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Article

New conformable fractional integral inequalities of Hermite-Hadamard type for convex functions

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Abstract: In this work, we established new inequalities of Hermite-Hadamard type for convex functions via conformable fractional integrals. Through the conformable fractional integral inequalities, we found out some new inequalities of Hermite-Hadamard type for convex functions in a form of classical integrals.

Keywords: Convex function; Integral inequalities; Hermite-Hadamard inequality; Conformable fractional integrals

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1. Introduction

A function $h : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on the interval $\mathcal{I}$, if the inequality

$$h(\zeta x + (1-\zeta)y) \leq \zeta h(x) + (1-\zeta)h(y) \tag{1.1}$$

holds for all $x, y \in \mathcal{I}$ and $\zeta \in [0,1]$. We say that h is concave if $-h$ is convex.

For convex functions, many equalities or inequalities have been established by many authors; for example Ostrowski type inequality [19], Hardy type inequality [20], Olsen type inequality [21] and Gagliardo-Nirenberg type inequality [22] but the most common and significant inequality is the Hermite-Hadamard type inequality [12,17], which is defined as:

$$h\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v h(x)dx \leq \frac{h(u)+h(v)}{2}, \tag{1.2}$$

where the function $h : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is assumed to be a convex function with $u < v$ and $u, v \in \mathcal{I}$.

A number of mathematicians in the field of applied and pure mathematics have dedicated their efforts to extend, generalize, counterpart and refine the Hermite–Hadamard inequality (1.2) for different classes of convex functions and mappings. For more recent results obtained on inequality (1.2); we refer the reader to [10–14].

Definition 1. [8] Suppose that $h \in L([u,v])$. The left and right Riemann-Liouville fractional integrals $J_{u+}^{\alpha}h$ and $J_{v-}^{\alpha}h$ of order $\alpha > 0$ are defined by

$$J_{u+}^{\alpha}h(x) = \frac{1}{\Gamma(\alpha)}\int_u^x (x-\zeta)^{\alpha-1}h(\zeta)d\zeta, \quad x > u,$$

and

$$J_{v-}^{\alpha}h(x) = \frac{1}{\Gamma(\alpha)}\int_x^v (t-x)^{\alpha-1}h(\zeta)d\zeta, \quad x < v,$$

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respectively, where $\Gamma(\alpha)$ is the standard gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-\zeta}\zeta^{\alpha-1}d\zeta$ and $J^0_{u^+}h(x) = J^0_{v^-}h(x) = h(x)$.

As for classical integrals, many Hermite–Hadamard type inequalities have been established for the Riemann-Liouville fractional integrals; for more details and interesting applications see [15–17].

Now, we give definition of conformable fractional derivative with its important properties which are useful in order to obtain our main results (see [1–7,18]). In our study, we use the Katugampola derivative formulation of conformable derivative which is explained in the following definition:

Definition 2 ([7])**.** Given a function $h : [0,\infty) \to \mathbb{R}$. Then the conformable fractional derivative of h of order α of h at ζ is defined by

$$D_\alpha(h)(\zeta) = \lim_{\epsilon\to 0}\frac{h\left(\zeta+\epsilon\zeta^{1-\alpha}\right)-h(\zeta)}{\epsilon}, \quad \alpha\in(0,1), \zeta>0. \tag{1.3}$$

If h is α-differentiable in some $(0,\alpha), \alpha > 0$, $\lim_{\zeta\to 0^+} h^{(\alpha)}(\zeta)$ exist, then define

$$h^{(\alpha)}(0) = \lim_{\zeta\to 0^+} h^{(\alpha)}(\zeta).$$

Also, note that if h is differentiable, then

$$D_\alpha(h)(\zeta) = \zeta^{1-\alpha}h'(\zeta), \quad \text{where } h'(\zeta) = \lim_{\epsilon\to 0}\frac{h(\zeta+\epsilon)-h(\zeta)}{\epsilon}. \tag{1.4}$$

We can write $h^{(\alpha)}(\zeta)$ for $D_\alpha(h)(\zeta)$ or $\frac{d_\alpha}{d_\alpha\zeta}(h(\zeta))$ to denote the conformable fractional derivatives of h of order α at ζ. In addition, if the conformable fractional derivative of h of order α exists, then we simply say h is α-differentiable.

Theorem 3 ([7])**.** *Let $\alpha\in(0,1]$ and h, g be α-differentiable at a point $\zeta > 0$. Then*

1. $D_\alpha(uf+vg) = uD_\alpha(h)+vD_\alpha(g)$ *for all* $u,v\in\mathbb{R}$,
2. $D_\alpha(hg) = hD_\alpha(g)+gD_\alpha(h)$,
3. $D_\alpha\left(\frac{h}{g}\right) = \frac{hD_\alpha(g)-gD_\alpha(h)}{g^2}$,
4. $D_\alpha(c) = 0$ *for all constant function* $h(\zeta) = c$,
5. $D_\alpha(1) = 0$,
6. $D_\alpha\left(\frac{1}{\alpha}\zeta^\alpha\right) = 1$.

Now, we give the definition of conformable fractional integral:

Definition 4 ([3])**.** Let $\alpha\in(0,1]$ and $0\le u\le v$. We say that a function $h : [u,v]\to\mathbb{R}$ is α-fractional integrable on $[u,v]$, if the integral

$$\int_u^v h(\zeta)d_\alpha\zeta = \int_u^v h(\zeta)\zeta^{\alpha-1}d\zeta \tag{1.5}$$

exists and is finite.

Remark 5.
(u) All α-fractional integrable functions on $[u,v]$ is indicated by $L^1_\alpha([u,v])$.
(v) For the usual Riemann improper integral and $\alpha\in(0,1]$, we have

$$I^u_\alpha(h)(\zeta) = I^a_1\left(\zeta^{\alpha-1}h\right) = \int_u^\zeta x^{\alpha-1}h(x)dx. \tag{1.6}$$

The aim of our article is to establish some new inequalities connected with the Hermite–Hadamard inequalities (1.2) via conformable fractional integral.

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2. Main Results

Our main results depend on the following equality:

Lemma 6. *Let $h : [u,v] \subseteq \mathbb{R} \to \mathbb{R}$ be an α-fractional differentiable mapping on (u,v) with $0 \leq u < v$. If $D_\alpha(h) \in L^1_\alpha([u,v])$, then the following identity for conformable fractional integral holds:*

$$\Phi_\alpha(u,v) = \sum_{i=1}^{4} \delta_i, \tag{2.1}$$

where

$$\begin{aligned}
\delta_1 &= \frac{v-u}{4}\int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^{2\alpha-1} - \left(\frac{3u+v}{4}\right)^{\alpha}\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^{\alpha-1}\right] \\
&\quad \times D_\alpha(h)\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right) d_\alpha\zeta, \\
\delta_2 &= \frac{v-u}{4}\int_0^1 \left[\left(\frac{3u+v}{4}\zeta + (1-\zeta)\frac{u+v}{2}\right)^{2\alpha-1} - \left(\frac{3u+v}{4}\right)^{\alpha}\left(\frac{3u+v}{4}\zeta + (1-\zeta)\frac{u+v}{2}\right)^{\alpha-1}\right] \\
&\quad \times D_\alpha(h)\left(\frac{3u+v}{4}\zeta + (1-\zeta)\frac{u+v}{2}\right) d_\alpha\zeta, \\
\delta_3 &= \frac{v-u}{4}\int_0^1 \left[\left(\frac{u+v}{2}\zeta + (1-\zeta)\frac{u+3v}{4}\right)^{2\alpha-1} - \left(\frac{u+3v}{4}\right)^{\alpha}\left(\frac{u+v}{2}\zeta + (1-\zeta)\frac{u+3v}{4}\right)^{\alpha-1}\right] \\
&\quad \times D_\alpha(h)\left(\frac{u+v}{2}\zeta + (1-\zeta)\frac{u+3v}{4}\right) d_\alpha\zeta, \\
\delta_4 &= \frac{v-u}{4}\int_0^1 \left[\left(\frac{u+3v}{4}\zeta + (1-\zeta)v\right)^{2\alpha-1} - \left(\frac{u+3v}{4}\right)^{\alpha}\left(\frac{u+3v}{4}\zeta + (1-\zeta)v\right)^{\alpha-1}\right] \\
&\quad \times D_\alpha(h)\left(\frac{u+3v}{4}\zeta + (1-\zeta)v\right) d_\alpha\zeta,
\end{aligned}$$

and

$$\begin{aligned}
\Phi_\alpha(u,v) &= \left[\left(\frac{3u+v}{4}\right)^{\alpha} - u^\alpha\right] h(u) + \left[v^\alpha - \left(\frac{u+3v}{4}\right)^{\alpha}\right] h(v) \\
&\quad + \left[\left(\frac{u+3v}{4}\right)^{\alpha} - \left(\frac{3u+v}{4}\right)^{\alpha}\right] h\left(\frac{u+v}{2}\right) - \alpha\int_u^v h(x)\, d_\alpha x.
\end{aligned}$$

Proof. By using definition of conformable fractional derivative (1.4), we have

$$\delta_1 = \frac{v-u}{4}\int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^{\alpha} - \left(\frac{3u+v}{4}\right)^{\alpha}\right] h'\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right) d\zeta.$$

On integrating by parts one can have

$$\delta_1 = \left[\left(\frac{3u+v}{4}\right)^{\alpha} - u^\alpha\right] h(u) - \alpha\int_0^1 \left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^{\alpha-1} h\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right) d\zeta.$$

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Using the change of the variable $x := u\zeta + (1-\zeta)\frac{3u+v}{4}, \zeta \in [0,1]$ and definition of conformable fractional integral (1.5), we obtain

$$\begin{aligned}\delta_1 &= \left[\left(\frac{3u+v}{4}\right)^{\alpha} - u^{\alpha}\right] h(u) - \alpha \int_u^{\frac{3u+v}{4}} x^{\alpha-1} h(x)dx \\ &= \left[\left(\frac{3u+v}{4}\right)^{\alpha} - u^{\alpha}\right] h(u) - \alpha \int_u^{\frac{3u+v}{4}} h(x)d_{\alpha}x.\end{aligned}$$

Similarly, we get

$$\delta_2 = \left[\left(\frac{u+v}{2}\right)^{\alpha} - \left(\frac{3u+v}{4}\right)^{\alpha}\right] h\left(\frac{u+v}{2}\right) - \alpha \int_{\frac{3u+v}{4}}^{\frac{u+v}{2}} h(x)d_{\alpha}x,$$

$$\delta_3 = \left[\left(\frac{u+3v}{4}\right)^{\alpha} - \left(\frac{u+v}{2}\right)^{\alpha}\right] h\left(\frac{u+v}{2}\right) - \alpha \int_{\frac{u+v}{2}}^{\frac{u+3v}{4}} h(x)d_{\alpha}x,$$

and

$$\delta_4 = \left[v^{\alpha} - \left(\frac{u+3v}{4}\right)^{\alpha}\right] h(v) - \alpha \int_{\frac{u+3v}{4}}^{v} h(x)d_{\alpha}x.$$

Adding $\delta_1, \delta_2, \delta_3$ and δ_4 together, we obtain the desired identity (2.1). This completes the proof of Lemma 6. □

Remark 7. With the similar assumptions of Lemma 6, if $\alpha = 1$, then identity (2.1) reduces to the following identity:

$$\begin{aligned}\Phi_1(u,v) &= \int_0^1 (1-\zeta)h'\left(\frac{3u+v}{4}\zeta + (1-\zeta)\frac{u+v}{2}\right)d\zeta - \int_0^1 t\, h'\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)d\zeta \\ &+ \int_0^1 (1-\zeta)h'\left(\frac{u+3v}{4}\zeta + (1-\zeta)v\right)d\zeta - \int_0^1 t\, h'\left(\frac{u+v}{2}\zeta + (1-\zeta)\frac{u+3v}{4}\right)d\zeta, \qquad (2.2)\end{aligned}$$

where

$$\Phi_1(u,v) = \frac{1}{2}\left[\frac{h(u)+h(v)}{2} + h\left(\frac{u+v}{2}\right)\right] - \frac{4}{v-u}\int_u^v h(x)dx$$

which is obtained by Shi et al. [15].

Theorem 8. *Let $h : [u,v] \subseteq \mathbb{R} \to \mathbb{R}$ be an α-fractional differentiable mapping on (u,v) with $0 \le u < v$. If $D_{\alpha}(h) \in L^1_{\alpha}([u,v])$ and $|h'|$ is convex on $[u,v]$, then the following inequality for conformable fractional integral holds:*

$$\begin{aligned}|\Phi_{\alpha}(u,v)| \le \frac{v-u}{4}\Bigg[&d_1(\alpha)|h'(u)| + d_2(\alpha)|h'(v)| \\ &+ d_3(\alpha)\left|h'\left(\frac{3u+v}{4}\right)\right| + d_4(\alpha)\left|h'\left(\frac{u+v}{2}\right)\right| + d_5(\alpha)\left|h'\left(\frac{u+3v}{4}\right)\right|\Bigg], \qquad (2.3)\end{aligned}$$

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where

$$d_1(\alpha) = \frac{1}{12}\left[4u^\alpha + \left(\frac{3u+v}{4}\right)u^{\alpha-1} + \left(\frac{3u+v}{4}\right)^{\alpha-1}a - 5\left(\frac{3u+v}{4}\right)^\alpha\right],$$

$$d_2(\alpha) = \frac{1}{12}\left[3v^\alpha + \left(\frac{3u+v}{4}\right)v^{\alpha-1} + \left(\frac{3u+v}{4}\right)^{\alpha-1}b - 5\left(\frac{3u+v}{4}\right)^\alpha\right],$$

$$d_3(\alpha) = \frac{1}{12}\Bigg[u^\alpha + \left(\frac{3u+v}{4}\right)u^{\alpha-1} + \left(\frac{3u+v}{4}\right)^{\alpha-1}a + \left(\frac{u+v}{2}\right)\left(\frac{3u+v}{4}\right)^{\alpha-1}$$
$$+ \left(\frac{u+v}{2}\right)^{\alpha-1}\left(\frac{3u+v}{4}\right) + \left(\frac{u+v}{2}\right)^\alpha - 5\left(\frac{3u+v}{4}\right)^\alpha\Bigg],$$

$$d_4(\alpha) = \frac{1}{12}\Bigg[7\left(\frac{u+v}{2}\right)^\alpha + \left(\frac{u+v}{2}\right)\left(\frac{3u+v}{4}\right)^{\alpha-1} + \left(\frac{u+v}{2}\right)^{\alpha-1}\left(\frac{3u+v}{4}\right)$$
$$+ \left(\frac{u+v}{2}\right)\left(\frac{u+3v}{4}\right)^{\alpha-1} + \left(\frac{u+v}{2}\right)^{\alpha-1}\left(\frac{u+3v}{4}\right) - 5\left(\frac{3u+v}{4}\right)^\alpha - 5\left(\frac{u+3v}{4}\right)^\alpha\Bigg],$$

$$d_5(\alpha) = \frac{1}{12}\Bigg[v^\alpha + \left(\frac{u+v}{2}\right)^\alpha + \left(\frac{u+3v}{4}\right)\left(\frac{u+v}{2}\right)^{\alpha-1} + \left(\frac{u+3v}{4}\right)^{\alpha-1}\left(\frac{u+v}{2}\right)$$
$$+ b\left(\frac{u+3v}{4}\right)^{\alpha-1} + v^{\alpha-1}\left(\frac{u+3v}{4}\right) - 5\left(\frac{u+3v}{4}\right)^\alpha\Bigg].$$

Proof. Using Lemma 6 and the property (1.4), we have

$$\Phi_\alpha(u,v) = \frac{v-u}{4}\Bigg\{\int_0^1\left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right]h'\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)d\zeta$$
$$+ \int_0^1\left[\left(\frac{3u+v}{4}\zeta + (1-\zeta)\frac{u+v}{2}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right]h'\left(\frac{3u+v}{4}\zeta + (1-\zeta)\frac{u+v}{2}\right)d\zeta$$
$$+ \int_0^1\left[\left(\frac{u+v}{2}\zeta + (1-\zeta)\frac{u+3v}{4}\right)^\alpha - \left(\frac{u+3v}{4}\right)^\alpha\right]h'\left(\frac{u+v}{2}\zeta + (1-\zeta)\frac{u+3v}{4}\right)d_\alpha\zeta$$
$$+ \int_0^1\left[\left(\frac{u+3v}{4}\zeta + (1-\zeta)v\right)^\alpha - \left(\frac{u+3v}{4}\right)^\alpha\right]h'\left(\frac{u+3v}{4}\zeta + (1-\zeta)v\right)d_\alpha\zeta\Bigg\}. \tag{2.4}$$

By using the convexity of $x^{\alpha-1}$ for $x > 0, \alpha \in (0,1]$, we have

$$\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha = \left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^{\alpha-1+1} - \left(\frac{3u+v}{4}\right)^\alpha$$
$$= \left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^{\alpha-1}\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right) - \left(\frac{3u+v}{4}\right)^\alpha$$
$$\leq \left[u^{\alpha-1}\zeta + (1-\zeta)\left(\frac{3u+v}{4}\right)^{\alpha-1}\right]\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right) - \left(\frac{3u+v}{4}\right)^\alpha, \tag{2.5}$$

$$\left(\frac{3u+v}{4}\zeta + (1-\zeta)\frac{u+v}{2}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha$$
$$\leq \left[t\left(\frac{3u+v}{4}\right)^{\alpha-1} + (1-\zeta)\left(\frac{u+v}{2}\right)^{\alpha-1}\right]\left(\frac{3u+v}{4}\zeta + (1-\zeta)\frac{u+v}{2}\right) - \left(\frac{3u+v}{4}\right)^\alpha, \tag{2.6}$$

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$$\begin{aligned}
&\left(\frac{u+v}{2}\zeta+(1-\zeta)\frac{u+3v}{2}\right)^{\alpha}-\left(\frac{u+3v}{4}\right)^{\alpha}\\
&\leq\left[t\left(\frac{u+v}{2}\right)^{\alpha-1}+(1-\zeta)\left(\frac{u+3v}{4}\right)^{\alpha-1}\right]\left(\frac{u+v}{2}\zeta+(1-\zeta)\frac{u+3v}{4}\right)-\left(\frac{u+3v}{4}\right)^{\alpha}, \qquad (2.7)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{u+3v}{4}\zeta+(1-\zeta)v\right)^{\alpha}-\left(\frac{u+3v}{4}\right)^{\alpha}\\
&\leq\left[t\left(\frac{u+3v}{4}\right)^{\alpha-1}+(1-\zeta)v^{\alpha-1}\right]\left(\frac{u+3v}{4}\zeta+(1-\zeta)v\right)-\left(\frac{u+3v}{4}\right)^{\alpha}. \qquad (2.8)
\end{aligned}$$

Using (2.5), (2.6), (2.7), and (2.8) in (2.4) and using the properties of modulus, we get

$$\begin{aligned}
&|\Phi_\alpha(u,v)|\leq\frac{v-u}{4}\Bigg\{\int_0^1\left[\left(u^{\alpha-1}\zeta+(1-\zeta)\left(\frac{3u+v}{4}\right)^{\alpha-1}\right)\left(u\zeta+(1-\zeta)\frac{3u+v}{4}\right)-\left(\frac{3u+v}{4}\right)^{\alpha}\right]\\
&\times\left|h'\left(u\zeta+(1-\zeta)\frac{3u+v}{4}\right)\right|d\zeta\\
&+\int_0^1\left[\left(\left(\frac{3u+v}{4}\right)^{\alpha-1}\zeta+(1-\zeta)\left(\frac{u+v}{2}\right)^{\alpha-1}\right)\left(\frac{3u+v}{4}\zeta+(1-\zeta)\frac{u+v}{2}\right)-\left(\frac{3u+v}{4}\right)^{\alpha}\right]\\
&\times\left|h'\left(\frac{3u+v}{4}\zeta+(1-\zeta)\frac{u+v}{2}\right)\right|d\zeta\\
&+\int_0^1\left[\left(\left(\frac{u+v}{2}\right)^{\alpha-1}\zeta+(1-\zeta)\left(\frac{u+3v}{4}\right)^{\alpha-1}\right)\left(\frac{u+v}{2}\zeta+(1-\zeta)\frac{u+3v}{4}\right)-\left(\frac{u+3v}{4}\right)^{\alpha}\right]\\
&\times\left|h'\left(\frac{u+v}{2}\zeta+(1-\zeta)\frac{u+3v}{4}\right)\right|d\zeta\\
&+\int_0^1\left[\left(\left(\frac{u+3v}{4}\right)^{\alpha-1}\zeta+(1-\zeta)v^{\alpha-1}\right)\left(\frac{u+3v}{4}\zeta+(1-\zeta)v\right)-\left(\frac{u+3v}{4}\right)^{\alpha}\right]\\
&\times\left|h'\left(\frac{u+3v}{4}\zeta+(1-\zeta)v\right)\right|d\zeta\Bigg\}. \qquad (2.9)
\end{aligned}$$

Since $|h'|$ is convex on $[u,v]$ for any $\zeta\in[0,1]$ so (2.9) becomes

$$\begin{aligned}
&|\Phi_\alpha(u,v)|\leq\frac{v-u}{4}\Bigg\{\int_0^1\left[\left(u^{\alpha-1}\zeta+(1-\zeta)\left(\frac{3u+v}{4}\right)^{\alpha-1}\right)\left(u\zeta+(1-\zeta)\frac{3u+v}{4}\right)-\left(\frac{3u+v}{4}\right)^{\alpha}\right]\\
&\times\left[\zeta\left|h'(u)\right|+(1-\zeta)\left|h'\left(\frac{3u+v}{4}\right)\right|\right]d\zeta+\int_0^1\left[\left(\left(\frac{3u+v}{4}\right)^{\alpha-1}\zeta+(1-\zeta)\left(\frac{u+v}{2}\right)^{\alpha-1}\right)\right.\\
&\times\left.\left(\frac{3u+v}{4}\zeta+(1-\zeta)\frac{u+v}{2}\right)-\left(\frac{3u+v}{4}\right)^{\alpha}\right]\left[\zeta\left|h'\left(\frac{3u+v}{4}\right)\right|+(1-\zeta)\left|h'\left(\frac{u+v}{2}\right)\right|\right]d\zeta\\
&+\int_0^1\left[\left(\left(\frac{u+v}{2}\right)^{\alpha-1}\zeta+(1-\zeta)\left(\frac{u+3v}{4}\right)^{\alpha-1}\right)\left(\frac{u+v}{2}\zeta+(1-\zeta)\frac{u+3v}{4}\right)-\left(\frac{u+3v}{4}\right)^{\alpha}\right]\\
&\times\left[\zeta\left|h'\left(\frac{u+v}{2}\right)\right|+(1-\zeta)\left|h'\left(\frac{u+3v}{4}\right)\right|\right]d\zeta+\int_0^1\left[\left(\left(\frac{u+3v}{4}\right)^{\alpha-1}\zeta+(1-\zeta)v^{\alpha-1}\right)\right.\\
&\times\left.\left(\frac{u+3v}{4}\zeta+(1-\zeta)v\right)-\left(\frac{u+3v}{4}\right)^{\alpha}\right]\left[\zeta\left|h'\left(\frac{u+3v}{4}\right)\right|+(1-\zeta)\left|h'(v)\right|\right]d\zeta\Bigg\}. \qquad (2.10)
\end{aligned}$$

Simple calculation gives

$$\begin{aligned}
|\Phi_\alpha(u,v)| \leq \frac{v-u}{4}\Bigg\{ &\left(\frac{1}{12}\left[4u^\alpha + \left(\frac{3u+v}{4}\right)u^{\alpha-1} + \left(\frac{3u+v}{4}\right)^{\alpha-1} a - 5\left(\frac{3u+v}{4}\right)^\alpha\right]\right)|h'(u)| \\
&+ \left(\frac{1}{12}\left[3v^\alpha + \left(\frac{3u+v}{4}\right)v^{\alpha-1} + \left(\frac{3u+v}{4}\right)^{\alpha-1} b - 5\left(\frac{3u+v}{4}\right)^\alpha\right]\right)|h'(v)| \\
&+ \left(\frac{1}{12}\left[u^\alpha + \left(\frac{3u+v}{4}\right)u^{\alpha-1} + \left(\frac{3u+v}{4}\right)^{\alpha-1} a + \left(\frac{u+v}{2}\right)\left(\frac{3u+v}{4}\right)^{\alpha-1}\right.\right. \\
&\left.\left. + \left(\frac{u+v}{2}\right)^{\alpha-1}\left(\frac{3u+v}{4}\right) + \left(\frac{u+v}{2}\right)^\alpha - 5\left(\frac{3u+v}{4}\right)^\alpha\right]\right)\left|h'\left(\frac{3u+v}{4}\right)\right| \\
&+ \left(\frac{1}{12}\left[7\left(\frac{u+v}{2}\right)^\alpha + \left(\frac{u+v}{2}\right)\left(\frac{3u+v}{4}\right)^{\alpha-1} + \left(\frac{u+v}{2}\right)^{\alpha-1}\left(\frac{3u+v}{4}\right)\right.\right. \\
&\left.\left. + \left(\frac{u+v}{2}\right)\left(\frac{u+3v}{4}\right)^{\alpha-1} + \left(\frac{u+v}{2}\right)^{\alpha-1}\left(\frac{u+3v}{4}\right) - 5\left(\frac{3u+v}{4}\right)^\alpha - 5\left(\frac{u+3v}{4}\right)^\alpha\right]\right)\left|h'\left(\frac{u+v}{2}\right)\right| \\
&+ \left(\frac{1}{12}\left[v^\alpha + \left(\frac{u+v}{2}\right)^\alpha + \left(\frac{u+3v}{4}\right)\left(\frac{u+v}{2}\right)^{\alpha-1} + \left(\frac{u+3v}{4}\right)^{\alpha-1}\left(\frac{u+v}{2}\right)\right.\right. \\
&\left.\left. + b\left(\frac{u+3v}{4}\right)^{\alpha-1} + v^{\alpha-1}\left(\frac{u+3v}{4}\right) - 5\left(\frac{u+3v}{4}\right)^\alpha\right]\right)\left|h'\left(\frac{u+3v}{4}\right)\right|\Bigg\}.
\end{aligned}$$

This completes the proof of Theorem 8. $\square$

Corollary 9. *With the similar assumptions of Theorem 8, if $\alpha = 1$, then*

$$\begin{aligned}
|\Phi_1(u,v)| \leq \frac{v-u}{4}\Bigg[&\frac{2u-v}{12}|h'(u)| + \frac{v-u}{12}|h'(v)| \\
&+ \frac{3u+v}{48}\left|h'\left(\frac{3u+v}{4}\right)\right| + \frac{u+v}{24}\left|h'\left(\frac{u+v}{2}\right)\right| + \frac{u+3v}{48}\left|h'\left(\frac{u+3v}{4}\right)\right|\Bigg].
\end{aligned}$$

Theorem 10. *Let $h : [u,v] \subseteq \mathbb{R} \to \mathbb{R}$ be an α-fractional differentiable mapping on (u,v) with $0 \leq u < v$. If $D_\alpha(h) \in L^1_\alpha([u,v])$ and $|h'|^q$ is convex on $[u,v]$, then the following inequality for conformable fractional integral holds:*

$$\begin{aligned}
|\Phi_\alpha(u,v)| \leq \frac{v-u}{4}\Bigg[&(A_1(\alpha))^{1-\frac{1}{q}}\left\{A_2(\alpha)|h'(u)|^q + A_3(\alpha)\left|h'\left(\frac{3u+v}{4}\right)\right|^q\right\}^{\frac{1}{q}} \\
&+ (B_1(\alpha))^{1-\frac{1}{q}}\left\{B_2(\alpha)\left|h'\left(\frac{3u+v}{4}\right)\right|^q + B_3(\alpha)\left|h'\left(\frac{u+v}{2}\right)\right|^q\right\}^{\frac{1}{q}} \\
&+ (C_1(\alpha))^{1-\frac{1}{q}}\left\{C_2(\alpha)\left|h'\left(\frac{u+v}{2}\right)\right|^q + C_3(\alpha)\left|h'\left(\frac{u+3v}{4}\right)\right|^q\right\}^{\frac{1}{q}} \\
&+ (D_1(\alpha))^{1-\frac{1}{q}}\left\{D_2(\alpha)\left|h'\left(\frac{u+3v}{4}\right)\right|^q + D_3(\alpha)|h'(v)|^q\right\}^{\frac{1}{q}}\Bigg], \qquad (2.11)
\end{aligned}$$

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where

$$
\begin{aligned}
A_1(\alpha) &= \frac{4u(3u+v)^\alpha - (4u)^{\alpha+1}}{4^\alpha(v-u)(\alpha+1)} - \frac{\alpha(3u+v)^\alpha}{4^\alpha},\\
A_2(\alpha) &= \frac{(3u+v)^{\alpha+2} - (4u)^{\alpha+1}[\alpha(v-u)+(u+v)]}{4^\alpha(v-u)^2(\alpha+1)(\alpha+2)} - \frac{(3u+v)^\alpha}{2^{2\alpha+1}},\\
A_3(\alpha) &= \frac{(4u)^{\alpha+2} - (3u+v)^{\alpha+2}(5u-v) - \alpha(3u+v)^\alpha(3u-v)(u+v)}{4^\alpha(v-u)^2(\alpha+1)(\alpha+2)} - \frac{(3u+v)^\alpha}{2^{2\alpha+1}},
\end{aligned}
$$

$$
\begin{aligned}
B_1(\alpha) &= \frac{(2(u+v))^{\alpha+2} - (3u+v)^{\alpha+1}}{4^\alpha(v-u)(\alpha+1)} - \frac{(3u+v)^\alpha}{4^\alpha},\\
B_2(\alpha) &= \frac{(2(u+v))^{\alpha+2} - (3u+v)^{\alpha+1}(u+3v) - \alpha(3u+v)^{\alpha+1}(v-u)}{4^\alpha(v-u)^2(\alpha+1)(\alpha+2)} - \frac{(3u+v)^\alpha}{2^{2\alpha+1}},\\
B_3(\alpha) &= \frac{(3u+v)^{\alpha+2} + 4(2(u+v))^{\alpha+1} + \alpha(2(u+v))^{\alpha+1}(v-u)}{4^\alpha(v-u)^2(\alpha+1)(\alpha+2)} - \frac{(3u+v)^\alpha}{2^{2\alpha+1}},
\end{aligned}
$$

$$
\begin{aligned}
C_1(\alpha) &= \frac{4(u+3v)^{\alpha+1} - (2(u+v))^{\alpha+2}}{4^\alpha(v-u)(\alpha+1)} - \frac{(u+3v)^\alpha}{4^\alpha},\\
C_2(\alpha) &= \frac{\alpha^2(v-u)^2(u+3v)^\alpha + 16b(u+v)(u+3v)^\alpha - 3\alpha\left(u^2+v^2\right)(u+3v)^\alpha}{2^{2\alpha+1}(v-u)^2(\alpha+1)(\alpha+2)}\\
&\quad - \frac{8b(2(u+v))^{\alpha+1} + 2\alpha(v-u)(2(u+v))^{\alpha+1}}{2^{2\alpha+1}(v-u)^2(\alpha+1)(\alpha+2)} - \frac{(u+3v)^\alpha}{2^{2\alpha+1}},\\
C_3(\alpha) &= \frac{(2(u+v))^{\alpha+2} + (u+3v)^{\alpha+2} + \alpha(u+3v)^{\alpha+2} - 5(u+v)(u+3v)^{\alpha+1}}{4^\alpha(v-u)^2(\alpha+1)(\alpha+2)},
\end{aligned}
$$

$$
\begin{aligned}
D_1(\alpha) &= \frac{(4v)^{\alpha+1} - (u+3v)^{\alpha+1}}{4^\alpha(\alpha+1)(v-u)} - \frac{(u+3v)^\alpha}{2^{2\alpha+1}},\\
D_2(\alpha) &= \frac{(4v)^{\alpha+2} + (u+3v)^{\alpha+1}(a-5v) + \alpha(u+3v)^{\alpha+1(v-u)}}{4^\alpha(v-u)^2(\alpha+1)(\alpha+2)},\\
D_3(\alpha) &= \frac{\alpha(4v)^{\alpha+2}(v-u) - 2(4v)^{\alpha+1}(u+v) + (u+3v)^{\alpha+2}}{4^\alpha(v-u)^2(\alpha+1)(\alpha+2)}.
\end{aligned}
$$

Proof. Using Lemma 6 and the property (1.4), we have

$$
\begin{aligned}
&|\Phi_\alpha(u,v)| \le \frac{v-u}{4}\left[\int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right]\left|h'\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)\right| d\zeta\right.\\
&+ \int_0^1 \left[\left(\frac{3u+v}{4}\zeta + (1-\zeta)\frac{u+v}{2}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right]\left|h'\left(\frac{3u+v}{4}\zeta + (1-\zeta)\frac{u+v}{2}\right)\right| d\zeta\\
&+ \int_0^1 \left[\left(\frac{u+v}{2}\zeta + (1-\zeta)\frac{u+3v}{4}\right)^\alpha - \left(\frac{u+3v}{4}\right)^\alpha\right]\left|h'\left(\frac{u+v}{2}\zeta + (1-\zeta)\frac{u+3v}{4}\right)\right| d\zeta\\
&+ \int_0^1 \left[\left(\frac{u+3v}{4}\zeta + (1-\zeta)v\right)^\alpha - \left(\frac{u+3v}{4}\right)^\alpha\right]\left|h'\left(\frac{u+3v}{4}\zeta + (1-\zeta)v\right)\right| d\zeta\\
&:= \frac{v-u}{4}(J_1+J_2+J_3+J_4). \qquad (2.12)
\end{aligned}
$$

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It follows from the power–mean inequality that

$$J_1 \le \left(\int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right] d\zeta\right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right] \left|h'\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)\right|^q d\zeta\right)^{\frac{1}{q}}.$$

Since $|h'|^q$ is convex on $[u,v]$ for any $\zeta \in [0,1]$, we obtain

$$J_1 \le \left(\int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right] d\zeta\right)^{1-\frac{1}{q}}$$
$$\times \left(|h'(u)|^q \int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right] t\, d\zeta\right.$$
$$\left. + \left|h'\left(\frac{3u+v}{4}\right)\right|^q \int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right] (1-\zeta) d\zeta\right)^{\frac{1}{q}}$$
$$= (A_1(\alpha))^{1-\frac{1}{q}} \left\{A_2(\alpha)|h'(u)|^q + A_3(\alpha)\left|h'\left(\frac{3u+v}{4}\right)\right|^q\right\}^{\frac{1}{q}},$$

where we have used the facts that

$$A_1(\alpha) := \int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right] d\zeta = \frac{4u(3u+v)^\alpha - (4u)^{\alpha+1}}{4^\alpha (v-u)(\alpha+1)} - \frac{\alpha(3u+v)^\alpha}{4^\alpha},$$

$$A_2(\alpha) := \int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right] t\, d\zeta$$
$$= \frac{(3u+v)^{\alpha+2} - (4u)^{\alpha+1}[\alpha(v-u) + (u+v)]}{4^\alpha (v-u)^2 (\alpha+1)(\alpha+2)} - \frac{(3u+v)^\alpha}{2^{2\alpha+1}},$$

and

$$A_3(\alpha) := \int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right] (1-\zeta) d\zeta$$
$$= \frac{(4u)^{\alpha+2} - (3u+v)^{\alpha+2}(5u-v) - \alpha(3u+v)^\alpha (3u-v)(u+v)}{4^\alpha (v-u)^2 (\alpha+1)(\alpha+2)} - \frac{(3u+v)^\alpha}{2^{2\alpha+1}}.$$

Analogously

$$J_2 \le (B_1(\alpha))^{1-\frac{1}{q}} \left\{B_2(\alpha)\left|h'\left(\frac{3u+v}{4}\right)\right|^q + B_3(\alpha)\left|h'\left(\frac{u+v}{2}\right)\right|^q\right\}^{\frac{1}{q}},$$
$$J_3 \le (C_1(\alpha))^{1-\frac{1}{q}} \left\{C_2(\alpha)\left|h'\left(\frac{u+v}{2}\right)\right|^q + C_3(\alpha)\left|h'\left(\frac{u+3v}{4}\right)\right|^q\right\}^{\frac{1}{q}},$$

and

$$J_4 \le (D_1(\alpha))^{1-\frac{1}{q}} \left\{D_2(\alpha)\left|h'\left(\frac{u+3v}{4}\right)\right|^q + D_3(\alpha)\,|h'(v)|^q\right\}^{\frac{1}{q}}.$$

Using J_1, J_2, J_3, and J_4 in (2.12), we obtain the desired inequality (2.11). This completes the proof of Theorem 10. □

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Corollary 11. *With the similar assumptions of Theorem 10, if $\alpha = 1$, then*

$$
\begin{aligned}
|\Phi_1(u,v)| \leq \frac{v-u}{4}\Bigg[&\left(\frac{u-v}{8}\right)^{1-\frac{1}{q}}\left\{\left(\frac{11u+v}{24}\right)|h'(u)|^q + \left(\frac{5u+v}{12}\right)\left|h'\left(\frac{3u+v}{4}\right)\right|^q\right\}^{\frac{1}{q}} \\
&+\left(\frac{v-u}{8}\right)^{1-\frac{1}{q}}\left\{\left(\frac{2u+v}{6}\right)\left|h'\left(\frac{3u+v}{4}\right)\right|^q + \left(\frac{7u+5b}{24}\right)\left|h'\left(\frac{u+v}{2}\right)\right|^q\right\}^{\frac{1}{q}} \\
&+\left(\frac{u-v}{8}\right)^{1-\frac{1}{q}}\left\{\left(\frac{u-v}{12}\right)\left|h'\left(\frac{u+v}{2}\right)\right|^q + \left(\frac{u-v}{24}\right)\left|h'\left(\frac{u+3v}{4}\right)\right|^q\right\}^{\frac{1}{q}} \\
&+\left(\frac{v-u}{8}\right)^{1-\frac{1}{q}}\left\{\left(\frac{u+5b}{12}\right)\left|h'\left(\frac{u+3v}{4}\right)\right|^q + \left(\frac{u+11b}{24}\right)|h'(v)|^q\right\}^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

Theorem 12. *Let $h : [u,v] \subseteq \mathbb{R} \to \mathbb{R}$ be an α-fractional differentiable mapping on (u,v) with $0 \leq u < v$. If $D_\alpha(h) \in L^1_\alpha([u,v])$ and $|h'|^q$ is concave on $[u,v]$, then the following inequality for conformable fractional integral holds:*

$$
\begin{aligned}
|\Phi_\alpha(u,v)| \leq \frac{v-u}{4}\Bigg[&A_1(\alpha)h'\left(\frac{P_1(\alpha)}{A_1(\alpha)}\right) \\
&+B_1(\alpha)h'\left(\frac{P_2(\alpha)}{B_1(\alpha)}\right) + C_1(\alpha)h'\left(\frac{P_3(\alpha)}{C_1(\alpha)}\right) + D_1(\alpha)h'\left(\frac{P_4(\alpha)}{D_1(\alpha)}\right)\Bigg], \qquad (2.13)
\end{aligned}
$$

where $A_1(\alpha), B_1(\alpha), C_1(\alpha)$, and $D_1(\alpha)$ are given in Theorem 10 and

$$
\begin{aligned}
P_1(\alpha) &= \frac{2^{1-\alpha}(3u+v)^{\alpha+2} - 2^{2\alpha+5} - 2^{2\alpha+1}(7u+6v)(v-u)(3u+v)^\alpha - \alpha(7u+v)(v-u)(3u+v)^\alpha}{2^{2\alpha+3}(v-u)(\alpha+2)}, \\
P_2(\alpha) &= \frac{2(2(u+v))^{\alpha+2} - 8(u+v)^2(3u+v)^\alpha - \alpha(5u+3v)(v-u)(3u+v)^\alpha}{2^{2\alpha+3}(v-u)(\alpha+2)}, \\
P_3(\alpha) &= \frac{8(u+v)^2(u+3v)^\alpha - 4\alpha(u+v)^\alpha(u+3v)^\alpha - \alpha(u+3v)^{\alpha+2} - 2^{\alpha+3}(u+v)^{\alpha+2}}{2^{2\alpha+1}(v-u)(\alpha+2)}, \\
P_4(\alpha) &= \frac{2^{2\alpha-1}\alpha(u+3v) + 4^\alpha(u+3v)^3 + 4(4v)^{\alpha+2} - 4^{\alpha+2}v^2(u+3v) - 4(u+3v)^{\alpha+2} - 2^{2\alpha+3}\alpha v^2(u+3v)}{4^{\alpha+2}(v-u)(\alpha+2)}.
\end{aligned}
$$

Proof. By using power-mean's inequality and the concavity of $|h'|^q$ for any $\zeta \in [0,1]$, we have

$$
\left|h'\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)\right|^q \geq \left(\zeta|h'(u)| + (1-\zeta)\left|\left(\frac{3u+v}{4}\right)\right|\right)^q.
$$

This implies that

$$
\left|h'\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)\right| \geq \zeta|h'(u)| + (1-\zeta)\left|\left(\frac{3u+v}{4}\right)\right|. \qquad (2.14)
$$

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This means that $|h'|$ is also concave. Using inequality (2.14) in (2.12) and then applying the Jensen's integral inequality, we get

$$\begin{aligned} J_1 &\leq \left(\int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right] d\zeta\right) \\ &\times h'\left(\frac{\int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right]\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right) d\zeta}{\int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right] d\zeta}\right) \\ &= A_1(\alpha)h'\left(\frac{P_1(\alpha)}{A_1(\alpha)}\right), \end{aligned}$$

where we have used the facts that

$$\begin{aligned} &\int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right] d\zeta = A_1(\alpha) = \frac{4u(3u+v)^\alpha - (4u)^{\alpha+1}}{4^\alpha(v-u)(\alpha+1)} - \frac{\alpha(3u+v)^\alpha}{4^\alpha}, \\ &\int_0^1 \left[\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right)^\alpha - \left(\frac{3u+v}{4}\right)^\alpha\right]\left(u\zeta + (1-\zeta)\frac{3u+v}{4}\right) d\zeta = P_1(\alpha) \\ &= \frac{2^{1-\alpha}(3u+v)^{\alpha+2} - 2^{2\alpha+5} - 2^{2\alpha+1}(7u+6v)(v-u)(3u+v)^\alpha - \alpha(7u+v)(v-u)(3u+v)^\alpha}{2^{2\alpha+3}(v-u)(\alpha+2)}. \end{aligned}$$

Similarly, we get $J_2 \leq B_1(\alpha)h'\left(\frac{P_2(\alpha)}{B_1(\alpha)}\right)$, $J_3 \leq C_1(\alpha)h'\left(\frac{P_3(\alpha)}{C_1(\alpha)}\right)$ and $J_4 \leq D_1(\alpha)h'\left(\frac{P_4(\alpha)}{D_1(\alpha)}\right)$.

Using J_1, J_2, J_3, and J_4 in (2.12), we obtain the required inequality (2.13). This completes the proof of Theorem 12. □

Corollary 13. *With the similar assumptions of Theorem 12, if $\alpha = 1$, then*

$$|\Phi_1(u,v)| \leq \frac{v-u}{4}\left[A_1(\alpha)h'\left(\frac{P_1(\alpha)}{A_1(\alpha)}\right) + B_1(\alpha)h'\left(\frac{P_2(\alpha)}{B_1(\alpha)}\right) + C_1(\alpha)h'\left(\frac{P_3(\alpha)}{C_1(\alpha)}\right) + D_1(\alpha)h'\left(\frac{P_4(\alpha)}{D_1(\alpha)}\right)\right],$$

where

$$A_1(1) = \frac{u-v}{8}, \quad B_1(1) = \frac{v-u}{8}, \quad D_1(1) = \frac{u-v}{8}, \quad D_1(1) = \frac{v-u}{8},$$

and

$$\begin{aligned} P_1(1) &= \frac{(u-v)(11u+v)}{96}, \quad P_2(1) = \frac{(v-u)(7u+5v)}{96}, \\ P_3(1) &= \frac{(u-v)(5u+7v)}{96}, \quad P_4(1) = \frac{(v-u)(u+11v)}{96}. \end{aligned}$$

3. Conclusion

In this work, we have established new conformable fractional integral inequalities of Hermite–Hadamard type for convex functions using the. As a special case, if we substitute $\alpha = 1$ into the general definition of conformable fractional integrals (Definition 2), we obtain the classical integrals. In view of this, we obtained some new inequalities of Hermite–Hadamard type for convex functions involving classical integrals.

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