The logic of pseudo-uninorms and their residua☆

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Abstract

Our method for density elimination is generalized to the non-commutative substructural logic **GpsUL***. Then the standard completeness of **HpsUL*** follows as a lemma by virtue of previous work by Metcalfe and Montagna. This result shows that **HpsUL*** is the logic of pseudo-uninorms and their residua and answered the question posed by Prof. Metcalfe, Olivetti, Gabbay and Tsinakis.

Keywords: Density elimination, Pseudo-uninorm logic, Standard completeness of **HpsUL***, Substructural logics, Fuzzy logic
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1. Introduction

In 2009, Prof. Metcalfe, Olivetti and Gabbay conjectured that the Hilbert system **HpsUL** is the logic of pseudo-uninorms and their residua [1]. Although **HpsUL** is the logic of bounded representable residuated lattices, it is not the case, as shown by Prof. Wang and Zhao in [2]. In 2013, we constructed the system **HpsUL*** by adding the weakly commutativity rule

$$(WCM) \vdash (A \rightsquigarrow t) \rightarrow (A \rightarrow t)$$

to **HpsUL** and conjectured that it is the logic of residuated pseudo-uninorms and their residua [3].

In this paper, we prove our conjecture by showing that the density elimination holds for the hypersequent system **GpsUL*** corresponding to **HpsUL***. Then the standard completeness of **HpsUL*** follows as a lemma by virtue of previous work by Metcalfe and Montagna [4]. This shows that **HpsUL*** is an axiomatization for the variety of residuated lattices generated by all dense residuated chains. Thus we also answered the question posed by Prof. Metcalfe and Tsinakis in [5] in 2017.

In proving the density elimination for **GpsUL***, we have to overcome several difficulties as follows. Firstly, cut-elimination doesn't holds for **GpsUL***. Note that (WCM) and the density rule(D) are formulated as

$$\frac{G|\Gamma, \Delta \Rightarrow t}{G|\Delta, \Gamma \Rightarrow t}, \quad \frac{G|\Pi \Rightarrow p|\Gamma, p, \Delta \Rightarrow B}{G|\Gamma, \Pi, \Delta \Rightarrow B}$$

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in **GpsUL***, respectively. Consider the following derivation fragment.

$$\dfrac{\dfrac{\vdots}{G_1|\Gamma_1,t,\Delta_1\Rightarrow A} \quad \dfrac{\dfrac{\vdots}{G_2|\Gamma_2,\Delta_2\Rightarrow t}}{G_2|\Delta_2,\Gamma_2\Rightarrow t}(WCM)}{G_1|G_2|\Gamma_1,\Delta_2,\Gamma_2,\Delta_1\Rightarrow A}(CUT).$$

By the induction hypothesis of the proof of cut-elimination, we get that $G_1|G_2|\Gamma_1,\Gamma_2,\Delta_2,\Delta_1\Rightarrow A$ from $G_2|\Gamma_2,\Delta_2\Rightarrow t$ and $G_1|\Gamma_1,t,\Delta_1\Rightarrow A$ by (CUT). But we can't deduce $G_1|G_2|\Gamma_1,\Delta_2,\Gamma_2,\Delta_1\Rightarrow A$ from $G_1|G_2|\Gamma_1,\Gamma_2,\Delta_2,\Delta_1\Rightarrow A$ by (WCM). We overcome this difficulty by introducing the following weakly cut rule into **GpsUL***

$$\dfrac{G_1|\Gamma,t,\Delta\Rightarrow A \quad G_2|\Pi\Rightarrow t}{G_1|G_2|\Gamma,\Pi,\Delta\Rightarrow A}(WCT).$$

Secondly, the proof of the density elimination for **GpsUL*** become troublesome even for some simple cases in **GUL** [4]. Consider the following derivation fragment

$$\dfrac{\dfrac{\dfrac{\vdots}{G_1|\Gamma_1,\Pi_1,\Sigma_1\Rightarrow A_1} \quad \dfrac{\vdots}{G_2|\Gamma_2,\Pi_2',p,\Pi_2'',\Sigma_2\Rightarrow p}}{G_1|G_2|\Gamma_1,\Pi_2',p,\Pi_2'',\Sigma_1\Rightarrow A_1|\Gamma_2,\Pi_1,\Sigma_2\Rightarrow p}(COM)}{G_1|G_2|\Gamma_1,\Pi_2',\Gamma_2,\Pi_1,\Sigma_2,\Pi_2'',\Sigma_1\Rightarrow A_1}(D).$$

Here, the major problem is how to extend (D) such that it is applicable to $G_2|\Gamma_2,\Pi_2',p,\Pi_2'',\Sigma_2\Rightarrow p$. By replacing p with t, we get $G_2|\Gamma_2,\Pi_2',t,\Pi_2'',\Sigma_2\Rightarrow t$. But there exists no derivation of $G_1|G_2|\Gamma_1,\Pi_2',\Gamma_2,\Pi_1,\Sigma_2,\Pi_2'',\Sigma_1\Rightarrow A_1$ from $G_2|\Gamma_2,\Pi_2',\Pi_2'',\Sigma_2\Rightarrow t$ and $G_1|\Gamma_1,\Pi_1,\Sigma_1\Rightarrow A_1$. Notice that Γ_2,Π_2' and Π_2'',Σ_2 in $G_2|\Gamma_2,\Pi_2',p,\Pi_2'',\Sigma_2\Rightarrow p$ are commutated simultaneously in $G_1|G_2|\Gamma_1,\Pi_2',\Gamma_2,\Pi_1,\Sigma_2,\Pi_2'',\Sigma_1\Rightarrow A_1$, which we can't obtain by (WCM). It seems that (WCM) can't be strengthened further in order to solve this difficulty. We overcome this difficulty by introducing a restricted subsystem $\mathbf{GpsUL}_\Omega$ of **GpsUL***. $\mathbf{GpsUL}_\Omega$ is a generalization of $\mathbf{GIUL}_\Omega$, which we introduced in [6] in order to solve a longstanding open problem, i.e., the standard completeness of **IUL**. Two new manipulations, which we call the derivation-splitting operation and derivation-splicing operation, are introduced in $\mathbf{GpsUL}_\Omega$.

The third difficulty we encounter is that the conditions of applying the restricted external contraction rule (EC_Ω) become more complex in $\mathbf{GpsUL}_\Omega$ because new derivation-splitting operations make the conclusion of the generalized density rule to be a set of hypersequents rather than one hypersequent. We continue to apply derivation-grafting operations in the separation algorithm of the multiple branches of $\mathbf{GIUL}_\Omega$ in [6] but we have to introduce a new construction method for $\mathbf{GpsUL}_\Omega$ by induction on the height of the complete set of maximal (pEC)-nodes other than on the number of branches.

2. GpsUL, GpsUL* and $\mathbf{GpsUL}_\Omega$

Definition 2.1. ([1]) **GpsUL** consist of the following initial sequents and rules:
Initial sequents

$$\dfrac{}{A\Rightarrow A}(ID) \quad \dfrac{}{\Rightarrow t}(t_r) \quad \dfrac{}{\Gamma,\bot,\Delta\Rightarrow A}(\bot_l) \quad \dfrac{}{\Gamma\Rightarrow\top}(\top_r)$$

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Structural Rules

$$\frac{G|\Gamma \Rightarrow A|\Gamma \Rightarrow A}{G|\Gamma \Rightarrow A}(EC) \quad \frac{G}{G|\Gamma \Rightarrow A}(EW)$$

$$\frac{G_1|\Gamma_1,\Pi_1,\Delta_1 \Rightarrow A_1 \quad G_2|\Gamma_2,\Pi_2,\Delta_2 \Rightarrow A_2}{G_1|G_2|\Gamma_1,\Pi_2,\Delta_1 \Rightarrow A_1|\Gamma_2,\Pi_1,\Delta_2 \Rightarrow A_2}(COM)$$

Logical Rules

$$\frac{G_1|\Gamma \Rightarrow A \quad G_2|\Delta \Rightarrow B}{G_1|G_2|\Gamma,\Delta \Rightarrow A \odot B}(\odot_r) \qquad \frac{G|\Gamma,A,B,\Delta \Rightarrow C}{G|\Gamma,A \odot B,\Delta \Rightarrow C}(\odot_l)$$

$$\frac{G_1|\Gamma,B,\Delta \Rightarrow C \quad G_2|\Pi \Rightarrow A}{G_1|G_2|\Gamma,\Pi,A \rightarrow B,\Delta \Rightarrow C}(\rightarrow_l) \qquad \frac{G|A,\Gamma \Rightarrow B}{G|\Gamma \Rightarrow A \rightarrow B}(\rightarrow_r)$$

$$\frac{G_1|\Pi \Rightarrow A \quad G_2|\Gamma,B,\Delta \Rightarrow C}{G_1|G_2|\Gamma,A \rightsquigarrow B,\Pi,\Delta \Rightarrow C}(\rightsquigarrow_l) \qquad \frac{G|\Gamma,A \Rightarrow B}{G|\Gamma \Rightarrow A \rightsquigarrow B}(\rightsquigarrow_r)$$

$$\frac{G_1|\Gamma,A,\Delta \Rightarrow C \quad G_2|\Gamma,B,\Delta \Rightarrow C}{G_1|G_2|\Gamma,A \vee B,\Delta \Rightarrow C}(\vee_l) \qquad \frac{G|\Gamma \Rightarrow A}{G|\Gamma \Rightarrow A \vee B}(\vee_{rr})$$

$$\frac{G_1|\Gamma \Rightarrow A \quad G_2|\Gamma \Rightarrow B}{G_1|G_2|\Gamma \Rightarrow A \wedge B}(\wedge_l) \qquad \frac{G|\Gamma \Rightarrow B}{G|\Gamma \Rightarrow A \vee B}(\vee_{rl})$$

$$\frac{G|\Gamma,A,\Delta \Rightarrow C}{G|\Gamma,A \wedge B,\Delta \Rightarrow C}(\wedge_{rr}) \qquad \frac{G|\Gamma,B,\Delta \Rightarrow C}{G|\Gamma,A \wedge B,\Delta \Rightarrow C}(\wedge_{rl})$$

$$\frac{G|\Gamma,\Delta \Rightarrow A}{G|\Gamma,t,\Delta \Rightarrow A}(t_l)$$

Cut Rule

$$\frac{G_1|\Gamma,A,\Delta \Rightarrow B \quad G_2|\Pi \Rightarrow A}{G_1|G_2|\Gamma,\Pi,\Delta \Rightarrow B}(CUT).$$

Definition 2.2. ([3]) **GpsUL*** is **GpsUL** plus the weakly commutativity rule

$$\frac{G|\Gamma,\Delta \Rightarrow t}{G|\Delta,\Gamma \Rightarrow t}(WCM).$$

Definition 2.3. **GpsUL**^{*D} is **GpsUL*** plus the density rule $\frac{G|\Pi \Rightarrow p|\Gamma,p,\Delta \Rightarrow B}{G|\Gamma,\Pi,\Delta \Rightarrow B}(D)$.

Lemma 2.4. *$G \equiv B \vee ((D \rightarrow B) \odot C \odot (C \rightarrow D) \odot A \rightarrow A)$ is not a theorem in* **HpsUL***.*

Proof. Let $\mathcal{A} = (\{0,1,2,3\},\wedge,\vee,\odot,\rightarrow,\rightsquigarrow,2,0,3)$ be an algebra, where $x \wedge y = \min(x,y)$, $x \vee y = \max(x,y)$ for all $x,y \in \{0,1,2,3\}$, and the binary operations $\odot$, $\rightarrow$ and $\rightsquigarrow$ are defined by the following tables (See [2]).

$\odot$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	3
3	0	3	3	3

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	3	3	3
2	0	1	2	3
3	0	0	0	3

$\rightsquigarrow$	0	1	2	3
0	3	3	3	3
1	0	2	2	3
2	0	1	2	3
3	0	1	1	3

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By easy calculation, we get that $\mathcal{A}$ is a linearly ordered **HpsUL**-algebra, where 0 and 3 are the least and the greatest element of $\mathcal{A}$, respectively, and 2 is its unit. Let $v(A) = v(B) = v(C) = v(D) = 1$. Then $v(G) = 1 \vee (3 \odot 1 \odot 3 \odot 1 \to 1) = 1 < 2$. Hence G is not a tautology in **HpsUL**. Therefore it is not a theorem in **HpsUL** by Theorem 9.27 in [1]. □

Theorem 2.5. *Cut-elimination doesn't holds for* **GpsUL***.

Proof. $G \equiv\Rightarrow B \vee ((D \to B) \odot C \odot (C \to D) \odot A \to A)$ is provable in **GpsUL***, as shown in Figure 1.

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{t, A \Rightarrow A}(t_l) \quad \dfrac{\dfrac{\dfrac{B \Rightarrow B \qquad \Rightarrow t}{\Rightarrow B \mid B \Rightarrow t}(COM) \quad \dfrac{C \Rightarrow C \qquad D \Rightarrow D}{C, C \to D \Rightarrow D}(\to_l)}{\Rightarrow B \mid C, C \to D, D \to B \Rightarrow t}(\to_l)}{\Rightarrow B \mid D \to B, C, C \to D \Rightarrow t}(WCM)}{\dfrac{\Rightarrow B \mid D \to B, C, C \to D, A \Rightarrow A}{\Rightarrow B \mid \ \Rightarrow (D \to B) \odot C \odot (C \to D) \odot A \to A}(\odot_l^*, \to_r)}(CUT)}{\Rightarrow B \vee ((D \to B) \odot C \odot (C \to D) \odot A \to A)}(\vee_{rr}, \vee_{rl}, EC).$$

Figure 1 A proof τ of G

Suppose that G has a cut-free proof ρ. Then there exists no occurrence of t in ρ by its subformula property. Thus there exists no application of (WCM) in ρ. Hence G is a theorem of **GpsUL**, which contradicts Lemma 2.4. □

Remark 2.6. Following the construction given in the proof of Theorem 53 in [4], (CUT) in the figure 1 is eliminated by the following derivation. However, the application of (WCM) in ρ is invalid, which illustrates the reason why the cut-elimination theorem doesn't hold in **GpsUL***.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B \qquad A \Rightarrow A}{\Rightarrow B \mid B, A \Rightarrow A}(COM) \quad \dfrac{C \Rightarrow C \qquad D \Rightarrow D}{C, C \to D \Rightarrow D}(\to_l)}{\Rightarrow B \mid C, C \to D, D \to B, A \Rightarrow A}(\to_l)}{\Rightarrow B \mid D \to B, C, C \to D, A \Rightarrow A}(WCM)}{\Rightarrow B \mid \ \Rightarrow (D \to B) \odot C \odot (C \to D) \odot A \to A}(\odot_l^*, \to_r)}{\Rightarrow B \vee ((D \to B) \odot C \odot (C \to D) \odot A \to A)}(\vee_{rr}, \vee_{rl}, EC)$$

Figure 2 A possible cut-free proof ρ of G

Definition 2.7. **GpsUL**** is constructed by replacing (CUT) in **GpsUL*** with

$$\dfrac{G_1 | \Gamma, t, \Delta \Rightarrow A \quad G_2 | \Pi \Rightarrow t}{G_1 | G_2 | \Gamma, \Pi, \Delta \Rightarrow A}(WCT).$$

We call it the weakly cut rule and, denote by (WCT).

Theorem 2.8. *If* $\vdash_{\mathbf{GpsUL}^*} G$*, then* $\vdash_{\mathbf{GpsUL}^{**}} G$.

Proof. It is proved by a procedure similar to that of Theorem 53 in [4] and omitted. □

Definition 2.9. ([6]) **GpsUL**$_\Omega$ is a restricted subsystem of **GpsUL*** such that

(i) p is designated as the unique eigenvariable by which we means that it does not be used to built up any formula containing logical connectives and only used as a sequent-formula.

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(ii) Each occurrence of p in a hypersequent is assigned one unique identification number i in $\mathbf{GpsUL}_\Omega$ and written as p_i. Initial sequent $p \Rightarrow p$ of $\mathbf{GpsUL}^*$ has the form $p_i \Rightarrow p_i$ in $\mathbf{GpsUL}_\Omega$. p doesn't occur in A, Γ or Δ for each initial sequent $\Gamma, \bot, \Delta \Rightarrow A$ or $\Gamma \Rightarrow \top$ in $\mathbf{GpsUL}_\Omega$.

(iii) Each sequent S of the form $\Gamma_0, p, \Gamma_1, \cdots, \Gamma_{\lambda-1}, p, \Gamma_\lambda \Rightarrow A$ in $\mathbf{GpsUL}^*$ has the form $\Gamma_0, p_{i_1}, \Gamma_1, \cdots, \Gamma_{\lambda-1}, p_{i_\lambda}, \Gamma_\lambda \Rightarrow A$ in $\mathbf{GpsUL}_\Omega$, where p does not occur in Γ_k for all $0 \leqslant k \leqslant \lambda$ and, $i_k \neq i_l$ for all $1 \leqslant k < l \leqslant \lambda$. Define $v_l(S) = \{i_1, \cdots, i_\lambda\}$, $v_r(S) = \{j_1\}$ if A is an eigenvariable with the identification number j_1 and, $v_r(S) = \varnothing$ if A isn't an eigenvariable.

Let G be a hypersequent of $\mathbf{GpsUL}_\Omega$ in the form $S_1|\cdots|S_n$ then $v_l(S_k) \bigcap v_l(S_l) = \varnothing$ and $v_r(S_k) \bigcap v_r(S_l) = \varnothing$ for all $1 \leqslant k < l \leqslant n$. Define $v_l(G) = \bigcup_{k=1}^n v_l(S_k)$, $v_r(G) = \bigcup_{k=1}^n v_r(S_k)$.

(iv) A hypersequent G of $\mathbf{GpsUL}_\Omega$ is called closed if $v_l(G) = v_r(G)$. Two hypersequents G' and G'' of $\mathbf{GpsUL}_\Omega$ are called disjoint if $v_l(G') \bigcap v_l(G'') = \varnothing$, $v_l(G') \bigcap v_r(G'') = \varnothing$, $v_r(G') \bigcap v_l(G'') = \varnothing$ and $v_r(G') \bigcap v_r(G'') = \varnothing$. G'' is a copy of G' if they are disjoint and there exist two bijections $\sigma_l : v_l(G') \to v_l(G'')$ and $\sigma_r : v_r(G') \to v_r(G'')$ such that G'' can be obtained by applying σ_l to antecedents of sequents in G' and σ_r to succedents of sequents in G'.

(v) A hypersequent $G|G_1|G_2$ can be contracted as $G|G_1$ in $\mathbf{GpsUL}_\Omega$ under certain condition given in Construction 3.15, which we called the constraint external contraction rule and denote by $\dfrac{G'|G_1|G_2}{G'|G_1}(EC_\Omega)$.

(vi) (EW) is forbidden in $\mathbf{GpsUL}_\Omega$ and, (EC) and (CUT) are replaced with (EC_Ω) and (WCT), respectively.

(vii) Two rules $(\wedge_r)$ and $(\vee_l)$ of $\mathbf{GL}$ are replaced with $\dfrac{G_1|\Gamma_1 \Rightarrow A \quad G_2|\Gamma_2 \Rightarrow B}{G_1|G_2|\Gamma_1 \Rightarrow A \wedge B|\Gamma_2 \Rightarrow A \wedge B}(\wedge_{rw})$ and $\dfrac{G_1|\Gamma_1, A, \Delta_1 \Rightarrow C_1 \quad G_2|\Gamma_2, B, \Delta_2 \Rightarrow C_2}{G_1|G_2|\Gamma_1, A \vee B, \Delta_1 \Rightarrow C_1|\Gamma_2, A \vee B, \Delta_2 \Rightarrow C_2}(\vee_{lw})$ in $\mathbf{GpsUL}_\Omega$, respectively.

(viii) $G_1|S_1$ and $G_2|S_2$ are closed and disjoint for each two-premise inference rule $\dfrac{G_1|S_1 \quad G_2|S_2}{G_1|G_2|H'}(II)$ of $\mathbf{GpsUL}_\Omega$ and, $G'|S'$ is closed for each one-premise inference rule $\dfrac{G'|S'}{G'|S''}(I)$.

Proposition 2.10. Let $\dfrac{G'|S'}{G'|S''}(I)$ and $\dfrac{G_1|S_1 \quad G_2|S_2}{G_1|G_2|H'}(II)$ be inference rules of $\mathbf{GpsUL}_\Omega$ then $v_l(G'|S'') = v_r(G'|S'') = v_r(G'|S') = v_l(G'|S')$ and $v_l(G_1|G_2|H') = v_l(G_1|S_1) \bigcup v_l(G_2|S_2) = v_r(G_1|G_2|H') = v_r(G_1|S_1) \bigcup v_r(G_2|S_2)$.

Proof. Although (WCT) makes t's in its premises disappear in its conclusion, it has no effect on identification numbers of the eigenvariable p in a hypersequent because t is a constant in $\mathbf{GpsUL}_\Omega$ and distinguished from propositional variables. □

Definition 2.11 (1)**.** Let G be a closed hypersequent of $\mathbf{GpsUL}_\Omega$ and $S \in G$. $[S]_G := \bigcap\{H : S \in H \subseteq G, v_l(H) = v_r(H)\}$ is called a minimal closed unit of G.

3. The generalized density rule $(\mathcal{D})$ for $\mathbf{GpsUL}_\Omega$

In this section, $\mathbf{GL}^{\mathbf{cf}}_\Omega$ is $\mathbf{G_{ps}UL\Omega}$ without (EC_Ω). Generally, $A, B, C, \cdots$, denote a formula other than an eigenvariable p_i.

Construction 3.1. *Given a proof τ^* of $H \equiv G|\Gamma, p_j, \Delta \Rightarrow p_j$ in $\mathbf{GL}^{\mathbf{cf}}_\Omega$, let $Th_{\tau^*}(p_j \Rightarrow p_j) = (H_0, \cdots, H_n)$, where $H_0 \equiv p_j \Rightarrow p_j$, $H_n \equiv H$. By $\Gamma_k, p_j, \Delta_k \Rightarrow p_j$ we denote the sequent containing*

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p_j in H_k. Then $\Gamma_0 = \varnothing$, $\Delta_0 = \varnothing$, $\Gamma_n = \Gamma$ and $\Delta_n = \Delta$. Hypersequents $\langle H_k\rangle_j^-$, $\langle H_k\rangle_j^+$ and their proofs $\langle\tau^*\rangle_j^-\left(\langle H_k\rangle_j^-\right)$, $\langle\tau^*\rangle_j^+\left(\langle H_k\rangle_j^+\right)$ are constructed inductively for all $0 \leqslant k \leqslant n$ in the following such that $\Gamma_k \Rightarrow t \in \langle H_k\rangle_j^-$, $\Delta_k \Rightarrow t \in \langle H_k\rangle_j^+$, and $\langle H_k\rangle_j^+ \backslash\{\Delta_k \Rightarrow t\}|\langle H_k\rangle_j^- \backslash\{\Gamma_k \Rightarrow t\} = H_k\backslash\{\Gamma_k, p_j, \Delta_k \Rightarrow p_j\}$.

(i) $\langle H_0\rangle_j^- := \langle H_0\rangle_j^+ :=\Rightarrow t$, $\langle\tau^*\rangle_j^-\left(\langle H_0\rangle_j^-\right)$ and $\langle\tau^*\rangle_j^+\left(\langle H_0\rangle_j^+\right)$ are built up with $\Rightarrow t$.

(ii) Let $\dfrac{G'|S' \quad G''|S''}{G'|G''|H'}(II)$ (or $\dfrac{G'|S'}{G'|S''}(I)$) be in τ^*, $H_k = G'|S'$ and $H_{k+1} = G'|G''|H'$ (accordingly $H_{k+1} = G'|S''$ for (I)) for some $0 \leqslant k \leqslant n-1$. There are three cases to be considered.

Case 1 *$S' = \Gamma_k, p_j, \Delta_k \Rightarrow p_j$. If all focus formula(s) of S' is (are) contained in Γ_k,*

$$\langle H_{k+1}\rangle_j^- := \left(\langle H_k\rangle_j^- \backslash\{\Gamma_k \Rightarrow t\}\right)|G''|H'\backslash\{\Gamma_{k+1}, p_j, \Delta_{k+1} \Rightarrow p_j\}|\Gamma_{k+1} \Rightarrow t$$

$$\langle H_{k+1}\rangle_j^+ := \langle H_k\rangle_j^+$$

(accordingly $\langle H_{k+1}\rangle_j^- = \langle H_k\rangle_j^- \backslash\{\Gamma_k \Rightarrow t\}|\Gamma_{k+1} \Rightarrow t$ for (I)) and, $\langle\tau^*\rangle_j^-\left(\langle H_{k+1}\rangle_j^-\right)$ is constructed by combining the derivation $\langle\tau^*\rangle_j^-\left(\langle H_k\rangle_j^-\right)$ and $\dfrac{\langle H_k\rangle_j^- \quad G''|S''}{\langle H_{k+1}\rangle_j^-}(II)$ (accordingly $\dfrac{\langle H_k\rangle_j^-}{\langle H_{k+1}\rangle_j^-}(I)$ for (I)) and, $\langle\tau^*\rangle_j^+\left(\langle H_{k+1}\rangle_j^+\right)$ is constructed by combining $\langle\tau^*\rangle_j^+\left(\langle H_k\rangle_j^+\right)$ and $\dfrac{\langle H_k\rangle_j^+}{\langle H_{k+1}\rangle_j^+}(ID_\Omega)$. The case of all focus formula(s) of S' contained in Δ_k is dealt with by a procedure dual to above and omitted.

Case 2 *$S' \in \langle H_k\rangle_j^-$. $\langle H_{k+1}\rangle_j^- := \left(\langle H_k\rangle_j^- \backslash\{S'\}\right)|G''|H'$ (accordingly $\langle H_{k+1}\rangle_j^- = \langle H_k\rangle_j^- \backslash\{S'\}|S''$ for (I)), $\langle H_{k+1}\rangle_j^+ := \langle H_k\rangle_j^+$ and $\langle\tau^*\rangle_j^-\left(\langle H_{k+1}\rangle_j^-\right)$ is constructed by combining the derivation $\langle\tau^*\rangle_j^-\left(\langle H_k\rangle_j^-\right)$ and $\dfrac{\langle H_k\rangle_j^- \quad G''|S''}{\langle H_{k+1}\rangle_j^-}(II)$ (accordingly $\dfrac{\langle H_k\rangle_j^-}{\langle H_{k+1}\rangle_j^-}(I)$ for (I)) and, $\langle\tau^*\rangle_j^+\left(\langle H_{k+1}\rangle_j^+\right)$ is constructed by combining $\langle\tau^*\rangle_j^+\left(\langle H_k\rangle_j^+\right)$ and $\dfrac{\langle H_k\rangle_j^+}{\langle H_{k+1}\rangle_j^+}(ID_\Omega)$.*

Case 3 *$S' \in \langle H_k\rangle_j^+$. It is dealt with by a procedure dual to Case 2 and omitted.*

Definition 3.2. The manipulation described in Construction 3.1 is called the derivation-splitting operation when it is applied to a derivation and, the splitting operation when applied to a hypersequent.

Corollary 3.3. *Let $\vdash_{\mathbf{GL}_\Omega^{\mathbf{cf}}} G|\Gamma, p_1, \Delta \Rightarrow p_1$. Then there exist two hypersequents G_1 and G_2 such that $G = G_1 \bigcup G_2$, $G_1 \bigcap G_2 = \varnothing$, $\vdash_{\mathbf{GL}_\Omega^{\mathbf{cf}}} G_1|\Gamma \Rightarrow t$ and $\vdash_{\mathbf{GL}_\Omega^{\mathbf{cf}}} G_2|\Delta \Rightarrow t$.*

Construction 3.4. *Given a proof τ^* of $H \equiv G|\Pi \Rightarrow p_j|\Gamma, p_j, \Delta \Rightarrow A$ in $\mathbf{GL}_\Omega^{\mathbf{cf}}$, let $Th_{\tau^*}(p_j \Rightarrow p_j) = (H_0, \cdots, H_n)$, where $H_0 \equiv p_j \Rightarrow p_j$ and $H_n \equiv H$. Then there exists $1 \leqslant m \leqslant n$ such that H_m is in the form $G'|\Pi' \Rightarrow p_j|\Gamma', p_j, \Delta' \Rightarrow A'$ and H_{m-1} is in the form $G''|\Gamma'', p_j, \Delta'' \Rightarrow p_j$. A proof of $G|\Gamma, \Pi, \Delta \Rightarrow A$ in $\mathbf{GL}_\Omega^{\mathbf{cf}}$ is constructed by induction on $n-m$ as follows.*

- *For the base step, let $n - m = 0$. Then $\dfrac{H_{n-1} \equiv G'|\Pi', \Gamma', p_j, \Delta', \Pi''' \Rightarrow p_j \quad G''|\Gamma'', \Pi'', \Delta'' \Rightarrow A}{H_n \equiv G'|G''|\Pi', \Pi'', \Pi''' \Rightarrow p_j|\Gamma'', \Gamma', p_j, \Delta', \Delta'' \Rightarrow A}(COM) \in \tau^*$, where $G'|G'' = G$ and $\Pi', \Pi'', \Pi''' = \Pi$ and $\Gamma'', \Gamma' = \Gamma$ and $\Delta', \Delta'' = \Delta$. It follows from Corollary 3.3 that*

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there exist G'_1 and G'_2 such that $G' = G'_1 \cup G'_2$, $G'_1 \cap G'_2 = \varnothing$, $\vdash_{\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}} G'_1|\Pi', \Gamma' \Rightarrow t$ and $\vdash_{\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}} G'_2|\Delta', \Pi''' \Rightarrow t$. Then $G|\Gamma, \Pi, \Delta \Rightarrow A$ is proved as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{G''|\Gamma'',\Pi'',\Delta'' \Rightarrow A}{G''|\Gamma'',t,\Pi'',\Delta'' \Rightarrow A}(t_l) \qquad \dfrac{G'_1|\Pi',\Gamma' \Rightarrow t}{G'_1|\Gamma',\Pi' \Rightarrow t}(WCM)}{G''|G'_1|\Gamma'',\Gamma',\Pi',\Pi'',\Delta'' \Rightarrow A}(WCT)}{G''|G'_1|\Gamma'',\Gamma',\Pi',\Pi'',t,\Delta'' \Rightarrow A}(t_l) \qquad \dfrac{G'_2|\Delta',\Pi''' \Rightarrow t}{G'_2|\Pi''',\Delta' \Rightarrow t}(WCM)}{G''|G'_1|G'_2|\Gamma'',\Gamma',\Pi',\Pi'',\Pi''',\Delta',\Delta'' \Rightarrow A}(WCT).$$

- *For the induction step, let $n - m > 0$. Then it is treated using applications of the induction hypothesis to the premise followed by an application of the relevant rule. For example, let $\dfrac{H_{n-1} = G'|\Pi \Rightarrow p_j|\Sigma',\Gamma'',p_j,\Delta'',\Sigma''' \Rightarrow A' \qquad G''|\Gamma',\Sigma'',\Delta' \Rightarrow A}{H_n = G'|\Pi \Rightarrow p_j|\Sigma',\Sigma'',\Sigma''' \Rightarrow A'|G''|\Gamma',\Gamma'',p_j,\Delta'',\Delta' \Rightarrow A}(COM) \in \tau^*$, where $G'|G''|\Sigma',\Sigma'',\Sigma''' \Rightarrow A' = G$ and $\Gamma',\Gamma'' = \Gamma$ and $\Delta'',\Delta' = \Delta$. By the induction hypothesis we obtain a derivation of $G|\Gamma,\Pi,\Delta \Rightarrow A$:*

$$\dfrac{G'|\Sigma',\Gamma'',\Pi,\Delta'',\Sigma''' \Rightarrow A' \qquad G''|\Gamma',\Sigma'',\Delta' \Rightarrow A}{G'|\Sigma',\Sigma'',\Sigma''' \Rightarrow A'|G''|\Gamma',\Gamma'',\Pi,\Delta'',\Delta' \Rightarrow A}(COM).$$

Definition 3.5. The manipulation described in Construction 3.4 is called the derivation-splicing operation when it is applied to a derivation and, the splicing operation when applied to a hypersequent.

Corollary 3.6. *If $\vdash_{\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}} G|\Pi \Rightarrow p_j|\Gamma, p_j, \Delta \Rightarrow A$, then $\vdash_{\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}} G|\Gamma, \Pi, \Delta \Rightarrow A$.*

Definition 3.7. (i) Let $\vdash_{\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}} H \equiv G|\Gamma, p_j, \Delta \Rightarrow p_j$. Define $\langle H \rangle_j^- = G_1|\Gamma \Rightarrow t$, $\langle H \rangle_j^+ = G_2|\Delta \Rightarrow t$ and $D_j(H) = \{G_1|\Gamma \Rightarrow t, G_2|\Delta \Rightarrow t\}$, where, G_1 and G_2 are determined by Corollary 3.3.

(ii) Let $\vdash_{\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}} H \equiv G|\Pi \Rightarrow p_j|\Gamma, p_j, \Delta \Rightarrow A$. Define$D_j(H) = \{G|\Gamma, \Pi, \Delta \Rightarrow A\} = \langle H \rangle_j$.

(iii) Let $\vdash_{\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}} G$. $D_j(G) = \{G\}$ if p_j does not occur in G.

(iv) Let $\vdash_{\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}} G_i$ for all $1 \leqslant i \leqslant n$. Define $D_j(\{G_1, \cdots, G_n\}) = D_j(G_1) \cup \cdots \cup D_j(G_n)$.

(v) Let $\vdash_{\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}} G$ and $K = \{1, \cdots, n\} \subseteq v(G)$. Define $D_K(G) = D_n(\cdots D_2(D_1(G))\cdots)$. Especially, define $\mathcal{D}(G) = D_{v_l(G)}(G)$.

Theorem 3.8. *Let $\vdash_{\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}} G$. Then $\vdash_{\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}} H$ for all $H \in \mathcal{D}(G)$.*

Proof. Immediately from Corollary 3.3, Corollary 3.6 and Definition 3.7. □

Lemma 3.9. *Let G' be a minimal closed unit of $G|G'$. Then G' has the form $\Gamma \Rightarrow A|\Gamma_{i_2} \Rightarrow p_{i_2}|\cdots|\Gamma_{i_n} \Rightarrow p_{i_n}$ if there exists one sequent $\Gamma \Rightarrow A \in G'$ such that A is not an eigenvariable otherwise G' has the form $\Gamma_{i_1} \Rightarrow p_{i_1}|\cdots|\Gamma_{i_n} \Rightarrow p_{i_n}$.*

Proof. Define $G_1 = \Gamma \Rightarrow A$ in Construction 5.2 in [6]. Then $\varnothing = v_r(G_1) \subseteq v_l(G_1)$. Suppose that G_k is constructed such that $v_r(G_k) \subseteq v_l(G_k)$. If $v_l(G_k) = v_r(G_k)$, the procedure terminates and $n := k$, otherwise $v_l(G_k) \backslash v_r(G_k) \neq \varnothing$ and define i_{k+1} to be an identification number in $v_l(G_k) \backslash v_r(G_k)$. Then there exists $\Gamma_{i_{k+1}} \Rightarrow p_{i_{k+1}} \in G \backslash G_k$ by $v_l(G) = v_r(G)$ and, define $G_{k+1} = G_k|\Gamma_{i_{k+1}} \Rightarrow p_{i_{k+1}}$. Thus $v_r(G_{k+1}) = v_r(G_k) \cup \{i_{k+1}\} \subseteq v_l(G_k) \subseteq v_l(G_{k+1})$. Hence there exists a sequence $i_2, \cdots, i_n$ of identification numberssuch that $v_r(G_k) \subseteq v_l(G_k)$ for all $1 \leqslant k \leqslant$ **n**, where $G_1 = \Gamma \Rightarrow A$, $G_k = \Gamma \Rightarrow A|\Gamma_{i_2} \Rightarrow p_{i_2}|\cdots|\Gamma_{i_k} \Rightarrow p_{i_k}$ for all $2 \leqslant k \leqslant n$. Therefore G' has the form $\Gamma \Rightarrow A|\Gamma_{i_2} \Rightarrow p_{i_2}|\cdots|\Gamma_{i_n} \Rightarrow p_{i_n}$. □

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Definition 3.10. Let G' be a minimal closed unit of $G|G'$. G' is a splicing unit if it has the form $\Gamma \Rightarrow A|\Gamma_{i_2} \Rightarrow p_{i_2}|\cdots|\Gamma_{i_n} \Rightarrow p_{i_n}$. G' is a splitting unit if it has the form $\Gamma_{i_1} \Rightarrow p_{i_1}|\cdots|\Gamma_{i_n} \Rightarrow p_{i_n}$.

Lemma 3.11. *Let G' be a splicing unit of $G|G'$ in the form $\Gamma \Rightarrow A|\Gamma_{i_2} \Rightarrow p_{i_2}|\cdots|\Gamma_{i_n} \Rightarrow p_{i_n}$ and $K = \{i_2, \cdots, i_n\}$. Then $|D_K(G|G')| = 1$.*

Proof. By the construction in the proof of Lemma 3.9, $i_k \in v_l(G_{k-1})$ for all $2 \leqslant k \leqslant n$. Then $p_{i_2} \in \Gamma$ and $D_{i_2}(G|G') = G|\Gamma[\Gamma_{i_2}] \Rightarrow A|\Gamma_{i_3} \Rightarrow p_{i_3}|\cdots|\Gamma_{i_n} \Rightarrow p_{i_n}$, where $\Gamma[\Gamma_{i_2}]$is obtained by replacing p_{i_2} in Γ with Γ_{i_2}. Then $p_{i_3} \in \Gamma[\Gamma_{i_2}]$. Repeatedly, we get $D_{i_2\cdots i_n}(G|G') = D_K(G|G') = G|\Gamma[\Gamma_{i_2}]\cdots[\Gamma_{i_n}] \Rightarrow A$. □

This shows that $D_K(G|G')$ is constructed by repeatedly applying splicing operations.

Definition 3.12. Let G' be a minimal closed unit of $G|G'$. Define $V_{G'} = v(G')$, $E_{G'} = \{(i, j)|\Gamma, p_i, \Delta \Rightarrow p_j \in G'\}$ and, j is called the child node of i for all $(i, j) \in E_{G'}$. We call $\Omega_{G'} = (V_{G'}, E_{G'})$ the Ω-graph of G'.

Let G' be a splitting unit of $G|G'$ in the form $\Gamma_1 \Rightarrow p_1|\cdots|\Gamma_n \Rightarrow p_n$. Then each node of $\Omega_{G'}$ has one and only one child node. Thus there exists one cycle in $\Omega_{G'}$ by $|V_{G'}| = n < \infty$. Assume that, without loss of generality, $(1, 2), (2, 3), \cdots, (i, 1)$ is the cycle of $\Omega_{G'}$. Then $p_1 \in \Gamma_2$, $p_2 \in \Gamma_3$, $\cdots, p_{i-1} \in \Gamma_i$ and $p_i \in \Gamma_1$. Thus $D_{i\cdots 2}(G|G') = G|\Gamma_1[\Gamma_i][\Gamma_{i-1}]\cdots[\Gamma_2] \Rightarrow p_1$ is in the form $G|\Gamma', p_1, \Delta' \Rightarrow p_1$. By a suitable permutation σ of $i + 1, \cdots, n$, we get $D_{i\cdots 2\sigma(i+1\cdots n)}(G|G') = G|\Gamma_1[\Gamma_i][\Gamma_{i-1}]\cdots[\Gamma_2][\Gamma_{\sigma(i+1)}]\cdots[\Gamma_{\sigma(n)}] \Rightarrow p_1 = G|\Gamma, p_1, \Delta \Rightarrow p_1$. This process also shows that there exists only one cycle in $\Omega_{G'}$. Then we introduce the following definition.

Definition 3.13. (i) $\Gamma_j \Rightarrow p_j$ is called a splitting sequent of G' and p_j its corresponding splitting variable for all $1 \leqslant j \leqslant i$.

(ii) Let $K = \{1, 2, \cdots, n\}$ and $D_1(G|\Gamma, p_1, \Delta \Rightarrow p_1) = \{G_1|\Gamma \Rightarrow t, G_2|\Delta \Rightarrow t\}$. Define $\langle G|G'\rangle^-_K = G_1|\Gamma \Rightarrow t$, $\langle G|G'\rangle^+_K = G_2|\Delta \Rightarrow t$ and $D_K(G|G') = \{\langle G|G'\rangle^+_K, \langle G|G'\rangle^-_K\}$.

Lemma 3.14. *If G' be a splitting unit of $G|G'$, $K = v(G')$ and k be a splitting variable of G'. Then $D_{K\backslash\{k\}}(G|G')$ is constructed by repeatedly applying splicing operations and only the last operation D_k is a splitting operation.*

Construction 3.15. *(***The constrained external contraction rule***)*
Let $H \equiv G'|\{[S]_H\}_1|\{[S]_H\}_2$, $\{[S]_H\}_1$ and $\{[S]_H\}_2$ be two copies of a minimal closed unit $[S]_H$, where we put two copies into $\{\}_1$ and $\{\}_2$ in order to distinguish them. For any splitting unit $[S']_H \subseteq G'$, $\{[S]_H\}_1|\{[S]_H\}_2 \subseteq \langle H\rangle^-_K$ or $\{[S]_H\}_1|\{[S]_H\}_2 \subseteq \langle H\rangle^+_K$, where $K = v([S']_H)$. Then $G''|\{[S]_H\}_1$ is constructed by cutting off $\{[S]_H\}_2$ and some sequents in G' as follows.

(i) If $\{[S]_H\}_1$ and $\{[S]_H\}_2$ are two splicing units, then $G'' := G'$;

(ii) If $\{[S]_H\}_1$ and $\{[S]_H\}_2$ are two splitting units and, k, k' their splitting variables, respectively, $K = v(\{[S]_H\}_1)$, $K' = v(\{[S]_H\}_2)$, $D_{K\backslash\{k\}}(\{[S]_H\}_1) = \Gamma, p_k, \Delta \Rightarrow p_k$, $D_{K'\backslash\{k'\}}(\{[S]_H\}_2) = \Gamma, p_{k'}, \Delta \Rightarrow p_{k'}$, $D_{K\cup K'}(H) = \{G'_1|\Gamma \Rightarrow t|\Gamma \Rightarrow t, G'_2|\Delta \Rightarrow t, G''_2|\Delta \Rightarrow t\}$ or $D_{K\cup K'}(H) = \{G'_1|\Delta \Rightarrow t|\Delta \Rightarrow t, G'_2|\Gamma \Rightarrow t, G''_2|\Gamma \Rightarrow t\}$, where $G'_1 \cup G'_2 \cup G''_2 = G'$ and G''_2 is a copy of G'_2. Then $G'' := G'\backslash G''_2$.

The above operation is called the constrained external contraction rule, denoted by $\langle EC^*_\Omega\rangle$ and written as $\dfrac{G'|\{[S]_H\}_1|\{[S]_H\}_2}{G''|\{[S]_H\}_1}\langle EC^*_\Omega\rangle$.

Lemma 3.16. *If $\vdash_{\mathbf{GL}^{\mathrm{cf}}_\Omega} H$ as above. Then $\vdash_{\mathbf{GpsUL}_\Omega} H'$ for all $H' \in \mathcal{D}(G''|\{[S]_H\}_1)$.*

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4. Density elimination for GpsUL*

In this section, we adapt the separation algorithm of branches in [6] to **GpsUL*** and prove the following theorem.

Theorem 4.1. *Density elimination holds for* **GpsUL***.

The proof of Theorem 4.1 runs as follows. It is sufficient to prove that the following strong density rule

$$\frac{G_0 \equiv G'|\{\Gamma_i, p, \Delta_i \Rightarrow A_i\}_{i=1\cdots n}|\{\Pi_j \Rightarrow p\}_{j=1\cdots m}}{\mathcal{D}_0(G_0) \equiv G'|\{\Gamma_i, \Pi_j, \Delta_i \Rightarrow A_i\}_{i=1\cdots n; j=1\cdots m}}(\mathcal{D}_0)$$

is admissible in **GpsUL***, where $n, m \geqslant 1$, p does not occur in $G', \Gamma_i, \Delta_i, A_i, \Pi_j$ for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$.

Let τ be a proof of G_0 in **GpsUL**** by Theorem 2.8. Starting with τ, we construct a proof τ^* of $G|G^*$ in $\mathbf{GL}^{\mathbf{cf}}_{\mathbf{\Omega}}$ by a preprocessing of τ described in Section 4 in [6].

In Step 1 of preprocessing of τ, a proof τ' is constructed by replacing inductively all applications of $(\wedge_r)$ and $(\vee_l)$ in τ with $(\wedge_{rw})$ and $(\vee_{lw})$ followed by an application of (EC), respectively. In Step 2, a proof τ'' is constructed by converting all $\dfrac{G_i'''|\{S_i^c\}^{m_i'}}{G_i'''|S_i^c}(EC^*) \in \tau'$ into $\dfrac{G_i''|\{S_i^c\}^{m_i'}}{G_i''|\{S_i^c\}^{m_i'}}(ID_\Omega)$, where $G_i''' \subseteq G_i''$. In Step 3, a proof τ''' is constructed by converting $\dfrac{G'}{G'|S'}(EW) \in \tau''$ into $\dfrac{G''}{G''}(ID_\Omega)$, where $G'' \subseteq G'$. In Step 4, a proof τ'''' is constructed by replacing some $G'|\Gamma', p, \Delta' \Rightarrow A' \in \tau'''$ (or $G'|\Gamma' \Rightarrow p \in \tau'''$) with $G'|\Gamma', \top, \Delta' \Rightarrow A'$ (or $G'|\Gamma' \Rightarrow \bot$). In Step 5, a proof τ^* is constructed by assigning the unique identification number to each occurrence of p in τ''''. Let $H_i^c = G_i'|\{S_i^c\}^{m_i}$ denote the unique node of τ^* such that $H_i^c \leqslant G_i''|\{S_i^c\}^{m_i}$and S_i^c is the focus sequent of H_i^c in τ^*. We call H_i^c, S_i^c the i-th (pEC)-node of τ^* and (pEC)-sequent, respectively. If we ignore the replacements from Step 4, each sequent of G is a copy of some sequent of G_0 and, each sequent of G^* is a copy of some contraction sequent in τ'.

Now, starting with $G|G^*$ and its proof τ^*, we construct a proof $\tau^{\star}$ of $G^{\star}$ in $\mathbf{GpsUL}_{\Omega}$ such that each sequent of $G^{\star}$ is a copy of some sequent of G. Then $\vdash_{\mathbf{GpsUL}_{\Omega}} \mathcal{D}(G^{\star})$ by Theorem 3.8 and Lemma 3.16. Then $\vdash_{\mathbf{GpsUL}^*} \mathcal{D}_0(G_0)$ by Lemma 9.1 in [6].

In [6], $G^{\star}$ is constructed by eliminating (pEC)-sequents in $G|G^*$ one by one. In order to control the process, we introduce the set $I = \{H_{i_1}^c, \cdots, H_{i_m}^c\}$ of maximal (pEC)-nodes of τ^* (See Definition 4.2) and the set $\mathbf{I}$ of the branches relative to I and construct $G_{\mathbf{I}}^{\star}$ such that $G_{\mathbf{I}}^{\star}$ doesn't contain the contraction sequents lower than any node in I, i.e., $S_j^c \in G_{\mathbf{I}}^{\star}$ implies $H_j^c \| H_i^c$ for all $H_i^c \in I$. The procedure is called the separation algorithm of branches in [6].

The problem we encounter in $\mathbf{GpsUL}_{\Omega}$ is that Lemma 7.11 of [6] doesn't hold because new derivation-splitting operations make the conclusion of $(\mathcal{D})$-rule to be a set of hypersequents rather than one hypersequent. Then $G_{\ddagger}^{m_{q'}}$ generally can't be contracted to $G_{\ddagger}$ in Step 2 of Stage 1 in Main algorithm in [6] and, $\{G_{\mathbf{I}_{l\backslash r}}^{\star}\}^{m_{q'}}$ can't be contracted to $G_{\mathbf{I}_{l\backslash r}}^{\star}$ in Step 2 of Stage 2. Furthermore, we sometimes can't construct some branches to I in $\mathbf{GpsUL}_{\Omega}$ before we construct $\tau_{\mathbf{I}}^{\star}$. Therefore we have to introduce a new induction strategy for $\mathbf{GpsUL}_{\Omega}$ and don't perform the induction on the number of branches. First we give some primary definitions and lemmas.

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Definition 4.2. A (pEC)-node H_i^c is maximal if no other (pEC)-node is higher than H_i^c. Define I_0 to be the set of maximal (pEC)-nodes in τ^*. A nonempty subset I of I_0 is complete if I contains all maximal (pEC)-nodes higher than or equal to the intersection node H_I^V of I. Define $H_I^V = H_i^c$ if $I = \{H_i^c\}$, i.e., the intersection node of a single node is itself.

Proposition 4.3. (i) $H_i^c \parallel H_j^c$ for all $i \neq j$, $H_i^c, H_j^c \in I_0$.

(ii) Let I be complete and $H_j^c \geqslant H_I^V$. Then $H_j^c \leqslant H_i^c$ for some $H_i^c \in I$.

(iii) I_0 is complete and $\{H_i^c\}$ is complete for all $H_i^c \in I_0$.

(iv) If $I \subseteq I_0$ is complete and $|I| > 1$, then I_l and I_r are complete, where I_l and I_r denote the sets of all maximal (pEC)-nodes in the left subtree and right subtree of $\tau^*(H_I^V)$, respectively.

(v) If $I_1, I_2 \subseteq I_0$ are complete, then $I_1 \subseteq I_2$, $I_2 \subseteq I_1$ or $I_1 \bigcap I_2 = \varnothing$.

Proof. (v) $I_1 \subseteq I_2$, $I_2 \subseteq I_1$ or $I_1 \bigcap I_2 = \varnothing$ holds by $H_{I_2}^V \leqslant H_{I_1}^V$, $H_{I_1}^V \leqslant H_{I_2}^V$ or $H_{I_2}^V \parallel H_{I_1}^V$, respectively. □

Definition 4.4. A labeled binary tree ρ is constructed inductively by the following operations.

(i) The root of ρ is labeled by I_0 and leaves labeled $\{H_i^c\} \subseteq I_0$.

(ii) If an inner node is labeled by I, then its parent nodes are labeled by I_l and I_r, where I_l and I_r are defined in Proposition 4.3 (iv).

Definition 4.5. We define the height $o(I)$ of $I \in \rho$ by letting $o(I) = 1$ for each leave $I \in \rho$ and, $o(I) = \max\{o(I_l), o(I_r)\} + 1$ for any non-leaf node.

Note that in Lemma 7.11 in [6] only uniqueness of $G_{H_1:G_2}^{\star(J)}|\widehat{S_2}$ in $G_{H_{i_k}^c}^{\star}$ doesn't hold in $\mathbf{GpsUL}_\Omega$ and the following lemma holds in $\mathbf{GpsUL}_\Omega$.

Lemma 4.6. *Let* $\dfrac{G_1|S_1 \quad G_2|S_2}{H_1 \equiv G_1|G_2|H''}(II) \in \tau^*$, $\tau^*_{G_b|S_j^c} \in \tau^{\star}_{H_i^c}$, $\dfrac{G_b|\langle G_1|S_1\rangle_{S_j^c} \quad G_2|S_2}{H_2 \equiv G_b|\langle G_1\rangle_{S_j^c}|G_2|H''}(II) \in \tau^*_{G_b|S_j^c}$. *Then H'' is separable in $\tau_{H_i^c}^{\star(J)}$ and there are some copies of $G_{H_1:G_2}^{\star(J)}|\widehat{S_2}$ in $G_{H_i^c}^{\star}$.*

Lemma 4.7. **(New main algorithm for $\mathbf{GpsUL}_\Omega$)** *Let I be a complete subset of I_0 and $\overline{I} = \{H_i^c : H_i^c \leqslant H_j^c$ for some $H_j^c \in I\}$. Then there exist one close hypersequent $G_I^{\star} \subseteq_c G|G^*$ and its derivation $\tau_I^{\star}$ in $\mathbf{GpsUL}_\Omega$ such that*

(i) $\tau_I^{\star}$ is constructed by initial hypersequent $\overline{\overline{G|G^*}}\langle\tau^*\rangle$, the fully constraint contraction rules of the form $\dfrac{G_2}{G_1}\langle EC_\Omega^*\rangle$ and elimination rule of the form

$$\frac{G_{b_1}|S_{j_1}^c \quad G_{b_2}|S_{j_2}^c \quad \cdots \quad G_{b_w}|S_{j_w}^c}{G^*_{\mathbf{I_j}} = \{G_{b_k}\}_{k=1}^{w}|G^*_{\mathcal{I}_\mathbf{j}}}\left\langle\tau^*_{\mathbf{I_j}}\right\rangle,$$

where $1 \leqslant w \leqslant |I|$, $H_{j_k}^c \nleftrightarrow H_{j_l}^c$ for all $1 \leqslant k < l \leqslant w$, $\mathbf{I_j} = \left\{H_{j_1}^c, \cdots, H_{j_w}^c\right\} \subseteq \overline{I}$, $\mathcal{I}_\mathbf{j} = \{S_{j_1}^c, S_{j_2}^c, \cdots, S_{j_w}^c\}$, $\mathbf{I_j} = \{G_{b_1}|S_{j_1}^c, G_{b_2}|S_{j_2}^c, \cdots, G_{b_w}|S_{j_w}^c\}$, $G_{b_k}|S_{j_k}^c$ is closed for all $1 \leqslant k \leqslant w$. Then $H_i^c \nleqslant H_j^c$ for each $S_j^c \in G^*_{\mathcal{I}_\mathbf{j}}$ and $H_i^c \in I$.

(ii) For all $H \in \overline{\tau}_I^{\star}$, let

$$\partial_{\tau_I^{\star}}(H) := \begin{cases} G|G^* & H \text{ is the root of } \tau_I^{\star} \text{ or } G_2 \text{ in } \dfrac{G_2}{G_1}\langle EC_\Omega^* \text{ or } ID_\Omega\rangle \in \overline{\tau}_I^{\star}, \\ H_{j_k}^c & G_{b_k}|S_{j_k}^c \text{ in } \tau^*_{\mathbf{I_j}} \in \overline{\tau}_I^{\star} \text{ for some } 1 \leqslant k \leqslant w, \end{cases}$$

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where, $\overline{\tau}_I^{\star}$ is the skeleton of $\tau_I^{\star}$, which is defined by Definition 7.13 [6]. Then $\partial_{\tau_I^{\star}}\left(G^*_{\mathbf{I_j}}\right) \leqslant \partial_{\tau_I^{\star}}\left(G_{b_k}|S^c_{j_k}\right)$ for some $1 \leqslant k \leqslant w$ in $\tau^*_{\mathbf{I_j}}$;

(iii) Let $H \in \overline{\tau}_I^{\star}$ and $G|G^* < \partial_{\tau_I^{\star}}(H) \leqslant H_I^V$ then $G^{\star(J)}_{H_I^V:H} \in \tau_I^{\star}$ and it is built up by applying the separation algorithm along H_I^V to H, and is an upper hypersequent of either $\langle EC^*_{\Omega}\rangle$ if it is applicable, or $\langle ID_{\Omega}\rangle$ otherwise.

(iv) $S^c_j \in G^{\star}_I$ implies $H^c_j \| H^c_i$ for all $H^c_i \in I$ and, $S^c_j \in G^*_{\mathcal{I}_{\mathbf{j}}}$ for some $\tau^*_{\mathbf{I_j}} \in \tau_I^{\star}$.

Proof. $\tau_I^{\star}$ is constructed by induction on $o(I)$. For the base case, let $o(I) = 1$, then $\tau_I^{\star}$ is built up by Construction 7.3 and 7.7 in [6]. For the induction case, suppose that $o(I) \geqslant 2$, $\tau_{I_l}^{\star}$ and $\tau_{I_r}^{\star}$ are constructed such that Claims from (i) to (iv) hold.

Let $\dfrac{G'|S' \quad G''|S''}{G'|G''|H'}(II) \in \tau^*$, where $G'|G''|H' {=} H_I^V$. Then I_l and I_r occur in the left subtree $\tau^*(G'|S')$ and right subtree $\tau^*(G''|S'')$ of $\tau^*(H_I^V)$, respectively. Here, almost all manipulations of the new main algorithm are same as those of the old main algorithm. There are some caveats need to be considered.

Firstly, all leaves $\dfrac{}{G|G^*}\langle\tau^*\rangle \in \overline{\tau}_{I_l}^{\star}$ are replaced with $\tau_{I_r}^{\star}$ in Step 3 at Stage 1 in old main algorithm and, $\dfrac{}{G|G^*}\langle\tau^*\rangle \in \overline{\tau}_{I_r}^{\star}$ are replaced with $\tau_{I_l}^{\star}$ in Step 3 at Stage 2. Secondly, we abandon the definitions of branch to I and Notation 8.1 in [6] and then the symbol $\mathbf{I}$ of the set of branches, which occur in $\tau_{\mathbf{I}}^{\star}$ in [6], is replaced with I in the new algorithm. We call the new algorithm the separation algorithm along I. We also replace Ω in $\tau_{\mathbf{I}}^{\Omega}$ with $\star$. Thirdly, under the new requirement that I is complete, we prove the following property.

Property (A) $G_{I_l}^{\star}$ contains at most one copy of $G^{\star(J)}_{H_I^V:G''}|\widehat{S''}$.

Proof. Suppose that there exist two copies $\left\{G^{\star(J)}_{H_I^V:G''}|\widehat{S''}\right\}_1$ and $\left\{G^{\star(J)}_{H_I^V:G''}|\widehat{S''}\right\}_2$ of $G^{\star(J)}_{H_I^V:G''}|\widehat{S''}$ in $G^{\star}_{\mathbf{I}_l}$ and, we put them into $\{\}_1$ and $\{\}_2$ in order to distinguish them. Let $[S]_{G^{\star}_{I_l}}$ be a splitting unit of $G^{\star}_{\mathbf{I}_l}$ and S its splitting sequent. Then $|v_l(S)| + |v_r(S)| \geqslant 2$. Thus S is a (pEC)-sequent and has the form S^c_i by $[S]_{G^{\star}_{I_l}} \subseteq_c G|G^*$. Then $[S]_{G^{\star}_{I_l}} = [S^c_i]_{G^{\star}_{I_l}}$, $H^c_i \parallel H^c_j$ for all $H^c_j \in I_l$ and $S^c_i \in G^*_{\mathcal{I}_{\mathbf{j}_l}}$ for some $\tau^*_{\mathbf{I}_{\mathbf{j}_l}} \in \tau^{\star}_{I_l}$ by Claim (iv). Since I_l is complete and $G'|S' \leqslant H^V_{I_l}$, then $H^c_i \parallel G'|S'$.

Let $\tau^*_{\mathbf{I}_{\mathbf{j}_l}}$ be in the form $\dfrac{G_{b_{l1}}|S^c_{j_{l1}} \quad G_{b_{l2}}|S^c_{j_{l2}} \quad \cdots \quad G_{b_{lu}}|S^c_{j_{lu}}}{G^*_{\mathbf{I}_{\mathbf{j}_l}} = \{G_{b_{lk}}\}_{k=1}^{u}|G^*_{\mathcal{I}_{\mathbf{j}_l}}}\left\langle\tau^*_{\mathbf{I}_{\mathbf{j}_l}}\right\rangle$, $\dfrac{G_1|S_1 \quad G_2|S_2}{H_1 \equiv G_1|G_2|H''}(II) \in \tau^*$, where $G_1|S_1 \leqslant G'|S'$, $G_2|S_2 \leqslant H^c_i$, $G_1|G_2|H''$ is the intersection node of H^c_i and $G'|S'$, as shown in Figure 3. Then $\dfrac{\{G_{b_{lk}}\}_{k=1}^{u}|\langle G_1|S_1\rangle_{\mathcal{I}_{\mathbf{j}_l}} \quad G_2|S_2}{H_2 \equiv \{G_{b_{lk}}\}_{k=1}^{u}|\langle G_1\rangle_{\mathcal{I}_{\mathbf{j}_l}}|G_2|H''}(II) \in \tau^*_{\mathbf{I}_{\mathbf{j}_l}}$ by $G_1|S_1 \leqslant G'|S' \leqslant H^V_{I_l}$ and $S^c_i \in G^*_{\mathcal{I}_{\mathbf{j}_l}}$. Since S_2 is separable in $G^{\star}_{I_l}$ by $G'|S' \leqslant H^V_{I_l}$, then $S^c_i \in G_2|S_2$ and S^c_i is not S_2.

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$$
\tau^{*}_{\mathbf{I}_{\mathbf{j}_l}}
\left\{
\begin{array}{c}
\vdots \qquad\qquad \vdots \qquad\qquad \cdots \qquad\qquad \vdots \\
G_{b_{l1}}|S^{c}_{j_{l1}} \quad G_{b_{l2}}|S^{c}_{j_{l2}} \quad \cdots \quad G_{b_{lu}}|S^{c}_{j_{lu}} \\
\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
\dfrac{\{G_{b_{lk}}\}^{u}_{k=1}\,|\,\langle G_1|S_1\rangle_{\mathcal{I}_{\mathbf{j}_l}} \qquad\qquad G_2|S_2}{H_2 \equiv \{G_{b_{lk}}\}^{u}_{k=1}\,|\,\langle G_1\rangle_{\mathcal{I}_{\mathbf{j}_l}}\,|G_2|H''}(II) \\
\vdots \\
G^{*}_{\mathbf{I}_{\mathbf{j}_l}} = \{G_{b_{lk}}\}^{u}_{k=1}|G^{*}_{\mathcal{I}_{\mathbf{j}_l}} \\
\vdots \\
G^{\star}_{I_l}
\end{array}
\right.
$$

Figure 3 A fragment of $\tau^{\star}_{I_l}$

Property (B) The set of splitting sequents of $[S^c_i]_{G^{\star}_{I_l}}$ is equal to that of $[S^c_i]_{G_2|S_2}$.

Proof. Let $\dfrac{G'_1|S'_1 \quad G'_2|S'_2}{H'_1 \equiv G'_1|G'_2|H'''}(II) \in \tau^*$, $G'_1|S'_1 \leqslant H_1$ and $S'_1 \in \langle G'_1|S'_1\rangle_{\mathcal{I}_{\mathbf{j}_l}}$. Then S'_1 and S'_2 are separable in $G^{\star}_{I_l}$. Thus $G^{\star(J)}_{H'_1:G'_2}|\widehat{S'_2} \subseteq G^{\star}_{I_l}$ is closed. Hence $G^{\star(J)}_{H_1:G_2}|\widehat{S}_2 - \bigcup_{G'_2|S'_2} G^{\star(J)}_{H'_1:G'_2}|\widehat{S'_2}$ is closed, where $G'_2|S'_2$ in $\bigcup_{G'_2|S'_2}$ runs over all $II \in \tau^*$ above such that $G^{\star(J)}_{H'_1:G'_2}|\widehat{S'_2} \subseteq G^{\star(J)}_{H_1:G_2}|\widehat{S}_2$. Therefore $v(G^{\star(J)}_{H_1:G_2}|\widehat{S}_2 - \bigcup_{G'_2|S'_2} G^{\star(J)}_{H'_1:G'_2}|\widehat{S'_2}) = v(G_2|S_2)$, $\{S^c_j : S^c_j \in G_2|S_2, H^c_j \geqslant G_2|S_2\} = \{S^c_j : S^c_j \in G^{\star(J)}_{H_1:G_2}|\widehat{S}_2 - \bigcup_{G'_2|S'_2} G^{\star(J)}_{H'_1:G'_2}|\widehat{S'_2}\}$ and $[S^c_i]_{G^{\star}_{I_l}} \subseteq G^{\star(J)}_{H_1:G_2}|\widehat{S}_2 - \bigcup_{G'_2|S'_2} G^{\star(J)}_{H'_1:G'_2}|\widehat{S'_2}$. Then the set of splitting sequents of $[S^c_i]_{G^{\star}_{I_l}}$ is equal to that of $[S^c_i]_{G_2|S_2}$ since each splitting sequent $S''' \in [S^c_i]_{G^{\star}_{I_l}}$ is a (pEC)-sequent by $|v_l(S''')| + |v_r(S''')| \geqslant 2$ and $S''' \in_c G|G^*$. This completes the proof of Property (B). □

We therefore assume that, without loss of generality, S^c_i is in the form $\Gamma, p_k, \Delta \Rightarrow p_k$ by Property (B), Lemma 3.16 and the observation that each derivation-splicing operation is local. There are two cases to be considered in the following.

Case 1 $S_1 \notin \langle G_1|S_1\rangle_{G_b|S^c_j}$ for all $\tau^*_{G_b|S^c_j} \in \tau^{\star(J)}_{H^V_I:G''}$, $G_1|S_1 \leqslant H^c_j \leqslant H^V_I$. Then $G^{\star(J)}_{H_1:G_2} \cap G^{\star(J)}_{H^V_I:G''} = \varnothing$. We assume that, without loss of generality, $\langle G_2|S_2\rangle^-_k = G'_2|\Gamma \Rightarrow t$, $\langle G_2|S_2\rangle^+_k = G''_2|S_2|\Delta \Rightarrow t$. Then $\left\langle G^{\star}_{I_l}\right\rangle^-_k = G^{\star(J)}_{H_2:G'_2}|\Gamma \Rightarrow t$ since $S = \Gamma, p_k, \Delta \Rightarrow p_k$ isn't a focus sequent at all nodes from $G_2|S_2$ to $G^{\star}_{I_l}$ in $\tau^{\star}_{I_l}$ and, $H^c_j \leqslant H_1$ or $H^c_j||G_1|S_1$ for all $S^c_j \in G'_2$ by Lemma 6.7 in [6]. Thus $\left\langle G^{\star}_{I_l}\right\rangle^-_k \backslash \Gamma \Rightarrow t \subseteq G^{\star(J)}_{H_2:G_2}$. Therefore $\left\{G^{\star(J)}_{H^V_I:G''}|\widehat{S''}\right\}_1 | \left\{G^{\star(J)}_{H^V_I:G''}|\widehat{S''}\right\}_2 \subseteq \left\langle G^{\star}_{I_l}\right\rangle^+_k$ because $[S]_{G^{\star}_{I_l}} \subseteq G^{\star(J)}_{H_2:G_2}|\widehat{S}_2$, $G^{\star(J)}_{H_2:G_2}|\widehat{S}_2 \cap (\left\{G^{\star(J)}_{H^V_I:G''}|\widehat{S''}\right\}_1 | \left\{G^{\star(J)}_{H^V_I:G''}|\widehat{S''}\right\}_2) = \varnothing$ and $\left\langle G^{\star}_{I_l}\right\rangle^-_k \backslash \{\Gamma \Rightarrow t\} | \left\langle G^{\star}_{I_l}\right\rangle^+_k \backslash \{\Delta \Rightarrow t\} | \Gamma, p_k, \Delta \Rightarrow p_k = G^{\star}_{I_l}$. This shows that any splitting unit $[S]_{G^{\star}_{I_l}}$ outside $G^{\star(J)}_{H^V_I:G''}|\widehat{S''}$ in $G^{\star}_{\mathbf{I}_l}$ doesn't take two copies of $G^{\star(J)}_{H^V_I:G''}|\widehat{S''}$ apart, i.e., the case of $\left\{G^{\star(J)}_{H^V_I:G''}|\widehat{S''}\right\}_1 \subseteq \left\langle G^{\star}_{I_l}\right\rangle^-_k$ and $\left\{G^{\star(J)}_{H^V_I:G''}|\widehat{S''}\right\}_2 \subseteq \left\langle G^{\star}_{I_l}\right\rangle^+_k$ doesn't happen.

Case 2 $S_1 \in \langle G_1|S_1\rangle_{G_b|S^c_j}$ for some $\tau^*_{G_b|S^c_j} \in \tau^{\star(J)}_{H^V_I:G''}$, $G_1|S_1 \leqslant H^c_j \leqslant H^V_I$. Then $G_b|\langle G_1\rangle_{S^c_j}|G_2|H'' \in \tau^*_{G_b|S^c_j}$. Thus $G^{\star(J)}_{H_1:G_2}|\widehat{S}_2 \subseteq G^{\star(J)}_{H^V_I:G''}|\widehat{S''}$. Hence $[S^c_i]_{G^{\star}_{I_l}} \subseteq G^{\star(J)}_{H^V_I:G''}|\widehat{S''}$. The case

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of $S_i^c \in G''$ is tackled with the same procedure as the following. Let $[S_i^c]_{G_{I_l}^{\star}} \subseteq \left\{G_{H_I^V:G''}^{\star(J)}|\widehat{S''}\right\}_1$. Then there exists a copy of $[S]_{G_{I_l}^{\star}}$ in $\left\{G_{H_I^V:G''}^{\star(J)}|\widehat{S''}\right\}_2$ and let $\Gamma, p_{k'}, \Delta \Rightarrow p_{k'}$ be its splitting sequent. We put two splitting units into $\{\}_k$ and $\{\}_{k'}$ in order to distinguish them. Then $\{[S]_{G_{I_l}^{\star}}\}_k \subseteq \left\{G_{H_I^V:G''}^{\star(J)}|\widehat{S''}\right\}_1$ and $\{[S]_{G_{I_l}^{\star}}\}_{k'} \subseteq \left\{G_{H_I^V:G''}^{\star(J)}|\widehat{S''}\right\}_2$. We assume that, without loss of generality, $\langle G_2|S_2\rangle_k^- = G_2'|\Gamma \Rightarrow t$, $\langle G_2|S_2\rangle_k^+ = G_2''|S_2|\Delta \Rightarrow t$. Then $\langle G_{I_l}^{\star}\rangle_k^- \backslash\{\Gamma \Rightarrow t\} \subseteq \left\{G_{H_I^V:G''}^{\star(J)}|\widehat{S''}\right\}_1$. Thus $\{[S]_{G_{I_l}^{\star}}\}_{k'} \subseteq \left\{G_{H_I^V:G''}^{\star(J)}|\widehat{S''}\right\}_2 \subseteq \langle G_{I_l}^{\star}\rangle_k^+$ by $\langle G_{I_l}^{\star}\rangle_k^- \backslash\{\Gamma \Rightarrow t\} \cup \langle G_{I_l}^{\star}\rangle_k^+ \backslash\{\Delta \Rightarrow t\} = G_{I_l}^{\star}\backslash\Gamma, p_k, \Delta \Rightarrow p_k$. Then $\left\langle\langle G_{I_l}^{\star}\rangle_k^+\right\rangle_{k'}^- = \langle G_{I_l}^{\star}\rangle_{k'}^-, \{\Delta \Rightarrow t\}_k|\{\Delta \Rightarrow t\}_{k'} \subseteq \left\langle\langle G_{I_l}^{\star}\rangle_k^+\right\rangle_{k'}^+$ where, we put two copies of $\Delta \Rightarrow t$ into $\{\}_k$ and $\{\}_{k'}$ in order to distinguish them. Then $\Gamma \Rightarrow t \in \langle G_{I_l}^{\star}\rangle_{k'}^-$, $\vdash_{\mathbf{GL}} \langle G_{I_l}^{\star}\rangle_k^-$, $\vdash_{\mathbf{GL}} \langle G_{I_l}^{\star}\rangle_{k'}^-$ and $\langle G_{I_l}^{\star}\rangle_{k'}^-$ is a copy of $\langle G_{I_l}^{\star}\rangle_k^-$. Then $\mathcal{D}(\langle G_{I_l}^{\star}\rangle_k^-) = \mathcal{D}(\langle G_{I_l}^{\star}\rangle_{k'}^-) \subseteq \mathcal{D}(G_{I_l}^{\star})$ could be cut off one of them because they are two same sets of hypersequents in $\mathcal{D}(G_{I_l}^{\star})$. Meanwhile, two copies of $\Delta \Rightarrow t$ in $\left\langle\langle G_{I_l}^{\star}\rangle_k^+\right\rangle_{k'}^+$ can't be taken apart by any splitting unit outside $G_{H_I^V:G''}^{\star(J)}|\widehat{S''}$ in $G_{I_l}^{\star}$ by the reason as shown in Case 1 and thus could be contracted into one by (EC) in $\mathcal{D}(G_{I_l}^{\star})$. Therefore two copies $\left\{G_{H_I^V:G''}^{\star(J)}|\widehat{S''}\right\}_1$ and $\left\{G_{H_I^V:G''}^{\star(J)}|\widehat{S''}\right\}_2$ of $G_{H_I^V:G''}^{\star(J)}|\widehat{S''}$ can be contracted into one in $G_{I_l}^{\star}$ by $\langle EC_{\Omega}^{*}\rangle$. This completes the proof of Property (A). □

With Property (A), all manipulations in the old main algorithm in [6] work well. This completes the construction of $\tau_I^{\star}$ and the proof of Theorem 4.1. □

Theorem 4.8. *The standard completeness holds for* **HpsUL***.

Proof. Let $\overset{i}{\longleftrightarrow}$ denote the i-th logical link of iff in the following. $\models_{\mathcal{K}} A$ means that $v(A) \geqslant t$ for every algebra $\mathcal{A}$ in $\mathcal{K}$ and valuation v on $\mathcal{A}$. Let **psUL***, LIN(**psUL***), **psUL**^{*D} and $[0,1]_{\mathbf{psUL}^*}$ denote the classes of all **psUL***-algebras, **psUL***-chain, dense **psUL***-chain and s-tandard **psUL***-algebras (i.e., their lattice reducts are $[0,1]$), respectively. We have an inference sequence, as shown in Figure 4.

$$\begin{array}{lcr}
\vdash_{\mathbf{HpsUL}^*} A \overset{1^\circ}{\longleftrightarrow} \vdash_{\mathbf{GpsUL}^*} \Rightarrow A \overset{2^\circ}{\longleftrightarrow} \vdash_{\mathbf{GpsUL}^{*D}} \Rightarrow A \overset{3^\circ}{\longleftrightarrow} \models_{\mathbf{psUL}^{*D}} A \\
\updownarrow 1 & & \updownarrow 4^\circ \\
\models_{\mathbf{psUL}^*} A \overset{2}{\longleftrightarrow} \models_{\mathrm{LIN}(\mathbf{psUL}^*)} A \overset{3}{\longleftrightarrow} \models_{\mathbf{psUL}^{*D}} A \overset{4}{\longleftrightarrow} \models_{[0,1]_{\mathbf{psUL}^*}} A
\end{array}$$

Figure 4 Two ways to prove standard completeness

Links from 1 to 4 show Jenei and Montagna's algebraic method to prove standard completeness and currently, it seems hopeless to built up the link 3, see [7~10]. Links from 1° to 4° show Metcalfe and Montagna's proof-theoretical method. Density elimination is at Link 2° in Figure 4 and other links are proved by standard procedures with minor revisions and omitted, see [1, 4, 11, 12]. □

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