*_{\mathcal{I}_\mathbf{j}}}\langle \tau^*_{\mathbf{I_j}}\rangle,$$

and the fully constraint contraction rules, say $\dfrac{G_2}{G_1}\langle EC^*_{\Omega}\rangle$, *where* $1 \leqslant w \leqslant m$, $H^c_{j_k} \nleftrightarrow H^c_{j_l}$ *for all* $1 \leqslant k < l \leqslant w$, $I_\mathbf{j} = \{H^c_{j_1}, \cdots, H^c_{j_w}\} \subseteq \overline{I}$, $\mathcal{I}_\mathbf{j} = \{S^c_{j_1}, \cdots, S^c_{j_w}\}$, $\mathbf{I_j} = \{G_{b_1}|S^c_{j_1}, \cdots, G_{b_w}|S^c_{j_w}\}$ *and* $G_{b_k}|S^c_{j_k}$ *is closed for all* $1 \leqslant k \leqslant w$. *Then* $H^c_i \nleqslant H^c_j$ *for all* $S^c_j \in G^*_{\mathcal{I}_\mathbf{j}}$ *and* $H^c_i \in I$.

(ii) For all $H \in \overline{\tau}^{\star}_{\mathbf{I}}$,

$$\partial_{\tau^{\star}_{\mathbf{I}}}(H) := \begin{cases} G|G^* & H \text{ is the root of } \overline{\tau}^{\star}_{\mathbf{I}} \text{ or } G_2 \text{ in } \dfrac{G_2}{G_1}\langle EC^*_{\Omega} \text{ or } ID_{\Omega}\rangle \in \overline{\tau}^{\star}_{\mathbf{I}}, \\ H^c_{j_k} & H \text{ is } G_{b_k}|S^c_{j_k} \text{ in } \tau^*_{\mathbf{I_j}} \in \overline{\tau}^{\star}_{\mathbf{I}} \text{ for some } 1 \leqslant k \leqslant w, \end{cases}$$

where, $\overline{\tau}^{\star}_{\mathbf{I}}$ *is the skeleton of* $\tau^{\star}_{\mathbf{I}}$ *which is defined as Definition 7.13. Then* $\partial_{\tau^{\star}_{\mathbf{I}}}(G^*_{\mathbf{I_j}}) \leqslant \partial_{\tau^{\star}_{\mathbf{I}}}(G_{b_k}|S^c_{j_k})$ *for some* $1 \leqslant k \leqslant w$ *in* $\tau^*_{\mathbf{I_j}}$.

(iii) Let $H \in \overline{\tau}^{\star}_{\mathbf{I}}$, $G|G^* < \partial_{\tau^{\star}_{\mathbf{I}}}(H) \leqslant H^V_I$, *then* $G^{\star(J)}_{H^V_I:H} \in \tau^{\star}_{\mathbf{I}}$ *and it is constructed by applying the separation algorithm along* H^V_I *to* H *and, is an upper hypersequent of either* $\langle EC^*_{\Omega}\rangle$ *if it is applicable, or* $\langle ID_{\Omega}\rangle$ *otherwise.*

(iv) $S^c_j \in G^{\star}_{\mathbf{I}}$ *implies* $H^c_j \| H^c_i$ *for all* $H^c_i \in I$ *and,* $S^c_j \in G^*_{\mathcal{I}_\mathbf{j}}$ *for some* $\tau^*_{\mathbf{I_j}} \in \tau^{\star}_{\mathbf{I}}$ *or* $S^c_j \in \lceil S^c_{i_k}\rceil_I$ *for some* $H^c_{i_k} \in I$ *satisfying* $H^c_j \nleqslant H^c_{i_k}$.

Note that in Claim (i), bold $\mathbf{j}$ in $I_\mathbf{j}, \mathcal{I}_\mathbf{j}$ or $\mathbf{I_j}$ indicates the w-tuple $(j_1, \cdots, j_w)$ in $S^c_{j_1}, \cdots, S^c_{j_w}$. Claim (iv) shows the final aim of Theorem 8.2, i.e., there exists no $S^c_j \in G^{\star}_{\mathbf{I}}$ such that $H^c_j \leqslant H^c_i$ for some $H^c_i \in I$. It is almost impossible to construct $\tau^{\star}_{\mathbf{I}}$ in a non-recursive way. Thus we use Claims (i), (ii) and (iii) in Theorem 8.2 to characterize the structure of $\tau^{\star}_{\mathbf{I}}$ in order to construct it recursively.

Proof. $\tau^{\star}_{\mathbf{I}}$ is constructed by induction on $|I|$. For the base case, let $|I| = 1$. Then $\tau^{\star}_{\mathbf{I}}$ is constructed by Construction 7.3 and 7.7. Here, Claim (i) holds by Lemma 7.8(ii), Lemma 7.4(i) and Lemma 6.6 (vi), Claim (ii) by Lemma 7.4(i), (iii) is clear and (iv) by Lemma 7.4(iv).

For the induction case, let $|I| \geqslant 2$. Let $\dfrac{G'|S' \quad G''|S''}{G'|G''|H'}(II) \in \tau^*$, where $G'|G''|H' = H^V_I$. Then $\{H^c_{i_1}, \cdots, H^c_{i_m}\}$ is divided into two subsets $I_l = \{H^c_{l_1}, \cdots, H^c_{l_{m(l)}}\}, I_r = \{H^c_{r_1}, \cdots, H^c_{r_{m(r)}}\}$, which occur in the left subtree $\tau^*(G'|S')$ and right subtree $\tau^*(G''|S'')$ of $\tau^*(H^V_I)$, respectively. Then $m(l) + m(r) = m$. Let $\mathbf{I}_l = \{\lceil S^c_{l_1}\rceil_I, \cdots, \lceil S^c_{l_{m(l)}}\rceil_I\}, \mathbf{I}_r = \{\lceil S^c_{r_1}\rceil_I, \cdots, \lceil S^c_{r_{m(r)}}\rceil_I\}$. Suppose that derivations $\tau^{\star}_{\mathbf{I}_l}$ of $G^{\star}_{\mathbf{I}_l}$ and $\tau^{\star}_{\mathbf{I}_r}$ of $G^{\star}_{\mathbf{I}_r}$ are constructed such that Claims from (i) to (iv) hold. There are three cases to be considered in the following.

Case 1 $S' \notin \langle G'|S'\rangle_{\mathcal{I}_{\mathbf{j}_l}}$ for all $\tau^*_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\star}_{\mathbf{I}_l}$. Then $\tau^{\star}_{\mathbf{I}} := \tau^{\star}_{\mathbf{I}_l}$ and $G^{\star}_{\mathbf{I}} := G^{\star}_{\mathbf{I}_l}$.

• For Claim (i), let $\tau^*_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\star}_{\mathbf{I}_l}$ and $S^c_j \in G^*_{\mathcal{I}_{\mathbf{j}_l}}$. By the induction hypothesis, $H^c_i \nleqslant H^c_j$ for all $H^c_i \in I_l$. Since $S' \notin \langle G'|S'\rangle_{\mathcal{I}_{\mathbf{j}_l}}$ then $G''|H' \cap \langle G'|G''|H'\rangle_{\mathcal{I}_{\mathbf{j}_l}} = \varnothing$. Thus $G^*_{H^V_I:G''|H'} \cap G^*_{\mathcal{I}_{\mathbf{j}_l}} = \varnothing$ by

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Lemma 6.3 and 6.4. Then $S_j^c \notin G^*_{H_I^V:G''|H'}$. Thus $G''|S'' \nleqslant H_j^c$ by Proposition 4.15(i). Hence, for all $H_i^c \in I_r$, $H_i^c \nleqslant H_j^c$ by $G''|S'' \leqslant H_i^c$. Then $H_i^c \nleqslant H_j^c$ for all $H_i^c \in I$. Claims (ii) and (iii) follow directly from the induction hypothesis.

• For Claim (iv), let $S_j^c \in G^{\star}_{\mathbf{I}}$. It follows from the induction hypothesis that $H_j^c \| H_i^c$ for all $H_i^c \in I_l$ and, $S_j^c \in G^*_{\mathcal{I}_{\mathbf{j}_l}}$ for some $\tau^*_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\star}_{\mathbf{I}_l}$ or $S_j^c \in \lceil S^c_{i_k} \rceil_I$ for some $H^c_{l_k} \in I_l, H_j^c \nleqslant H^c_{l_k}$. Then $H_j^c \nleqslant H_I^V$ by $H_j^c \| H^c_{l_1}, H_I^V < H^c_{l_1}$.

If $S_j^c \in \lceil S^c_{l_k} \rceil_I$ for some $H^c_{l_k} \in I_l, H_j^c \nleqslant H^c_{l_k}$ then $H_j^c \| H_i^c$ for all $H_i^c \in I$ by the definition of branches to I. Thus we assume that $S_j^c \in G^*_{\mathcal{I}_{\mathbf{j}_l}}$ for some $\tau^*_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\star}_{\mathbf{I}_l}$ in the following. If $G'|S' \leqslant H_j^c$ then $H_j^c \| H_i^c$ for all $H_i^c \in I_r$ thus $H_j^c \| H_i^c$ for all $H_i^c \in I$. Thus let $G'|S' \nleqslant H_j^c$ in the following. By the proof of Claim (i) above, $G''|S'' \nleqslant H_j^c$. Then $H_I^V \nless H_j^c$ by $G'|S' \nleqslant H_j^c$ and $G''|S'' \nleqslant H_j^c$. Thus $H_j^c \| H_I^V$. Hence $H_j^c \| H_i^c$ for all $H_i^c \in I$.

Case 2 $S'' \notin \langle G''|S'' \rangle_{\mathcal{I}_{\mathbf{j}_r}}$ for all $\tau^*_{\mathbf{I}_{\mathbf{j}_r}} \in \tau^{\star}_{\mathbf{I}_r}$. Then $\tau^{\star}_{\mathbf{I}} := \tau^{\star}_{\mathbf{I}_r}$ and $G^{\star}_{\mathbf{I}} := G^{\star}_{\mathbf{I}_r}$. This case is proved by a procedure similar to that of Case 1 and omitted.

Case 3 $S' \in \langle G'|S' \rangle_{\mathcal{I}_{\mathbf{j}_l}}$ for some $\tau^*_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\star}_{\mathbf{I}_l}$ and $S'' \in \langle G''|S'' \rangle_{\mathcal{I}^{\star}_{\mathbf{j}_r}}$ for some $\tau^*_{\mathbf{I}_{\mathbf{j}_r}} \in \tau^{\star}_{\mathbf{I}_r}$.

Given

$$\frac{G_{b_{r1}}|S^c_{j_{r1}} \quad G_{b_{r2}}|S^c_{j_{r2}} \quad \cdots \quad G_{b_{rv}}|S^c_{j_{rv}}}{G_r \equiv \{G_{b_{rk}}\}_{k=1}^{v}|G^*_{\mathcal{I}_{\mathbf{j}_r}}} \left\langle \tau^*_{\mathbf{I}_{\mathbf{j}_r}} \right\rangle \in \tau^{\star}_{\mathbf{I}_r}$$

such that $S'' \in \langle G''|S'' \rangle_{\mathcal{I}_{\mathbf{j}_r}}$ and $H^c_{j_{rk}} > H_I^V$ for all $1 \leqslant k \leqslant v$, where, $1 \leqslant v \leqslant m(r)$, $G_{b_{rk}}|S^c_{j_{rk}}$ is closed for all $1 \leqslant k \leqslant v$, $I_{\mathbf{j}_r} = \{H^c_{j_{r1}}, H^c_{j_{r2}}, \cdots, H^c_{j_{rv}}\} \subseteq \overline{I_r}$, $\mathcal{I}_{\mathbf{j}_r} = \{S^c_{j_{r1}}, S^c_{j_{r2}}, \cdots, S^c_{j_{rv}}\}$, $\mathbf{I}_{\mathbf{j}_r} = \{G_{b_{r1}}|S^c_{j_{r1}}, \cdots, G_{b_{rv}}|S^c_{j_{rv}}\}$. Then $H^V_{I_{\mathbf{j}_r}} \geqslant G''|S''$ by $I_{\mathbf{j}_r} \subseteq \overline{I_r}$ and $H^c_{j_{rk}} > H_I^V$ for all $1 \leqslant k \leqslant v$. Thus $H_j^c \rightsquigarrow H_i^c$ for all $H_j^c \in I_{\mathbf{j}_r}$ and $H_i^c \in I_l$ by $S'' \in \langle G''|S'' \rangle_{\mathcal{I}_{\mathbf{j}_r}}$ and Construction 6.11.

For each $\tau^*_{\mathbf{I}_{\mathbf{j}_r}} \in \tau^{\star}_{\mathbf{I}_r}$ above, we construct a derivation $\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I}_{\mathbf{j}_r}})$ in which you may regard $\tau^{\star}_{\mathbf{I}_l}$ as a subroutine, and $\tau^*_{\mathbf{I}_{\mathbf{j}_r}}$ as its input in the following stage 1. Then a derivation $\tau^{\star}_{\mathbf{I}_r}(\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I}_{\mathbf{j}_r}}))$ is constructed by calling $\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I}_{\mathbf{j}_r}})$ in Stage 2, in which you may regard $\tau^{\star}_{\mathbf{I}_r}(\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I}_{\mathbf{j}_r}}))$ as a routine and $\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I}_{\mathbf{j}_r}})$ as its subroutine.

Firstly, we present some properties of $\tau^{\star}_{\mathbf{I}}$ which are derived from Claims (i) ~ (iv) and applicable to $\tau^{\star}_{\mathbf{I}_l}$ or $\tau^{\star}_{\mathbf{I}_r}$ under the induction hypothesis.

Notation 8.3. *Let*

$$G_{\dagger} := \widehat{S''}|G^{\star(J)}_{H_I^V:G''}|G^{\star(J)}_{H_I^V:H'} \backslash \{\widehat{S'}|\widehat{S''}\} \text{ and}$$

$$G_{\ddagger} := \{G_{b_{rk}}\}_{k=1}^{v}|\widehat{S''}|G^{\star(J)}_{H_I^V:\langle G'' \rangle_{\mathcal{I}_{\mathbf{j}_r}}}|G^{\star(J)}_{H_I^V:H'} \backslash \{\widehat{S'}|\widehat{S''}\}$$

be two close hypersequents, $G_{\dagger} \subseteq H$ for some $H \in \tau^{\star}_{\mathbf{I}_l}$ and $G_{\ddagger} \backslash \{G_{b_{rk}}\}_{k=1}^{v} \subseteq H$ for some $H \in \tau^{\star}_{\mathbf{I}_r}$.

Generally, $\widehat{S''} \subseteq G_{\dagger}$ is a copy of $\widehat{S''} \subseteq G_{\ddagger}$, i.e., eigenvariables in $\widehat{S''} \subseteq G_{\dagger}$ have different identification numbers with those in $\widehat{S''} \subseteq G_{\ddagger}$, so are H', G'', S'.

Lemma 8.4. *$S_j^c \in G_{\dagger}$ implies $H_j^c \parallel G'|S'$.*

Proof. Let $S_j^c \in G_{\dagger} \subseteq G^{\star(J)}_{H_I^V:G''|H'}$. Then $H_j^c \nleqslant H_I^V$ by Lemma 7.6(i). Thus $H_j^c > H_I^V$ or $H_j^c \| H_I^V$. If $H_j^c \| H_I^V$ then $H_j^c \parallel G'|S'$ by $H_I^V < G'|S'$ and Proposition 2.12(ii). If $H_j^c > H_I^V$ then $S_j^c \in H_I^V$ by Proposition 4.15(i). Thus $S_j^c \in G''$ by Lemma 6.3, Lemma 6.7(i). Hence $H_j^c \parallel G'|S'$ by $H_j^c \geqslant G''|S''$, $G'|S' \| G''|S''$. □

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Lemma 8.5. *(1) $\bar{\tau}^{\star}_{\mathbf{I}}$ is an m-ary tree and, $\tau^{\star}_{\mathbf{I}}$ is a binary tree;*

(2) Let $H \in \bar{\tau}^{\star}_{\mathbf{I}}$ then $\partial_{\tau^{\star}_{\mathbf{I}}}(H) \leqslant H^{c}_{i_k}$ for some $1 \leqslant k \leqslant m$;

(3) Let $H \in \bar{\tau}^{\star}_{\mathbf{I}}$ then $H^{V}_{I} \nparallel \partial_{\tau^{\star}_{\mathbf{I}}}(H)$;

(4) Let $w > 1$ in $\tau^{*}_{\mathbf{I_j}} \in \tau^{\star}_{\mathbf{I}}$ then $H^{V}_{I} < H^{c}_{j_k}$ for all $1 \leqslant k \leqslant w$.

(5) Let $\tau^{*}_{\mathbf{I_j}} \in \tau^{\star}_{\mathbf{I}}$, $\partial_{\tau^{\star}_{\mathbf{I}}}(G_{b_k}|S^{\,c}_{j_k}) \leqslant H^{V}_{I}$ for some $1 \leqslant k \leqslant w$. Then $w = 1$.

Proof. (1) is immediately from Claim (i). (2) holds by $G|G^{*} \leqslant H^{c}_{j_k}$ and $H^{c}_{j_k} \leqslant H^{c}_{i_k}$ for some $H^{c}_{i_k} \in I$ by $I_{\mathbf{j}} \subseteq \bar{I}$. (3) holds by Proposition 2.12(iii), (2) and $H^{V}_{I} \leqslant H^{c}_{i_k}$.

For (4), let $w > 1$. Then $H^{c}_{j_1} \| H^{c}_{j_k}$ for each $2 \leqslant k \leqslant w$, $H^{c}_{j_1} \leqslant H^{c}_{i_g}$ and $H^{c}_{j_k} \leqslant H^{c}_{i_h}$ for some $H^{c}_{i_g}, H^{c}_{i_h} \in I$ by (2). Thus $H^{c}_{j_1} \| H^{c}_{i_h}$ and $H^{c}_{j_k} \| H^{c}_{i_g}$ by Proposition 2.12(ii). Hence $H^{c}_{j_1} \nleqslant H^{V}_{i_g i_h}$ by $H^{V}_{i_g i_h} < H^{c}_{i_h}$, and $H^{c}_{j_k} \nleqslant H^{V}_{i_g i_h}$ by $H^{V}_{i_g i_h} < H^{c}_{i_g}$. Thus $H^{V}_{I} < H^{c}_{j_1}$ and $H^{V}_{I} < H^{c}_{j_k}$ by (3), $H^{V}_{I} \leqslant H^{V}_{j_1 j_k}$. Hence $H^{V}_{I} < H^{c}_{j_k}$ for all $1 \leqslant k \leqslant w$. (5) is from (4). □

Lemma 8.6. *Let $\dfrac{H_{i,1} \ \cdots \ H_{i,w_i}}{H_{i-1,1}} \left\langle \tau^{*}_{\mathbf{I_j}(i)} \right\rangle \in \tau^{\star}_{\mathbf{I}}$ for all $1 \leqslant i \leqslant n$ such that $\partial_{\tau^{\star}_{\mathbf{I}}}(H_{0,1}) = G|G^{*}$ and $\partial_{\tau^{\star}_{\mathbf{I}}}(H_{n,1}) \leqslant H^{V}_{I}$. Then $\partial_{\tau^{\star}_{\mathbf{I}}}(H_{i,1}) \leqslant H^{V}_{I}$ and $w_i = 1$ for all $1 \leqslant i \leqslant n$.*

Proof. The proof is by induction on n. Let $n = 1$ then $w_1 = 1$ by Lemma 8.5(5) and $\partial_{\tau^{\star}_{\mathbf{I}}}(H_{1,1}) \leqslant H^{V}_{I}$. For the induction step, let $\partial_{\tau^{\star}_{\mathbf{I}}}(H_{i,1}) \leqslant H^{V}_{I}$ for some $1 < i \leqslant n$ then $w_i = 1$ by Lemma 8.5(5). Since $\dfrac{H_{i,1} \ \cdots \ H_{i,w_i}}{H_{i-1,1}} \left\langle \tau^{*}_{\mathbf{I_j}(i)} \right\rangle \in \tau^{\star}_{\mathbf{I}}$ then $\partial_{\tau^{\star}_{\mathbf{I}}}(H_{i-1,1}) \leqslant \partial_{\tau^{\star}_{\mathbf{I}}}(H_{i,k})$ for some $1 \leqslant k \leqslant w_i$ by Claim (ii). Then $\partial_{\tau^{\star}_{\mathbf{I}}}(H_{i-1,1}) \leqslant \partial_{\tau^{\star}_{\mathbf{I}}}(H_{i,1}) \leqslant H^{V}_{I}$ by $w_i = 1$. Thus $w_{i-1} = 1$ by Lemma 8.5(5). □

Definition 8.7. Let $\dfrac{G_2}{G_1} \langle EC^{*}_{\Omega} \rangle \in \tau^{\star}_{\mathbf{I}}$. The module of $\tau^{\star}_{\mathbf{I}}$ at G_2, which we denote by $\tau^{\star}_{\mathbf{I}:G_2}$, is defined as follows: (1) $G_2 \in \tau^{\star}_{\mathbf{I}:G_2}$; (2) $\dfrac{H_1 \cdots H_u}{H_0} \left\langle \tau^{*}_{\mathbf{I_j}} \right\rangle \in \tau^{\star}_{\mathbf{I}:G_2}$ if $H_0 \in \tau^{\star}_{\mathbf{I}:G_2}$; (3) $H_1 \notin \tau^{\star}_{\mathbf{I}:G_2}$ if $\dfrac{H_1}{H_0} \langle EC^{*}_{\Omega} \rangle \in \tau^{\star}_{\mathbf{I}}$, $H_0 \in \tau^{\star}_{\mathbf{I}:G_2}$.

Each node of $\tau^{\star}_{\mathbf{I}:G_2}$ is determined bottom-up, starting with G_2, whose root is G_2 and leaves may be branches, leaves of τ^{*} or lower hypersequents of $\langle EC^{*}_{\Omega} \rangle$-applications. While each node of $\tau^{*}_{H:H'}$ is determined top-down, starting with H', whose root is a subset of $G|G^{*}$ and leaves contain H' and some leaves of τ^{*}.

Lemma 8.8. *(1) $\tau^{\star}_{\mathbf{I}:G_2}$ is a derivation without $\langle EC^{*}_{\Omega} \rangle$ in* $\mathbf{GL_{\Omega}}$.

(2) Let $H' \in \bar{\tau}^{\star}_{\mathbf{I}:G_2}$ and $\partial_{\tau^{\star}_{\mathbf{I}}}(H') > H^{V}_{I}$. Then $\partial_{\tau^{\star}_{\mathbf{I}}}(H) > H^{V}_{I}$ for all $H \in \bar{\tau}^{\star}_{\mathbf{I}:G_2}$ and $H \geqslant H'$.

Proof. (1) is clear and (2) immediately from Lemma 8.6. □

Stage 1 Construction of Subroutine $\tau^{\star}_{\mathbf{I}_l}(\tau^{*}_{\mathbf{I_{j}}_r})$. Roughly speaking, $\tau^{\star}_{\mathbf{I}_l}(\tau^{*}_{\mathbf{I_{j}}_r})$ is constructed by replacing some nodes $\tau^{*}_{\mathbf{I_{j}}_l} \in \tau^{\star}_{\mathbf{I}_l}$ with $\tau^{*}_{\mathbf{I_{j}}_l \cup \mathbf{I_{j}}_r}$ in post-order. However, the ordinal postorder-traversal algorithm cannot be used to construct $\tau^{\star}_{\mathbf{I}_l}(\tau^{*}_{\mathbf{I_{j}}_r})$ because the tree structure of $\tau^{\star}_{\mathbf{I}_l}(\tau^{*}_{\mathbf{I_{j}}_r})$ is

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generally different from that of $\tau^{\star}_{\mathbf{I}_l}$ at some nodes $H \in \tau^{\star}_{\mathbf{I}_l}$ satisfying $\partial_{\tau^{\star}_{\mathbf{I}_l}}(H) < H^V_{\mathbf{I}_l}$. Thus we construct a sequence $\tau^{\star(q)}_{\mathbf{I}_l}$ of trees for all $q \geqslant 0$ inductively as follows.

For the base case, we mark all $\langle EC^*_\Omega \rangle$-applications in $\tau^{\star}_{\mathbf{I}_l}$ as unprocessed and define such marked derivation to be $\tau^{\star(0)}_{\mathbf{I}_l}$. For the induction case, let $\tau^{\star(q)}_{\mathbf{I}_l}$ be constructed. If all applications of $\langle EC^*_\Omega \rangle$ in $\tau^{\star(q)}_{\mathbf{I}_l}$ are marked as processed, we firstly delete the root of the tree resulting from the procedure and then, apply $\langle EC^*_\Omega \rangle$ to the root of the resulting derivation if it is applicable otherwise add an $\langle ID_\Omega \rangle$-application to it and finally, terminate the procedure. Otherwise we select one of the outermost unprocessed $\langle EC^*_\Omega \rangle$-applications in $\tau^{\star(q)}_{\mathbf{I}_l}$, say, $\dfrac{G^{\circ\circ}_{q+1}}{G^{\circ}_{q+1}} \langle EC^*_\Omega \rangle^{\circ}_{q+1}$, and perform the following steps to construct $\tau^{\star(q+1)}_{\mathbf{I}_l}$ in which $\dfrac{G^{\circ\circ}_{q+1}}{G^{\circ}_{q+1}} \langle EC^*_\Omega \rangle^{\circ}_{q+1}$ be revised as $\dfrac{G^{\cdot\cdot}_{q+1}}{G^{\circ}_{q+1}} \langle EC^*_\Omega \rangle^{\cdot}_{q+1}$ such that

(a) $\tau^{\star(q+1)}_{\mathbf{I}_l}$ is constructed by locally revising $\tau^{\star(q)}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}}$ and leaving other nodes of $\tau^{\star(q)}_{\mathbf{I}_l}$ unchanged, particularly including G°_{q+1};

(b) $\tau^{\star(q+1)}_{\mathbf{I}_l}(G^{\cdot\cdot}_{q+1})$ is a derivation in $\mathbf{GL_\Omega}$;

(c) $G^{\cdot\cdot}_{q+1} = G^{\circ\circ}_{q+1}$ if $S' \notin \langle G' | S' \rangle_{\mathcal{I}_{\mathbf{j}_l}}$ for all $\tau^{*}_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\star}_{\mathbf{I}_l}(G^{\circ\circ}_{q+1})$ otherwise $G^{\cdot\cdot}_{q+1} = G^{\circ\circ}_{q+1} \backslash G^{m_{q+1}}_{\dagger} | G^{m_{q+1}}_{\ddagger}$ for some $m_{q+1} \geqslant 1$.

Remark 8.9. By two superscripts $\circ$ and $\cdot$ in $\langle EC^*_\Omega \rangle^{\circ}_{q+1}$ or $\langle EC^*_\Omega \rangle^{\cdot}_{q+1}$, we indicate the unprocessed state and processed state, respectively. This procedure determines an ordering for all $\langle EC^*_\Omega \rangle$-applications in $\tau^{\star}_{\mathbf{I}_l}$ and the subscript $q+1$ indicates that it is the $q+1$-th application of $\langle EC^*_\Omega \rangle$ in a post-order transversal of $\tau^{\star}_{\mathbf{I}_l}$. $G^{\circ\circ}_{q+1}$ and G°_{q+1} ($G^{\cdot\cdot}_{q+1}$ and $G^{\cdot}_{q+1}$) are the premise and conclusion of $\langle EC^*_\Omega \rangle^{\circ}_{q+1}$ ($\langle EC^*_\Omega \rangle^{\cdot}_{q+1}$), respectively.

Step 1 (Delete). Take the module $\tau^{\star(q)}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}}$ out of $\tau^{\star(q)}_{\mathbf{I}_l}$. Since $\langle EC^*_\Omega \rangle^{\circ}_{q+1}$ is the unique unprocessed $\langle EC^*_\Omega \rangle$-applications in $\tau^{\star(q)}_{\mathbf{I}_l}(G^{\circ}_{q+1})$ by its choice criteria, $\tau^{\star(q)}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}}$ is the same as $\tau^{\star}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}}$ by Claim (a). Thus it is a derivation. If $\partial_{\tau^{\star}_{\mathbf{I}_l}}(H) \leqslant H^V_I$ for all $H \in \tau^{\star(q)}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}}$, delete all internal nodes of $\tau^{\star(q)}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}}$. Otherwise there exists

$$\frac{G_{b_{l'1}} | S^c_{j_{l'1}} \quad G_{b_{l'2}} | S^c_{j_{l'2}} \quad \cdots \quad G_{b_{l'u'}} | S^c_{j_{l'u'}}}{G_{l'} \equiv \{G_{b_{l'k}}\}^{u'}_{k=1} | G^*_{\mathcal{I}_{\mathbf{j}_{l'}}}} \left\langle \tau^*_{\mathbf{I}_{\mathbf{j}_{l'}}} \right\rangle \in \tau^{\star(q)}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}}$$

such that $\partial_{\tau^{\star}_{\mathbf{I}_l}}(G_{b_{l'k}} | S^c_{j_{l'k}}) > H^V_I$ for all $1 \leqslant k \leqslant u'$ and $\partial_{\tau^{\star}_{\mathbf{I}_l}}(G_{l'}) \leqslant H^V_I$ by Lemma 8.8(2) and $\partial_{\tau^{\star}_{\mathbf{I}_l}}(G^{\circ\circ}_{q+1}) = G | G^* \leqslant H^V_I$, then delete all $H \in \tau^{\star(q)}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}}$, $G^{\circ\circ}_{q+1} \leqslant H < G_{l'}$. We denote the structure resulting from the deletion operation above by $\tau^{\star(q)}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}(1)}$. Since $\partial_{\tau^{\star}_{\mathbf{I}_l}}(G_{l'}) \leqslant H^V_I$ then $\tau^{\star(q)}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}(1)}$ is a tree by Lemma 8.6. Thus it is also a derivation.

Step 2 (Update). For each $G^{\circ}_{q'} \in \tau^{\star(q)}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}(1)}$ which satisfies $\dfrac{G^{\cdot\cdot}_{q'}}{G^{\circ}_{q'}} \langle EC^*_\Omega \rangle^{\cdot}_{q'} \in \tau^{\star(q)}_{\mathbf{I}_l}$ and $S' \in \langle G' | S' \rangle_{\mathcal{I}_{\mathbf{j}_l}}$ for some $\tau^{*}_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\star}_{\mathbf{I}_l}(G^{\circ\circ}_{q'})$, we replace H with $H \backslash G_{\dagger} | G_{\ddagger}$ for each $H \in \tau^{\star(q)}_{\mathbf{I}_l : G^{\circ\circ}_{q+1}(1)}$, $G_{l'} \leqslant$

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$H \leqslant G^{\circ}_{q'}$.

Since $\dfrac{G^{\cdot\cdot}_{q'}}{G^{\circ}_{q'}} \langle EC^{*}_{\Omega}\rangle^{\cdot}_{q'} \in \tau^{\star(q)}_{I_l}(G^{\circ\circ}_{q+1})$ and $\langle EC^{*}_{\Omega}\rangle^{\circ}_{q+1}$ is the outermost unprocessed $\langle EC^{*}_{\Omega}\rangle$-application in $\tau^{\star(q)}_{\mathbf{I}_l}$ then $q' \leqslant q$ and $\langle EC^{*}_{\Omega}\rangle^{\cdot}_{q'}$ has been processed. Thus Claims (b) and (c) hold for $\tau^{\star(q)}_{\mathbf{I}_l}(G^{\cdot}_{q'})$ by the induction hypothesis. Then $\dfrac{G^{\cdot\cdot}_{q'}}{G^{\cdot}_{q'}}$ is a valid $\langle EC^{*}_{\Omega}\rangle$-application since $\dfrac{G^{\circ\circ}_{q'}}{G^{\circ}_{q'}}$, $\dfrac{G^{m_{q'}}_{\dagger}}{G_{\dagger}}$ and $\dfrac{G^{m_{q'}}_{\ddagger}}{G_{\ddagger}}$ are valid, where $G^{\cdot\cdot}_{q'} = G^{\circ\circ}_{q'} \backslash G^{m_{q'}}_{\dagger} | G^{m_{q'}}_{\ddagger}$, $G^{\cdot}_{q'} = G^{\circ}_{q'} \backslash G_{\dagger} | G_{\ddagger}$.

Lemma 8.10. *Let $G_{l'} < H \leqslant G^{\circ}_{q'}$. Then $\partial_{\tau^{\star}_{\mathbf{I}_l}}(H) \geqslant G'|S'$.*

Proof. Since $G_{l'} < H$ then $G_{b_{l'k}}|S^{c}_{j_{l'k}} \leqslant H$ for some $1 \leqslant k \leqslant u'$. If $\partial_{\tau^{\star}_{\mathbf{I}_l}}(H) \geqslant H^{V}_{I_l}$ then $\partial_{\tau^{\star}_{\mathbf{I}_l}}(H) \geqslant G'|S'$. Otherwise all applications between $G_{l'}$ and H are one-premise rules by Lemma 8.6. Then $H^{c}_{j_{l'k}} \leqslant \partial_{\tau^{\star}_{\mathbf{I}_l}}(H)$ by Claim (ii). Thus $\partial_{\tau^{\star}_{\mathbf{I}_l}}(H) \geqslant G'|S'$ by $H^{V}_{I} < H^{c}_{j_{l'k}}$, $\partial_{\tau^{\star}_{\mathbf{I}_l}}(H) \leqslant H^{c}_{l_{k'}}$ for some $1 \leqslant k' \leqslant m(l)$ by Claim (i). □

Since $\partial_{\tau^{\star}_{\mathbf{I}_l}}(H) \geqslant G'|S'$ by Lemma 8.10 and $H^{c}_{j}\|G'|S'$ for each $S^{c}_{j} \in G_{\dagger}$ by Lemma 8.4, then $G_{\dagger} \subseteq H$ as side-hypersequent of H. Thus this step updates the revision of $G^{\cdot\cdot}_{q'}$ downward to $G_{l'}$.

Let m' be the number of $G^{\circ}_{q'}$ satisfying the above conditions, $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(1)}$, $G_{l'}$ and $G_{b_{l'k}}|S^{c}_{j_{l'k}}$ for all $1 \leqslant k \leqslant u'$ be updated as $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(2)}$, $G_{l''}$, $G'_{b_{l'k}}|S^{c}_{j_{l'k}}$, respectively. Then $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(2)}$ is a derivation and $G_{l''} = G_{l'} \backslash G^{m'}_{\dagger} | G^{m'}_{\ddagger}$.

Step 3 (Replace). All $\tau^{*}_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(2)}$ are processed in post-order. If $H^{c}_{i} \rightsquigarrow H^{c}_{j}$ for all $H^{c}_{i} \in I_{\mathbf{j}_l}$ and $H^{c}_{j} \in I_{\mathbf{j}_r}$ it proceeds by the following procedure otherwise it remains unchanged. Let $\tau^{*}_{\mathbf{I}_{\mathbf{j}_l}}$ be in the form

$$\dfrac{G_{b_{l1}}|S^{c}_{j_{l1}} \quad G_{b_{l2}}|S^{c}_{j_{l2}} \quad \cdots \quad G_{b_{lu}}|S^{c}_{j_{lu}}}{G_{l} \equiv \{G_{b_{lk}}\}^{u}_{k=1}|G^{*}_{\mathcal{I}_{\mathbf{j}_l}}}.$$

Then $H^{c}_{j_{lk}} \geqslant G'|S'$ for all $1 \leqslant k \leqslant u$ by Lemma 8.10, $G_{b_{lk}}|S^{c}_{j_{lk}} > G_{l''}$.

Firstly, replace $\tau^{*}_{\mathbf{I}_{\mathbf{j}_l}}$ with $\tau^{*}_{\mathbf{I}_{\mathbf{j}_l} \cup \mathbf{I}_{\mathbf{j}_r}}$. We may rewrite the roots of $\tau^{*}_{\mathbf{I}_{\mathbf{j}_l}}$ and $\tau^{*}_{\mathbf{I}_{\mathbf{j}_l} \cup \mathbf{I}_{\mathbf{j}_r}}$ as

$$G_{l} = \{G_{b_{lk}}\}^{u}_{k=1}|G^{*}_{H^{V}_{I}:\langle G'\rangle_{\mathcal{I}_{\mathbf{j}_l}}}|G^{*}_{H^{V}_{I}:G''|H'} \text{ and}$$

$$G_{l,r} \equiv \{G_{b_{lk}}\}^{u}_{k=1}|G^{*}_{H^{V}_{I}:\langle G'\rangle_{\mathcal{I}_{\mathbf{j}_l}}}|\{G_{b_{rk}}\}^{v}_{k=1}|G^{*}_{H^{V}_{I}:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}|H'},$$

respectively.

Let $G_{l''} < H \leqslant G_{l}$. By Lemma 8.10, $\partial_{\tau^{\star}_{\mathbf{I}_l}}(H) \geqslant G'|S'$. By Lemma 6.7, $H^{c}_{j} \leqslant H^{V}_{I} < G'|S'$ or $H^{c}_{j}\|G'|S'$ for all $S^{c}_{j} \in G^{*}_{H^{V}_{I}:G''|H'}$. Thus $G^{*}_{H^{V}_{I}:G''|H'} \subseteq H$. Secondly, we replace H with $H \backslash G^{*}_{H^{V}_{I}:G''|H'}|\{G_{b_{rk}}\}^{v}_{k=1}|G^{*}_{H^{V}_{I}:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}|H'}$ for all $G_{l'} \leqslant H \leqslant G_{l}$. Let m'' be the number of $\tau^{*}_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\Omega(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(2)}$ satisfying the replacement conditions above, $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(2)}$, $G_{l''}$ and $G'_{b_{l'k}}|S^{c}_{j_{l'k}}$ for all $1 \leqslant k \leqslant u'$ be

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updated as $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(3)}$, $G_{l'''}$, $G''_{b_{l'k}}|S^c_{j_{l'k}}$, respectively. Then $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(3)}$ is a derivation of $G_{l'''}$ and $G_{l'''}=G_{l''}\backslash\{G^{*}_{H^V_I:G''|H'}\}^{m''}|\{\{G_{b_{rk}}\}^{v}_{k=1}|G^{*}_{H^V_I:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}|H'}\}^{m''}$.

Step 4 (Separation along H^V_I). Apply the separation algorithm along H^V_I to $G_{l'''}$ and denote the resulting derivation by $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(4)}$ whose root is labeled by $G^{\cdot\cdot}_{q+1}$. Then all $G^{*}_{H^V_I:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}|H'}$ in $G_{l'''}$ are transformed into $G^{\star(J)}_{H^V_I:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}|H'}$ in $\tau^{\Omega(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(4)}$. Since $\dfrac{G'|S' \quad G''|S''}{H^V_I=G'|G''|H'}(II)\in\tau^{*}$,

$$\frac{\{G_{b_{lk}}\}^{u}_{k=1}|\langle G'|S'\rangle_{\mathcal{I}_{\mathbf{j}_l}} \qquad \{G_{b_{rk}}\}^{v}_{k=1}|\langle G''|S''\rangle_{\mathcal{I}_{\mathbf{j}_r}}}{\{G_{b_{lk}}\}^{u}_{k=1}|\{G_{b_{rk}}\}^{v}_{k=1}|\langle G'\rangle_{\mathcal{I}_{\mathbf{j}_l}}|\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}|H'}(II)\in\tau^{*}_{\mathbf{I}_{\mathbf{j}_l}\cup\mathbf{I}_{\mathbf{j}_r}}\in\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(3)},$$

H', S' and S'' are separable in $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(4)}$ by a procedure similar to that of Lemma 7.11. Let S' and S'' be separated into $\widehat{S'}$ and $\widehat{S''}$, respectively. By Claim (iii), $G^{\star(J)}_{H^V_I:G_{l'}}=G^{\circ\circ}_{q+1}$.

$$\begin{aligned}
G^{\star(J)}_{H^V_I:G_{l''}} &= G^{\circ\circ}_{q+1}\backslash G^{m'}_{\dagger}|G^{m'}_{\ddagger} \quad by \;\; Lemma \;\; 7.6(iv),\\
G^{\cdot\cdot}_{q+1} &= G^{\star(J)}_{H^V_I:G_{l'''}}\\
&= G^{\star(J)}_{H^V_I:G_{l''}}\backslash\{G^{\star(J)}_{H^V_I:G''|H'}\}^{m''}|\{\{G_{b_{rk}}\}^{v}_{k=1}|G^{\star(J)}_{H^V_I:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}|H'}\}^{m''}\\
&= G^{\star(J)}_{H^V_I:G_{l''}}\backslash\{\widehat{S'}|\widehat{S''}|G^{\star(J)}_{H^V_I:G''}|G^{\star(J)}_{H^V_I:H'}\backslash\{\widehat{S'}|\widehat{S''}\}\}^{m''}|\\
&\{\{G_{b_{rk}}\}^{v}_{k=1}|\widehat{S'}|\widehat{S''}|G^{\star(J)}_{H^V_I:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}}|G^{\star(J)}_{H^V_I:H'}\backslash\{\widehat{S'}|\widehat{S''}\}\}^{m''}\\
&= G^{\star(J)}_{H^V_I:G_{l''}}\backslash G^{m''}_{\dagger}|G^{m''}_{\ddagger}\\
&= \{G^{\circ\circ}_{q+1}\backslash G^{m'}_{\dagger}|G^{m'}_{\ddagger}\}\backslash G^{m''}_{\dagger}|G^{m''}_{\ddagger}\\
&= G^{\circ\circ}_{q+1}\backslash G^{m'+m''}_{\dagger}|G^{m'+m''}_{\ddagger}\\
&= G^{\circ\circ}_{q+1}\backslash G^{m_{q+1}}_{\dagger}|G^{m_{q+1}}_{\ddagger}
\end{aligned}$$

where $m_{q+1}:=m'+m''$.

Step 5 (Put back). Replace $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}}$ in $\tau^{\star(q)}_{\mathbf{I}_l}$ with $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(4)}$ and mark $\dfrac{G^{\cdot\cdot}_{q+1}}{G^{\circ}_{q+1}}\langle EC^{*}_{\Omega}\rangle^{\circ}_{q+1}$ as processed, i.e., revise $\langle EC^{*}_{\Omega}\rangle^{\circ}_{q+1}$ as $\langle EC^{*}_{\Omega}\rangle^{\cdot}_{q+1}$. Among leaves of $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}}$, all $G^{\circ}_{q'}$ are updated as $G^{\cdot}_{q'}$ and others keep unchanged in $\tau^{\star(q)}_{\mathbf{I}_l:G^{\circ\circ}_{q+1}(4)}$. Then this replacement is feasible, especially, $G^{\circ\circ}_{q+1}$ be replaced with $G^{\cdot\cdot}_{q+1}$. Define the tree resulting from Step 5 to be $\tau^{\star(q+1)}_{\mathbf{I}_l}$. Then Claims (a), (b) and (c) hold for $q+1$ by the above construction.

Finally, we construct a derivation of $G^{\star}_{\mathbf{I}_l}\backslash G_{\dagger}|G_{\ddagger}$ from $[S^c_{l_1}]_I,\cdots,[S^c_{l_{m(l)}}]_I, G_{b_{r1}}|S^c_{j_{r1}},\cdots,G_{b_{rv}}|S^c_{j_{rv}}$ in $\mathbf{GL}_{\mathbf{\Omega}}$, which we denote by $\tau^{\star}_{\mathbf{I}_l}(\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}})$.

Remark 8.11. All elimination rules used in constructing $\tau^{\star}_{\mathbf{I}_l}$ are extracted from τ^{*}. Since $\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}}$ is a derivation in $\mathbf{GL}_{\mathbf{\Omega}}$ without (EC_{Ω}), we may extract elimination rules from $\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}}$ which we may use to construct $\tau^{\star}_{\mathbf{I}_l}(\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}})$ by a procedure similar to that of constructing $\tau^{\star}_{\mathbf{I}_l}$ with minor revision

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at every node H that $\partial_{\tau^{\star}_{\mathbf{I}_l}}(H) \leqslant H^V_l$. Note that updates and replacements in Steps 2 and 3 are essentially inductive operations but we neglect it for simplicity.

We may also think of constructing $\tau^{\star}_{\mathbf{I}_l}(\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}})$ as grafting $\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}}$ in $\tau^{\star}_{\mathbf{I}_l}$ by adding $\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}}$ to some $\tau^{*}_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\star}_{\mathbf{I}_l}$. Since the rootstock $\tau^{\star}_{\mathbf{I}_l}$ of the grafting process is invariant in Stage 2, we encapsulate $\tau^{\star}_{\mathbf{I}_l}(\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}})$ as an rule in $\mathbf{GL_{\Omega}}$ whose premises are $G_{b_{r1}}|S^{\,c}_{j_{r1}}, G_{b_{r2}}|S^{\,c}_{j_{r2}}, \cdots, G_{b_{rv}}|S^{\,c}_{j_{rv}}$ and conclusion is $\widehat{S''}|\{G_{b_{rk}}\}^{v}_{k=1}|G^{\star(J)}_{H^V_l:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}}|G^{\star(J)}_{H^V_l:H'}\backslash\{\widehat{S'}|\widehat{S''}\}|G^{\star}_{\mathbf{I}_{\backslash r}}$, i.e.,

$$\frac{G_{b_{r1}}|S^{\,c}_{j_{r1}} \quad G_{b_{r2}}|S^{\,c}_{j_{r2}} \quad \cdots \quad G_{b_{rv}}|S^{\,c}_{j_{rv}}}{\widehat{S''}|\{G_{b_{rk}}\}^{v}_{k=1}|G^{\star(J)}_{H^V_l:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}}|G^{\star(J)}_{H^V_l:H'}\backslash\{\widehat{S'}|\widehat{S''}\}|G^{\star}_{\mathbf{I}_{\backslash r}}}\left\langle \tau^{\star}_{\mathbf{I}_l}(\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}})\right\rangle,$$

where, $G^{\star}_{\mathbf{I}_{\backslash r}} = G^{\star}_{\mathbf{I}_l}\backslash G_{\dagger}$ is closed.

Stage 2 Construction of Routine $\tau^{\star}_{\mathbf{I}_r}(\tau^{\star}_{\mathbf{I}_l}(\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}}))$. A sequence $\tau^{\star(q)}_{\mathbf{I}_r}$ of trees for all $q \geqslant 0$ is constructed inductively as follows. $\tau^{\star(0)}_{\mathbf{I}_r}$, $\tau^{\star(q)}_{\mathbf{I}_r}$, $\dfrac{G^{\circ\circ}_{q+1}}{G^{\circ}_{q+1}}\langle EC^{*}_{\Omega}\rangle^{\circ}_{q+1}$ are defined as those of Stage 1.

Then we perform the following steps to construct $\tau^{\star(q+1)}_{\mathbf{I}_r}$ in which $\dfrac{G^{\circ\circ}_{q+1}}{G^{\circ}_{q+1}}\langle EC^{*}_{\Omega}\rangle^{\circ}_{q+1}$ be revised as $\dfrac{G^{\cdot\cdot}_{q+1}}{G^{\circ}_{q+1}}\langle EC^{*}_{\Omega}\rangle^{\cdot}_{q+1}$ such that Claims (a) and (b) are same as those of Stage 1 and (c) $G^{\cdot\cdot}_{q+1} = G^{\circ\circ}_{q+1}$ if $S'' \notin \langle G''|S''\rangle_{\mathcal{I}^{(\mathbf{t}_r)}_{\mathbf{j}_r}}$ for all $\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}} \in \tau^{\star}_{\mathbf{I}_r}(G^{\circ\circ}_{q+1})$ otherwise

$G^{\cdot\cdot}_{q+1} = G^{\circ\circ}_{q+1}\backslash\{\widehat{S'}|G^{\star(J)}_{H^V_l:G'}\}^{m_{q+1}}|\{G^{\star}_{\mathbf{I}_{\backslash r}}\}^{m_{q+1}}$ for some $m_{q+1} \geqslant 1$.

Step 1 (Delete). $\tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}}$ and $\tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(1)}$ are defined as before.

$$\frac{G_{b_{r'1}}|S^{\,c}_{j_{r'1}} \quad G_{b_{r'2}}|S^{\,c}_{j_{r'2}} \quad \cdots \quad G_{b_{r'v'}}|S^{\,c}_{j_{r'v'}}}{G_{r'} \equiv \{G_{b_{r'k}}\}^{v'}_{k=1}|G^{*}_{\mathcal{I}_{\mathbf{j}_{r'}}}}\left\langle \tau^{*}_{\mathbf{I}_{\mathbf{j}_{r'}}}\right\rangle \in \tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}}$$

satisfies $\partial_{\tau^{\star}_{\mathbf{I}_r}}(G_{b_{r'k}}|S^{\,c}_{j_{r'k}}) > H^V_l$ for all $1 \leqslant k \leqslant v'$ and $\partial_{\tau^{\star}_{\mathbf{I}_r}}(G_{r'}) \leqslant H^V_l$.

Step 2 (Update). For all $G^{\circ}_{q'} \in \tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(1)}$ which satisfy $\dfrac{G^{\cdot\cdot}_{q'}}{G^{\circ}_{q'}}\langle EC^{*}_{\Omega}\rangle^{\cdot}_{q'} \in \tau^{\star(q)}_{\mathbf{I}_r}$ and $S'' \in \langle G''|S''\rangle_{\mathcal{I}_{\mathbf{j}_r}}$ for some $\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}} \in \tau^{\star}_{\mathbf{I}_r}(G^{\circ\circ}_{q'})$, we replace H with $H\backslash\{\widehat{S'}|G^{\star(J)}_{H^V_l:G'}\}|G^{\star}_{\mathbf{I}_{\backslash r}}$ for all $H \in \tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(1)}$, $G_{r'} \leqslant H \leqslant G^{\circ}_{q'}$. Then Claims (a) and (b) are proved by a procedure as before. Let m' be the number of $G^{\cdot}_{q'}$ satisfying the above conditions. $\tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(1)}$, $G_{r'}$ and $G_{b_{r'k}}|S^{\,c}_{j_{r'k}}$ for all $1 \leqslant k \leqslant v'$ be updated as $\tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(2)}$, $G_{r''}$, $G'_{b_{r'k}}|S^{\,c}_{j_{r'k}}$, respectively. Then $\tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(2)}$ is a derivation and $G_{r''} = G_{r'}\backslash\{\widehat{S'}|G^{\star(J)}_{H^V_l:G'}\}^{m'}|\{G^{\star}_{\mathbf{I}_{\backslash r}}\}^{m'}$.

Step 3 (Replace). All $\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}} \in \tau^{\Omega(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(2)}$ are processed in post-order. If $H^c_i \rightsquigarrow H^c_j$ for all $H^c_i \in I_{\mathbf{j}_r}$ and $H^c_j \in I_l$ it proceeds by the following procedure otherwise it remains unchanged. Let $\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}}$ be in

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the form

$$\frac{G_{b_{r1}}|S^c_{j_{r1}} \quad G_{b_{r2}}|S^c_{j_{r2}} \quad \cdots \quad G_{b_{rv}}|S^c_{j_{rv}}}{G_r \equiv \{G_{b_{rk}}\}_{k=1}^{v}|G^*_{\mathcal{I}_{\mathbf{j}_r}}}.$$

Then there exists the unique $1 \leqslant k' \leqslant v'$ such that $G_{r''} < G_{b_{r'k'}}|S^c_{j_{r'k'}} \leqslant G_r$.

Firstly, we replace $\tau^*_{\mathbf{I}_{\mathbf{j}_r}}$ with $\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I}_{\mathbf{j}_r}})$. We may rewrite the roots of $\tau^*_{\mathbf{I}_{\mathbf{j}_r}}$, $\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I}_{\mathbf{j}_r}})$ as $G_r = \{G_{b_{rk}}\}_{k=1}^{v}|G^*_{H_I^V:\langle G''\rangle_{\mathcal{I}_{j_r}}}|G^*_{H_I^V:G'|H'}$, $G_{I\backslash r} \equiv \{G_{b_{rk}}\}_{k=1}^{v}|\widehat{S''}|G^{\star(J)}_{H_I^V:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}}|G^{\star(J)}_{H_I^V:H'}\backslash\{\widehat{S'}|\widehat{S''}\}|G^{\star}_{\mathbf{I}_{I\backslash r}}$, respectively.

Let $G_{r''} < H \leqslant G_r$. Then $\partial_{\tau^{\star}_{\mathbf{I}_r}}(H) \geqslant G''|S''$ by Lemma 8.10. Thus $G^{\star(0)}_{H_I^V:G'|H'} \subseteq H$, $\{S^c_j : S^c_j \in G^*_{H_I^V:\langle G''\rangle_{\mathcal{I}_{j_r}}}, H^c_j \geqslant G''|S''\} = \{S^c_j : S^c_j \in G^{\star(J)}_{H_I^V:\langle G''\rangle_{\mathcal{I}_{j_r}}}, H^c_j \geqslant G''|S''\}$. Define $G^{**}_H = \{S^c_j : S^c_j \in G^*_{H_I^V:\langle G''\rangle_{\mathcal{I}_{j_r}}}, S^c_j$ be the focus sequent of some $H' \in \tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(2)}, H \leqslant H' \leqslant G_r\}$.

Then we replace H with

$$H\backslash\{G^*_{H_I^V:\langle G''\rangle_{\mathcal{I}_{j_r}}}\backslash G^{**}_H|G^*_{H_I^V:G'|H'}\}|\widehat{S''}|\{G^{\star(J)}_{H_I^V:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}}\backslash G^{**}_H\}|G^{\star(J)}_{H_I^V:H'}\backslash\{\widehat{S'}|\widehat{S''}\}|G^{\star}_{\mathbf{I}_{I\backslash r}}$$

for all $G_{b_{r'k'}}|S^c_{j_{r'k'}} \leqslant H \leqslant G_r$.

Let m'' be the number of $\tau^*_{\mathbf{I}_{\mathbf{j}_r}} \in \tau^{\Omega(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(2)}$ satisfying the replacement conditions as above, $\tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(2)}$, $G_{r''}$ and $G'_{b_{r'k}}|S^c_{j_{r'k}}$ for all $1 \leqslant k \leqslant v'$ be updated as $\tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(3)}$, $G_{r'''}$, $G''_{b_{r'k}}|S^c_{j_{r'k}}$, respectively. Then $\tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(3)}$ is a derivation and $G_{r'''} = G_{r''}\backslash H_1^{m''}|H_2^{m''}$, where

$$\begin{aligned} H_0 &= G^{**}_{G'_{b_{r'k}}|S^c_{j_{r'k}}}, \\ H_1 &= G^*_{H_I^V:\langle G''\rangle_{\mathcal{I}_{j_r}}}\backslash H_0|G^*_{H_I^V:G'|H'}, \\ H_2 &= \widehat{S''}|G^{\star(J)}_{H_I^V:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}}\backslash H_0|G^{\star(J)}_{H_I^V:H'}\backslash\{\widehat{S'}|\widehat{S''}\}|G^{\star}_{\mathbf{I}_{I\backslash r}}. \end{aligned}$$

Step 4 (Separation along H_I^V). Apply the separation algorithm along H_I^V to $G_{r'''}$ and denote the resulting derivation by $\tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(4)}$ whose root is labeled by $G^{\cdot\cdot}_{q+1}$.

By Claim (iii), $G^{\star(J)}_{H_I^V:G_{r'}} = G^{\circ\circ}_{q+1}$.

$$\begin{aligned} G^{\star(J)}_{H_I^V:G_{r''}} &= G^{\circ\circ}_{q+1}\backslash\{G^{\star(J)}_{H_I^V:G'}|\widehat{S'}\}^{m'}|\{G^{\star}_{\mathbf{I}_{I\backslash r}}\}^{m'}, \\ G^{\star(J)}_{H_I^V:H_1} &= G^{\star(J)}_{H_I^V:\langle G''\rangle_{\mathcal{I}_{j_r}}}\backslash G^{\star(J)}_{H_I^V:H_3}|G^{\star(J)}_{H_I^V:G'|H'}, \\ G^{\star(J)}_{H_I^V:H_2} &= \widehat{S''}|G^{\star(J)}_{H_I^V:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}}\backslash G^{\star(J)}_{H_I^V:H_3}|G^{\star(J)}_{H_I^V:H'}\backslash\{\widehat{S'}|\widehat{S''}\}|G^{\star}_{\mathbf{I}_{I\backslash r}}. \end{aligned}$$

Then

$$\begin{aligned} G^{\star(J)}_{H_I^V:G_{r'''}} &= G^{\star(J)}_{H_I^V:G_{r''}}\backslash\{G^{\star(J)}_{H_I^V:G'|H'}\}^{m''}|\{\widehat{S''}|G^{\star(J)}_{H_I^V:H'}\backslash\{\widehat{S'}|\widehat{S''}\}|G^{\star}_{\mathbf{I}_{I\backslash r}}\}^{m''} \\ &= G^{\star(J)}_{H_I^V:G_{r''}}\backslash\{G^{\star(J)}_{H_I^V:G'}|\widehat{S'}\}^{m''}|\{G^{\star}_{\mathbf{I}_{I\backslash r}}\}^{m''}. \end{aligned}$$

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Then

$$G^{\cdot\cdot}_{q+1} = G^{\star(J)}_{H^V_I:G_{r'''}} = G^{\circ\circ}_{q+1}\backslash\{\widehat{S'}|G^{\star(J)}_{H^V_I:G'}\}^{m_{q+1}}|\{G^{\star}_{\mathbf{I}_{\backslash r}}\}^{m_{q+1}}$$

where $m_{q+1} := m' + m''$.

Step 5 (Put back). Replace $\tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}}$ in $\tau^{\star(q)}_{\mathbf{I}_r}$ with $\tau^{\star(q)}_{\mathbf{I}_r:G^{\circ\circ}_{q+1}(4)}$ and revise $\dfrac{G^{\cdot\cdot}_{q+1}}{G^{\circ}_{q+1}}\langle EC^*_\Omega\rangle^{\circ}_{q+1}$ as $\dfrac{G^{\cdot\cdot}_{q+1}}{G^{\circ}_{q+1}}\langle EC^*_\Omega\rangle^{\cdot}_{q+1}$. Define the resulting tree from Step 5 to be $\tau^{\star(q+1)}_{\mathbf{I}_r}$ then Claims (a), (b) and (c) hold for $q+1$ by the above construction.

Finally, we construct a derivation of $G^{\star}_{\mathbf{I}_r}\backslash\{\widehat{S'}|G^{\star(J)}_{H^V_I:G'}\}|G^{\star}_{\mathbf{I}_{\backslash r}}$ from $\lceil S^c_{i_1}\rceil_I, \ldots, \lceil S^c_{i_m}\rceil_I$ in $\mathbf{GL_\Omega}$. Since the major operation of Stage 2 is to replace $\tau^*_{\mathbf{I_{j}}_r}$ with $\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I}^{(t_r)}_{\mathbf{j}_r}})$ for all $\tau^*_{\mathbf{I_{j}}_r} \in \tau^{\star}_{\mathbf{I}_r}$ satisfying $S'' \in \langle G''|S''\rangle_{\mathcal{I}^{(t_r)}_{\mathbf{j}_r}}$, then we denote the resulting derivation from Stage 2 by $\tau^{\star}_{\mathbf{I}_r}(\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I_j}_r}))$.

In the following, we prove that the claims from (i) to (iv) hold if $\tau^{\star}_{\mathbf{I}} := \tau^{\star}_{\mathbf{I}_r}(\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I_j}_r}))$ and $G^{\star}_{\mathbf{I}} := G^{\star}_{\mathbf{I}_r}\backslash\{\widehat{S'}|G^{\star(J)}_{H^V_I:G'}\}|G^{\star}_{\mathbf{I}_{\backslash r}}$.

• For Claim (i), (ii): Let $\dfrac{H_1 \ \cdots \ H_w}{H_0}\left\langle\tau^*_{\mathbf{I_j}}\right\rangle \in \tau^{\star}_{\mathbf{I}}$ and $S^c_j \in G^*_{\mathcal{I}_\mathbf{j}}$. Then $\partial_{\tau^{\star}_{\mathbf{I}}}(H_k) \nleqslant H^c_j$ for all $1 \leqslant k \leqslant w$ by Lemma 6.13(iv).

If $\partial_{\tau^{\star}_{\mathbf{I}}}(H_{k'}) \leqslant H^V_I$ for some $1 \leqslant k' \leqslant w$, then $H^c_i \nleqslant H^c_j$ for all $H^c_i \in I$ by $\partial_{\tau^{\star}_{\mathbf{I}}}(H_{k'}) \leqslant H^V_I \leqslant H^c_i$. Thus Claim (i) holds and Claim (ii) holds by Lemma 8.5(5) and Lemma 7.6(i). Note that Lemma 8.5(5) is independent of Claims from (ii) to (iv).

Otherwise $\tau^*_{\mathbf{I_j}}$ is built up from $\tau^*_{\mathbf{I_j}_r} \in \tau^{\star}_{\mathbf{I}_r}$, $\tau^*_{\mathbf{I_j}_l}$ or $\tau^*_{\mathbf{I_j}_l\cup\mathbf{I_j}_r} \in \tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I_j}_r})$ by keeping their focus and principal sequents unchanged and making their side-hypersequents possibly to be modified, but which has no effect on discussing Claim (ii) and then Claim (ii) holds for $\tau^{\star}_{\mathbf{I}}$ by the induction hypothesis on Claim (ii) of $\tau^{\star}_{\mathbf{I}_l}$ or $\tau^{\star}_{\mathbf{I}_r}$.

If $\tau^*_{\mathbf{I_j}}$ is from $\tau^*_{\mathbf{I_j}_l\cup\mathbf{I_j}_r}$ then $S' \in \langle G'|S'\rangle_{\mathcal{I}_{\mathbf{j}_l}}$ and $S'' \in \langle G''|S''\rangle_{\mathcal{I}_{\mathbf{j}_r}}$ by the choice of $\tau^*_{\mathbf{I_j}_l}$ and $\tau^*_{\mathbf{I_j}_r}$ at Stage 1. By the induction hypothesis, $H^c_i \nleqslant H^c_j$ for all $S^c_j \in G^*_{\mathcal{I}_{\mathbf{j}_l}}$, $H^c_i \in I_l$ and $H^c_i \nleqslant H^c_j$ for all $S^c_j \in G^*_{\mathcal{I}_{\mathbf{j}_r}}$, $H^c_i \in I_r$. Then $H^c_i \nleqslant H^c_j$ for all $S^c_j \in G^*_{\mathcal{I}_\mathbf{j}} = G^*_{\mathcal{I}_{\mathbf{j}_l}\cup\mathcal{I}_{\mathbf{j}_r}}$, $H^c_i \in I$ by $G^*_{\mathcal{I}_{\mathbf{j}_l}\cup\mathcal{I}_{\mathbf{j}_r}} = G^*_{\mathcal{I}_{\mathbf{j}_l}} \cap G^*_{\mathcal{I}_{\mathbf{j}_r}}$, $I = I_l \cup I_r$.

If $\tau^*_{\mathbf{I_j}}$ is from $\tau^*_{\mathbf{I_j}_l}$ then $S' \notin \langle G'|S'\rangle_{\mathcal{I}_{\mathbf{j}_l}}$ by Step 3 at Stage 1. Then $\langle G'|G''|H'\rangle_{\mathcal{I}_{\mathbf{j}_l}} \cap (G''|H') = \varnothing$. Thus $S^c_j \notin G^*_{H^V_I:G''|H'}$. Hence $G''|S'' \nleqslant H^c_j$. Therefore $H^c_i \nleqslant H^c_j$ for all $H^c_i \in I_r$ by $G''|S'' \leqslant H^c_i$. Thus $H^c_i \nleqslant H^c_j$ for all $H^c_i \in I$ by $S^c_j \in G^*_{\mathcal{I}_\mathbf{j}} = G^*_{\mathcal{I}_{\mathbf{j}_l}}$ and the induction hypothesis from $\tau^*_{\mathbf{I_j}_l} \in \tau^{\star}_{\mathbf{I}_l}$. The case of $\tau^*_{\mathbf{I_j}}$ built up from $\tau^*_{\mathbf{I_j}_r}$ is proved by a procedure similar to above and omitted.

• Claim (iii) holds by Step 4 at Stage 1 and 2. Note that in the whole of Stage 1, we treat $\{G_{b_{rk}}\}^v_{k=1}$ as a side-hypersequent. But it is possible that there exists $S^c_j \in \{G_{b_{rk}}\}^v_{k=1}$ such that $H^c_j \leqslant H^V_I$. Since we haven't applied the separation algorithm to $\{G_{b_{rk}}\}^v_{k=1}$ in Step 4 at Stage 1, then it could make Claim (iii) invalid. But it is not difficult to find that we just move the separation of such S^c_j to Step 4 at Stage 2. Of course, we can move it to Step 4 at Stage 1, but which make the discussion complicated.

• For Claim (iv), we prove (1) $H^c_i \| H^c_j$ for all $S^c_j \in G^{\star}_{\mathbf{I}_{\backslash r}}$ and $H^c_i \in I$, (2) $H^c_i \| H^c_j$ for all $S^c_j \in G^{\star}_{\mathbf{I}_r}\backslash\{\widehat{S'}|G^{\star(J)}_{H^V_I:G'}\}$ and $H^c_i \in I$. Only (1) is proved as follows and (2) by a similar procedure and omitted.

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Let $S_j^c \in G^{\star}_{\mathbf{I}_{l\backslash r}}$. Then $S_j^c \in G^{\star}_{\mathbf{I}_l}$ and $S_j^c \notin \widehat{S''}|G^{\star(J)}_{H_I^V:G''}|G^{\star(J)}_{H_I^V:H'}\backslash\{\widehat{S'}|\widehat{S''}\}$ by the definition of $G^{\star}_{\mathbf{I}_{l\backslash r}}$. By a procedure similar to that of Claim (iv) in Case 1, we get $H_j^c \nleqslant H_I^V$ and assume that $S_j^c \in G^*_{\mathcal{I}_{\mathbf{j}_l}}$ for some $\tau^*_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\star}_{\mathbf{I}_l}$ and let $G'|S' \nleqslant H_j^c$ in the following.

Suppose that $G''|S'' \leqslant H_j^c$. Then $S_j^c \in G^*_{H_I^V:G''}$ and $S' \in \langle G'|S'\rangle_{\mathcal{I}_{\mathbf{j}_l}}$ by $S_j^c \in G^*_{\mathcal{I}_{\mathbf{j}_l}}$. Hence $S_j^c \in G^{\star(J)}_{H_I^V:G''}$ by $H_j^c \geqslant G''|S'' > H_I^V$. Therefore $S_j^c \in \widehat{S''}|G^{\star(J)}_{H_I^V:G''}|G^{\star(J)}_{H_I^V:H'}\backslash\{\widehat{S'}|\widehat{S''}\}$, a contradiction thus $G''|S'' \nleqslant H_j^c$. Then $H_I^V \nless H_j^c$ by $G'|S' \nleqslant H_j^c$ and $G''|S'' \nleqslant H_j^c$. Thus $H_j^c \| H_I^V$. Hence $H_j^c \| H_i^c$ for all $H_i^c \in I$. This completes the proof of Theorem 8.2.

□

Definition 8.12. The manipulation described in Theorem 8.2 is called derivation-grafting operation.

9. The proof of Main theorem

Recall that in Main theorem $G_0 \equiv G'|\{\Gamma_i, p \Rightarrow \Delta_i\}_{i=1\cdots n}|\{\Pi_j \Rightarrow p, \Sigma_j\}_{j=1\cdots m}$.

Lemma 9.1. *(i) If $G_2 = G_0\backslash\{\Gamma_1, p \Rightarrow \Delta_1\}$ and $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_2)$ then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$;*
(i') If $G_2 = G_0\backslash\{\Pi_1 \Rightarrow p, \Sigma_1\}$ and $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_2)$ then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$;
(ii) If $G_2 = G_0|\{\Gamma_1, p \Rightarrow \Delta_1\}$ and $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_2)$ then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$;
(ii') If $G_2 = G_0|\{\Pi_1 \Rightarrow p, \Sigma_1\}$ and $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_2)$ then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$;
(iii) If $G_2 = G_0\backslash\{\Gamma_1, p \Rightarrow \Delta_1\}|\{\Gamma_1, \top \Rightarrow \Delta_1\}$ and $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_2)$ then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$;
(iii') If $G_2 = G_0\backslash\Pi_1 \Rightarrow p, \Sigma_1|\Pi_1 \Rightarrow \bot, \Sigma_1$ and $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_2)$ then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$.

Proof. (i) Since $\mathcal{D}_0(G_2) = G'|\{\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j\}_{i=2\cdots n; j=1\cdots m} \subseteq G'|\{\Gamma_1, \Pi_j \Rightarrow \Delta_1, \Sigma_j\}_{j=1\cdots m}|$ $\{\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j\}_{i=2\cdots n; j=1\cdots m} = \mathcal{D}_0(G_0)$ then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$ holds. If $n = 1$, we replace all p in $\Pi_j \Rightarrow p, \Sigma_j$ with $\bot$. Then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$ holds by applying (CUT) to $\Gamma_1, \bot \Rightarrow \Delta_1$ and $G'|\{\Pi_j \Rightarrow \bot, \Sigma_j\}_{j=1\cdots m}$.

(ii) Since $\mathcal{D}_0(G_2) = G'|\{\Gamma_1, \Pi_j \Rightarrow \Delta_1, \Sigma_j\}_{j=1\cdots m}|\{\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j\}_{i=1\cdots n; j=1\cdots m}$ then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$ holds by applying (EC^*) to $\mathcal{D}_0(G_2)$.

(iii) Since $\mathcal{D}_0(G_2) = G'|\Gamma_1, \top \Rightarrow \Delta_1|\{\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j\}_{i=2\cdots n; j=1\cdots m}$ then $\vdash_{\mathbf{GL}} G'' \equiv G'|\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1|\{\Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j\}_{i=2\cdots n; j=1\cdots m}$ holds by applying (CUT) to $\Gamma_1, \top \Rightarrow \Delta_1$ in $\mathcal{D}_0(G_2)$ and $\Pi_1 \Rightarrow \top, \Sigma_1$. Thus $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$ holds by applying (EW) to G''.

(i'), (ii') and (iii') are proved by a procedure respectively similar to those of (i), (ii) and (iii) and omitted. □

Let $I = \{H_{i_1}^c, \cdots, H_{i_m}^c\} \subseteq \{H_1^c, \cdots, H_N^c\}$, G_I denote a closed hypersequent such that $G_I \subseteq_c G|G^*$ and $H_j^c \| H_i^c$ for all $S_j^c \in G_I$ and $H_i^c \in I$.

Lemma 9.2. *There exists G_I such that $\vdash_{\mathbf{GL_\Omega}} G_I$ for all $I \subseteq \{H_1^c, \cdots, H_N^c\}$.*

Proof. The proof is by induction on m. For the base step, let $m = 0$, then $I = \varnothing$ and $G_I := G|G^*$ and $\vdash_{\mathbf{GL_\Omega}} G_I$ by Lemma 4.17(v).

For the induction step, suppose that $m \geqslant 1$ and there exists G_I such that $\vdash_{\mathbf{GL_\Omega}} G_I$ for all $|I| \leqslant m - 1$. Then there exist $G_{I\backslash\{H_{i_k}^c\}}$ for all $1 \leqslant k \leqslant m$ such that $\vdash_{\mathbf{GL_\Omega}} G_{I\backslash\{H_{i_k}^c\}}$ and $H_j^c \| H_i^c$ for all $S_j^c \in G_{I\backslash\{H_{i_k}^c\}}$ and $H_i^c \in I\backslash\{H_{i_k}^c\}$.

If $H_j^c \| H_{i_k}^c$ for all $S_j^c \in G_{I\backslash\{H_{i_k}^c\}}$ then $G_I := G_{I\backslash\{H_{i_k}^c\}}$ and the claim holds clearly. Otherwise there exists $S_j^c \in G_{I\backslash\{H_{i_k}^c\}}$ such that $H_j^c \leqslant H_{i_k}^c$ or $H_j^c > H_{i_k}^c$ then we rewrite $G_{I\backslash\{H_{i_k}^c\}}$ as $\lceil S_{i'_k}^c \rceil_{\{H_{i'_k}^c\}\cup I\backslash\{H_{i_k}^c\}}$,

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where we define $H^c_{i'_k}$ such that $S^c_{i'_k} \in G_{I\backslash\{H^c_{i_k}\}}$ and, $S^c_j \in G_{I\backslash\{H^c_{i_k}\}}$ implies $H^c_j \leqslant H^c_{i'_k}$ or $H^c_j \| H^c_i$ for all $H^c_i \in \{H^c_{i'_k}\} \cup I\backslash\{H^c_{i_k}\}$. If we can't define G_I to be $G_{I\backslash\{H^c_{i_k}\}}$ for each $1 \leqslant k \leqslant m$, let $I' := \{H^c_{i'_1}, \cdots, H^c_{i'_m}\}$. Then $G_{I'}$ is constructed by applying the separation algorithm of multiple branches (or one branch if $m = 1$) to $\lceil S^c_{i'_1} \rceil_{I'}, \cdots, \lceil S^c_{i'_m} \rceil_{I'}$ Then $\vdash_{\mathbf{GL}_\Omega} G_{I'}$ by $\vdash_{\mathbf{GL}_\Omega} \lceil S^c_{i'_1} \rceil_{I'}, \cdots, \vdash_{\mathbf{GL}_\Omega} \lceil S^c_{i'_m} \rceil_{I'}$, Theorem 8.2 (or Lemma 7.8 (i) for one branch). Let $G_I := G_{I'}$ then $\vdash_{\mathbf{GL}_\Omega} G_I$ clearly. □

The proof of Main theorem: Let $I = \{H^c_1, \cdots, H^c_N\}$ in Lemma 9.2. Then there exists G_I such that $\vdash_{\mathbf{GL}_\Omega} G_I$, $G_I \subseteq_c G|G^*$ and $H^c_j \| H^c_i$ for all $S^c_j \in G_I$ and $H^c_i \in I$. Then $\vdash_{\mathbf{GL}} \mathcal{D}(G_I)$ by Lemma 5.6.

Suppose that $S^c_j \in G_I$. Then $H^c_j \| H^c_i$ for all $H^c_i \in I$. Thus $H^c_j \| H^c_j$ by $H^c_j \in I$, a contradiction with $H^c_j \leqslant H^c_j$ and hence there doesn't exist $S^c_j \in G_I$. Therefore $G_I \subseteq_c G$ by $G_I \subseteq_c G|G^*$.

By removing the identification number of each occurrence of p in G, we obtain the sub-hypersequent G_2 of $G_2|G^*_2$, which is the root of τ^4 resulting from Step 4 in Section 4. Then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_2)$ by $\vdash_{\mathbf{GL}} \mathcal{D}(G_I)$ and $G_I \subseteq_c G$. Since G_2 is constructed by adding or removing some $\Gamma_i, p \Rightarrow \Delta_i$ or $\Pi_j \Rightarrow p, \Sigma_j$ from G_0, or replacing $\Gamma_i, p \Rightarrow \Delta_i$ with $\Gamma_i, \top \Rightarrow \Delta_i$, or $\Pi_j \Rightarrow p, \Sigma_j$ with $\Pi_j \Rightarrow \bot, \Sigma_j$, then $\vdash_{\mathbf{GL}} \mathcal{D}_0(G_0)$ by Lemma 9.1. This completes the proof of Main theorem.□

Theorem 9.3. *Density elimination holds for all* **GL** *in* $\{$**GUL**, **GIUL**, **GMTL**, **GIMTL**$\}$.

Proof. It follows immediately from Main theorem. □

10. Final remarks and open problems

Recently, we have generalized our method described in this paper to the non-commutative substructural logic $\mathbf{GpsUL}^*$ in [24]. This result shows that $\mathbf{GpsUL}^*$ is the logic of pseudo-uninorms and their residua and answered the question posed by Prof. Metcalfe, Olivetti, Gabbay and Tsinakis in [17, 18].

It has often been the case in the past that metamathematical proofs of the standard completeness have the corresponding algebraic ones, and vise verse. In particular, Baldi and Terui [3] had given an algebraic proof of the standard completeness of **UL** and their method had also been extended by Galatos and Horcik [11]. A natural problem is whether there is an algebraic proof corresponding to our proof-theoretic one. It seems difficult to obtain it by using the insights gained from the approach described in this paper because ideas and syntactic manipulations introduced here are complicated and specialized. In addition, Baldi and Terui [3] also mentioned some open problems. Whether our method could be applied to their problems is another research direction.

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Appendices

A.1 Why do we adopt Avron-style hypersequent calculi?

A hypersequent calculus is called Pottinger-style if its two-premise rules are in the form of $\dfrac{G|S' \quad G|S''}{G|H'}(II)$ and, Avron-style if in the form of $\dfrac{G'|S' \quad G''|S''}{G'|G''|H'}(II)$. In the viewpoint of Avron-style systems, each application of two-premise rules contains implicitly applications of (EC) in Pottinger-style systems, as shown in the following.

$$\dfrac{G|S' \quad G|S''}{G|H'}(II) \xrightarrow[\text{in Avron-style system}]{\text{corresponds to}} \dfrac{\dfrac{G|S' \quad G|S''}{G|G|H'}(II)}{G|H'}(EC^*)$$

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The choice of the underlying system of hypersequent calculus is vital to our purpose and it gives the background or arena. In Pottinger-style system, G_0 in Section 3 is proved without application of (EC) as follows. But it seems helpless to prove that H_0 is a theorem of **IUL**.

$$\dfrac{\dfrac{C\Rightarrow C}{C\Rightarrow C|\Rightarrow p,B|B\Rightarrow p,\neg A\odot\neg A}\quad \dfrac{\dfrac{B\Rightarrow B}{B\Rightarrow B\,|\,p,p\Rightarrow A\odot A}\quad \dfrac{\dfrac{\dfrac{\dfrac{p\Rightarrow p\quad A\Rightarrow A}{A\Rightarrow p\,|\,p\Rightarrow A}\quad \dfrac{p\Rightarrow p\quad A\Rightarrow A}{A\Rightarrow p\,|\,p\Rightarrow A}}{A\Rightarrow p\,|\,p,p\Rightarrow A\odot A}}{\Rightarrow p,\neg A\,|\,p,p\Rightarrow A\odot A}\quad \dfrac{\dfrac{\dfrac{p\Rightarrow p\quad A\Rightarrow A}{A\Rightarrow p\,|\,p\Rightarrow A}\quad \dfrac{p\Rightarrow p\quad A\Rightarrow A}{A\Rightarrow p\,|\,p\Rightarrow A}}{A\Rightarrow p\,|\,p,p\Rightarrow A\odot A}}{\Rightarrow p,\neg A\,|\,p,p\Rightarrow A\odot A}}{\Rightarrow p,p,\neg A\odot\neg A\,|\,p,p\Rightarrow A\odot A}}{\Rightarrow p,B|B\Rightarrow p,\neg A\odot\neg A\,|\,p,p\Rightarrow A\odot A}}{\Rightarrow p,B|B\Rightarrow p,\neg A\odot\neg A\,|\,p\Rightarrow C|C,p\Rightarrow A\odot A}$$

The peculiarity of our method is not only to focus on controlling the role of the external contraction rule in the hypersequent calculus but also introduce other syntactic manipulations. For example, we label occurrences of the eigenvariable p introduced by an application of the density rule in order to be able to trace these occurrences from the leaves (axioms) of the derivation to the root (the derived hypersequent).

A.2 Why do we need the constrained external contraction rule?

We use the example in Section 3 to answer this question. Firstly, we illustrate Notation 4.14 as follows. In Figure 4, let $S^c_{11} = A \Rightarrow p_2; S^c_{12} = A \Rightarrow p_1; S^c_{21} = A \Rightarrow p_4; S^c_{22} = A \Rightarrow p_3; S^c_{31} = p_1, p_2 \Rightarrow A\odot A$; $S^c_{32} = p_3, p_4 \Rightarrow A\odot A; G'_1 = p_1, p_2 \Rightarrow A\odot A; G'_2 = p_3, p_4 \Rightarrow A\odot A$; $G'_3 = A \Rightarrow p_1 | \Rightarrow p_2, B|B \Rightarrow p_4, \neg A\odot\neg A|A \Rightarrow p_3$. Then $H^c_i = G'_i|S^c_{i1}|S^c_{i2}$ for $i = 1, 2, 3$. H^c_i are (pEC)-nodes and, S^c_{i1} and S^c_{i2} are (pEC)-sequents.

Let $G^*_{H^c_1:A\Rightarrow p_2} \Longrightarrow p_2, B|B \Rightarrow p_4, \neg A\odot\neg A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A\odot A$. We denote the derivation $\tau^*_{H^c_1:A\Rightarrow p_2}$ of $G^*_{H^c_1:A\Rightarrow p_2}$ from $A \Rightarrow p_2$ by $\dfrac{A\Rightarrow p_2}{G^*_{H^c_1:A\Rightarrow p_2}}\left\langle\tau^*_{H^c_1:A\Rightarrow p_2}\right\rangle$. Since we focus on sequents in G^* in the separation algorithm, we abbreviate $\dfrac{A\Rightarrow p_2}{G^*_{H^c_1:A\Rightarrow p_2}}\left\langle\tau^*_{H^c_1:A\Rightarrow p_2}\right\rangle$ to $\dfrac{S^c_{11}}{S^c_{22}|S^c_{32}}\left\langle\tau^*_{S^c_{11}}\right\rangle$ and further to $\dfrac{1}{2|3}\langle\tau^*_1\rangle$. Then the separation algorithm $\tau^{\star}_{H^c_1:G|G^*}$ is abbreviated as

$$\dfrac{\dfrac{\dfrac{1|2|3}{2'|3'|2|3}\langle\tau^*_1\rangle}{2'|2}\langle\tau^*_3,\tau^*_{3'}\rangle}{2}\langle EC_\Omega\rangle$$

where $2'$ and $3'$ are abbreviations of $A \Rightarrow p_5$ and $p_5, p_6 \Rightarrow A\odot A$, respectively. We also write $2'$ and $3'$ respectively as 2 and 3 for simplicity. Then the whole separation derivation is given as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{1|2|3}{2|3|2|3}\langle\tau^*_1\rangle}{2|2}\langle\tau^*_3,\tau^*_3\rangle}{2}\langle EC_\Omega\rangle \qquad \dfrac{\dfrac{\dfrac{1|2|3}{1|1|3}\langle\tau^*_2\rangle}{1|1}\langle\tau^*_3\rangle}{1}\langle EC_\Omega\rangle}{\varnothing}\langle\tau^*_{\{1,2\}}\rangle$$

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where $\varnothing$ is an abbreviation of G'' in Page 14 and means that all sequents in it are copies of sequents in G_0. Note that the simplified notations become intractable when we decide whether $\langle EC_\Omega \rangle$ is applicable to resulting hypersequents. If no application of $\langle EC_\Omega \rangle$ is used in it, all resulting hypersequents fall into the set $\{1|2|\underbrace{3|\cdots|3}_{l},\ 2|2|\underbrace{3|\cdots|3}_{m},\ 1|1|\underbrace{3|\cdots|3}_{n} : l \geq 0, m \geq 0, n \geq 0\}$ and $\varnothing$ is never obtained.

A.3 Why do we need the separation of branches?

In Figure 11, p_1 and p_2 in the premise of $\dfrac{p_1, p_2 \Rightarrow A \odot A}{p_1 \Rightarrow C | C, p_2 \Rightarrow A \odot A} \left\langle \tau^*_{S^c_{31}} \right\rangle$ could be viewed as being tangled in one sequent $p_1, p_2 \Rightarrow A \odot A$ but in the conclusion of $\left\langle \tau^*_{S^c_{31}} \right\rangle$ they are separated into two sequents $p_1 \Rightarrow C$ and $C, p_2 \Rightarrow A \odot A$, which are copies of sequents in G_0. In Figure 5, p_2 in $A \Rightarrow p_2$ falls into $\Rightarrow p_2, B$ in the root of $\tau^*_{H^c_1 : A \Rightarrow p_2}$ and $\Rightarrow p_2, B$ is a copy of a sequent in G_0. The same is true for p_4 in $A \Rightarrow p_4$ in Figure 8. But it's not the case.

Lemma 6.6 (vi) shows that in the elimination rule $\dfrac{S^c_{11}}{G^*_{S^c_{11}}} \left\langle \tau^*_{S^c_{11}} \right\rangle$, $S^c_j \in G^*_{S^c_{11}}$ implies $H^c_j < H^c_i$ or $H^c_j \parallel H^c_i$. If there exists no $S^c_j \in G^*_{S^c_{11}}$ such that $H^c_j < H^c_i$, then $S^c_j \in G^*_{S^c_{11}}$ implies $H^c_j \parallel H^c_i$ and, thus each occurrence of $p's$ in S^c_{11} is fell into a unique sequent which is a copy of a sequent in G_0. Otherwise there exists $S^c_j \in G^*_{S^c_{11}}$ such that $H^c_j < H^c_i$, then we apply $\left\langle \tau^*_{S^c_j} \right\rangle$ to S^c_j in $G^*_{S^c_{11}}$ and the whole operations can be written as

$$\dfrac{\dfrac{S^c_{11}}{G^{\star(0)}_{S^c_{11}} \equiv G^*_{S^c_{11}} \backslash \{S^c_j\} | S^c_j} \left\langle \tau^*_{S^c_{11}} \right\rangle}{G^{\star(1)}_{S^c_{11}} \equiv G^*_{S^c_{11}} \backslash \{S^c_j\} | G^*_{S^c_j}} \left\langle \tau^*_{S^c_j} \right\rangle .$$

Repeatedly we can get $G^{\star(J)}_{S^c_{11}}$ such that $S^c_j \in G^{\star(J)}_{S^c_{11}}$ implies $H^c_j \parallel H^c_1$. Then each occurrence of $p's$ in S^c_{11} is fell into a unique sequent in $G^{\star(J)}_{S^c_{11}}$ which is a copy of a sequent in G_0. In such case, we call occurrences of $p's$ in S^c_{11} are separated in $G^{\star(J)}_{S^c_{11}}$ and call such a procedure the separation algorithm. It is the starting point of the separation algorithm. We introduce branches in order to tackle the case of multiple-premise separation derivations for which it is necessary to apply (EC_Ω) to the resulting hypersequents.

A.4 Some questions about Theorem 8.2

In Theorem 8.2, $\tau^\star_{\mathbf{I}}$ is constructed by induction on the number $|I|$ of branches. As usual, we take the algorithm of $|I| - 1$ branches as the induction hypothesis. Why do we take $\tau^\star_{\mathbf{I}_l}$ and $\tau^\star_{\mathbf{I}_r}$ as the induction hypothesises?

Roughly speaking, it degenerates the case of $|I|$ branches into the case of two branches in the following sense. The subtree $\tau^*(G''|S'')$ of τ^* is as a whole contained in $\tau^*_{\mathcal{I}_{\mathbf{j}_l}}$ or not in it. Similarly, $\tau^*(G'|S')$ of τ^* is as a whole contained in $\tau^*_{\mathcal{I}_{\mathbf{j}_r}}$ or not in it. It is such a division of I into I_l and I_r that makes the whole algorithm possible.

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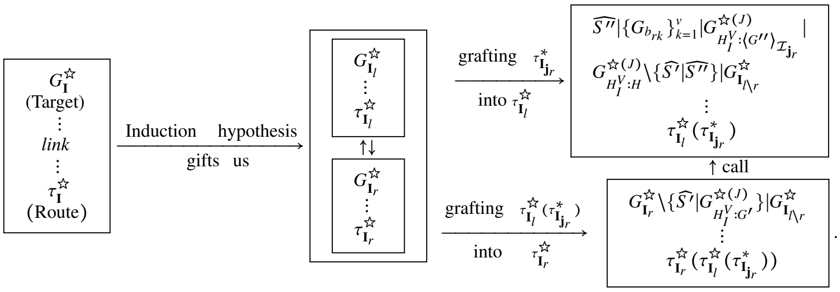
Claim (i) of Theorem 8.2 asserts that $H_i^c \nleqslant H_j^c$ for all $S_j^c \in G^*_{\mathcal{I}_{\mathbf{j}}}$ and $H_i^c \in I$. It guarantees that $\tau^*_{\mathcal{I}_{\mathbf{j}}}$ is not far from the final aim of Theorem 8.2 but roughly close to it if we define some complexity to calculate it. If $H_i^c \leqslant H_j^c$, the complexity of $G^*_{\mathbf{I_j}}$ is more than or equal to that of $\lceil S_i^c \rceil_I$ under such a definition of complexity and thus such an application of $\tau^*_{\mathcal{I}_{\mathbf{j}}}$ is redundant at least. Claim (iii) of Theorem 8.2 guarantees the validity of the step 4 of Stage 1 and 2.

The tree structure of the skeleton of $\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I}_{\mathbf{j}_r}})$ can be obtained by deleting some node $H \in \bar{\tau}^{\star}_{\mathbf{I}_l}$ satisfying $\partial_{\tau^{\star}_{\mathbf{I}_l}}(H) \leqslant H^V_{\mathbf{I}}$. The same is true for $\tau^{\star}_{\mathbf{I}}$ if $\tau^{\star}_{\mathbf{I}_l}(\tau^*_{\mathbf{I}_{\mathbf{j}_r}})$ is treated as a rule or a subroutine whose premises are same as ones of $\tau^*_{\mathbf{I}_{\mathbf{j}_r}}$. However, it is incredibly difficult to imagine or describe the structure of $\tau^{\star}_{\mathbf{I}}$ if you want to expand it as a normal derivation, a binary tree.

All syntactic manipulations in constructing $\tau^{\star}_{\mathbf{I}}$ are performed on the skeletons of $\tau^{\star}_{\mathbf{I_l}}$ or $\tau^{\star}_{\mathbf{I_r}}$. The structure of the proof of Theorem 8.2 is depicted in the following figure.



A.5 Illustrations of notations and algorithms

We use the example in Section 3 to illustrate some notations and algorithms in this paper.

A.5.1 Illustration of two cases of (COM) in the proof of Lemma 5.6

Let $\dfrac{G' \quad G''}{G'''}(COM)$ be $\dfrac{p_1 \Rightarrow p_1 \quad A \Rightarrow A}{A \Rightarrow p_1 | p_1 \Rightarrow A}(COM)$, where $G' = S_1 = p_1 \Rightarrow p_1$; $G'' = S_2 = A \Rightarrow A$; $S_3 = A \Rightarrow p_1$; $S_4 = p_1 \Rightarrow A$ and $G''' = S_3|S_4$. Then $[S_3]_{G'''} = [S_4]_{G'''}$; $\mathcal{D}_{G'}(S_1) \Longrightarrow t$; $\mathcal{D}_{G''}(S_2) = A \Rightarrow A$; $\mathcal{D}_{G'''}(S_3|S_4) = A \Rightarrow A$. Thus the proof of $\dfrac{\mathcal{D}_{G'}(S_1) \quad \mathcal{D}_{G'}(S_2)}{\mathcal{D}_{G'}(S_3|S_4)}$ is constructed by $\dfrac{\Rightarrow t \quad \dfrac{A \Rightarrow A}{A, t \Rightarrow A}(t_l)}{A \Rightarrow A}(CUT)$.

Let $\dfrac{G' \quad G''}{G'''}(COM)$ be $\dfrac{B \Rightarrow B \begin{pmatrix} \Rightarrow p_2, p_4, \neg A \odot \neg A | p_1, p_2 \Rightarrow A \odot A | \\ A \Rightarrow p_1 | A \Rightarrow p_3 | p_3, p_4 \Rightarrow A \odot A \end{pmatrix}}{\begin{pmatrix} \Rightarrow p_2, B | B \Rightarrow p_4, \neg A \odot \neg A | A \Rightarrow p_1 | \\ p_1, p_2 \Rightarrow A \odot A | A \Rightarrow p_3 | p_3, p_4 \Rightarrow A \odot A \end{pmatrix}}(COM)$,
where $G' = S_1 = B \Rightarrow B$; $G_2 = p_1, p_2 \Rightarrow A \odot A | A \Rightarrow p_1 | A \Rightarrow p_3 | p_3, p_4 \Rightarrow A \odot A$; $S_2 \Longrightarrow p_2, p_4, \neg A \odot \neg A$; $G'' = G_2|S_2$; $S_3 \Longrightarrow p_2, B$; $S_4 = B \Rightarrow p_4, \neg A \odot \neg A$ and $G''' = G_2|S_3|S_4$. Then $\mathcal{D}_{G'}(S_1) = B \Rightarrow B$; $\mathcal{D}_{G''}(S_2) = A, A \Rightarrow A \odot A, \neg A \odot \neg A, A \odot A$; $\mathcal{D}_{G'''}(S_3) = A \Rightarrow B, A \odot A$; $\mathcal{D}_{G'''}(S_4) = A, B \Rightarrow A \odot A, \neg A \odot \neg A$; $\mathcal{D}_{G'''}(S_3|S_4) = \mathcal{D}_{G'''}(S_4)|\mathcal{D}_{G'''}(S_4)$.

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Thus the proof of $\dfrac{\mathcal{D}_{G'}(S_1) \quad \mathcal{D}_{G'}(S_2)}{\mathcal{D}_{G'}(S_3|S_4)}$ is constructed by

$$\dfrac{B \Rightarrow B \quad A, A \Rightarrow A \odot A, \neg A \odot \neg A, A \odot A}{A \Rightarrow B, A \odot A|A, B \Rightarrow A \odot A, \neg A \odot \neg A}(COM).$$

A.5.2 Illustration of Construction 6.1

Let τ^* be

$$\dfrac{\dfrac{\dfrac{H_8 \equiv B \Rightarrow B \; H_9 \equiv A \Rightarrow A}{H_4 \equiv A \Rightarrow B|B \Rightarrow A}(COM) \quad \dfrac{H_{10} \equiv B \Rightarrow B \; H_{11} \equiv A \Rightarrow A}{H_5 \equiv A \Rightarrow B|B \Rightarrow A}(COM)}{H_2 \equiv A \Rightarrow B|A \Rightarrow B|B, B \Rightarrow A \odot A}(\odot_r)}{H_1 \equiv A \Rightarrow B| \Rightarrow B, \neg A|B, B \Rightarrow A \odot A}(\neg_r).$$

By Construction 6.1, τ^{**} is then given as follows.

$$\dfrac{\dfrac{\dfrac{(B \Rightarrow B; 8, 0) \; (A \Rightarrow A; 9, 0)}{(A \Rightarrow B; 4, 1)|(B \Rightarrow A; 4, 2)}(COM) \quad \dfrac{(B \Rightarrow B; 10, 0) \; (A \Rightarrow A; 11, 0)}{(A \Rightarrow B; 5, 1)|(B \Rightarrow A; 5, 2)}(COM)}{(A \Rightarrow B; 4, 1)|(A \Rightarrow B; 5, 1)|(B, B \Rightarrow A \odot A; 2, 0)}(\odot_r)}{(A \Rightarrow B; 4, 1)|(\Rightarrow B, \neg A; 1, 0)|(B, B \Rightarrow A \odot A; 2, 0)}(\neg_r).$$

As an example, we calculate $\wp(H_8)$. Since $Th(H_8) = (H_8, H_4, H_2, H_1)$, then $b_3 = 1$, $b_2 = b_1 = b_0 = 0$ by Definition 2.13. Thus $\wp(H_8) = b_0 2^0 + b_1 2^1 + b_2 2^2 + b_3 2^3 = 8$.

Note that we can't distinguish the one from the other for two $A \Rightarrow B$'s in $H_2 \in \tau^*$. If we divide H_2 into $H'|H''$, where $H' \equiv A \Rightarrow B$ and $H'' \equiv A \Rightarrow B|B, B \Rightarrow A \odot A$, then $H' \bigcap H'' = \{A \Rightarrow B\}$ in the conventional meaning of hypersequents. Thus only in the sense that we treat τ^* as τ^{**}, the assertion that $H' \bigcap H'' = \varnothing$ for any $H'|H'' \subseteq H$ in Proposition 6.2 holds.

A.5.3 Illustration of Notation 6.10 and Construction 6.11

Let $I = \{H_1^c, H_2^c\}, I_l = \{H_1^c\}, I_r = \{H_2^c\}, \mathcal{I} = \{S_{11}^c, S_{21}^c\}, \mathcal{I}_l = \{S_{11}^c\}, \mathcal{I}_r = \{S_{21}^c\}$,

$$\dfrac{G'|S' \quad G''|S''}{G'|G''|H'}(\odot_r) \in \tau^*,$$

where $G'|G''|H' = H_I^V; G' \equiv A \Rightarrow p_1|p_1, p_2 \Rightarrow A \odot A; S' \equiv \Rightarrow p_2, \neg A;$
$G'' \equiv A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A; S'' \equiv \Rightarrow p_4, \neg A; H' \equiv \Rightarrow p_2, p_4, \neg A \odot \neg A$
(See Figure 4).

$\langle G'|S'\rangle_{\mathcal{I}_l} \equiv \Rightarrow p_2, \neg A; \langle G'\rangle_{\mathcal{I}_l} = \varnothing; \langle G'|G''|H'\rangle_{\mathcal{I}_l} = A \Rightarrow p_3| \Rightarrow p_2, p_4, \neg A \odot \neg A|p_3, p_4 \Rightarrow A \odot A; \langle G|G^*\rangle_{\mathcal{I}_l} = G^*_{\mathcal{I}_l} = G^*_{S_{11}^c} \equiv \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|A \Rightarrow p_3|p_3, p_4 \Rightarrow A \odot A$ (See Figure 5).

$\langle G''|S''\rangle_{\mathcal{I}_r} \equiv \Rightarrow p_4, \neg A; \langle G'|G''|H'\rangle_{\mathcal{I}_r} = A \Rightarrow p_1| \Rightarrow p_2, p_4, \neg A \odot \neg A|p_1, p_2 \Rightarrow A \odot A;$
$\langle G|G^*\rangle_{\mathcal{I}_r} = G^*_{\mathcal{I}_r} = G^*_{S_{21}^c} = A \Rightarrow p_1| \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A$ (See Figure 8).

$\langle G'|G''|H'\rangle_{\mathcal{I}} \equiv \Rightarrow p_2, p_4, \neg A \odot \neg A; \langle G|G^*\rangle_{\mathcal{I}} = G^*_{\mathcal{I}} = G^*_{\{S_{11}^c, S_{21}^c\}} = G^*_{\mathcal{I}_l} \bigcap G^*_{\mathcal{I}_r} \equiv \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A$ (See Figure 10).

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A.5.4 Illustration of Theorem 8.2

Note that sequents in [] are principal sequents of elimination rules in the following. Let I, I_r, I_l be the same as in A.5.3 and, $\mathbf{I} = \{\lceil S_1^c \rceil_I, \lceil S_2^c \rceil_I\}, \mathbf{I}_l = \{\lceil S_1^c \rceil_I\}, \mathbf{I}_r = \{\lceil S_2^c \rceil_I\}$,

$$\lceil S_1^c \rceil_I = G^{\star}_{H_2^c:G|G^*} = A \Rightarrow p_5| \Rightarrow p_6, B|B \Rightarrow p_8, \neg A \odot \neg A|p_5 \Rightarrow C| \\ C, p_6 \Rightarrow A \odot A|B \Rightarrow p_7, \neg A \odot \neg A|p_7 \Rightarrow C|C, p_8 \Rightarrow A \odot A,$$

$$\lceil S_2^c \rceil_I = G^{\star}_{H_1^c:G|G^*} = \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A| \\ A \Rightarrow p_3| \Rightarrow p_1, B|p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A.$$

$$\tau^{\star}_{\mathbf{I}_l} = \cfrac{\cfrac{\cfrac{\lceil S_1^c \rceil_I}{G^{\star(1)}_{\mathbf{I}_l}} \left\langle \tau^*_{H_1^c:A\Rightarrow p_5} \right\rangle}{G^{\star(2)}_{\mathbf{I}_l}} \left\langle \tau^*_{H_3^c:p_9,p_{10}\Rightarrow A\odot A} \right\rangle}{G^{\star}_{\mathbf{I}_l}} \langle EC^*_{\Omega} \rangle,$$

where $G^{\star(1)}_{\mathbf{I}_l} = [\Rightarrow p_5, B|B \Rightarrow p_{10}, \neg A \odot \neg A|A \Rightarrow p_9|p_{10}, p_9 \Rightarrow A \odot A] | \Rightarrow p_6, B|$
$B \Rightarrow p_8, \neg A \odot \neg A|p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A|B \Rightarrow p_7, \neg A \odot \neg A|$
$p_7 \Rightarrow C|C, p_8 \Rightarrow A \odot A,$

$G^{\star(2)}_{\mathbf{I}_l} = \Rightarrow p_5, B|B \Rightarrow p_{10}, \neg A \odot \neg A|A \Rightarrow p_9|[p_9 \Rightarrow C|C, p_{10} \Rightarrow A \odot A]|$
$\Rightarrow p_6, B|B \Rightarrow p_8, \neg A \odot \neg A|p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A|$
$B \Rightarrow p_7, \neg A \odot \neg A|p_7 \Rightarrow C|C, p_8 \Rightarrow A \odot A,$

$G^{\star}_{\mathbf{I}_l} = \Rightarrow p_5, B|A \Rightarrow p_9|p_9 \Rightarrow C| \Rightarrow p_6, B|B \Rightarrow p_8, \neg A \odot \neg A|$
$p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A|B \Rightarrow p_7, \neg A \odot \neg A|p_7 \Rightarrow C|C, p_8 \Rightarrow A \odot A,$

$G^{\star(J)}_{H_I^V:G''} = A \Rightarrow p_9|p_9 \Rightarrow C|C, p_{10} \Rightarrow A \odot A$; $\widehat{S''} = B \Rightarrow p_{10}, \neg A \odot \neg A$; $\widehat{S'} = \Rightarrow p_5, B$; $G^{\star(J)}_{H_I^V:H'} = G^*_{H_I^V:H'} = \widehat{S'}|\widehat{S''}$; $G_{\dagger} = A \Rightarrow p_9|p_9 \Rightarrow C|C, p_{10} \Rightarrow A \odot A|B \Rightarrow p_{10}, \neg A \odot \neg A.$

$$\tau^{\star}_{\mathbf{I}_r} = \cfrac{\cfrac{\lceil S_2^c \rceil_I}{G^{\star(1)}_{\mathbf{I}_r}} \left\langle \tau^*_{H_2^c:A\Rightarrow p_3} \right\rangle}{G^{\star}_{\mathbf{I}_r}} \langle EC^*_{\Omega} \rangle,$$

where $G^{\star(1)}_{\mathbf{I}_r} = \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A|$
$\Rightarrow p_1, B|p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A|[A \Rightarrow p_{11}| \Rightarrow p_{12}, B|$
$B \Rightarrow p_3, \neg A \odot \neg A|p_{11} \Rightarrow C|C, p_{12} \Rightarrow A \odot A],$

$G^{\star}_{\mathbf{I}_r} = \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A|$
$\Rightarrow p_1, B|p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A|A \Rightarrow p_{11}|B \Rightarrow p_3, \neg A \odot \neg A|p_{11} \Rightarrow C.$

Since there is only one elimination rule in $\tau^{\star}_{\mathbf{I}_r}$, the case we need to process is $\tau^*_{H_2^c:A\Rightarrow p_3}$, i.e.,

$$\tau^*_{\mathbf{I}_{\mathbf{j}_r}} = \cfrac{\lceil S_2^c \rceil_I}{G^{\star(1)}_{H_2^c:\lceil S_2^c \rceil_I}} \left\langle \tau^*_{H_2^c:A\Rightarrow p_3} \right\rangle.$$

Then $v = 1$, $S^c_{j_{r1}} = A \Rightarrow p_3$; $G_{b_{r1}} = \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|$
$C, p_2 \Rightarrow A \odot A| \Rightarrow p_1, B|p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A$ in $\tau^*_{\mathbf{I}_{\mathbf{j}_r}}$.

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$$\tau_{\mathbf{I}_l}^{\star(0)} = \cfrac{\cfrac{\cfrac{\lceil S_1^c \rceil_I}{G_{\mathbf{I}_l}^{\star(1)}} \left\langle \tau^*_{H_1^c:A\Rightarrow p_5} \right\rangle}{G_{\mathbf{I}_l}^{\star(2)}} \left\langle \tau^*_{H_3^c:p_9,p_{10}\Rightarrow A\odot A} \right\rangle}{G_{\mathbf{I}_l}^{\star}} \langle EC_\Omega^* \rangle_1^\circ,$$

where $\partial_{\tau_{\mathbf{I}_l}^{\star}}(\lceil S_1^c \rceil_I) = H_1^c$, $\partial_{\tau_{\mathbf{I}_l}^{\star}}(G_{\mathbf{I}_l}^{\star(1)}) = H_3^c < H_I^V$,$\partial_{\tau_{\mathbf{I}_l}^{\star}}(G_{\mathbf{I}_l}^{\star(2)}) = \partial_{\tau_{\mathbf{I}_l}^{\star}}(G_{\mathbf{I}_l}^{\star}) = G|G^*$, $G_1^{\circ\circ} = G_{\mathbf{I}_l}^{\star(2)}$, $G_1^\circ = G_{\mathbf{I}_l}^{\star}$.

$$\tau_{\mathbf{I}_l:G_1^{\circ\circ}}^{\star(0)} = \cfrac{\cfrac{\lceil S_1^c \rceil_I}{G_{\mathbf{I}_l}^{\star(1)}} \left\langle \tau^*_{H_1^c:A\Rightarrow p_5} \right\rangle}{G_{\mathbf{I}_l}^{\star(2)}} \left\langle \tau^*_{H_3^c:p_9,p_{10}\Rightarrow A\odot A} \right\rangle,$$

$$\tau_{\mathbf{I}_l:G_1^{\circ\circ}(1)}^{\star(0)} = \tau_{\mathbf{I}_l:G_1^{\circ\circ}(2)}^{\star(0)} = \cfrac{\lceil S_1^c \rceil_I}{G_{\mathbf{I}_l}^{\star(1)}} \left\langle \tau^*_{H_1^c:A\Rightarrow p_5} \right\rangle.$$

Since there is only one elimination rule in $\tau_{\mathbf{I}_l:G_1^{\circ\circ}(2)}^{\star(0)}$, the case we need to process is $\tau^*_{H_1^c:A\Rightarrow p_5}$, i.e.,

$$\tau^*_{\mathbf{I}_{\mathbf{j}_l}} = \cfrac{\lceil S_1^c \rceil_I}{G_{\mathbf{I}_l}^{\star(1)}} \left\langle \tau^*_{H_1^c:A\Rightarrow p_5} \right\rangle.$$

Then $u = 1$, $S^c_{j_{l1}^{(\iota_{l1})}} = A \Rightarrow p_5$; $G_{b_{l1}} \Longrightarrow p_6, B|B \Rightarrow p_8, \neg A \odot \neg A|$
$p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A|B \Rightarrow p_7, \neg A \odot \neg A|p_7 \Rightarrow C|C, p_8 \Rightarrow A \odot A$ in $\tau^*_{\mathbf{I}_{\mathbf{j}_l}}$.

$\tau^*_{\mathbf{I}_{\mathbf{j}_l}}$ is replaced with $\tau^*_{\mathbf{I}_{\mathbf{j}_l}\cup\mathbf{I}_{\mathbf{j}_r}}$ in Step 3 of Stage 1, i.e.,

$\cfrac{\lceil S_1^c \rceil_I \quad \lceil S_2^c \rceil_I}{G_{l,r}} \left\langle \tau^*_{\{H_1^c:A\Rightarrow p_5, H_2^c:A\Rightarrow p_3\}} \right\rangle = \tau_{\mathbf{I}_l:G_1^{\circ\circ}(3)}^{\star(0)} = \tau_{\mathbf{I}_l:G_1^{\circ\circ}(4)}^{\star(0)}$, where

$$\begin{aligned} G_{l,r} \Longrightarrow p_5, B|B \Rightarrow p_3, \neg A \odot \neg A|G_{b_{r1}}|G_{b_{l1}} = \\ \Rightarrow p_2, B|B \Rightarrow p_4, \neg A \odot \neg A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A| \Rightarrow p_1, B| \\ p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A| \Rightarrow p_6, B|B \Rightarrow p_8, \neg A \odot \neg A| \\ p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A|B \Rightarrow p_7, \neg A \odot \neg A|p_7 \Rightarrow C| \\ \Rightarrow C, p_8 \Rightarrow A \odot A|p_5, B|B \Rightarrow p_3, \neg A \odot \neg A. \end{aligned}$$

Replacing $\tau_{\mathbf{I}_l:G_1^{\circ\circ}}^{\star(0)}$ in $\tau_{\mathbf{I}_l}^{\star(0)}$ with $\tau_{\mathbf{I}_l:G_1^{\circ\circ}(4)}^{\star(0)}$, then deleting $G_{\mathbf{I}_l}^{\star}$ and after that applying $\langle EC_\Omega^* \rangle$ to $G_{l,r}$ and keeping $G_{b_{r1}}$ unchanged, we get

$$\tau_{\mathbf{I}_l}^{\star}(\tau^*_{\mathbf{I}_{\mathbf{j}_r}}) = \cfrac{\cfrac{\lceil S_1^c \rceil_I \quad \lceil S_2^c \rceil_I}{G_{l,r}} \left\langle \tau^*_{\{H_1^c:A\Rightarrow p_5, H_2^c:A\Rightarrow p_3\}} \right\rangle}{\widehat{S''}|G_{b_{r1}}|G^{\star(J)}_{H_I^V:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}}|G^{\star(J)}_{H_I^V:H'} \backslash \{\widehat{S'}|\widehat{S''}\}|G^{\star}_{\mathbf{I}_{l\backslash r}}} \langle EC_\Omega^* \rangle,$$

where $G^{\star(J)}_{H_I^V:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}} = G^*_{H_I^V:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}} = \varnothing$; $\widehat{S'} \Longrightarrow p_5, B$;
$\widehat{S''} = B \Rightarrow p_3, \neg A \odot \neg A$; $G_\ddagger = G_{b_{r1}}|\widehat{S''}$; $G^{\star(J)}_{H_I^V:H'} = G^*_{H_I^V:H'} = \widehat{S'}|\widehat{S''}$;

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$G^{\star}_{\mathbf{I}_{l\backslash r}} \Longrightarrow p_5, B| \Rightarrow p_6, B|B \Rightarrow p_7, \neg A \odot \neg A|p_5 \Rightarrow C|C, p_6 \Rightarrow A \odot A|$
$p_7 \Rightarrow C|C, p_8 \Rightarrow A \odot A|B \Rightarrow p_8, \neg A \odot \neg A.$

Stage 2 $\tau^{\star(0)}_{\mathbf{I}_r:G_1^{\circ\circ}} = \tau^{\star(0)}_{\mathbf{I}_r:G_1^{\circ\circ}(1)} = \tau^{\star(0)}_{\mathbf{I}_r:G_1^{\circ\circ}(2)} = \frac{\lceil S_2^c \rceil_I}{G^{\star(1)}_{\mathbf{I}_r}} \left\langle \tau^{*}_{H_2^c:A\Rightarrow p_3} \right\rangle,$

$$\tau^{\star(0)}_{\mathbf{I}_r:G_1^{\circ\circ}(3)} = \tau^{\star(0)}_{\mathbf{I}_r:G_1^{\circ\circ}(4)} = \frac{\lceil S_1^c \rceil_I \qquad\qquad \lceil S_2^c \rceil_I}{\widehat{S''}|G_{b_{r1}}|G^{\star(J)}_{H_I^V:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}} \; |G^{\star(J)}_{H_I^V:H'} \backslash \{\widehat{S'}|\widehat{S''}\}|G^{\star}_{\mathbf{I}_{l\backslash r}}} \left\langle \tau^{\star}_{\mathbf{I}_l}(\tau^{*}_{\mathbf{I}_{\mathbf{j}_r}}) \right\rangle.$$

Replacing $\tau^{\star(0)}_{\mathbf{I}_r:G_1^{\circ\circ}}$ in $\tau^{\star(0)}_{\mathbf{I}_r}$ with $\tau^{\star(0)}_{\mathbf{I}_r:G_1^{\circ\circ}(4)}$, then deleting $G^{\star}_{\mathbf{I}_r}$ and after that applying $\langle EC^{*}_{\Omega}\rangle$ to $\widehat{S''}|G_{b_{r1}}|G^{\star(J)}_{H_I^V:\langle G''\rangle_{\mathcal{I}_{\mathbf{j}_r}}} \; |G^{\star(J)}_{H_I^V:H'} \backslash \{\widehat{S'}|\widehat{S''}\}|G^{\star}_{\mathbf{I}_{l\backslash r}}$, we get $\tau^{\star}_{\mathbf{I}}$.