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The line element approach for the geometry of Poincaré disk

Jong Ryul Kim

Abstract The geometry of Poincaré disk itself is interpreted without any mapping to different spaces. Our approach might be one of the shortest and is intended for educational contribution.

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Email: kimjr0@kunsan.ac.kr

Address: Department of Mathematics, Kunsan National University, Kunsan, 573-701, Republic of Korea

1. Introduction

For vectors $v_p = (v_1, v_2)$ and $w_p = (w_1, w_2)$ in 2-dimensional Euclidean space $\mathbb{R}^2$, the norm of a vector $|v_p| = \sqrt{v_p \cdot v_p} = \sqrt{v_1^2 + v_2^2}$ which is the Pythagorean theorem is defined by the dot product

$$v_p \cdot w_p = v_1 w_1 + v_2 w_2$$

and the angle θ formed by two vector $v, w \in \mathbb{R}^2$ is given by $\cos\theta = \frac{v \cdot w}{|v||w|}$. The arc length of a differentiable curve $\alpha(t)$ in $\mathbb{R}^2$ from $\alpha(0)$ to $\alpha(1)$ is given by

$$\int_0^1 |\alpha'(t)|\, dt = \int_0^1 \sqrt{\alpha'(t) \cdot \alpha'(t)}\, dt$$

and the arc length of a piecewise differentiable curve is the sum of the arc length of differentiable parts. The distance from $\alpha(0)$ to $\alpha(1)$ is defined by the shortest arc length among all curves. We can easily show that the straight line from $\alpha(0)$ to $\alpha(1)$ is the shortest arc length when the dot product is given on $\mathbb{R}^2$. Thus by considering an inner product $g(v, w)$ on a vector space $V \subset \mathbb{R}^2$ and defining the arc length of a curve $\alpha(t)$ by

$$\int_0^1 |\alpha'(t)|\, dt = \int_0^1 \sqrt{g(\alpha'(t), \alpha'(t))}\, dt,$$

we can have a distance different from Euclidean geometry. A geometry where four Euclidean postulates except for the Parallel one hold is known as absolute geometry ([6]). A non-Euclidean geometry with an inner product g on Poincaré disk $D_P = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2<1\}$ satisfying the following three observations is going to be determined.

• First, if $g(v_p,w_p) = f(p)(v_p \cdot w_p)$, then the angle defined by an inner product is equal to the angle defined by the dot product

$$\begin{aligned}\cos\theta &= \frac{g(v_p,w_p)}{\sqrt{g(v_p,v_p)}\sqrt{g(w_p,w_p)}} \\ &= \frac{f(p)(v_p \cdot w_p)}{\sqrt{f(p)(v_p\cdot v_p)}\sqrt{f(p)(w_p\cdot w_p)}} \\ &= \frac{v_p\cdot w_p}{|v_p||w_p|} \\ &= \cos\theta\,.\end{aligned}$$

• Second, the geometry of Poincaré disk D_P is assumed to be rotationally symmetric, that is, the geometry of a neighborhood at p is isometric to that of a neighborhood at any point q related to p by rotation of any angle. It means that a function $f(r,\theta) = f(x,y)$ in $g(v_p,w_p) = f(p)(v_p \cdot w_p)$ depends only on r for the polar coordinates $p = (x,y) = (r\cos\theta, r\sin\theta) \in D_P$.

• Third, the Euclidean norm of a vector $v_p = (v_1,v_2)$ at $p = (p_1,p_2) \in D_P$ with $(p_1+v_1, p_2+v_2) \in D_P$ must be scaled to infinity as p goes to the boundary of D_P, since the boundary is considered to be a circle of radius ∞.

Under these three assumptions, one of the simplest candidates for an inner product g on Poincaré disk D_P could be

$$g(v_p,w_p) = \frac{2(v_p\cdot w_p)}{1-(x^2+y^2)}$$

for all points $p=(x,y) \in D_P$ and scaling constant 2. The line element ds^2 of Poincaré disk is

$$ds^2 = \frac{4(dx^2+dy^2)}{(1-(x^2+y^2))^2}.$$

Let $\alpha(t) = (x(t),y(t))$ be a differentiable curve from $\alpha(0)$ to $\alpha(1)$ in D_P. The arc length of $\alpha(t)$ from $\alpha(0) =$ to $\alpha(1)$ is

$$\int_0^1 \sqrt{g(\alpha'(t),\alpha'(t))}\,dt = \int_0^1 \frac{2|\alpha'(t)|}{1-|\alpha(t)|^2}\,dt,$$

where $|\alpha'(t)| = \sqrt{\alpha'(t)\cdot\alpha'(t)}$ and $|\alpha(t)|^2 = x(t)^2+y(t)^2$. The distance $d(\alpha(0),\alpha(1))$ is the shortest arc length among all curve from $\alpha(0)$ to $\alpha(1)$

$$d(\alpha(0),\alpha(1)) = \inf_\alpha \int_0^1 \frac{2|\alpha'(t)|}{1-|\alpha(t)|^2}\,dt\ .$$

We are going to find a shortest path between any two points in Poincaré disk by using a distance-preserving biholomorphic mapping on Poincaré disk or a linear fractional transformation which preserves the cross ratio and the distance. We also show that Poincaré Disk is isometric to one connected component of two-sheeted hyperboloid $-x^2+y^2+z^2=-1$ in 3-dimensional Minkowski space-time and the sum of the interior angles of a triangle, a Saccheri quadrilateral on Poincaré disk is less than π, 2π, respectively.

There are plenty lecture notes, papers([1],[5], [7]) and books ([2] [3],[4], [6]) on the hyperbolic geometry. The picture of the hyperbolic geometry is well-known. Here we suggest intuitive and direct approaches for the effective understanding of the hyperbolic geometry.

2. A distance-preserving biholomorphic mapping on Poincaré disk

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A function $f : D_P \longrightarrow D_P \subset \mathbb{C}$ is said to be holomorphic if $f(x,y) = u(x,y) + iv(x,y)$ satisfies the Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

A biholomorphic function is a holomorphic function f which is bijective and whose inverse f^{-1} is also holomorphic. Let $\alpha(t) = x(t) + iy(t)$ be a differentiable curve from $\alpha(0) = z_1$ to $\alpha(1) = z_2$ in D_P. For $f(z_1) = (f \circ \alpha)(0)$ and $f(z_2) = (f \circ \alpha)(1)$, the distance $d(f(z_1), f(z_2))$ is

$$d(f(z_1), f(z_2)) = \inf_{f(\alpha)} \int_0^1 \frac{2|(f \circ \alpha)'(t)|}{1 - |(f \circ \alpha)(t)|^2}\, dt\ .$$

We show that if f is a holomorphic function on D_P, then $d(f(z_1), f(z_2)) \le d(z_1, z_2)$.

Schwarz-Pick Theorem For a holomorphic function $f : D_P \longrightarrow D_P$, it holds that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}\ .$$

Proof. Define

$$g(z) = \frac{z_1 - z}{1 - \overline{z_1} z}, \quad h(z) = \frac{f(z_1) - z}{1 - \overline{f(z_1)} z} \quad z_1 \in D_P.$$

Since $g_{-1}(0) = z_1$, we have $h\Big(f(g^{-1}(0))\Big) = 0$. Using the Schwarz Lemma, we get

$$\Big|h\Big(f(g^{-1}(z))\Big)\Big| \le |z|.$$

Hence

$$\Big|h\Big(f(z)\Big)\Big| = \Big|h\Big(f(g^{-1}(g(z)))\Big)\Big| \le |g(z)|$$

$$\left|\frac{f(z_1) - f(z)}{1 - \overline{f(z_1)} f(z)}\right| \le \left|\frac{z_1 - z}{1 - \overline{z_1} z}\right|$$

$$\lim_{z \to z_1} \left|\frac{\frac{f(z_1) - f(z)}{z_1 - z}}{1 - \overline{f(z_1)} f(z)}\right| \le \lim_{z \to z_1} \left|\frac{1}{1 - \overline{z_1} z}\right|\ .$$

So we have

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}\ .$$

For $z_1 = \alpha(0)$, $z_2 = \alpha(1)$ and $f(z_1) = (f \circ \alpha)(0)$, $f(z_2) = (f \circ \alpha)(1)$, we get

$$\inf_{f(\alpha)} \int_0^1 \frac{2|f'(\alpha(t))||\alpha'(t)|}{1 - |f(\alpha(t))|^2}\, dt \le \inf_{\alpha} \int_0^1 \frac{2|\alpha'(t)|}{1 - |\alpha(t)|^2}\, dt = d(z_1, z_2)$$

by Schwarz-Pick Theorem. So we get ([5])

$$d(f(z_1), f(z_2)) \le d(z_1, z_2). \tag{1}$$

If f^{-1} is a holomorphic function on D_P, then we have

$$d(z_1, z_2) = d(f^{-1}(f(z_1)), f^{-1}(f(z_2))) \le d(f(z_1), f(z_2))$$

by applying the above (1). Hence we obtain

$$d(f(z_1), f(z_2)) = d(z_1, z_2) \tag{2}$$

for a biholomorphic function f on D_P.

3. A shortest path between any two points in Poincaré disk

We show that a straight line connecting the origin and an arbitrary point $p \in D_P$ is a shortest path. Let $\alpha(t) = (x(t), y(t)) = (r(t)\cos\theta(t), r(t)\sin\theta(t))$ be a differentiable curve from $\alpha(0) = (0,0)$ to $\alpha(1) = (x(1), y(1)) = p$ for the polar coordinates. Calculations show that

$$\alpha'(t) = (r'(t)\cos\theta(t) - r(t)\sin\theta(t)\theta'(t),\ r'(t)\sin\theta(t) + r(t)\cos\theta(t)\theta'(t))$$

$$\begin{aligned}\int_0^1 \sqrt{g(\alpha'(t),\alpha'(t))}\,dt &= \int_0^1 \frac{2\sqrt{(r'(t))^2 + (r(t))^2(\theta'(t))^2}}{1-r(t)^2}\,dt \\ &\geq \int_0^1 \frac{2\sqrt{(r'(t))^2}}{1-r(t)^2}\,dt \\ &= \int_0^1 \frac{2|r'(t)|}{1-r(t)^2}\,dt,\end{aligned}$$

where we use the inequality

$$\sqrt{(r'(t))^2 + (r(t))^2(\theta'(t))^2} \geq \sqrt{(r'(t))^2}.$$

It means that the length of an arbitrary curve connecting the origin and an arbitrary point $p \in D_P$ is greater than equal to the length of a straight line which is the case of a constant $\theta_0 = \theta(t)$ connecting the origin and an arbitrary point $p \in D_P$. Fix two constants $\cos\theta(t) = a$ and $\sin\theta(t) = b$. Let $\alpha(t) = (at, bt)$ be a straight line from $\alpha(0) = (0,0)$ to $\alpha(1) = (a,b)$ for $c = \sqrt{a^2+b^2} < 1$. Then we have

$$\begin{aligned}\int_0^1 \sqrt{g(\alpha'(t),\alpha'(t))}\,dt &= \int_0^1 \frac{2\sqrt{a^2+b^2}}{1-(a^2+b^2)t^2}\,dt \\ &= \int_0^1 \frac{2c}{1-c^2t^2}\,dt \\ &= \int_0^1 \frac{c}{1+ct} - \frac{-c}{1-ct}\,dt \\ &= \ln\frac{1+c}{1-c}.\end{aligned}$$

Also let $\alpha(t) = (0, ct)$ be a straight line from $\alpha(0) = (0,0)$ to $\alpha(1) = (0,c)$ for $c < 1$. Then we have

$$\begin{aligned}\int_0^1 \sqrt{g(\alpha'(t),\alpha'(t))}\,dt &= \int_0^1 \frac{2c}{1-c^2t^2}\,dt \\ &= \int_0^1 \frac{c}{1+ct} - \frac{-c}{1-ct}\,dt \\ &= \ln\frac{1+c}{1-c}.\end{aligned}$$

A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ of the form

$$f(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)$$

is called a linear fractional transformation. Note that a Möbius transformation f

$$f(z) = \frac{z+a}{\bar{a}z+1} \tag{3}$$

for $|a|^2 = a\bar{a} < 1$ and $a \in \mathbb{C}$ is a linear fractional transformation which is a bijective mapping on D_P, since

$$|z+a|^2 - |\bar{a}z+1|^2 = -(1-|a|^2)(1-|z|^2) < 0$$

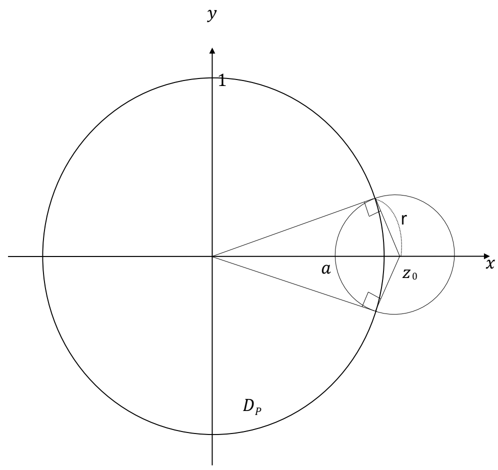

FIGURE 1. A shortest path through a point $(a, 0)$ on Poincaré disk

and $f(0) = a$. It is clear that

$$f(z) = \frac{z+a}{\bar{a}z+1} = \frac{1}{\bar{a}} + \frac{a-\frac{1}{\bar{a}}}{\bar{a}z+1}$$

is biholomorphic, since $f(z) = az + b$ for $a, b \in \mathbb{C}$ and $f(z) = \frac{1}{z}$ are holomorhic.

Let C be a circle with center $z_0 = (x_0, y_0)$ and radius r on the complex plane $\mathbb{C}$, that is,

$$(z - z_0)\overline{(z - z_0)} = r^2$$

$$z\bar{z} - \bar{z}_0 z - z_0\bar{z} + z_0\bar{z}_0 - r^2 = 0. \tag{4}$$

We show that a part of C in D_P which meets orthogonally at two points of the boundary of Poincaré disk is a shortest path. Rewrite (4) as

$$z\bar{z} + \delta z + \bar{\delta}\bar{z} + \gamma = 0 \quad \text{for } \delta = -\overline{z_0} \text{ and } |z_0|^2 - r^2 = \gamma\ . \tag{5}$$

Let L be a line through the origin on the complex plane $\mathbb{C}$

$$L = \{(x, y) \mid cx + dy = 0\}. \tag{6}$$

Rewrite (6) as

$$\beta z + \bar{\beta}\bar{z} = 0 \quad \text{for } \beta = \frac{c - id}{2} \text{ and } z = x + iy\,. \tag{7}$$

We find the image of $y-$axis (that is, $\bar{\beta} = \beta$ by (7)) in D_P by a biholomorphic function

$$f(z) = \frac{z+a}{az+1} \quad \text{for a real } a \in D_P.$$

Recall that $d(f(z_1), f(z_2)) = d(z_1, z_2)$ (2). $\bar{\beta} = \beta$ implies $d = 0$. So we get $x = 0$. Put $w = \frac{z+a}{az+1}$. Then we get $z = \frac{w-a}{-aw+1}$. From the equation (7)

$$\beta\frac{w-a}{-aw+1} + \beta\overline{\left(\frac{w-a}{-aw+1}\right)} = 0,$$

we get a circle equation (5)

$$(-2a\beta)w\bar{w} + (\beta + a^2\beta)w + (\beta + a^2\beta)\bar{w} + (-2a\beta) = 0$$

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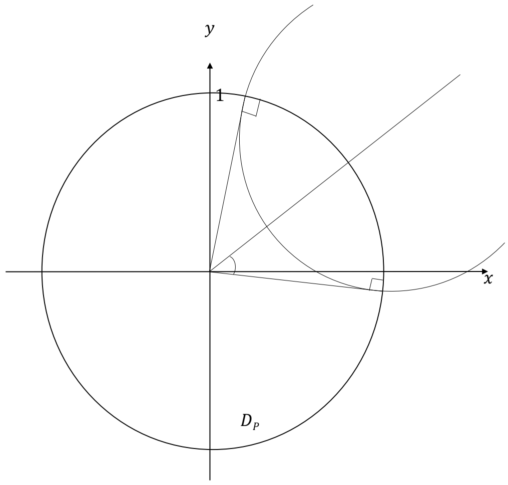

FIGURE 2. A rotation on Poincaré disk

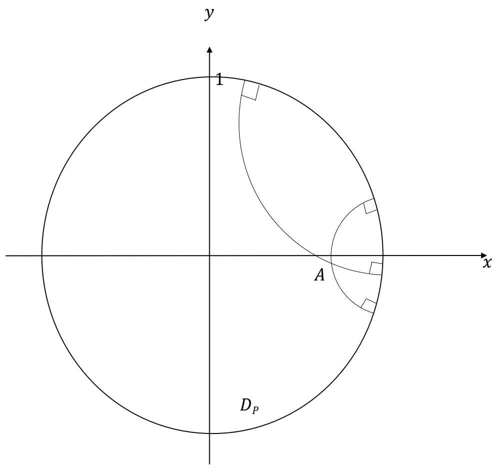

FIGURE 3. Two parallels to y-axis through a point A on Poincaré disk

$$w\bar{w} - \frac{1}{2}\Big(a + \frac{1}{a}\Big)w - \frac{1}{2}\Big(a + \frac{1}{a}\Big)\bar{w} + 1 = 0$$

with

$$-\delta = \overline{z_0} = z_0 = \frac{1}{2}\Big(a + \frac{1}{a}\Big) > \sqrt{a\,\frac{1}{a}} = 1 \tag{8}$$

and

$$|z_0|^2 - r^2 = \gamma = 1. \tag{9}$$

Two circles meet at two points with $x = \frac{1}{z_0}$, since

$$x^2 + y^2 = 1, \quad (x - z_0)^2 + y^2 = r^2$$

$$1 - x^2 = r^2 - (x - z_0)^2 = r^2 - x^2 + 2z_0x - z_0^2 = -1 - x^2 + 2z_0x$$

or

$$x^2 + y^2 = 1, \quad (x - z_0)^2 + y^2 = z_0^2 - 1$$
$$1 - x^2 = z_0^2 - 1 - (x - z_0)^2, \quad 2 = 2z_0 x.$$

As a goes to 0, $\frac{1}{z_0}$ goes to 0 by (8). The equations (8) and (9) imply that the center and radius of a circle depending on a are on the hyperbola $|z_0|^2 - r^2 = 1$. As a goes to 1, $\frac{1}{z_0}$ goes to 1 and the norm of radius of a circle goes to zero by $|z_0|^2 - r^2 = 1$.

Two circles meet orthogonally at two points by the following two facts. The equation $|z_0|^2 = r^2 + 1$ implies that the triangle consisting of three points (the origin, z_0 and one of the two meeting points) is a right triangle. Note that the position vector from the origin to a point p of the boundary of Poincaré disk is always orthogonal to the tangent vector at a point p (Fig. 1).

Finally, a rotation $f(z) = e^{i\theta}z$ on D_P is also holomorphic, it preserves the distance by (2) (Fig. 2).

4. The cross ratio and the distance between two points on Poincaré disk

Let $\alpha(t) = (t, 0) \subset D_P$ be a differentiable curve from $\alpha(0) = (0, 0)$ to $\alpha(x)$. The shortest arc length of $\alpha(t)$ from $\alpha(0)$ to $\alpha(x)$ is

$$\begin{aligned}\int_0^x \sqrt{g(\alpha'(t), \alpha'(t))}\, dt &= \int_0^x \frac{2}{1 - t^2}\, dt \\ &= \int_0^x \frac{1}{1 + t} - \frac{-1}{1 - t}\, dt \\ &= \ln \frac{1 + x}{1 - x}. \end{aligned} \tag{10}$$

The cross ratio $[z_0, z_1, w_1, w_0]$ for four points $z_0, z_1, w_1, w_0 \in \mathbb{C}$ is defined by

$$[z_0, z_1, w_1, w_0] = \frac{(z_0 - w_1)(z_1 - w_0)}{(z_1 - w_1)(z_0 - w_0)}.$$

Put

$$T(z) = [z, z_1, w_1, w_0] = \frac{(z - w_1)(z_1 - w_0)}{(z_1 - w_1)(z - w_0)}.$$

Then we see

$$T(z_1) = 1, \quad T(w_1) = 0, \quad T(w_0) = \infty.$$

For a real number $0 < x < 1$, we have

$$T(x) = [x, 0, 1, -1] = \frac{(x - 1)(0 - (-1))}{(0 - 1)(x + 1)} = \frac{1 - x}{1 + x} \tag{11}$$

$$T(0) = 1, \quad T(1) = 0, \quad T(-1) = \infty. \tag{12}$$

From (10), (11) and (12), we can define the distance from x to the origin O of D_P by

$$d(x, O) = \left| \ln \frac{1 - x}{1 + x} \right|.$$

For $0 < iy < i$, we have

$$T(iy) = [iy, 0, i, -i] = \frac{(iy - i)(0 - (-i))}{(0 - i)(iy + i)} = \frac{1 - y}{1 + y}$$

So we get

$$d(iy, O) = \left| \ln \frac{1 - y}{1 + y} \right|.$$

We find the image of $y-$axis by a linear fractional transformation

$$f(z) = \frac{z + a}{az + 1}$$

for a real a with $|a| < 1$. We get the same results as in section 3. A linear fractional transformation f preserves the cross ratio

$$[f(z_0), f(z_1), f(w_1), f(w_0)] = [z_0, z_1, w_1, w_0].$$

So we can define the distance by

$$d(z_1, z_2) = d(f(z_1), f(z_2)) = \Big|\ln[z_1, z_2, w_1, w_0]\Big|,$$

where $w_1, w_0 \in \partial D$. It is easy to check that

$$d(z_1, z_2) = d(z_2, z_1), \quad d(z_1, z_2) + d(z_2, z_3) \geq d(z_1, z_3)$$
$$d(z_1, z_2) \geq 0 \quad \text{and} \quad d(z_1, z_2) = 0 \text{ if and only if } z_1 = z_2$$

for all $z_1, z_2, z_3 \in D_P$.

Remark. Let us denote by $\mathbb{H}$ the Poincaré upper half plane. Take a bijective mapping $h : \mathbb{H} \to D_P$

$$h(z) = \frac{z - i}{iz - 1}$$

with the inverse $g(w) = \frac{i-w}{1-iw}$. Then we have $g'(w) = \frac{-2}{(1-iw)^2}$ and the imaginary part of $g(w)$ is

$$\mathrm{Im}(g(w)) = \frac{1 - |w|^2}{|1 - iw|^2}.$$

Since $|w^2|^2 = ww\bar{w}\bar{w} = w\bar{w}w\bar{w} = |w|^4$, we have $|w^2| = |w|^2$. So we obtain the following relation

$$\int_0^1 \frac{2|\alpha'(t)|}{1 - |\alpha(t)|^2}\, dt = \int_0^1 \frac{|\frac{-2}{(1-i\alpha(t))^2}||\alpha'(t)|}{\frac{1-|\alpha(t)|^2}{|1-i\alpha(t)|^2}}\, dt = \int_0^1 \frac{|(g \circ \alpha)'(t)|}{\mathrm{Im}(g(\alpha(t)))}\, dt.$$

Therefore we can define a inner product on $\mathbb{H}$ as

$$g(v_p, w_p) = \frac{1}{y}(v_p \cdot w_p)$$

for all $p \in \mathbb{H}$.

5. The line element

Let S be a regular surface in 3-dimensional Euclidean space $\mathbb{R}^3$. The first fundamental form

$$I_p : T_pS \times T_pS \longrightarrow \mathbb{R}, \quad I_p(v, w) = v \cdot w$$

is the inner product on the tangent space T_pS at $p \in S$ of a surface S induced by the dot product of $\mathbb{R}^3$. Let $\mathbf{X} : U \subseteq \mathbb{R}^2 \longrightarrow S \subset \mathbb{R}^3$ be a coordinate chart of a surface S, that is,

$$\mathbf{X}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

Let $\alpha : [t_0, t_1] = I \longrightarrow S \subset \mathbb{R}^3$ be a curve in S such that

$$\alpha(t) = \mathbf{X}(u(t), v(t))$$

for a curve $c(t) = (u(t), v(t) \subset U \subseteq \mathbb{R}^2$. The length of the curve from $\alpha(t_0)$ to $\alpha(t)$ is

$$s(t) = \int_{t_0}^{t} |\alpha'(r)|\, dr = \int_{t_0}^{t} \sqrt{I_{\alpha(r)}\big(\alpha'(r), \alpha'(r)\big)}\, dr.$$

Since

$$\alpha'(r) = \mathbf{X}_u \frac{du}{dr} + \mathbf{X}_v \frac{dv}{dr}$$
$$I_{\alpha(r)}\big(\alpha'(r), \alpha'(r)\big) = I_{\alpha(r)}\Big(\mathbf{X}_u \frac{du}{dr} + \mathbf{X}_v \frac{dv}{dr},\ \mathbf{X}_u \frac{du}{dr} + \mathbf{X}_v \frac{dv}{dr}\Big)$$

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$$= \Big(\mathbf{X}_u \cdot \mathbf{X}_u\Big)\Big(\frac{du}{dr}\Big)^2 + 2\Big(\mathbf{X}_u \cdot \mathbf{X}_v\Big)\Big(\frac{du}{dr}\Big)\Big(\frac{dv}{dr}\Big) + \Big(\mathbf{X}_v \cdot \mathbf{X}_v\Big)\Big(\frac{dv}{dr}\Big)^2$$
$$= E\Big(\frac{du}{dr}\Big)^2 + 2F\Big(\frac{du}{dr}\Big)\Big(\frac{dv}{dr}\Big) + G\Big(\frac{dv}{dr}\Big)^2,$$

where we put $E = \mathbf{X}_u \cdot \mathbf{X}_u$, $F = \mathbf{X}_u \cdot \mathbf{X}_v$ and $G = \mathbf{X}_v \cdot \mathbf{X}_v$, we get

$$\frac{ds}{dt} = \sqrt{E\Big(\frac{du}{dt}\Big)^2 + 2F\Big(\frac{du}{dt}\Big)\Big(\frac{dv}{dt}\Big) + G\Big(\frac{dv}{dr}\Big)^2}.$$

Therefore we have the so-called line element

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

Note that the line element ds^2 of 2-dimensional Euclidean space $\mathbb{R}^2$ with the dot product is

$$ds^2 = dx^2 + dy^2.$$

For a vector $v = (v_1, v_2) \in \mathbb{R}^2$, the notations

$$ds^2(v) = |v|^2,\ dx^2(v) = dx(v)dx(v) = v_1v_1 = v_1^2,\ dy^2(v) = dy(v)dy(v) = v_2v_2 = v_2^2$$

imply the Pythagorean Theorem. Using the polar coordinates, we get

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2d\theta\,,$$

since

$$(x, y) = (r\cos\theta, r\sin\theta)$$
$$(dx, dy) = (dr\cos\theta - r\sin\theta d\theta, dr\sin\theta + r\cos\theta d\theta).$$

The line element ds^2 of Poincaré upper half plane, Poincaré disk is

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad ds^2 = \frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2},$$

respectively. Let us find the line element ds^2 of Poincaré disk with respect to the polar coordinates. By (10), we can put for real $0 \le r < 1$

$$\bar{r} = \ln\frac{1 + r}{1 - r}.$$

Then we have

$$\sinh\bar{r} = \frac{e^{\bar{r}} - e^{-\bar{r}}}{2} = \frac{\frac{1+r}{1-r} - \frac{1-r}{1+r}}{2} = \frac{2r}{1 - r^2}.$$

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - r^2)^2} = \frac{4(dr^2 + r^2d\theta^2)}{(1 - r^2)^2} = \Big(\frac{2}{1 - r^2}\Big)^2 dr^2 + \Big(\frac{2r}{1 - r^2}\Big)^2 d\theta^2$$

It follows from $\bar{r}(r) = \ln\frac{1+r}{1-r}$ that

$$d\bar{r} = \frac{2}{1 - r^2}dr.$$

So we get

$$ds^2 = d\bar{r}^2 + \Big(\frac{2r}{1 - r^2}\Big)^2 d\theta^2 = d\bar{r}^2 + \sinh^2\bar{r}d\theta^2.$$

A rotation $f(z) = e^{i\theta}z$ on D_P preserves the isometry. Rewrite it on D_P

$$ds^2 = dr^2 + \sinh^2 rd\theta^2.$$

So

$$E = 1, \quad F = 0, \quad G = \sinh^2 r.$$

Then we get Gaussian curvature $K = -1$

$$K = \frac{-1}{2\sqrt{EG}}\Big(\frac{\partial}{\partial\theta}\Big(\frac{\frac{\partial E}{\partial\theta}}{\sqrt{EG}}\Big) + \frac{\partial}{\partial r}\Big(\frac{\frac{\partial G}{\partial r}}{\sqrt{EG}}\Big)\Big) = -1.$$

By the Gauss-Bonnet theorem, the sum of the interior angles of a triangle on D_P is less than π. A quadrilateral $\square ABCD$ which has two right angles $\angle DAB \cong \angle CBA$

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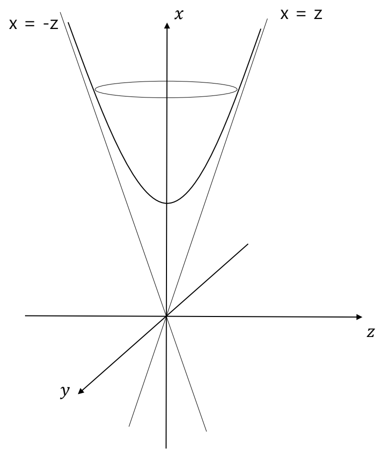

FIGURE 4. One connected component of two-sheeted hyperboloid

and two congruent sides $\overline{AD} \cong \overline{BC}$ without assuming the Parallel postulate is called a Saccheri quadrilateral. $\overline{AB}$ is called the base of the quadrilateral. Consider a Saccheri quadrilateral $\square YOXP$ with the base $\overline{OX}$ on Poincaré disk D_P, where X, Y is a point in x-axis, y-axis, respectively. Since we have two triangles $\triangle YOX$ and $\triangle XYP$, the sum of the interior angles of a Saccheri quadrilateral on Poincaré disk is less than 2π.

5-1. The line element of a surface of revolution

Let $\mathbb{R}^3_1$ be Minkowski space-time with metric

$$g(v, w) = -v_1 w_1 + v_2 w_2 + v_3 w_3.$$

Consider the surface of revolution of a curve $\alpha(r) = (\cosh r, 0, \sinh r)$ of the hyperbola $x^2 - z^2 = 1$ in the xz-plane with a coordinate chart

$$\mathbf{X}(r, \theta) = (\cosh r, \sinh r \cos\theta, \sinh r \sin\theta) \subset \mathbb{R}^3_1,$$

which is $-x^2 + y^2 + z^2 = -1$ (Fig. 4). So we have

$$\mathbf{X}_r = (\sinh r, \cosh r \cos\theta, \cosh r \sin\theta), \quad \mathbf{X}_\theta = (0, -\sinh r \sin\theta, \sinh r \cos\theta)$$

$$E = g(\mathbf{X}_r, \mathbf{X}_r) = -\sinh^2 r + \cosh^2 r = 1$$

$$F = g(\mathbf{X}_r, \mathbf{X}_\theta) = 0,\ G = g(\mathbf{X}_\theta, \mathbf{X}_\theta) = \sinh^2 r.$$

Hence we get

$$ds^2 = dr^2 + \sinh^2 r d\theta^2.$$

So Poincaré Disk is isometric to the above surface of revolution of a curve $\alpha(r) = (\cosh r, 0, \sinh r)$ of the hyperbola $x^2 - z^2 = 1$ in the xz-plane in Minkowski space-time $\mathbb{R}^3_1$.

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References

[1] G. S. Birman and A. A. Ungar, The hyperbolic derivative in the Poincaré ball model of hyperbolic geometry, J. Math. Anal. Appl. 254 (2001) 321–333
[2] N. V. Efimov, Higher geometry, Mir Publishers, 1980
[3] John McCleary, Geometry from a Differentiable Viewpoint, Cambrige university press, 1994
[4] Barrett O' Neill, Elementary Differential Geometry 2nd edition, Academic press, 1997
[5] G. Popescu, Hyperbolic geometry on noncommutative polyballs, J. Math. Anal. Appl. 456 (2017) 576–607
[6] James R. Smart, Modern geometries 5th edition, Brooks/Cole Publishing Company, 1998
[7] A. A. Ungar, Hyperbolic trigonometry and its application in the Poincaré ball model of hyperbolic geometry, Comput. Math. Appl., 41 (2001) 135–147