doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at *Entropy* **2019**, *21*, 315; doi:10.3390/e21030315

# Confidence interval estimation for precipitation quantiles based on principle of maximum entropy

Ting Wei  Songbai Song*

College of Water Resources and Architectural Engineering, Northwest A&F University, Yangling, 712100, China.

* Correspondence: songsb33@163.com (S.S.).

**Abstract:** Confidence interval of is an interval corresponding to a specified confidence and including the true value. It can be used to describe the precision of a statistical quantity and quantify its uncertainty. Although the principle of maximum entropy (POME) has been used for a variety of applications in hydrology, the confidence intervals of the POME quantile estimators have not been available. In this study, the calculation formulas of asymptotic variances and confidence intervals of quantiles based on POME for Gamma, Pearson type 3 (P3) and Extreme value type 1 (EV1) distributions were derived. Monte Carlo Simulation experiments were performed to evaluate the performance of derived formulas for finite samples. Using four data sets for annual precipitation at the Weihe River basin in China, the derived formulas were applied for calculating the variances and confidence intervals of precipitation quantiles for different return periods and the results were compared with those of the methods of moments (MOM) and of maximum likelihood (ML) method. It is shown that POME yields the smallest standard errors and the narrowest confidence intervals of quantile estimators among the three methods, and can reduce the uncertainty of quantile estimators.

**Keywords:** Principle of maximum entropy; quantile estimation; confidence interval; Monte Carlo simulation; precipitation frequency analysis

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

# 1 Introduction

One of the objectives of hydrological frequency analysis is to estimate the magnitude of a hydrologic event with a given return period [1,2]. The short record length of sample data and the application of inappropriate distribution inevitably involve some uncertainties in this estimation [3]. Hence, a point estimate of quantile corresponding to a return period is usually not enough because it cannot adequately express the precision of the estimation. Confidence interval can be used to describe the precision of a statistical quantity and it provides more information than a point estimate. The calculation of confidence interval requires the results of standard errors. Different estimation methods may yield different standard errors of the estimated quantities, and thus the confidence intervals are different as well. The most efficient method is the one that gives the smallest error and the narrowest confidence interval width [1]. Therefore, it is necessary to compare the performance of different methods and to choose the most efficient method for improving the calculation accuracy of hydrological quantile estimators.

There have been a multitude of studies related to the estimation of variances and confidence intervals of estimated quantile estimators. Nash and Amorocho obtained the expressions for the standard errors of flood quantile estimators for a given return period from random samples of normal and double exponential population [4]. Bobee derived the formulas of sampling error of *T*-year flood quantile computed by fitting a log-Pearson type-3 distribution [5]. Assuming the design events are normally distributed, Kite derived the confidence limits of design events using data generation experiments [6]. Condie gave the maximum likelihood estimators for the parameters of a log-Pearson type 3 distribution and derived the expressions for asymptotic standard error of a $T$-year event and he concluded that the maximum likelihood method is markedly superior to method of moments in the estimation of asymptotic standard error of $T$-year event [7]. Lawless reviewed the methods for calculating the confidence interval for parameters or other characteristics of the Weibull and extreme value distributions, and compared performances of the various methods [8]. Hoshi and Barges developed an approximation technique to compute the derivative of a standard gamma quantile with respect to the distribution shape parameter for estimating the sampling variance of a specified quantile [9]. They also derived the expressions for calculating the sampling variances and covariances of log-Pearson type 3 distribution parameters as well as the sampling variance of $T$-year flood event using the method of moments [10]. Phien derived the explicit formulas for the variances and covariances of the parameter estimates of log-Pearson type 3 distribution when the method of direct and mixed moments was used for parameter estimation [11]. Stedinger (1983), Ashkar and Bobee (1988) also investigated several methods for calculating the confidence intervals for P3 and log-Pearson type 3 quantiles and tested their performance [12,13]. Lu and Stedinger derived the simple formulas for estimating the asymptotic variance of probability weighted moments (PWM) quantile estimators for generalized extreme value (GEV) distribution when the location and scale parameters were estimated with a fixed regional shape parameter or all three parameters are estimated. They compared the variances of quantile estimators for the two cases and examined the performance of the approximate confidence intervals obtained using the derived formulas [14]. Sampling variance of normalized GEV/PWM quantile estimators were also investigated by Lu and Stedinger [15]. Heo and Salas derived the confidence limits of population quantiles for log-Gumbel distribution by using the asymptotic variances estimated

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

from the method of moments (MOM), PWM method and maximum likelihood (ML) method, and compared the applicability of the three estimation methods by using observed flood data. They concluded that the MOM yielded narrower confidence limits than did the ML and PWM methods especially for high return periods [16]. Ashkar presented an approximate procedure for calculating confidence intervals of quantiles for gamma and generalized gamma distributions, and tested its suitability by numerical applications. They found that the approximate method performs well for hydrological applications where the data record is short [17]. Heo et al. derived and compared the asymptotic variances and confidence intervals of quantiles for the two- and three-parameter weibull distributions based on the MOM, ML, and PWM methods [18]. Su investigated two parametric methods for computing the confidence intervals of quantiles for generalized lambda distribution [19]. Shin summarized the parameter estimation procedure for the generalized logistic distribution based on the MOM, ML, and PWM, and derived the asymptotic variances of the MOM, ML, and PWM quantile estimators for the generalized logistic distribution [20]. The confidence intervals of quantiles for two-parameter Kappa distribution and heavy-tailed generalized Pareto distributions have also been investigated [21,22].

Shannon defined the concept of entropy as a measure of uncertainty of a random variable or its probability distribution [23]. Jaynes later formulated the principle of maximum entropy (POME) which provides a rational approach to choose the most unbiased probability distribution for hydrologic frequency analysis [24]. POME also provides a way to estimate parameters of a given distribution from the specified constraints. The Shannon entropy and POME have been widely applied in hydrology. Sonuga developed a minimally biased probability distribution appropriate for hydrologic frequency analysis in the absence of a large amount of data [25]. Singh developed a procedure for derivation of a number of frequency distributions used in hydrology using POME, including the two- and three-parameter lognormal distributions, extreme value type I and III distributions, generalized extreme value distribution, Weibull distribution, P3 distribution, log-Pearson type 3 distribution, beta distribution, two- and three-parameter log-logistic distributions, two- and three-parameter generalized Pareto distributions and two-component extreme value distribution, etc [26]. He also summarized the entropy method for parameter estimation for these distributions and indicated that the entropy method is reasonable and efficient for parameter estimation [27]. Fiorentino derived a parameter estimation method for the two component extreme value distribution based on the POME and indicated that the POME method is suitable for both site-specific and regional estimation [28]. Singh and Song applied POME to estimate parameters of the four-parameter kappa distribution and the four-parameter exponential gamma distribution. The results of the estimation show that the POME enables these two distributions to fit the data better than the other estimation methods [29,30]. Chen derived the generalized distribution for flood and extreme rainfall frequency analysis and she concluded that the entropy-based generalized distributions are superior or comparable to other traditional distributions [31,32]. Hao constructed the bivariate joint distribution of drought duration and severity using bivariate entropy and results show that the entropy-based joint distribution is capable for bivariate drought analysis. In recent years, an integration of entropy and copula has been developed to construct the joint distribution function [33]. Hao proposed maximum entropy copula for multisite monthly streamflow simulation and he found that the simulated streamflow from

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

maximum entropy copula is very close to the observed streamflow [34]. Kong integrated POME with Gumbel-Hougaard copula for monthly streamflow simulation. The simulation shows that the variability of simulated monthly streamflow is similar to that of observed monthly streamflow [35]. AghaKouchak reviewed the currently available maximum entropy copula models and discussed their potential applications in hydrology and climatology. He applied the entropy copula to bivariate flood frequency analysis to illustrate the construction of the bivariate joint distribution using the entropy copula [36].

In variances estimation, Phien provided the formulas for calculating the approximate variances and covariances of the parameter estimators and the approximate variance of the *T*-year event obtained by POME for the extreme value type-1 (EV1) distribution. Through applications of the formulas to simulated data, he concluded that the approximations for the variance of estimates of the *T*-year event are of sufficient accuracy [37]. The sampling properties of the maximum entropy estimators for P3 distribution has been investigated as well [38]. The investigation concluded that the efficiency of the POME can be improved by using biased estimator of the variance of variable. Despite the advances mentioned above, the POME based confidence intervals of quantile estimators have not been investigated and the performance of POME, MOM and ML in the estimation has not been compared.

The objective of this study is therefore to derive the formulas for calculating the confidence intervals of quantiles based on POME for Gamma, P3 and EV1 distributions; evaluate the performance of derived confidence intervals for finite samples using simulation experiments; compute the confidence of annual precipitation quantiles for different return periods; and compare these confidence intervals with those estimated when MOM and ML were employed.

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

# 2 Confidence interval estimation of the quantile

The standard error and confidence interval are two measures to describe the uncertainty of a statistical quantity, such as the $T$-year quantile estimator $\hat{x}_T$. The standard error $s_T$ measures the standard deviation of an estimated quantile from a sample about the true value, and the confidence interval of a quantile estimated by a sample is an interval that corresponds to a specified confidence and includes the true value.

## 2.1 Estimation of quantile

A general form for calculating $\hat{x}_T$ of a given distribution can be written in terms of the distribution moments and the frequency factor $K_T$ [39]:

$$\hat{x}_T = \hat{\mu}_1' + K_T\sqrt{\hat{\mu}_2} \tag{1}$$

where $\hat{\mu}_1'$ and $\hat{\mu}_2$ are the mean and the standard error of the population, respectively, and they will equal to the sample moments only when the MOM is used for parameter estimation; $K_T$ is the frequency factor specific to the chosen distribution, which can be derived from the distribution parameters, sample size, and return period $T$ or cumulative probability of exceedance of the design event. Expressions of $K_T$ for different distributions are commonly given in statistics texts [1].

## 2.2 Calculation of confidence interval

The confidence interval is a convenient approach for quantifying the uncertainty and indicating the accuracy of quantiles [1,40]. The estimation of the confidence interval of an unknown parameter $\theta$ (or of some characteristic value of a distribution function) consists of calculating an interval which by repeated sampling will contain the estimate $\hat{\theta}$ with a given probability. Thus, the confidence interval will be defined by its two limits, and a certain

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

cumulative probability results for each limit from the probability corresponding to the interval.

Using the statistical asymptotical theory that the distribution of quantile estimator $\hat{x}_T$ is asymptotically normal with mean $\bar{x}_T$ and variance $s_T^2$ as the sample size $n \to \infty$. Thus an approximate $(1-\alpha)$ confidence interval for $\hat{x}_T$ can be written by:

$$\hat{x}_L = \hat{x}_T \pm u_{1\text{-}\frac{\alpha}{2}} s_T \tag{2}$$

where $\hat{x}_L$ is the confidence interval; $u_{1\text{-}\frac{\alpha}{2}}$ is the quantile of the standard normal distribution for confidence levels equal to $1-\frac{\alpha}{2}$; $\hat{x}_T$ is the design value for the return period *T*; $s_T$ is the standard error of $\hat{x}_T$.

To calculate the confidence intervals of the quantile estimators, the asymptotic variance should be estimated first. The quantile can be written in the following form:

$$\hat{x}_T = f\left(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3\right) \tag{3}$$

where $\hat{\theta}_i, i = 1,2,3$ denotes the estimators of either moments or distribution parameters. Then the asymptotic variance of $\hat{x}_T$ can be expressed as [2,6]:

$$\begin{aligned} s_T^2 = \operatorname{var}(\hat{x}_T) = &\left(\frac{\partial x_T}{\partial \theta_1}\right)^2 \operatorname{var}\left(\hat{\theta}_1\right) + \left(\frac{\partial x_T}{\partial \theta_2}\right)^2 \operatorname{var}\left(\hat{\theta}_2\right) + \left(\frac{\partial x_T}{\partial \theta_3}\right)^2 \operatorname{var}\left(\hat{\theta}_3\right) \\ &+ 2\frac{\partial x_T}{\partial \theta_1}\frac{\partial x_T}{\partial \theta_2}\operatorname{cov}\left(\hat{\theta}_1, \hat{\theta}_2\right) + 2\frac{\partial x_T}{\partial \theta_2}\frac{\partial x_T}{\partial \theta_3}\operatorname{cov}\left(\hat{\theta}_2, \hat{\theta}_3\right) + 2\frac{\partial x_T}{\partial \theta_1}\frac{\partial x_T}{\partial \theta_3}\operatorname{cov}\left(\hat{\theta}_1, \hat{\theta}_3\right) \end{aligned} \tag{4}$$

where $\operatorname{var}\left(\hat{\theta}_i\right)$ is the variance of $\theta_i$; $\operatorname{cov}\left(\hat{\theta}_i, \hat{\theta}_j\right)$ is the covariance of $\hat{\theta}_i$ and $\hat{\theta}_j$; $i, j = 1,2,3$.

The calculation of the terms in the right-hand side of Equation (4) depends in general on the method of parameter estimation. In this paper, the method of moments (MOM), maximum likelihood (ML) method, and principle of maximum entropy (POME) were considered and the asymptotic variances estimated by these methods are described below.

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

#### 2.2.1 Method of moments (MOM)

For MOM, the $\theta_1, \theta_2$ and $\theta_3$ in Equations (3) and (4) denote the first three moments $\mu_1', \mu_2$, and $\mu_3$ of a given distribution and the quantile can be written as $\hat{x}_T = f(\hat{\mu}_1', \hat{\mu}_2, \hat{\mu}_3)$. Then the variance of $\hat{x}_T$ is calculated by using the relation between the parameters and the moments. Using the first three sample moments, the asymptotic variance of $\hat{x}_T$ can be expressed as [1]:

$$\begin{aligned} s_T^2 = \frac{\mu_2}{n}\Bigg\{ & 1+\gamma_1 K_T + \frac{K_T^2}{4}(\gamma_2 - 1) + \frac{\partial K_T}{\partial \gamma_1}\left[2\gamma_2 - 3\gamma_1^2 - 6 + K_T\left(\gamma_3 - \frac{3}{2}\gamma_1\gamma_2 - \frac{5}{2}\gamma_1\right)\right] \\ & + \left(\frac{\partial K_T}{\partial \gamma_1}\right)^2\left(\gamma_4 - 3\gamma_1\gamma_3 - 6\gamma_2 + \frac{9}{4}\gamma_1^2\gamma_2 + \frac{35}{4}\gamma_1^2 + 9\right)\Bigg\} \end{aligned} \tag{5}$$

where $\gamma_j, j = 1,2,3,4$ are the cumulants which are given by:

$$\gamma_1 = \mu_3 / \mu_2^{3/2} = C_S \tag{6}$$

$$\gamma_2 = \mu_4 / \mu_2^2 \tag{7}$$

$$\gamma_3 = \mu_5 / \mu_2^{5/2} \tag{8}$$

$$\gamma_4 = \mu_6 / \mu_2^3 \tag{9}$$

where $\mu_r, r = 1,2,\ldots,6$ are the *r*th central moments.

For a two-parameter distribution, the frequency factor $K_T$ does not depend on $\gamma_1$, then $\partial K_T / \partial \gamma_1 = 0$ in the above equation and the expression simplifies to:

$$s_T^2 = \frac{\mu_2}{n}\left[1+\gamma_1 K_T + \frac{K_T^2}{4}(\gamma_2 - 1)\right] \tag{10}$$

#### 2.2.2 Maximum likelihood (ML) method

ML is a probability distribution-related method that requires the log-likelihood function of the probability density function (pdf) of a specific distribution. The ML parameters estimators of the commonly used distributions in hydrology are available in the literature in the literature

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

[1].

The asymptotic variance of the ML quantile estimators can be obtained by replacing $\theta_1, \theta_2$ and $\theta_3$ in Equation (4) with the distribution parameters. The asymptotic variance and covariance terms for the ML parameter estimators are the elements of the inverse of the following information matrix [41]:

$$\mathbf{I} = \begin{bmatrix} E\left(-\frac{\partial^2 \log L}{\partial \theta_1^2}\right) & E\left(-\frac{\partial^2 \log L}{\partial \theta_1 \partial \theta_2}\right) & E\left(-\frac{\partial^2 \log L}{\partial \theta_1 \partial \theta_3}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial \theta_2 \partial \theta_1}\right) & E\left(-\frac{\partial^2 \log L}{\partial \theta_2^2}\right) & E\left(-\frac{\partial^2 \log L}{\partial \theta_2 \partial \theta_3}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial \theta_3 \partial \theta_1}\right) & E\left(-\frac{\partial^2 \log L}{\partial \theta_3 \partial \theta_2}\right) & E\left(-\frac{\partial^2 \log L}{\partial \theta_3^2}\right) \end{bmatrix} \tag{11}$$

where $L$ is the likelihood function, $E$ represents the expected value. The inverse matrix of $\mathbf{I}$ can be written as:

$$\mathbf{I}^{-1} = \begin{bmatrix} \operatorname{var}(\hat{\theta}_1) & \operatorname{cov}(\hat{\theta}_1, \hat{\theta}_2) & \operatorname{cov}(\hat{\theta}_1, \hat{\theta}_3) \\ \operatorname{cov}(\hat{\theta}_2, \hat{\theta}_1) & \operatorname{var}(\hat{\theta}_2) & \operatorname{cov}(\hat{\theta}_2, \hat{\theta}_3) \\ \operatorname{cov}(\hat{\theta}_3, \hat{\theta}_1) & \operatorname{cov}(\hat{\theta}_3, \hat{\theta}_2) & \operatorname{var}(\hat{\theta}_3) \end{bmatrix} \tag{12}$$

Differentiating Equation (1) with distribution parameters, one obtains the derivatives of $x_T$ with respect to the parameters. Substituting the derivative terms and the asymptotic variances and covariances of the ML parameter estimators into Equation (4) yields the asymptotic variance of the ML quantile estimators.

### 2.2.3 Principle of maximum entropy (POME) method

POME involves essentially five steps in the estimation of the distribution parameters: (1) specification of constraints from the given information; (2) derivation of the probability density function of the maximum entropy distribution; (3) derivation of the relationship between Lagrange multipliers and constraints; (4) derivation of the relationship between Lagrange

*Preprints* (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at *Entropy* **2019**, *21*, 315; doi:10.3390/e21030315

multipliers and distribution parameters; and (5) derivation the relationship between distribution parameters and constraints [27,30]. Relations between parameters and constraints for many commonly used distributions, such as P3, GEV, and Weibull distributions, are given in the existing literatures [27].

The constraints in POME can be expressed in terms of moments [27], therefore, the variance and covariances of the parameters can be obtained from the relationship between the variance and covariances of the moments and that of the parameter estimates. Let $P$, $Q$ and $R$ denote the three moments, one can approximately write the vector of variance and covariances of $P$, $Q$ and $R$ of a three-parameter distribution as [37,38]:

$$V_M = D V_P \tag{13}$$

where $V_M$ and $V_P$ are the vectors of variance and covariances of the moments and parameter estimators respectively:

$$V_M = \begin{bmatrix} \mathrm{var}(P) \\ \mathrm{var}(Q) \\ \mathrm{var}(R) \\ \mathrm{cov}(P,Q) \\ \mathrm{cov}(Q,R) \\ \mathrm{cov}(P,R) \end{bmatrix}, V_P = \begin{bmatrix} \mathrm{var}(\hat{\theta}_1) \\ \mathrm{var}(\hat{\theta}_2) \\ \mathrm{var}(\hat{\theta}_3) \\ \mathrm{cov}(\hat{\theta}_1,\hat{\theta}_2) \\ \mathrm{cov}(\hat{\theta}_2,\hat{\theta}_3) \\ \mathrm{cov}(\hat{\theta}_1,\hat{\theta}_3) \end{bmatrix} \tag{14}$$

and $\theta_1, \theta_2$, and $\theta_3$ are the distribution parameters; $D$ is the matrix with elements $d_{ij}$ ($1 \le i, j \le 6$), which are the partial derivatives of the moments with respect to the distribution parameters. For example:

$$\begin{gathered} d_{11} = (\partial P/\partial\theta_1)^2,\ d_{12} = (\partial P/\partial\theta_2)^2,\ d_{13} = (\partial P/\theta_3)^2, \\ d_{14} = 2(\partial P/\partial\theta_1)(\partial P/\partial\theta_2),\ d_{15} = (\partial P/\partial\theta_2)(\partial P/\theta_3),\ d_{16} = 2(\partial P/\partial\theta_1)(\partial P/\partial\theta_3),\ \ldots\ldots \end{gathered} \tag{15}$$

Consequently, the $V_p$ can be calculated using Equation (16) as long as the elements of $D$ and the $V_M$ have been calculated.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

$$V_p = D^{-1} V_M \tag{16}$$

where $D^{-1}$ is the inverse matrix of $D$.

Substituting the elements of $V_p$ and the partial derivatives of $x_T$ with respect to distribution parameters into Equation (4), one can obtain the variances of quantile estimator.s

## 3 Asymptotic variances of quantile estimators for different distributions

Three commonly used distributions: Gamma distribution, P3 distribution and Extreme value type 1 distribution (EV1) were considered in this study.

### 3.1 Gamma distribution

The pdf of Gamma distribution is given by:

$$f(x) = \frac{1}{\alpha^{\beta}\Gamma(\beta)} x^{\beta-1} \exp(-x/\alpha) \tag{17}$$

where $\alpha$ and $\beta$ are the scale and shape parameters, and $\Gamma(\cdot)$ is the gamma function, and $0 < x < \infty$.

For Gamma distribution, the mean and variance are given by:

$$E(x) = \alpha\beta \tag{18}$$

$$\mu_2 = \mathrm{var}(x) = \alpha^2 \beta \tag{19}$$

Introduction of Equation (18) and (19) in Equation (1) yields the $T$-year quantile of Gamma distribution:

$$\hat{x}_T = \hat{\alpha}\hat{\beta} + K_T \sqrt{\hat{\alpha}^2 \hat{\beta}} \tag{20}$$

Differentiation of Equation (20) with respect to $\alpha$ and $\beta$ yields:

$$\frac{\partial x_T}{\partial \alpha} = \beta + K_T \sqrt{\beta} \frac{\alpha}{|\alpha|} \tag{21}$$

$$\frac{\partial x_T}{\partial \beta} = \alpha + \frac{K_T}{2} \sqrt{\alpha^2/\beta} - \frac{\sqrt{\alpha^2}}{\beta} \frac{\partial K_T}{\partial C_S} \tag{22}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

where $\partial K_T / \partial C_S$ can be calculated by using Wilson-Hilferty transformation [1].

The variances of quantile estimators for Gamma distribution estimated using MOM, ML and POME are described below.

3.1.1 Estimation of asymptotic variances by MOM and ML

Based on MOM, the standard error of $\hat{x}_T$ for Gamma distribution can be calculated directly by [5]:

$$s_T^2 = \frac{\mu_2}{n}\left[(1+K_T C_v)^2 + \frac{1}{2}\left(K_T + 2C_v \frac{\partial K_T}{\partial \gamma_1}\right)^2 (1+C_v^2)\right] \tag{23}$$

where $C_v$ is the coefficient of variation, $C_v = \mu_2^{1/2} / \mu_1'$; $\gamma_1 = C_s$.

For the ML parameter estimators, the asymptotic variance and covariances are the elements of the inverse of the information matrix **I**:

$$\begin{bmatrix} \mathrm{var}(\alpha) & \mathrm{cov}(\alpha,\beta) \\ \mathrm{cov}(\beta,\alpha) & \mathrm{var}(\beta) \end{bmatrix} = \begin{bmatrix} E\left(-\dfrac{\partial^2 \log L}{\partial^2 \alpha}\right) & E\left(-\dfrac{\partial^2 \log L}{\partial \alpha \partial \beta}\right) \\ E\left(-\dfrac{\partial^2 \log L}{\partial \beta \partial \alpha}\right) & E\left(-\dfrac{\partial^2 \log L}{\partial^2 \beta}\right) \end{bmatrix}^{-1} = \mathbf{I}^{-1}(\alpha,\beta) \tag{24}$$

The expected values of the second partial derivatives of the Gamma distribution in Equation (24) are given in Appendix A. Therefore, the asymptotic variances and covariances are obtained as:

$$\begin{bmatrix} \mathrm{var}(\alpha) & \mathrm{cov}(\alpha,\beta) \\ \mathrm{cov}(\beta,\alpha) & \mathrm{var}(\beta) \end{bmatrix} = \begin{bmatrix} \alpha^2 \psi' D & -\alpha D \\ -\alpha D & \beta D \end{bmatrix} \tag{25}$$

where $\psi' = \psi'(\beta) = \dfrac{d^2 \log \Gamma(\beta)}{d\beta^2}$ is the tri-gamma function; $D = \dfrac{1}{\beta\psi' - 1}$.

Substituting Equations (21) and (22) and the variances and covariances terms in Equation (25) into Equation (4) yields the variance of the ML quantile estimator.

Peer-reviewed version available at *Entropy* **2019**, *21*, 315; doi:10.3390/e21030315

#### 3.1.2 Estimation of asymptotic variances by POME

When the POME is used to estimate the parameters of Gamma distribution, the relation between parameters and constraints can be expressed as [27]:

$$\begin{cases} E(x) = \alpha\beta \\ E[\ln(x)] = \ln(\alpha) + \psi \end{cases} \tag{26}$$

where $\psi = \psi(\beta) = \dfrac{d \log\Gamma(\beta)}{d\beta}$ is digamma function.

Defining $W = \ln(x)$, and replacing the expectations of Equation (26) by their sample estimates. Thus, the estimators $\hat{\alpha}$ and $\hat{\beta}$ of parameters are obtained by solving the following equations:

$$\begin{cases} \alpha\beta = \overline{X} \\ \ln(\alpha) + \psi(\beta) = \overline{W} \end{cases} \tag{27}$$

It is noted that $\overline{X}$ and $\overline{W}$ in Equation (27) correspond to the *P* and *Q* in Equation (14) respectively; $\alpha$ and $\beta$ correspond to $\theta_1$ and $\theta_2$ respectively. Then $V_M$ and $V_p$ are written by:

$$V_M = \begin{bmatrix} \mathrm{var}(\overline{X}) \\ \mathrm{var}(\overline{W}) \\ \mathrm{cov}(\overline{X}, \overline{W}) \end{bmatrix}, V_p = \begin{bmatrix} \mathrm{var}(\hat{\alpha}) \\ \mathrm{var}(\hat{\beta}) \\ \mathrm{cov}(\hat{\alpha}, \hat{\beta}) \end{bmatrix} \tag{28}$$

The terms of $V_M$ are obtained according to Equations (27) and the following equations [42]:

$$\mathrm{var}(\overline{W}) = \mathrm{var}(W)/n = \left(E(W^2) - [E(W)]^2\right)/n \tag{29}$$

$$\mathrm{cov}(\overline{X}, \overline{W}) = \mathrm{cov}(X, W)/n = [E(xW) - E(x)E(W)]/n \tag{30}$$

For Gamma distribution, $\mu_2 = \alpha^2\beta$, we obtain:

$$\mathrm{var}(\overline{X}) = \alpha^2\beta/n \tag{31}$$

Since the $E(x)$ and $E(W)$ are known (Equations (27)), only the expressions for the

 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

$E(W^2)$ and $E(xW)$ are needed respectively for the calculation of the variance of $\overline{W}$ and the covariance of $\overline{X}$ and $\overline{W}$. According to the pdf of Gamma distribution, one obtains:

$$\begin{aligned} E(W^2) &= E\left[(\ln x)^2\right] = \int_0^\infty [\ln x]^2 \frac{1}{\alpha^\beta \Gamma(\beta)} x^{\beta-1}\exp(-x/\alpha)dx \\ &= \frac{(\ln\alpha)^2}{\Gamma(\beta)}\int_0^\infty y^{\beta-1}e^{(-y)}dy + \frac{2\ln\alpha}{\Gamma(\beta)}\int_0^\infty(\ln y)y^{\beta-1}e^{(-y)}dy + \frac{1}{\Gamma(\beta)}\int_0^\infty(\ln y)^2 y^{\beta-1}e^{(-y)}dy \end{aligned} \tag{32}$$

Let $y = x/\alpha$, $x = \alpha y$, $dx = \alpha dy$. Substituting the above quantities in Equation (32) and changing the integral limits, we obtain:

$$E(W^2) = \frac{(\ln\alpha)^2}{\Gamma(\beta)}\int_0^\infty y^{\beta-1}e^{(-y)}dy + \frac{2\ln\alpha}{\Gamma(\beta)}\int_0^\infty(\ln y)y^{\beta-1}e^{(-y)}dy + \frac{1}{\Gamma(\beta)}\int_0^\infty(\ln y)^2 y^{\beta-1}e^{(-y)}dy \tag{33}$$

Using the property of the gamma function $d^k\Gamma(\beta)/d\beta^k = \int_0^\infty (\ln y)^k y^{\beta-1}e^{-y}dy$, Then Equation (33) can be expressed as:

$$\begin{aligned} E(W^2) &= \frac{(\ln\alpha)^2}{\Gamma(\beta)}\int_0^\infty y^{\beta-1}e^{(-y)}dy + \frac{2\ln\alpha}{\Gamma(\beta)}\int_0^\infty(\ln y)y^{\beta-1}e^{(-y)}dy + \frac{1}{\Gamma(\beta)}\int_0^\infty(\ln y)^2 y^{\beta-1}e^{(-y)}dy \\ &= (\ln\alpha)^2 + 2\ln\alpha\psi + \psi' + \psi^2 \end{aligned} \tag{34}$$

Substitution of the second equation in Equation (27), and Equation (34) into Equation (29) yields:

$$\mathrm{var}(\overline{W}) = \left((\ln\alpha)^2 + 2\ln\alpha\psi + \psi' + \psi^2 - [\ln(\alpha)+\psi]^2\right)/n = \psi'/n \tag{35}$$

$E(xW)$ is written by:

$$\begin{aligned} E(xW) &= \int_0^\infty x\ln x \frac{1}{\alpha^\beta\Gamma(\beta)} x^{\beta-1}\exp(-x/\alpha)dx \\ &= \frac{\alpha\ln\alpha}{\Gamma(\beta)}\int_0^\infty y^\beta e^{(-y)}dy + \frac{\alpha}{\Gamma(\beta)}\int_0^\infty (\ln y)y^\beta e^{(-y)}dy \\ &= \alpha + \alpha\beta[\ln\alpha + \psi(b)] \end{aligned} \tag{36}$$

Substitution of the first equation in Equation (27), and Equation (36) into Equation (30) yields:

$$\mathrm{cov}(\overline{X}, \overline{W}) = \alpha/n \tag{37}$$

Consequently, one obtains:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

$$V_M = \frac{1}{n}\begin{bmatrix} \alpha^2\beta \\ \psi' \\ \alpha \end{bmatrix} \tag{38}$$

Additionally, taking the partial derivates of $\bar{X}$ and $\bar{W}$ with respect to $\alpha$ and $\beta$ one can obtain the matrix *D*:

$$D = \begin{bmatrix} \beta^2 & \alpha^2 & 2\alpha\beta \\ 1/\alpha^2 & \psi'^2 & 2\psi'/\alpha \\ \beta/\alpha & \alpha\psi' & 1+\beta\psi' \end{bmatrix} \tag{39}$$

where $\psi' = \psi'(\beta)$ is the tri-gamma function.

Thus all the components of $V_M$ and *D* are obtained. Substituting $V_M$ and *D* (Equations (38) and (39)) into Equation (16) yields $V_P$. The variance of the quantile estimator can then be obtained by substituting the terms of $V_P$ and Equations (21) and (22) into Equation (4).

### 3.2 Pearson type 3 (P3) distribution

The pdf of P3 distribution can be written as:

$$f(x \mid \alpha, \beta, \gamma) = \frac{1}{\alpha\Gamma(\beta)}\left(\frac{x-\gamma}{\alpha}\right)^{\beta-1} e^{-\frac{x-\gamma}{\alpha}}, \gamma < x < \infty \tag{40}$$

where $\alpha, \beta$ and $\gamma$ are the scale, shape and location parameters, respectively, and $\gamma < x < \infty$.

For P3 distribution, the mean and variance are given by:

$$E(x) = \alpha\beta + \gamma \tag{41}$$

$$\mu_2 = \mathrm{var}(x) = \alpha^2\beta \tag{42}$$

Introduction of Equations (41) and (42) in Equation (1) yields the *T*-year quantile of P3 distribution:

$$\hat{x}_T = \hat{\alpha}\hat{\beta} + \hat{\gamma} + K_T\sqrt{\hat{\alpha}^2\hat{\beta}} \tag{43}$$

Taking partial derivatives of Equation (43) with respect to $\alpha, \beta, \gamma$ yields:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1
Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

$$\frac{\partial x_T}{\partial \alpha} = \beta + K_T\sqrt{\beta}\frac{\alpha}{|\alpha|} \tag{44}$$

$$\frac{\partial x_T}{\partial \beta} = \alpha + \frac{K_T}{2}\sqrt{\frac{\alpha^2}{\beta}} - \frac{\sqrt{\alpha^2}}{\beta}\frac{\partial K_T}{\partial C_S} \tag{45}$$

$$\frac{\partial x_T}{\partial \gamma} = 1 \tag{46}$$

The variances of quantiles of P3 distribution estimated using MOM, ML and POME are described below.

3.2.1 Estimation of asymptotic variances by MOM and ML

For P3 distribution, the cumulants shown in Equations (7)-(9) can be further written as [5]:

$$\gamma_2 = 3\left(1 + \frac{\gamma_1^2}{2}\right) \tag{47}$$

$$\gamma_3 = \gamma_1\left(10 + 3\gamma_1^2\right) \tag{48}$$

$$\gamma_4 = 5\left(3 + \frac{13}{2}\gamma_1^2 + \frac{3}{2}\gamma_1^4\right) \tag{49}$$

Substituting Equations (47)-(49) into Equation (5) yields the asymptotic variance of MOM quantile estimator:

$$s_T^2 = \frac{\mu_2}{n}\left[1 + \gamma_1 K_T + \frac{K_T^2}{2}\left(\frac{3}{4}\gamma_1^2 + 1\right) + 3K_T\frac{\partial K_T}{\partial \gamma_1}\left(\gamma_1 + \frac{\gamma_1^3}{4}\right) + 3\left(\frac{\partial K_T}{\partial \gamma_1}\right)^2\left(2 + 3\gamma_1^2 + \frac{5\gamma_1^4}{8}\right)\right] \tag{50}$$

where $\gamma_1 = C_s$.

When the ML is used for parameter estimation, the asymptotic variance and covariances of parameter estimators are given by:

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

$$\begin{bmatrix} \mathrm{var}(\alpha) & \mathrm{cov}(\alpha,\beta) & \mathrm{cov}(\alpha,\gamma) \\ \mathrm{cov}(\beta,\alpha) & \mathrm{var}(\beta) & \mathrm{cov}(\beta,\gamma) \\ \mathrm{cov}(\gamma,\alpha) & \mathrm{cov}(\gamma,\beta) & \mathrm{var}(\gamma) \end{bmatrix}$$

$$= \begin{bmatrix} E\left(-\frac{\partial^2 \log L}{\partial^2 \alpha}\right) & E\left(-\frac{\partial^2 \log L}{\partial\alpha\partial\beta}\right) & E\left(-\frac{\partial^2 \log L}{\partial\alpha\partial\gamma}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial\beta\partial\alpha}\right) & E\left(-\frac{\partial^2 \log L}{\partial^2 \beta}\right) & E\left(-\frac{\partial^2 \log L}{\partial\beta\partial\gamma}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial\gamma\partial\alpha}\right) & E\left(-\frac{\partial^2 \log L}{\partial\gamma\partial\beta}\right) & E\left(-\frac{\partial^2 \log L}{\partial^2 \gamma}\right) \end{bmatrix}^{-1} = \mathbf{I}^{-1}(\alpha,\beta,\gamma) \tag{51}$$

According to the derivation in Appendix A, the variances and covariances are written as:

$$\begin{bmatrix} \mathrm{var}(\alpha) & \mathrm{cov}(\alpha,\beta) & \mathrm{cov}(\alpha,\gamma) \\ \mathrm{cov}(\beta,\alpha) & \mathrm{var}(\beta) & \mathrm{cov}(\beta,\gamma) \\ \mathrm{cov}(\gamma,\alpha) & \mathrm{cov}(\gamma,\beta) & \mathrm{var}(\gamma) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{n\alpha^2 D}\left[\frac{\psi'(\beta)}{(\beta-2)} - \frac{1}{(\beta-1)^2}\right] & -\frac{1}{n\alpha^3 D}\left(\frac{1}{\beta-2} - \frac{1}{\beta-1}\right) & \frac{1}{n\alpha^2 D}\left[\frac{1}{\beta-1} - \psi'(\beta)\right] \\ -\frac{1}{n\alpha^3 D}\left(\frac{1}{\beta-2} - \frac{1}{\beta-1}\right) & \frac{2}{\alpha^2(\beta-2)} & -\frac{1}{n\alpha^3 D}\left[\frac{\beta}{(\beta-1)} - 1\right] \\ \frac{1}{n\alpha^2 D}\left[\frac{1}{\beta-1} - \psi'(\beta)\right] & -\frac{1}{n\alpha^3 D}\left[\frac{\beta}{(\beta-1)} - 1\right] & \frac{1}{n\alpha^2 D}\left[\beta\psi'(\beta) - 1\right] \end{bmatrix} \tag{52}$$

where $\psi' = \psi'(\beta) = \frac{d^2 \log\Gamma(\beta)}{d\beta^2}$ is the tri-gamma function; $D = \frac{1}{(\beta-2)\alpha^4}\left[2\psi' - \frac{2\beta-3}{(\beta-1)^2}\right]$.

Substituting Equations (44)-(48) and the variance and covariance terms in Equation (52) into Equation (4) yields the asymptotic variance of the quantile estimator.

3.2.2 Estimation of asymptotic variances by POME

On the basis of POME, the relation between parameters and constraints for P3 distribution is given by [27]:

$$\begin{cases} E(x) = \alpha\beta + \gamma \\ E[\ln(x-\gamma)] = \ln(\alpha) + \psi \\ \mathrm{var}(x) = \alpha^2\beta \end{cases} \tag{53}$$

where $\psi = \psi(\beta)$ is digamma function.

Defining $W = \ln(x-\gamma)$, and replacing the expectations of Equation (67) by their sample

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

estimates. Thus the estimators $\hat{\alpha},\hat{\beta}$ and $\hat{\gamma}$ of the parameters can be obtained by solving the following equations:

$$\begin{cases} \alpha\beta+\gamma=\overline{X} \\ \ln(\alpha)+\psi(\beta)=\overline{W} \\ \alpha^2\beta=S^2 \end{cases} \tag{54}$$

Replacing the $P$, $Q$ and $R$ in Equation (14) by $\overline{X},\overline{W}$ and $S^2$, and the $\theta_1$, $\theta_2$ and $\theta_3$ by $\alpha$, $\beta$ and $\gamma$ yields $V_M$ and $V_p$, respectively:

$$V_M=\begin{bmatrix} \operatorname{var}(\overline{X}) \\ \operatorname{var}(S^2) \\ \operatorname{var}(\overline{W}) \\ \operatorname{cov}(\overline{X},S^2) \\ \operatorname{cov}(S^2,\overline{W}) \\ \operatorname{cov}(\overline{X},\overline{W}) \end{bmatrix}, V_p=\begin{bmatrix} \operatorname{var}(\hat{\alpha}) \\ \operatorname{var}(\hat{\beta}) \\ \operatorname{var}(\hat{\gamma}) \\ \operatorname{cov}(\hat{\alpha},\hat{\beta}) \\ \operatorname{cov}(\hat{\beta},\hat{\gamma}) \\ \operatorname{cov}(\hat{\alpha},\hat{\gamma}) \end{bmatrix} \tag{55}$$

For P3 distribution, the elements of $V_M$ are given by [38]:

$$V_M=\frac{1}{n}\begin{bmatrix} \alpha^2\beta \\ 2\alpha^4\beta(\beta+3) \\ \psi'(\beta) \\ 2\alpha^3\beta \\ \alpha^2 \\ \alpha \end{bmatrix} \tag{56}$$

Taking partial derivatives of $\overline{X},\overline{W}$ and $S^2$ with respect to $\alpha$, $\beta$ and $\gamma$ yields the matrix $D$:

$$D=\begin{bmatrix} \beta^2 & \alpha^2 & 1 & 2\alpha\beta & 2\alpha & 2\beta \\ 4\alpha^2\beta^2 & \alpha^4 & 0 & 4\alpha^3\beta^2 & 0 & 0 \\ 1/\alpha^2 & \psi'^2 & 0 & 2\psi'/\alpha & 0 & 0 \\ 2\alpha\beta & \alpha^3 & 0 & 3\alpha^2\beta & \alpha^2 & 2\alpha\beta \\ 2\beta & \alpha^2\psi' & 0 & \alpha(1+2\beta\psi') & 0 & 0 \\ \beta/\alpha & \alpha\psi' & 0 & 1+\beta\psi' & \psi' & 1/\alpha \end{bmatrix} \tag{57}$$

where $\psi'=\psi'(\beta)=\dfrac{d^2\log\Gamma(\beta)}{d\beta^2}$ is the tri-gamma function.

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

Thus all the terms of $V_M$ and $D$ are obtained; and so are the terms of $V_p$. Accordingly, the asymptotic variance of quantile estimator can be obtained.

### 3.3 Extreme value type 1 (EV1) distribution

The pdf and the cumulative distribution function of EV1 distribution can be expressed respectively as:

$$f(x)=\frac{1}{\alpha}\exp\left[-\frac{x-u}{\alpha}-\exp\left(-\frac{x-u}{\alpha}\right)\right] \tag{58}$$

$$F(x)=\exp\left[-\exp\left(-\frac{x-u}{\alpha}\right)\right] \tag{59}$$

where $\alpha$ and $u$ are the scale and shape parameters, respectively; and $-\infty < x < \infty$.

For EV1 distribution, the mean and variance are given by [43]:

$$E(x)=u+\alpha\varepsilon \tag{60}$$

$$\mu_2=\mathrm{var}(x)=\alpha^2\left(\frac{\pi^2}{6}\right) \tag{61}$$

where $\varepsilon$ is Euler constant, $\varepsilon=0.5772157$.

The *T*-year quantile of EV1 distribution can be obtained from Equation (59) by substituting $F(x)=1-1/T$ and solving for *x*:

$$\hat{x}_T=\hat{u}-\hat{\alpha}\log(-\log(1-1/T)) \tag{62}$$

Differentiating Equation (62) with $\alpha$ and *u* yields the derivatives of $x_T$ with respect to $\alpha$ and *u*:

$$\frac{\partial x_T}{\partial\alpha}=-\log(-\log(1-1/T)) \tag{63}$$

$$\frac{\partial x_T}{\partial u}=1 \tag{64}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

### 3.3.1 Estimation of asymptotic variance by MOM and ML

The cumulants of EV1 distribution are given by:

$$\gamma_1 = C_S = 1.1396 \tag{65}$$

$$\gamma_2 = C_k = 5.4002 \tag{66}$$

Introduction of Equation (65) and (66) in Equation (10) produces the asymptotic variance of MOM quantile estimator:

$$s_T^2 = \frac{\mu_2}{n}\left(1 + 1.1396K_T + 1.1K_T^2\right) \tag{67}$$

where $K_T$ can be obtained by substituting Equations (60)-(62) into Equation (1) as:

$$K_T = -\frac{\sqrt{6}}{\pi}\left[\log\left(-\log\left(1-\frac{1}{T}\right)\right) + \varepsilon\right] \tag{68}$$

For ML, according to the derivation in Appendix A, the information matrix of EV1 distribution is given by:

$$\mathbf{I} = -\begin{bmatrix} E\left(\frac{\partial^2 \log L}{\partial^2 \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial u}\right) \\ E\left(\frac{\partial^2 \log L}{\partial u \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial^2 u}\right) \end{bmatrix} = \frac{n}{\alpha^2}\begin{bmatrix} 1 & -0.4228 \\ -0.4228 & 1.8237 \end{bmatrix} \tag{69}$$

The variance and covariance terms for the ML parameter estimators are the elements of the inverse of the information matrix **I**:

$$\begin{bmatrix} \mathrm{var}(\alpha) & \mathrm{cov}(\alpha,u) \\ \mathrm{cov}(u,\alpha) & \mathrm{var}(u) \end{bmatrix} = \mathbf{I}^{-1}(\alpha,u) = \frac{\alpha^2}{n}\begin{bmatrix} 0.8046 & 0.2287 \\ 0.2287 & 1.1128 \end{bmatrix} \tag{70}$$

Substituting Equations (63) and (64), and the variance and covariance terms in Equation (70) into Equation (4) yields the asymptotic variance of the quantile estimator.

### 3.3.2 Estimation of asymptotic variances by POME

The relation between parameters and constraints of EV1 distribution can be expressed as

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

[27]:

$$\begin{cases} E\left[\dfrac{(x-u)}{\alpha}\right] = \varepsilon \\ E\left[\exp\left(-\dfrac{x-u}{\alpha}\right)\right] = 1 \end{cases} \tag{71}$$

Defining $y = (x-u)/\alpha$ and $V = \exp(-y)$, and replacing the expectations of Equation (71) by their sample estimates. Thus the estimators $\hat{\alpha}$ and $\hat{u}$ of the parameters can be obtained by solving the following equations:

$$\begin{cases} \overline{Y} = \varepsilon \\ \overline{V} = 1 \end{cases} \tag{72}$$

The variances and covariances of the moments and parameter estimators are written respectively as:

$$V_M = \begin{bmatrix} \mathrm{var}(\overline{Y}) \\ \mathrm{var}(\overline{V}) \\ \mathrm{cov}(\overline{Y},\overline{V}) \end{bmatrix}, V_p = \begin{bmatrix} \mathrm{var}(\hat{\alpha}) \\ \mathrm{var}(\hat{u}) \\ \mathrm{cov}(\hat{\alpha},\hat{u}) \end{bmatrix} \tag{73}$$

According the derivations in the reference [37], the $V_M$ is given by:

$$V_M = \frac{1}{n}\begin{bmatrix} \pi^2/6 \\ 1 \\ -1 \end{bmatrix} \tag{74}$$

Taking partial derivatives of $\overline{Y}$ and $\overline{V}$ with respect to $\alpha$, and $u$ yields:

$$D = \alpha^{-2}\begin{bmatrix} \varepsilon^2 & 1 & 2\varepsilon \\ \overline{W}^2 & 1 & 2\overline{W} \\ -\varepsilon\overline{W} & -1 & -(\varepsilon+\overline{W}) \end{bmatrix} \tag{75}$$

where $W = y\exp(-y)$.

Substituting $V_M$ and $D$ (Equations (74) and (75)) into Equation (28) yields $V_P$. The asymptotic variance of quantile estimator can then be obtained by substituting the terms of $V_P$ and Equations (63) and (64) into Equation (4).

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

## 4 Simulation experiments

In this section, Monte Carlo simulation experiments were performed to evaluate the performance of POME in calculation of the asymptotic variances and confidence intervals of quantiles and to compare it with the MOM, and ML methods. *N*s=10000 sets of data were generated for samples with size *n* (*n*=30, 50, 80) from a Wakeby distribution with certain parameters ( $\xi = 0, \alpha = 2.5, \beta = 2.5, \gamma = 0.2$ and $\delta = 0.02$ ) [44,45]. The inverse function of the Wakeby distribution is given by [46]:

$$x(F) = \xi + \frac{\alpha}{\beta}\left[1-(1-F)^{\beta}\right] - \frac{\gamma}{\delta}\left[1-(1-F)^{-\delta}\right] \tag{76}$$

The purpose of the simulation is to evaluate the performance of POME, thus only the EV1 distribution was choose as a candidate distribution to fit the generate data. For each generated data set, the distribution parameters were then estimated for EV1 distribution and the asymptotic variances and confidence intervals of quantiles $\hat{x}_T$ corresponding to different return periods (*T*=10, 20, 50, 100, 200, and 500) were calculated. Table 1 lists the median values of the estimated quantiles and confidence intervals by each method.

From Table 1, generally for all cases with sample size *n*=30, 50, 80, it was observed that the median of quantiles estimated by the three methods were very close. For example, for the sample size *n*=50, the median of quantiles corresponding to the 100-year return period estimated by MOM, ML and POME are 2.64, 2.67 and 2.68, respectively. Meanwhile, compared with the results of MOM and ML, the quantiles estimated by POME have the highest lower limits, the lowest upper limits, and the narrowest confidence intervals. It indicates that POME performs better than MOM and ML in reducing the uncertainty of quantile estimators.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

For example, for the case with $n$=50, the lower limits of 100 years quantile estimators are 2.03, 2.18, and 2.29 when the MOM, ML and POME were used respectively, the corresponding upper limits are 3.26, 3.15 and 3.06, and the interval widths are 1.23, 0.96 and 0.77. In addition, generally for all return periods, it can be observed that the confidence intervals of quantile estimators, including the lower and upper limits and the interval width, decrease with the increasing sample size $n$. For example, for $n$=30, 50 and 80, the median values of interval widths for a return period of 200 years estimated by POME are 0.89, 0.87 and 0.80, respectively.

**Table 1** Median of estimated quantiles and 95% confidence intervals with generated data

| Sample size *n* | Return period *T* | MOM | | | | ML | | | | POME | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Quantile | Confidence interval | | | Quantile | Confidence interval | | | Quantile | Confidence interval | | |
| | | | Lower limit | Upper limit | Interval width | | Lower limit | Upper limit | Interval width | | Lower limit | Upper limit | Interval width |
| 30 | 10 | 1.51 | 1.18 | 1.84 | 0.67 | 1.51 | 1.25 | 1.77 | 0.52 | 1.51 | 1.31 | 1.72 | 0.41 |
| | 20 | 1.76 | 1.34 | 2.18 | 0.84 | 1.77 | 1.44 | 2.10 | 0.66 | 1.77 | 1.51 | 2.03 | 0.52 |
| | 50 | 2.09 | 1.55 | 2.63 | 1.08 | 2.10 | 1.68 | 2.52 | 0.84 | 2.10 | 1.77 | 2.44 | 0.67 |
| | 100 | 2.34 | 1.71 | 2.96 | 1.26 | 2.35 | 1.86 | 2.84 | 0.98 | 2.35 | 1.97 | 2.74 | 0.78 |
| | 200 | 2.58 | 1.87 | 3.30 | 1.44 | 2.59 | 2.04 | 3.15 | 1.12 | 2.60 | 2.16 | 3.05 | 0.89 |
| | 500 | 2.91 | 2.07 | 3.74 | 1.67 | 2.92 | 2.27 | 3.57 | 1.31 | 2.93 | 2.41 | 3.45 | 1.04 |
| 50 | 10 | 1.65 | 1.30 | 2.00 | 0.70 | 1.66 | 1.38 | 1.93 | 0.55 | 1.66 | 1.44 | 1.88 | 0.44 |
| | 20 | 1.95 | 1.52 | 2.38 | 0.86 | 1.97 | 1.63 | 2.30 | 0.67 | 1.97 | 1.70 | 2.24 | 0.54 |
| | 50 | 2.35 | 1.81 | 2.88 | 1.07 | 2.37 | 1.95 | 2.79 | 0.84 | 2.38 | 2.04 | 2.71 | 0.67 |
| | 100 | 2.64 | 2.03 | 3.26 | 1.23 | 2.67 | 2.18 | 3.15 | 0.96 | 2.68 | 2.29 | 3.06 | 0.77 |
| | 200 | 2.94 | 2.24 | 3.63 | 1.39 | 2.97 | 2.42 | 3.51 | 1.09 | 2.98 | 2.54 | 3.41 | 0.87 |
| | 500 | 3.33 | 2.53 | 4.13 | 1.60 | 3.36 | 2.73 | 3.99 | 1.26 | 3.37 | 2.87 | 3.87 | 1.00 |
| 80 | 10 | 1.60 | 1.27 | 1.92 | 0.66 | 1.60 | 1.34 | 1.86 | 0.52 | 1.61 | 1.40 | 1.81 | 0.41 |
| | 20 | 1.89 | 1.49 | 2.29 | 0.80 | 1.90 | 1.58 | 2.21 | 0.63 | 1.90 | 1.65 | 2.15 | 0.50 |
| | 50 | 2.26 | 1.77 | 2.76 | 0.99 | 2.28 | 1.89 | 2.67 | 0.78 | 2.29 | 1.98 | 2.60 | 0.62 |
| | 100 | 2.54 | 1.98 | 3.11 | 1.13 | 2.57 | 2.12 | 3.01 | 0.89 | 2.58 | 2.22 | 2.93 | 0.71 |
| | 200 | 2.83 | 2.19 | 3.47 | 1.28 | 2.85 | 2.35 | 3.35 | 1.00 | 2.86 | 2.47 | 3.26 | 0.80 |
| | 500 | 3.20 | 2.46 | 3.94 | 1.47 | 3.23 | 2.65 | 3.81 | 1.15 | 3.24 | 2.78 | 3.70 | 0.92 |

Figure 1 shows the median of standard errors of the quantiles estimators. Apparently, for all return periods, the POME yields the smallest standard errors of the quantile estimators regardless of the sample size. It could be said that the uncertainty of POME estimators is less

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

than that of MOM and ML estimators. Generally for all estimation methods, the standard errors of quantile estimators show a decreasing trend when the sample size increases from 30 to 80. Additionally, the decrease in standard errors is relatively smaller when POME was used. This might imply that the POME is less affected by sample size. Therefore, the performance of the POME is found to be superior to MOM and ML, and it is more robust since it is less affected sample size.

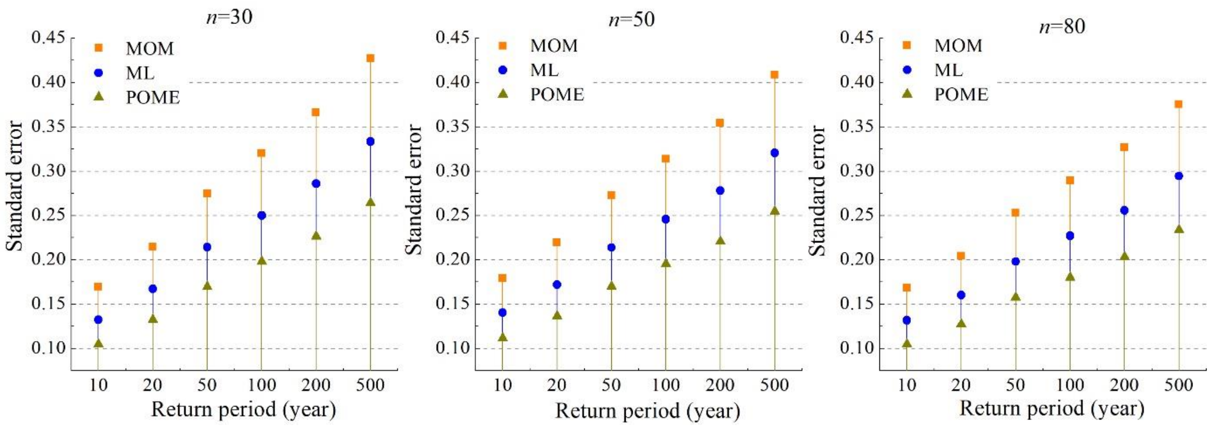


**Figure 1** Median of standard errors of quantile estimators for sample size $n$=30, 50 and 80

## 5 Application

The annual precipitation data from four gauging stations at the Weihe River basin in China were considered as case study. All data were obtained from the National Climate of China Meteorological Administration and were complete. The detailed information of these data is given in Table 2.

**Table 2** Basic information on each of the monthly precipitation series used in this study

| Station name | Record length (year) | Mean | Standard Deviation | Coefficient of Variation | Skewness | First-Order Serial Correlation Coefficient |
|---|---|---|---|---|---|---|
| Changwu | 51 | 580.6 | 131.8177 | 0.2270 | 0.5070 | 3.2153 |
| Lintong | 50 | 579.5 | 129.2014 | 0.2230 | 0.6299 | 3.7670 |
| Meixian | 50 | 578.0 | 129.7214 | 0.2245 | 0.5828 | 3.4614 |
| Tongguan | 52 | 605.5 | 143.4648 | 0.2369 | 0.5771 | 3.6438 |

The Gamma, P3 and EV1 distributions were used to fit the data set, and the MOM, ML

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

and POME were used to estimate the parameters of these distributions, as given in Table 3. It can be seen that the parameters of Gamma distribution estimated by MOM, ML and POME are very close, and so are the EV1 distribution, while those of the P3 distribution depart significantly.

**Table 3** Parameter values of each distribution estimated by the three methods

| Station name | Method | Gamma | | P3 | | | EV1 | |
|---|---|---|---|---|---|---|---|---|
| | | $\alpha$ | $\beta$ | $\alpha$ | $\beta$ | $\gamma$ | $\alpha$ | $u$ |
| Changwu | MOM | 29.9282 | 19.3993 | 15.5623 | 33.4147 | 60.5782 | 102.7783 | 521.2619 |
| | ML | 29.0510 | 19.9851 | 16.0097 | 32.5878 | 58.8660 | 114.9989 | 518.3959 |
| | POME | 29.3458 | 19.7893 | 11.1017 | 39.5620 | 141.3794 | 111.0142 | 516.5091 |
| Lintong | MOM | 28.8063 | 20.1169 | 10.0804 | 40.6938 | 169.2817 | 100.7383 | 521.3449 |
| | ML | 27.6873 | 20.9299 | 17.1488 | 30.6908 | 53.1797 | 113.0664 | 519.0634 |
| | POME | 28.6849 | 20.2192 | 10.3139 | 40.2305 | 164.5577 | 108.5086 | 516.8608 |
| Meixian | MOM | 29.1161 | 19.8498 | 11.7762 | 37.8015 | 132.7913 | 101.1438 | 519.5689 |
| | ML | 28.0875 | 20.5768 | 17.7271 | 30.3383 | 40.1383 | 113.7362 | 517.1681 |
| | POME | 28.7575 | 20.1089 | 10.8002 | 39.4726 | 151.6378 | 109.1477 | 514.9499 |
| Tongguan | MOM | 33.9927 | 17.8122 | 12.0090 | 41.3991 | 108.3228 | 111.8595 | 540.9202 |
| | ML | 32.9535 | 18.3740 | 15.0343 | 36.5741 | 55.6191 | 125.0433 | 537.9242 |
| | POME | 33.7243 | 17.9653 | 10.3756 | 44.5388 | 143.3688 | 120.6552 | 535.8444 |

To evaluate and compare the performance of the three methods and the distributions, the ordinary least square (OLS) criterion, Akaike information criterion (AIC), and Quasi-optimal deterministic coefficient test (QD) were employed that can be defined as:

$$OLS=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{x}_i)^2} \tag{77}$$

$$AIC=n\ln\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{x}_i)^2\right)+2m \tag{78}$$

$$QD=1-\frac{\sum_{i=1}^{n}(x_i-\hat{x}_i)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} \tag{79}$$

where $x_i$ and $\hat{x}_i$ are the observed data and the predicted values of a given ($i$-th) quantile, respectively, $\bar{x}$ is the mean value of observed data, $m$ is the number of parameters of a given

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

model, and $n$ is the sample size.

The OLS criterion is recommended as a curve optimization rule for measuring the difference between empirical and theoretical values in hydrological frequency analysis in China. The smaller OLS values represent the better performance of the model. The AIC is more appropriate for the comparison of models have different number of parameters. Given a set of candidate models for the data, the best model is the one with the minimum AIC value. QD is used to describe the fitting degree of observed values and theoretical values and the best fit model is the one that gets the QD value closest to 1. The OLS, AIC, and QD were calculated as given in Table 4.

It is seen from Table 4 that the selected best parameter estimation method for each distribution by the three criterions is coincident and the result of the best fitted distribution for each station by the three criterions is the same as well. Take the Changwu station in Table 4 for example. According to the smallest OLS and AIC values and the largest QD values, the POME, MOM and POME are suggested to be the best methods for parameter estimation for Gamma, P3 and EV1 distributions, respectively. And the best fitted distribution for Changwu station recommended by the OLS, AIC and QD criteria is P3 distribution. Additionally, according to the results given in Table 4, the best fitted distributions for the gauging stations Meixian, Tongguan and Lintong recommended by the OLS, AIC and QD methods, is EV1 distribution with the parameters estimated by POME. Thus the best estimation method for each station is POME and this is coincident with the results of the simulation experiments in Section 4, which shows that the performance of POME is better than MOM and ML. The bold values in the table denote the smallest OLS and AIC values and the largest QD values.

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

**Table 4** OLS, AIC and QD of five distributions for each data series calculated by MOM, ML and POME

| Station name | Method | Gamma | | | P3 | | | EV1 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | OLS | AIC | QD | OLS | AIC | QD | OLS | AIC | QD |
| Changwu | MOM | 16.6696 | 292.9861 | 0.9837 | 16.3741 | 291.1613 | 0.9843 | 19.3927 | 308.4197 | 0.9779 |
| | ML | 17.5392 | 298.1729 | 0.9819 | 17.0194 | 295.1041 | 0.983 | 16.8854 | 294.2978 | 0.9833 |
| | POME | 17.2128 | 296.2565 | 0.9826 | **16.0997** | **289.4375** | **0.9848** | 16.6415 | 292.8135 | 0.9837 |
| Lintong | MOM | 19.6949 | 304.0361 | 0.9763 | 18.516 | 297.8638 | 0.979 | 18.8765 | 299.792 | 0.9782 |
| | ML | 20.6502 | 308.7725 | 0.9739 | 20.1246 | 306.1944 | 0.9752 | 16.3975 | 285.7129 | 0.9836 |
| | POME | 19.7466 | 304.2982 | 0.9762 | 18.5442 | 298.0155 | 0.979 | **16.0775** | **283.742** | **0.9842** |
| Meixian | MOM | 17.9613 | 294.8219 | 0.9804 | 16.9804 | 289.2061 | 0.9825 | 17.8405 | 294.1469 | 0.9807 |
| | ML | 18.9359 | 300.106 | 0.9783 | 18.5121 | 297.8426 | 0.9792 | 15.0757 | 277.3083 | 0.9862 |
| | POME | 18.2446 | 296.3869 | 0.9798 | 16.8553 | 288.4664 | 0.9828 | **14.6984** | **274.7736** | **0.9869** |
| Tongguan | MOM | 20.419 | 319.7126 | 0.9793 | 20.0141 | 317.6294 | 0.9802 | 21.9677 | 327.3156 | 0.9761 |
| | ML | 21.2629 | 323.9242 | 0.9776 | 20.8105 | 321.6875 | 0.9785 | 19.5845 | 315.373 | 0.981 |
| | POME | 20.5825 | 320.542 | 0.979 | 19.9542 | 317.3178 | 0.9803 | **19.2731** | **313.7059** | **0.9816** |

Bold values indicate the smallest OLS and AIC values and the largest QD values.

The quantiles along with the standard errors and 95% confidence intervals for 10, 20, 50, 100, 200 and 500 year return periods of the best fitted distribution based on the parameters estimated by POME are given in Table 5. For the sake of comparison, the quantiles, standard errors and 95% confidence interval widths based on MOM and ML are also given in Table 5. The results show that the standard errors and confidence interval widths of quantile estimators obtained by POME are smaller than those obtained by MOM and ML methods, excepting the results of $T$=10 at Changwu station, which indicates that the POME yields more precise parameters and quantiles estimations.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 16 January 2019 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

**Table 5** Quantile estimators, standard error and 95% confidence interval widths based on MOM, ML and POME for the annual precipitation (mm)

| Station name | Best fitted distribution | Return period (year) | MOM | | | ML | | | POME | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | Quantile | Standard error | Confidence interval width | Quantile | Standard error | Confidence interval width | Quantile | Standard error | Confidence interval width |
| Changwu | P3 | 10 | 755.0 | 31.2 | 122.1 | 753.1 | 30.5 | 119.6 | 755.7 | 31.7 | 124.3 |
| | | 20 | 814.7 | 40.7 | 159.7 | 811.9 | 38.9 | 152.3 | 817.5 | 38.4 | 150.7 |
| | | 50 | 885.6 | 56.1 | 220.0 | 881.9 | 51.7 | 202.5 | 891.6 | 47.3 | 185.6 |
| | | 100 | 935.3 | 69.1 | 270.7 | 930.8 | 62.2 | 243.8 | 943.8 | 54.0 | 211.6 |
| | | 200 | 982.3 | 82.8 | 324.5 | 977.1 | 73.3 | 287.2 | 993.5 | 60.6 | 237.5 |
| | | 500 | 1041.4 | 101.8 | 398.9 | 1035.3 | 88.5 | 347.0 | 1056.3 | 69.2 | 271.2 |
| Lintong | EV1 | 10 | 704.4 | 31.8 | 124.7 | 724.5 | 32.2 | 126.1 | 714.0 | 30.6 | 120.0 |
| | | 20 | 748.0 | 37.5 | 146.9 | 773.5 | 37.0 | 144.9 | 761.0 | 35.0 | 137.1 |
| | | 50 | 820.6 | 47.4 | 185.7 | 854.9 | 45.2 | 177.3 | 839.2 | 42.5 | 166.7 |
| | | 100 | 984.8 | 70.7 | 277.1 | 1039.2 | 64.6 | 253.3 | 1016.0 | 60.3 | 236.2 |
| | | 200 | 1054.8 | 80.8 | 316.9 | 1117.8 | 73.1 | 286.4 | 1091.5 | 68.0 | 266.5 |
| | | 500 | 1147.3 | 94.3 | 369.7 | 1221.6 | 84.3 | 330.3 | 1191.1 | 78.2 | 306.6 |
| Meixian | EV1 | 10 | 703.3 | 31.9 | 125.2 | 723.8 | 32.4 | 126.9 | 713.3 | 30.8 | 120.7 |
| | | 20 | 747.2 | 37.6 | 147.5 | 773.1 | 37.2 | 145.8 | 760.6 | 35.2 | 137.9 |
| | | 50 | 820.0 | 47.6 | 186.4 | 855.0 | 45.5 | 178.3 | 839.1 | 42.8 | 167.6 |
| | | 100 | 984.8 | 71.0 | 278.3 | 1040.4 | 65.0 | 254.8 | 1017.0 | 60.6 | 237.6 |
| | | 200 | 1055.2 | 81.2 | 318.2 | 1119.5 | 73.5 | 288.1 | 1093.0 | 68.4 | 268.0 |
| | | 500 | 1148.0 | 94.7 | 371.2 | 1223.9 | 84.8 | 332.3 | 1193.2 | 78.7 | 308.4 |
| Tongguan | EV1 | 10 | 744.2 | 34.6 | 135.7 | 765.1 | 34.9 | 136.8 | 755.1 | 33.5 | 131.5 |
| | | 20 | 792.6 | 40.8 | 160.0 | 819.3 | 40.1 | 157.1 | 807.4 | 38.4 | 150.4 |
| | | 50 | 873.2 | 51.6 | 202.2 | 909.3 | 49.0 | 192.2 | 894.2 | 46.7 | 183.1 |
| | | 100 | 1055.5 | 77.0 | 301.8 | 1113.1 | 70.1 | 274.7 | 1090.9 | 66.3 | 259.9 |
| | | 200 | 1133.3 | 88.0 | 345.1 | 1200.1 | 79.2 | 310.6 | 1174.8 | 74.8 | 293.3 |
| | | 500 | 1236.0 | 102.7 | 402.5 | 1314.9 | 91.4 | 358.2 | 1285.5 | 86.1 | 337.6 |

To better understand the performance of the different methods, the differences in the uncertainty reductions for the standard errors and 95% confidence interval widths of the quantile estimators were given in terms of relative deviation as shown in Table 6. For the relatively long return period ($T \geq 50$), there are significant reductions in the standard errors and 95% confidence interval widths obtained by POME compared to MOM. For example, for a return period of $T$=500, the reductions in standard errors and the confidence interval widths are of about 32%, 17%, 17%, and 16% for Changwu, Lintong, Meixian and Tongguan,

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

respectively.

It can also be seen from Table 6 that, for Changwu station, the reduction in standard errors and 95% confidence interval widths obtained by POME are significant when compared to ML. For example, the reductions in the standard errors and confidence interval widths of a 500-year quantile is about 19%. For Lintong, Meixian and Tongguan stations, the reductions are relatively smaller, they are about 6%, 6% and 5% respectively. Overall, the POME provides more accurate quantile estimators.

**Table 6** Reduction in the uncertainty in quantile estimators based on POME compared with the MOM and ML methods (%)

| Station name | Return period (year) | POME to MOM | | | POME to ML | | |
|---|---|---|---|---|---|---|---|
| | | Quantile | Standard error | Confidence interval width | Quantile | Standard error | Confidence interval width |
| Changwu | 10 | 0.09 | 1.74 | 1.74 | 0.34 | 3.78 | 3.78 |
| | 20 | 0.34 | -5.63 | -5.63 | 0.68 | -1.02 | -1.02 |
| | 50 | 0.67 | -15.66 | -15.66 | 1.10 | -7.68 | -7.68 |
| | 100 | 0.91 | -21.83 | -21.83 | 1.39 | -11.88 | -11.88 |
| | 200 | 1.14 | -26.82 | -26.82 | 1.67 | -15.33 | -15.33 |
| | 500 | 1.43 | -32.02 | -32.02 | 2.02 | -18.99 | -18.99 |
| Lintong | 10 | 1.37 | -3.74 | -3.74 | -1.49 | -4.93 | -4.93 |
| | 20 | 1.74 | -6.67 | -6.67 | -1.67 | -5.29 | -5.29 |
| | 50 | 2.27 | -10.23 | -10.23 | -1.92 | -5.70 | -5.70 |
| | 100 | 3.17 | -14.76 | -14.76 | -2.35 | -6.18 | -6.18 |
| | 200 | 3.48 | -15.92 | -15.92 | -2.50 | -6.29 | -6.29 |
| | 500 | 3.82 | -17.05 | -17.05 | -2.66 | -6.41 | -6.41 |
| Meixian | 10 | 1.41 | -3.58 | -3.58 | -1.50 | -4.95 | -4.95 |
| | 20 | 1.79 | -6.51 | -6.51 | -1.68 | -5.32 | -5.32 |
| | 50 | 2.34 | -10.08 | -10.08 | -1.93 | -5.73 | -5.73 |
| | 100 | 3.27 | -14.62 | -14.62 | -2.37 | -6.21 | -6.21 |
| | 200 | 3.58 | -15.78 | -15.78 | -2.51 | -6.33 | -6.33 |
| | 500 | 3.93 | -16.92 | -16.92 | -2.68 | -6.44 | -6.44 |
| Tongguan | 10 | 1.47 | -3.14 | -3.14 | -1.35 | -3.92 | -3.92 |
| | 20 | 1.86 | -5.98 | -5.98 | -1.51 | -4.20 | -4.20 |
| | 50 | 2.41 | -9.45 | -9.45 | -1.73 | -4.53 | -4.53 |
| | 100 | 3.35 | -13.88 | -13.88 | -2.11 | -4.92 | -4.92 |
| | 200 | 3.66 | -15.00 | -15.00 | -2.23 | -5.02 | -5.02 |
| | 500 | 4.01 | -16.12 | -16.12 | -2.37 | -5.11 | -5.11 |

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

## 6 Conclusions

In this study, the POME method was applied for the estimation of the asymptotic variances and confidence intervals of quantiles and the corresponding calculation formulas for Gamma, P3 and EV1 distributions based on the POME method were deduced. The calculation procedures of MOM and ML methods were also reviewed briefly for comparison. Monte Carlo simulation experiments were carried out to evaluate the performance of the POME method and compare it with MOM and ML methods. In addition, annual precipitation data from four stations at the Weihe River basin in China were selected as case study. The following conclusions are drawn from this study:

(1) The calculation formulas of the asymptotic variances and confidence intervals of quantiles for three distributions based on POME are given. The results of simulation experiments and case study show that the POME method can provide an effective way for reducing the uncertainty of quantile estimators.

(2) Results of the simulation experiments demonstrate that the POME method yields the smallest standard errors and the narrowest confidence intervals of quantile estimators compared with the results of MOM and ML. Thus, the POME can give more accurate estimates. Furthermore, the standard errors of quantile estimators decrease as the sample size becomes large. The decreases in standard errors of the POME quantile estimators are not as significant as the decreases in the standard errors of the MOM and ML quantile estimators, which indicates that the POME is less affected by sample size.

(3) Results of the case study indicate that when using different criteria for distribution selection, the results are coincident, and the POME is the optimal method for parameter

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

estimation. Furthermore, the POME can give a more reliable precipitation quantiles since the standard errors and 95% confidence interval widths of precipitation quantiles obtained by POME are smaller than those obtained by MOM and ML methods.

Since the POME based confidence intervals of quantile estimators have not been investigated, in this study, the calculation formulas of asymptotic variances and confidence intervals based on POME were deduced for three commonly used distributions, and this study compares the performance of three methods for estimating the asymptotic variances and confidence intervals of quantiles. In addition, It is meaningful to carry out investigations about POME based asymptotic variances and confidence intervals of quantiles for more distributions.

**Acknowledge**

The present study is financially supported by the National Natural Science Foundation of China (Grant Nos. 51479171, 41501022 and 51409222). The authors would like to appreciate the editor and anonymous reviewers for their constructive comments which greatly improve the quality of this manuscript.

**Conflicts of Interest:** The authors declare no conflict of interest.

**Author Contributions:** Songbai Song and Ting Wei designed the computations; Songbai Song and Ting Wei wrote the paper. All authors have read and approved the final manuscript.

## Appendix A Derivation of information matrix

(1) Gamma distribution

$$E\left[\frac{\partial^2 \log L}{\partial \alpha^2}\right] = nE\left[\frac{\partial \log f(x)}{\partial \alpha}\frac{\partial \log f(x)}{\partial \alpha}\right] = nE\left[\frac{\partial^2 \log f(x)}{\partial^2 \alpha}\right] \tag{A1}$$

The $\log f(x)$ of Gamma distribution is

Peer-reviewed version available at *Entropy* **2019**, *21*, 315; doi:10.3390/e21030315

$$\log f(x) = -\beta\log\alpha - \log\Gamma(\beta) + (\beta-1)\log(x) - \frac{x}{\alpha} \tag{A2}$$

Differentiating Equation (A2) with respect to $\alpha$ and $\beta$, we obtain the derivatives of $\log f(x)$ with respect to $\alpha$ and $\beta$,

$$\frac{\partial \log f(x)}{\partial\alpha} = -\frac{\beta}{\alpha} + \frac{x}{\alpha^2}, \frac{\partial^2 \log f(x)}{\partial^2\alpha} = \frac{\beta}{\alpha^2} - \frac{2x}{\alpha^3}, \frac{\partial^2 \log f(x)}{\partial\alpha\partial\beta} = -\frac{1}{\alpha},$$

$$\frac{\partial \log f(x)}{\partial\beta} = -\log\alpha - \psi(\beta) + \log(x), \frac{\partial^2 \log f(x)}{\partial\beta\partial\alpha} = -\frac{1}{\alpha}, \frac{\partial^2 \log f(x)}{\partial\beta^2} = -\psi'(\beta).$$

For Gamma distribution, $E(x) = \alpha\beta$.

$$E\left(\frac{\partial^2 \log f(x)}{\partial^2\alpha}\right) = E\left(\frac{\beta}{\alpha^2} - \frac{2x}{\alpha^3}\right) = \frac{\beta}{\alpha^2} - \frac{2}{\alpha^3}E(x) = \frac{\beta}{\alpha^2} - \frac{2\alpha\beta}{\alpha^3} = -\frac{\beta}{\alpha^2} \tag{A3}$$

$$E\left(\frac{\partial^2 \log f(x)}{\partial^2\beta}\right) = E(-\psi'(\beta)) = -\psi'(\beta) \tag{A4}$$

$$E\left(\frac{\partial^2 \log f(x)}{\partial\alpha\partial\beta}\right) = E\left(-\frac{1}{\alpha}\right) = -\frac{1}{\alpha} \tag{A5}$$

The information matrix is then obtained

$$\begin{aligned}\mathbf{I} &= -\begin{bmatrix} E\left(\frac{\partial^2 \log L}{\partial^2\alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial\alpha\partial\beta}\right) \\ E\left(\frac{\partial^2 \log L}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial^2\beta}\right) \end{bmatrix} \\ &= -n\begin{bmatrix} E\left(\frac{\partial^2 \log f(x)}{\partial^2\alpha}\right) & E\left(\frac{\partial^2 \log f(x)}{\partial\alpha\partial\beta}\right) \\ E\left(\frac{\partial^2 \log f(x)}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 \log f(x)}{\partial^2\beta}\right) \end{bmatrix} = n\begin{bmatrix} \frac{\beta}{\alpha^2} & \frac{1}{\alpha} \\ \frac{1}{\alpha} & \psi'(\beta) \end{bmatrix}\end{aligned} \tag{A6}$$

where $\psi'(\beta) = \frac{\partial^2 \log\Gamma(\beta)}{\partial^2\beta}$.

(2) Pearson type 3 (P3) distribution

The $\log f(x)$ of P3 distribution is

 doi:10.20944/preprints201901.0164.v1

Peer-reviewed version available at Entropy 2019, 21, 315; doi:10.3390/e21030315

$$\log f(x) = -\log\Gamma(\beta) + (\beta-1)\log(x-\gamma) - \beta\log(\alpha) - \frac{1}{\alpha}(x-\gamma) \tag{A7}$$

Differentiating Equation (A7) with respect to $\alpha, \beta$ and $\gamma$ we obtain the derivatives of $\log f(x)$ with respect to $\alpha, \beta$ and $\gamma$,

$$\frac{\partial \log f(x)}{\partial\alpha} = -\frac{\beta}{\alpha} + \frac{x-\gamma}{\alpha^2}, \frac{\partial^2 \log f(x)}{\partial\alpha\partial\beta} = -\frac{1}{\alpha}, \frac{\partial^2 \log f(x)}{\partial\alpha\partial\gamma} = -\frac{1}{\alpha^2},$$

$$\frac{\partial^2 \log f(x)}{\partial^2\alpha} = \frac{\beta}{\alpha^2} - \frac{2(x-\gamma)}{\alpha^3}; \frac{\partial \log f(x)}{\partial\beta} = -\log\alpha - \psi(\beta) + \log(x-\gamma),$$

$$\frac{\partial^2 \log f(x)}{\partial\beta^2} = -\psi'(\beta), \frac{\partial^2 \log f(x)}{\partial\beta\partial\alpha} = -\frac{1}{\alpha}, \frac{\partial^2 \log f(x)}{\partial\beta\partial\gamma} = -\frac{1}{x-\gamma};$$

$$\frac{\partial \log f(x)}{\partial\gamma} = -\frac{\beta-1}{(x-\gamma)} + \frac{1}{\alpha}, \frac{\partial \log f(x)}{\partial\gamma\partial\alpha} = -\frac{1}{\alpha^2}, \frac{\partial \log f(x)}{\partial\gamma\partial\beta} = -\frac{1}{x-\gamma}, \frac{\partial^2 \log f(x)}{\partial^2\gamma} = -\frac{\beta-1}{(x-\gamma)^2}.$$

Then the expectations can be derived by using the characters of P3 distribution and the properties of gamma function: $E(x) = \alpha\beta + \gamma$, $\mathrm{var}(x) = \alpha^2\beta$, $\Gamma(\beta+1) = \beta\Gamma(\beta)$.

$$\begin{aligned} E\left(\frac{\partial^2 \log f(x)}{\partial^2\alpha}\right) &= E\left(\frac{\beta}{\alpha^2} - \frac{2(x-\gamma)}{\alpha^3}\right) = \frac{\beta}{\alpha^2} - \frac{2}{\alpha^3}E(x) + \frac{2\gamma}{\alpha^3} \\ &= \frac{\beta}{\alpha^2} - \frac{2}{\alpha^3}(\alpha\beta+\gamma) + \frac{2\gamma}{\alpha^3} = -\frac{\beta}{\alpha^2} \end{aligned} \tag{A8}$$

$$E\left(\frac{\partial^2 \log f(x)}{\partial^2\beta}\right) = E(-\psi'(\beta)) = -\psi'(\beta) \tag{A9}$$

$$E\left(\frac{\partial^2 \log f(x)}{\partial\alpha\partial\beta}\right) = E\left(-\frac{1}{\alpha}\right) = -\frac{1}{\alpha} \tag{A10}$$

$$E\left(\frac{\partial^2 \log f(x)}{\partial\alpha\partial\gamma}\right) = E\left(-\frac{1}{\alpha^2}\right) = -\frac{1}{\alpha^2} \tag{A11}$$

$$E\left(\frac{\partial^2 \log f(x)}{\partial\beta\partial\gamma}\right) = E\left(-\frac{1}{x-\gamma}\right) = -\frac{1}{\alpha(\beta-1)} \tag{A12}$$

$$E\left(\frac{\partial^2 \log f(x)}{\partial^2\gamma}\right) = E\left(-\frac{\beta-1}{(x-\gamma)^2}\right) = -\frac{1}{\alpha^2(\beta-2)} \tag{A13}$$

The information matrix is then obtained

Peer-reviewed version available at *Entropy* **2019**, *21*, 315; doi:10.3390/e21030315

$$\mathbf{I}(\alpha,\beta,\gamma)=-\begin{bmatrix} E\left(\frac{\partial^2 \log L}{\partial^2\alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 \log L}{\partial\alpha\partial\gamma}\right) \\ E\left(\frac{\partial^2 \log L}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial^2\beta}\right) & E\left(\frac{\partial^2 \log L}{\partial\beta\partial\gamma}\right) \\ E\left(\frac{\partial^2 \log L}{\partial\gamma\partial\alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial\gamma\partial\beta}\right) & E\left(\frac{\partial^2 \log L}{\partial^2\gamma}\right) \end{bmatrix} = n\begin{bmatrix} \frac{\beta}{\alpha^2} & \frac{1}{\alpha} & \frac{1}{\alpha^2} \\ \frac{1}{\alpha} & \psi'(\beta) & \frac{1}{\alpha(\beta-1)} \\ \frac{1}{\alpha^2} & \frac{1}{\alpha(\beta-1)} & \frac{1}{\alpha^2(\beta-2)} \end{bmatrix} \tag{A14}$$

(3) Extreme value type 1 (EV1) distribution

The $\log f(x)$ of EV1 distribution is

$$\log f(x) = -\log\alpha - \frac{x-u}{\alpha} - \exp\left(-\frac{x-u}{\alpha}\right) \tag{80}$$

Differentiating Equation (A15) with respect to $\alpha$ and $u$, we obtain the derivatives of $\log f(x)$ with respect to $\alpha$ and $u$,

$$\frac{\partial \log f(x)}{\partial\alpha} = -\frac{1}{\alpha} + \frac{x-u}{\alpha^2} - \frac{x-u}{\alpha^2}e^{\left(-\frac{x-u}{\alpha}\right)},$$

$$\frac{\partial^2 \log f(x)}{\partial\alpha\partial u} = -\frac{1}{\alpha^2} - \frac{x}{\alpha^3}e^{\left(-\frac{x-u}{\alpha}\right)} + \frac{\beta}{\alpha^3}e^{\left(-\frac{x-u}{\alpha}\right)} + \frac{1}{\alpha^2}e^{\left(-\frac{x-u}{\alpha}\right)},$$

$$\frac{\partial^2 \log f(x)}{\partial^2\alpha} = \frac{1}{\alpha^2} - \frac{2x}{\alpha^3} + \frac{2u}{\alpha^3} - \frac{x^2}{\alpha^4}e^{\left(-\frac{x-u}{\alpha}\right)} + \frac{2ux}{\alpha^4}e^{\left(-\frac{x-u}{\alpha}\right)} - \frac{u^2}{\alpha^4}e^{\left(-\frac{x-u}{\alpha}\right)} + \frac{2x}{\alpha^3}e^{\left(-\frac{x-u}{\alpha}\right)} - \frac{2u}{\alpha^3}e^{\left(-\frac{x-u}{\alpha}\right)};$$

$$\frac{\partial \log f(x)}{\partial u} = \frac{1}{\alpha} - \frac{1}{\alpha}e^{\left(-\frac{x-u}{\alpha}\right)}, \frac{\partial^2 \log f(x)}{\partial u\partial\alpha} = \frac{\partial^2 \log f(x)}{\partial\alpha\partial u}, \frac{\partial^2 \log f(x)}{\partial u^2} = -\frac{1}{\alpha^2}e^{\left(-\frac{x-u}{\alpha}\right)}.$$

For EV1 distribution, some relations have been derived in advance.

$$\begin{aligned} &E(x) = u + \alpha\varepsilon \\ &E\left(e^{\left(-\frac{x-u}{\alpha}\right)}\right) = 1 \\ &E\left(xe^{\left(-\frac{x-u}{\alpha}\right)}\right) = u - \alpha\psi(2) \\ &E\left(x^2e^{\left(-\frac{x-u}{\alpha}\right)}\right) = u^2 - 2\alpha u\psi(2) + \alpha^2\left\{[\psi(2)]^2 + \zeta(2,2)\right\} \end{aligned} \tag{A16}$$

where $\varepsilon$ is Euler constant, $\varepsilon = 0.5772$; $\zeta(2,2)$ is Hurwitz zeta function. Then the

Peer-reviewed version available at *Entropy* **2019**, *21*, 315; doi:10.3390/e21030315

expectations terms of information matrix can be obtained:

$$E\left(\frac{\partial^2 \log f(x)}{\partial^2 u}\right) = -\frac{1}{\alpha^2} E\left(e^{\left(-\frac{x-u}{\alpha}\right)}\right) = -\frac{1}{\alpha^2} \tag{A17}$$

$$\begin{aligned} E\left(\frac{\partial^2 \log f(x)}{\partial \alpha \partial u}\right) &= E\left(-\frac{1}{\alpha^2} - \frac{x}{\alpha^3} e^{\left(-\frac{x-u}{\alpha}\right)} + \frac{\beta}{\alpha^3} e^{\left(-\frac{x-u}{\alpha}\right)} + \frac{1}{\alpha^2} e^{\left(-\frac{x-u}{\alpha}\right)}\right) \\ &= -\frac{1}{\alpha^2} - \frac{1}{\alpha^3} E\left(x e^{\left(-\frac{x-u}{\alpha}\right)}\right) + \left(\frac{u}{\alpha^3} + \frac{1}{\alpha^2}\right) E\left(e^{\left(-\frac{x-u}{\alpha}\right)}\right) \\ &= \frac{\psi(2)}{\alpha^2} = \frac{0.4228}{\alpha^2} \end{aligned} \tag{A18}$$

$$\begin{aligned} & E\left(\frac{\partial^2 \log f(x)}{\partial^2 \alpha}\right) \\ &= E\left(\frac{1}{\alpha^2} - \frac{2x}{\alpha^3} + \frac{2u}{\alpha^3} - \frac{x^2}{\alpha^4} e^{\left(-\frac{x-u}{\alpha}\right)} + \frac{2ux}{\alpha^4} e^{\left(-\frac{x-u}{\alpha}\right)} - \frac{u^2}{\alpha^4} e^{\left(-\frac{x-u}{\alpha}\right)} + \frac{2x}{\alpha^3} e^{\left(-\frac{x-u}{\alpha}\right)} - \frac{2u}{\alpha^3} e^{\left(-\frac{x-u}{\alpha}\right)}\right) \\ &= \frac{1}{\alpha^2} - \frac{2}{\alpha^3} E(x) + \frac{2u}{\alpha^3} - \frac{1}{\alpha^4} E\left(x^2 e^{\left(-\frac{x-u}{\alpha}\right)}\right) \\ &\quad + \left(\frac{2u}{\alpha^4} + \frac{2}{\alpha^3}\right) E\left(x e^{\left(-\frac{x-u}{\alpha}\right)}\right) - \left(\frac{u^2}{\alpha^4} + \frac{2u}{\alpha^3}\right) E\left(e^{\left(-\frac{x-u}{\alpha}\right)}\right) \\ &= \frac{1}{\alpha^2} - \frac{2\varepsilon}{\alpha^2} - \frac{1}{\alpha^4} \alpha^2 \left([\psi(2)]^2 + \zeta(2,2)\right) - \frac{2\psi(2)}{\alpha^2} = \frac{1}{\alpha^2}\left(1 - 2\varepsilon - [\psi(2)]^2 - \zeta(2,2) - 2\psi(2)\right) \\ &= \frac{1}{\alpha^2}\left(1 - 2*0.5772157 - 0.4228^2 - 0.6450 - 2*0.4228\right) = -\frac{1.8237}{\alpha^2} \end{aligned} \tag{A19}$$

The information matrix is then obtained:

$$\begin{aligned} \mathbf{I} &= -\begin{bmatrix} E\left(\frac{\partial^2 \log L}{\partial^2 \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial u}\right) \\ E\left(\frac{\partial^2 \log L}{\partial u \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial^2 u}\right) \end{bmatrix} \\ &= -n \begin{bmatrix} E\left(\frac{\partial^2 \log f(x)}{\partial^2 \alpha}\right) & E\left(\frac{\partial^2 \log f(x)}{\partial \alpha \partial u}\right) \\ E\left(\frac{\partial^2 \log f(x)}{\partial u \partial \alpha}\right) & E\left(\frac{\partial^2 \log f(x)}{\partial^2 u}\right) \end{bmatrix} = \frac{n}{\alpha^2} \begin{bmatrix} 1 & -0.4228 \\ -0.4228 & 1.8237 \end{bmatrix} \end{aligned} \tag{A20}$$

Peer-reviewed version available at *Entropy* **2019**, *21*, 315; doi:10.3390/e21030315

# References:


1. Hamed K.; Rao A.R. *Flood Frequency Analysis*. Taylor & Francis Inc: Bosa Roca, United States, 1999.
2. Haan C.T. *Statistical Methods in Hydrology*. Iowa State University Press: Ames, Iowa, 2002.
3. Hosking J.R.M.;Wallis J.R. *Regional Frequency Analysis*. Cambridge University Press: New York, 1997.
4. Nash J.E.; Amorocho J. The Accuracy of the Prediction of Floods of High Return Period. *Water Resour. Res.* **1966**,2,191-198.
5 Bobee B. Sample error of T−year events commuted by fitting a Pearson type 3 distribution. *Water Resour. Res.* **1973**,9,1264-1270.
6. Kite G.W. Confidence Limits for Design Events. *Water Resour. Res.* **1975**,11,48-53.
7. Condie R. The Log Pearson Type 3 Distribution: The T-Year Event and Its Asymptotic Standard Error by Maximum Likelihood Theory. *Water Resour. Res.* **1977**,13,987-991.
8. Lawless J.F. Confidence Interval Estimation for the Weibull and Extreme Value Distributions. *Technometrics*. **1978**,20,355-364.
9. Hoshi K.; Burges S.J. Approximate estimation of the derivative of a standard gamma quantile for use in confidence interval estimates. *J. Hydrol.* **1981**,53,317-325.
10. Hoshi K.; Burges S.J. Sampling properties of parameter estimates for the log Pearson type 3 distribution, using moments in real space. *J. Hydrol.* **1981**,53,305-316.
11. Phien H.N. Variance of the T-year event in the log Pearson type-3 distribution. *J. Hydrol.* **1985**,77,141-158.
12. Stedinger J. Confidence Intervals for Design Events. *J. Hydraul. Eng.* **1983**,109,13-27.
13. Ashkar F.; Bobée B.; Fortier L. *Confidence Intervals for Design Flood Events under Different Statistical Flood Models*. Springer Netherlands, 1987.
14. Lu L.H.; Stedinger J.R. Variance of two- and three-parameter GEV/PWM quantile estimators: formulae, confidence intervals, and a comparison. *J. Hydrol.* **1992**,138,247-267.
15. Lu L.H.; Stedinger J.R. Sampling variance of normalized GEV/PWM quantile estimators and a regional homogeneity test. *J. Hydrol.* **1992**,138,223-245.
16. Heo J.H.; Salas J.D. Estimation of quantiles and confidence intervals for the log-Gumbel distribution. *Stochastic Hydrology & Hydraulics*. **1996**,10,187-207.
17. Ashkar F.; Ouarda T.B. M. J. Approximate Confidence Intervals for Quantiles of Gamma and Generalized Gamma Distributions. *J. Hydrol. Eng.* **1998**,3,43-51.
18. Heo J.H.; Salas D.J.; Kim K.D. Estimation of confidence intervals of quantiles for the Weibull distribution. *Stoch. Env. Res. Risk a.* **2001**,15,284-309.
19. Su S. Confidence intervals for quantiles using generalized lambda distributions. *Comput. Stat. Data an.* **2009**,53,3324-3333.
20. Shin H.; Kim T.; Kim S.; Heo J.H. Estimation of asymptotic variances of quantiles for the generalized logistic distribution. *Stoch. Environ. Res. Risk Assess*. 2010,24,183-197.
21. Tajvidi N. Confidence Intervals and Accuracy Estimation for Heavy-Tailed Generalized Pareto Distributions. *Extremes*. **2003**,6,111-123.
22. Ashkar F.; Adlouni S.E. Correcting confidence intervals for quantiles of a heavy-tailed distribution: Case of the two-parameter Kappa distribution. *J. Hydrol.* **2014**,512,498-505.
23. Shannon E.C. A mathematical theory of communication. *Bell Labs Technical Journal*. **1948**,4,379-423.
24. Jaynes E.T. On the rationale of maximum-entropy methods. *P. Ieee*. **1982**,70,939-952.
25. Sonuga J.O. Principle of maximum entropy in hydrologic frequency analysis. *J. Hydrol.* **1972**,17,177-191.
26. Singh V.P.; Rajagopal A.K.; Singh K. Derivation of some frequency distributions using the principle of maximum entropy (POME). *Adv. Water Resour.* **1986**,9,91-106.

Peer-reviewed version available at *Entropy* **2019**, *21*, 315; doi:10.3390/e21030315

27. Singh V.P. Entropy-based parameter estimation in hydrology. Kluwer Academic Publishers: Boston, MA, USA, 1998.
28. Fiorentino M.; Arora K.; Singh V.P. The two-component extreme value distribution for flood frequency analysis: Derivation of a new estimation method. *Stochastic Hydrol. & Hydraul*. **1987**,1,199-208.
29. Singh V.P.; Deng Z.Q. Entropy-Based Parameter Estimation for Kappa Distribution. *J. Hydrol. Eng.* **2003**,8,81-92.
30. Song S.; Song X.; Kang Y. Entropy-Based Parameter Estimation for the Four-Parameter Exponential Gamma Distribution. *Entropy-Switz.* **2017**,19,189.
31. Chen L.; Singh V.; Xiong F. An Entropy-Based Generalized Gamma Distribution for Flood Frequency Analysis. *Entropy-Switz.* **2017**,19,239.
32. Lu C.; Singh V.P. Entropy-based derivation of generalized distributions for hydrometeorological frequency analysis. *J. Hydrol.* **2018**,557,699-712.
33. Hao Z.; Singh V.P. Entropy-Based Method for Bivariate Drought Analysis. *J. Hydrol. Eng.* **2013**,18,780-786.
34 Hao Z.; Singh V.P. Modeling multisite streamflow dependence with maximum entropy copula. *Water Resour. Res.* **2013**,49,7139-7143.
35. Kong X.M.; Huang G.H.; Fan Y.R.; Li Y.P. Maximum entropy-Gumbel-Hougaard copula method for simulation of monthly streamflow in Xiangxi river, China. *Stoch. Env. Res. Risk a.* **2015**,29,833-846.
36. Aghakouchak A. Entropy−Copula in Hydrology and Climatology. *J. Hydrometeorol.* **2014**,15,2176-2189.
37. Phien H.N. Sampling properties of the maximum entropy estimators for the extreme-value type-1 distribution. *J. Hydrol.* **1986**,86,391-398.
38. Phien H.N.; Nguyen V.T.V. Variances and covariances of the maximum entropy estimators for the Pearson type-3 distribution. *Can. J. Civil Eng.* **1990**,17,590-596.
39. Chow V.T. A General Formula For Hydrologic Frequency Analysis. *Eos Trans. Am. Geophys. Union*. **1951**,32,231-237.
40. Silva A.T.; Portela M.M.; Baez J.; Naghettini M. Construction of confidence intervals for extreme rainfall quantiles. In 8th International Conference on Simulation in Risk Analysis and Hazard Mitigation, RISK 2012. September 19, 2012-September 21, 2012. WITPress: Island of Brac, Croatia, 2012:293-304.
41. Mood A.; Graybill F. *Introduction to the theory of statistics*. McGraw-Hill: New York, 1974.
42. Kendall M.; Stuart A. *The Advanced Theory of Statistics*. Charles Griffin and Go. Ltd.: London, United Kingdom,1979.
43. Jowitt P.W. The extreme-value type-1 distribution and the principle of maximum entropy. *J. Hydrol.* **1979**,42,23-38.
44. Xiong F.; Guo S.; Chen L.; Yin J.; Liu P. Flood Frequency Analysis Using Halphen Distribution and Maximum Entropy. *J. Hydrol. Eng.* **2018**,23,04018012.
45. Landwehr J.M.; Matalas N.C.; Wallis J.R. Quantile Estimation With More or Less Floodlike Distributions. *Water Resour. Res.* **1980**,16,547-555.
46. Houghton J.C. Birth of a Parent: The Wakeby Distribution for Modeling Flood Flows. *Water Resour. Res.* **1978**,14,1105-1109.