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On Sliced Spaces: Global Hyperbolicity revisited

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Abstract

We give a topological condition for a generic sliced space to be globally hyperbolic, without any hypothesis on the lapse function, shift function and spatial metric.

1 Preliminaries.

We begin with the definition of a *sliced space*, that one can read in [3], as a continuation of a study in [1] and [2] on systems of Einstein equations.

Let $V = M \times I$, where M is an n-dimensional smooth manifold and I is an interval of the real line, $\mathbb{R}$. We equip V with a $n+1$-dimensional Lorentz metric g, which splits in the following way:

$$g = -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j,$$

where $\theta^0 = dt$, $\theta^i = dx^i + \beta^i dt$, $N = N(t, x^i)$ is the *lapse function*, $\beta^i(t, x^j)$ is the *shift function* and $M_t = M \times \{t\}$, spatial slices of V, are spacelike submanifolds equipped with the time-dependent spatial metric $g_t = g_{ij}dx^i dx^j$. Such a product space V is called a *sliced space*.

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Intuitively, the lapse and shift functions indicate how to relate coordinates between two slices M_t and $M_{t'}$; the lapse measuring time and the shift measuring changes in spatial coordinates.

For simplicity, we will consider $I = \mathbb{R}$.

The author in [3] considered sliced spaces with uniformly bounded lapse, shift and spatial metric; by this hypothesis, it is ensured that parametre t measures up to a positive factor bounded (below and above) the time along the normals to spacelike slices M_t, the g_t norm of the shift vector β is uniformly bounded by a number and the time-dependent metric $g_{ij}dx^i dx^j$ is uniformly bounded (below and above) for all $t \in I(= \mathbb{R})$, respectively.

Given the above hypothesis, in the same article the following theorem is proved.

Theorem 1.1 (Cotsakis)**.** *Let (V, g) be a sliced space with uniformly bounded lapse N, shift β and spatial metric g_t. Then, the following are equivalent:*

1. *(M_0, γ) a complete Riemannian manifold.*

2. *The spacetime (V, g) is globally hyperbolic.*

In this article we review global hyperbolicity of sliced spaces, in terms of the product topology defined on the space $M \times \mathbb{R}$, for some finite dimensional smooth manifold M.

2 Strong Causality of Sliced Spaces.

Let $V = (M \times \mathbb{R}, g)$ be a sliced space. Consider the product topology T_P, on V. Since M is finite-dimensional, a base for T_P consists of all sets of the form $A \times B$, where $A \in T_M$ and $B \in T_{\mathbb{R}}$. Here T_M denotes the natural topology of the manifold M where, for an appropriate Riemann metric h, it has a base consisting of open balls $B^h_\epsilon(x)$ and $T_{\mathbb{R}}$ is the usual topology on the real line, with a base consisting of open intervals (a, b). For trivial topological reasons, we can restrict our discussion on T_P to basic-open sets $B^h_\epsilon(x) \times (a, b)$, which can be intuitively called as "open cylinders" in V.

We remind the the Alexandrov topology T_A (see [4]) has a base consisting of open sets of the form $< x, y >= I^+(x) \cap I^-(y)$, where $I^+(x) = \{z \in V : x \ll z\}$ and $I^-(y) = \{z \in V : z \ll y\}$, where $\ll$ is the *chronological order* defined as $x \ll y$ iff there exists a future

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oriented timelike curve, joining x with y. By $J^+(x)$ one denotes the topological closure of $I^+(x)$ and by $J^-(y)$ that one of $I^-(y)$.

We use the definition of global hyperbolicity from [4].

Definition 2.1. *A spacetime is globally hyperbolic, iff it is strongly causal and the "causal diamonds" $J^+(x) \cap J^-(y)$ are compact.*

We prove the following theorem.

Theorem 2.1. *Let (V, g) be a Hausdorff sliced space. Then, the following are equivalent.*

1. *V is strongly causal.*
2. *$T_A \equiv T_P$.*
3. *T_A is Hausdorff.*

Proof. 2. implies 3. is obvious and that 3. implies 1. can be found in [4].

For 1. implies 2. we consider two events $X, Y \in V$, such that $X \neq Y$; we note that each $X \in V$ has two coordinates, say (x_1, x_2), where $x_1 \in M$ and $x_2 \in \mathbb{R}$. Obviously, $X \in M_x = M \times \{x\}$ and $Y \in M_y = M \times \{y\}$. Then, $< X, Y >= I^+(X) \cap I^-(Y) \in T_A$. Let also $A \in M_a = M \times \{a\}$, where $a < x$ ($<$ is the natural order on $\mathbb{R}$) and $B \in M_b = M \times \{b\}$, where $y < b$. Consider some $\epsilon > 0$, such that $B^h_\epsilon(A) \in M$. Obviously, $B^h_\epsilon(A) \times (a, b) \in T_P$ and, for $\epsilon > 0$ sufficiently large enough, $< X, Y >\subset B^h_\epsilon(A) \times (a, b)$. Thus, $< X, Y >\in T_P$.

For 2. implies 1. we consider $\epsilon > 0$, such that $B^h_\epsilon(A) \in T_M$, so that $B^h_\epsilon(A) \times (a, b) = B \in T_P$. We let strong causality hold at an event P and consider $P \in B \in T_P$. We show that there exists $< X, Y >\in T_A$, such that $P \in< X, Y >\subset B$. Now, consider a simple region R in $< X, Y >$ which contains P and $P \in Q$, where Q is a causally convex-open subset of R. Thus, we have $U, V \in Q$, such that $P \in< U, V >\subset Q$. Finally, $P \in< U, V >\subset Q \subset B$ and this completes the proof. □

3 Global Hyperbolicity of Sliced Spaces, Revisited.

For the following theorem, we use Nash's theorem, that finite-dimensional manifolds satisfy the Heine-Borel theorem (see [5]).

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Theorem 3.1. *Let (V, g) be a Hausdorff sliced space, where $V = M \times R$, M is an n-dimensional manifold and g the $n+1$ Lorentz metric in V. Then, (V, g) is globally hyperbolic, iff $T_P = T_A$, in V.*

Proof. Given the proof of Theorem 2.1, strong causality in V holds iff $T_P = T_A$ and given Nash's theorem, the closure of $B^h_\epsilon(x) \times (a, b)$ will be compact. □

We note that neither in Theorem 2.1 nor in Theorem 3.1 we made any hypothesis on the lapse function, shift function or on the spatial metric.

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