MDPI

Article

CONTINUOUS WAVELET TRANSFORM OF SCHWARTZ TEMPERED DISTRIBUTIONS IN $S'(\mathbb{R}^n)$

J.N. Pandey [1,‡], Jay Singh Maurya [2,‡], S.K. Upadhyay [2,‡,*] and H.M. Srivastava [3,‡]

1 School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6, Canada; jimpandey1936@gmail.com

2 Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi-221005, India; jaysinghmaurya.rs.mat17@itbhu.ac.in

3 Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada

and

Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, Republic of China; harimsri@math.uvic.ca

* Correspondence: sk_upadhyay2001@yahoo.com; Tel.:+91-94501-12714

Abstract: In this paper we define a continuous wavelet transform of a Schwartz tempered distribution $f \in S^{'}(\mathbb{R}^n)$ with wavelet kernel $\psi \in S(\mathbb{R}^n)$ and derive the corresponding wavelet inversion formula interpreting convergence in the weak topology of $S^{'}(\mathbb{R}^n)$. It turns out that the wavelet transform of a constant distribution is zero and our wavelet inversion formula is not true for constant distribution, but it is true for a non-constant distribution which is not equal to the sum of a non-constant distribution with a non-zero constant distribution.

Keywords: Function spaces and their duals; Distributions; Generalized functions; Distribution space; Wavelet transform of generalized functions.

---

1. Introduction

As defined in [1,7,8,11–15,18–20]; we define a Schwartz testing function space $S(\mathbb{R}^n)$ to consist of C^∞ functions ϕ defined on $\mathbb{R}^n$ and satisfying the conditions

$$\sup_{x\in\mathbb{R}^n, k\in\mathbb{R}^n}\left|x_n^{m_n}\dots x_2^{m_2}x_1^{m_1}\frac{\partial^{k_n}}{\partial x_n}\frac{\partial^{k_{n-1}}}{\partial x_{n-1}}\dots\frac{\partial^{k_2}}{\partial x_2}\frac{\partial^{k_1}}{\partial x_1}\phi(x_1,x_2,\dots x_n)\right| < \infty \tag{1}$$

$|m|, |k| = 0, 1, 2, \dots.$
The topology over $S(\mathbb{R}^n)$ is generated by the sequence of semi-norms $\{\gamma_{m,k}\}_{|m|,|k|=0}^{\infty}$ where,

$$\gamma_{m,k}(\phi) = \sup_{x\in\mathbb{R}^n}\left||x^m|\,\phi^{(k)}(x)\right| \tag{2}$$

$$|m| = m_1 + m_2 + \dots + m_n$$

$$|k| = k_1 + k_2 + \dots + k_n$$

$$|x^m| = |x_1^{m_1}x_2^{m_2}\dots x_n^{m_n}|$$

$$\phi^{(k)}(x) = \frac{\partial^{k_n}}{\partial x_n}\dots\frac{\partial^{k_2}}{\partial x_2}\frac{\partial^{k_1}}{\partial x_1}\phi(x)$$

 doi:10.20944/preprints201901.0131.v1

Peer-reviewed version available at Symmetry 2019, 11, 235; doi:10.3390/sym11020235

These collections of semi-norms in (2) are separating which means that an element $\phi \in S(\mathbb{R}^n)$ is non-zero if and only if there exists at least one of the semi-norms $\gamma_{m,k}$ satisfying $\gamma_{m,k}(\phi) \neq 0$. A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ in $S(\mathbb{R}^n)$ tends to ϕ in $S(\mathbb{R}^n)$ if and only if $\gamma_{m,k}(\phi_\nu - \phi) \to 0$ as ν goes to ∞ for each of the subscripts $|m|, |k| = 0, 1, 2, \dots$, are as defined above. Now one can verify that the function $e^{-(t_1^2+t_2^2+\dots+t_n^2)} \in S(\mathbb{R}^n)$ and the sequence $\frac{\nu-1}{\nu}e^{-(t_1^2+t_2^2+\dots+t_n^2)} \to e^{-(t_1^2+t_2^2+\dots+t_n^2)}$ in $S(\mathbb{R}^n)$ as $\nu \to \infty$. One can check that $\delta(t_1, t_2, \dots, t_n)$ is a continuous linear functional on $S(\mathbb{R}^n)$. A regular distribution generated by a locally integrable function is an element of $S'(\mathbb{R}^n)$.
Now our objective is to find an element $\psi \in S(\mathbb{R}^n)$ which is a wavelet so as to be able to define the wavelet transform of $f \in S'(\mathbb{R}^n)$ with respect to this kernel.
A function $\psi \in L^2(\mathbb{R}^n)$ is a window function if it satisfies the conditions

$$x_i\psi(x), x_ix_j\psi(x), \dots, x_1x_2x_3\dots x_n\psi(x) \tag{3}$$

belong to $L^2(\mathbb{R}^n)$. Here $i, j, k, \dots$ all assume values 1,2,3,... and all the lower suffixes in a term in (3) are different. It has been proved by J.N. Pandey et al. [6,8] that a window function which is an element of $L^2(\mathbb{R}^n)$ belongs to $L^1(\mathbb{R}^n)$. It is easy to verify that every element of $S(\mathbb{R}^n)$ is a window function.
A window function ψ belonging to $L^2(\mathbb{R}^n)$ and satisfying the condition

$$\int_{-\infty}^{\infty} \psi(x_1, x_2, \dots, x_i, \dots, x_n)dx_i = 0 \tag{4}$$

for each $i = 1, 2, 3, \dots, n$ satisfies the admissibility condition

$$\int_{\Lambda^n} \frac{|\hat{\psi}(\Lambda)|^2}{|\Lambda|} d\Lambda < \infty \tag{5}$$

where $\hat{\psi}(\Lambda) = \hat{\psi}(\lambda_1, \lambda_2, \dots, \lambda_n), |\Lambda| = |\lambda_1\lambda_2\dots\lambda_n|$ and $\hat{\psi}(\Lambda)$ is the Fourier transform of $\psi(x) \equiv \psi(x_1, x_2, \dots, x_n)$; clearly ψ in (4) is a wavelet [6]. As an example one can easily verify that the function $\psi(x) = x_1x_2\dots x_n e^{-(x_1^2+x_2^2+\dots+x_n^2)}$ is a wavelet belonging to $S(\mathbb{R}^n)$. Let $s(\mathbb{R}^n)$ be a subspace of $S(\mathbb{R}^n)$ such that every element $\phi \in s(\mathbb{R}^n)$ satisfies (4). Clearly every element of $s(\mathbb{R}^n)$ is a wavelet [8].
Now if $f \in S'(\mathbb{R}^n)$ and ψ is a wavelet belonging to $S(\mathbb{R}^n)$ the wavelet transform of f can be defined by

$$W_f(a,b) = \left\langle f(x), \frac{1}{\sqrt{|a|}}\psi(\frac{x-b}{a}) \right\rangle$$

where, (a, b) is argument of W,

$$\psi(\frac{x-b}{a}) = \psi(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, \dots, \frac{x_n-b_n}{a_n})$$

$$a_i \neq 0, \quad \forall\, i = 1, 2 \dots n.$$

and

$$|a| = |a_1a_2\dots a_n|$$

Our objective is to prove the inversion formula

$$\left\langle \frac{1}{C_\psi} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W_f(a,b)\psi(\frac{t-b}{a})\frac{dbda}{\sqrt{|a|}a^2}, \phi(t) \right\rangle \to \langle f, \phi \rangle, \quad \phi \in S(\mathbb{R}^n) \tag{6}$$

interpreting convergence in $S'(\mathbb{R}^n)$. Here $C_\psi = (2\pi)^n \int_{R^n} \frac{|\hat{\psi}(\wedge)|^2}{|\wedge|} d\wedge$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 January 2019 doi:10.20944/preprints201901.0131.v1

Peer-reviewed version available at *Symmetry* **2019**, *11*, 235; doi:10.3390/sym11020235

The derivation of the inversion formula given by (6) is difficult; we however, make an easy approach. The work on the multidimensional wavelet transform with positive scale $[a > 0]$ was done by Daubechies 1990 [2], Meyer 1992 [5], Pathak 2009 [9] and some others. Motivated from [12,14,20], J.N. Pandey et al. [8] did the generalization of these works and extended the multidimensional wavelet transform with real scale $[a \neq 0]$. In the year 1995 Holschneider [3] extended the multidimensional wavelet transform to Schwartz tempered distributions with positive scales $[a > 0]$. Recently F. Weisz [16,17] studied the inversion formula for the continuous wavelet transform and found its convergence in $\mathbb{L}^p$ and Wiener amalgam spaces. Eugene B. Postnikov et al. [10] studied computational implementation of the inverse continuous wavelet transform without a requirement of the admissibility condition.

Our objective is to extend the wavelet transform to Schwartz tempered distributions with real scale $[a \neq 0]$. The standard cut off of negative frequencies (which is required to apply continuous wavelet transform with $a > 0$) may result in a loss of information if the transformed functions were non-symmetric (in the Fourier space) mixture of real and imaginary frequency components. Our proposed and proven inversion formula is free from the mentioned defect. The main advantage of our work is a possible further practical utility of the proven result and the simplicity of calculation, besides our extension of the multidimensional wavelet inversion formula is the most general. In [8] it is proved that a window function $\psi(x) \in L^2(\mathbb{R}^n)$ is a wavelet if and only if the integral of ψ along each of the axes is zero; therefore, any $\psi(x) \in s(\mathbb{R}^n)$ is a wavelet. Hence, the wavelet transform of a constant distribution is zero.

Thus we realize that two elements of $S'(\mathbb{R}^n)$ having equal wavelet transform will differ by a constant in general. Holschneider uses the wavelet inversion formula for $f \in S'(\mathbb{R}^n)$ but he does not mention the wavelet inversion formula and its convergence in $S'(\mathbb{R}^n)$. Perhaps, he takes it for granted, as such an inversion formula is valid for elements of $L^2(\mathbb{R}^n)$ interpreting convergence in $L^2(\mathbb{R}^n)$. So our objective in this paper is to fill up all these gaps. We will prove the inversion formula (6) in section 3.

2. Structure of generalized functions of slow growth

Elements of $S'(\mathbb{R}^n)$ are called tempered distributions or distributions of slow growth.

Definition 1. *A function $f(x)$ is said to be a function of slow growth in $\mathbb{R}^n$ if for $m \geq 0$ we have*

$$\int_{\mathbb{R}^n} |f(x)| \, (1+|x|)^{-m} \, dx < \infty$$

and it determines a regular functional f in $S'(\mathbb{R}^n)$ by the formula

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx, \phi \epsilon S(\mathbb{R}^n) \tag{7}$$

It is easy to verify that the functional f defined by (7) exists $\forall \ \phi \in S(\mathbb{R}^n)$ and that it is linear as well as continuous on $S(\mathbb{R}^n)$.

We now quote a theorem of V.S. Vladimirov proved in his book [14].

Theorem 1. *If $f \in S'(\mathbb{R}^n)$ then there exists a continuous function g of slow growth in $\mathbb{R}^n$ and an integer $m \geq 0$ such that*

$$f(x) = D_1^m D_2^m ... D_n^m g(x), \frac{\partial}{\partial x_i} \equiv D_i \tag{8}$$

Peer-reviewed version available at Symmetry 2019, 11, 235; doi:10.3390/sym11020235

The relation (8) can be written as

$$f(x) = D^m g(x); D = D_1 D_2 ... D_n \tag{9}$$

The n-dimensional wavelet inversion formula for tempered distributions will now be proved very simply by using the structure formula (9). This structure formula enables us to reduce the wavelet analysis problem relating to tempered distributions to the classical wavelet analysis problem of $L^2(\mathbb{R}^n)$ functions. Our wavelet inversion formula of $L^2(\mathbb{R}^n)$ functions will be used quite successfully to derive the wavelet inversion formula for the wavelet transform of tempered distributions.

3. Wavelet transform of tempered distributions in $\mathbb{R}^n$ and its inversion

From now we assume that $a \neq 0$ implies each of the component $a_i \neq 0$ for all $i = 1, 2, 3, ..., n$ and $a > 0$ means each of the component a_i of a is greater than zero. $|a| > \epsilon$ will mean that $|a_i| > \epsilon$ for all $i = 1, 2, 3, ..., n$.

Let $\psi(x) = \psi(x_1, x_2, ... x_n) \in S(\mathbb{R}^n)$ then $\psi(x)$ is a window function and is a wavelet if and only if $\int\limits_{-\infty}^{\infty} \psi(x_1, x_2, ... x_n) dx_i = 0$

$$\forall i = 1, 2, ... n. \tag{10}$$

We define $\psi\left(\frac{x-b}{a}\right) \equiv \psi\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}, ... \frac{x_n-b_n}{a_n}\right)$ where a_i, b_i are real numbers and none of the a_i is zero. The wavelet transform $W_f(a, b)$ of f with respect to the Kernel $\frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right)$ is defined by

$$W_f(a, b) = \left\langle f(x), \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right)\right\rangle \tag{11}$$

Here $|a| = |a_1 a_2 ... a_n|$; none of a_i's is zero.

We now prove the following Lemmas to be used to prove the main inversion formula.

Lemma 3a: Let $\phi \in S(\mathbb{R}^n)$ and ψ be a wavelet belonging to $S(\mathbb{R}^n)$, then

$$\frac{1}{C_\psi} \int\limits_{a\epsilon\mathbb{R}^n} \int\limits_{b\epsilon\mathbb{R}^n} \int\limits_{t\epsilon\mathbb{R}^n} (-D_t)^m \phi(t) \bar{\psi}\left(\frac{t-b}{a}\right) \psi\left(\frac{x_0-b}{a}\right) \frac{dt\, db\, da}{a^2|a|}$$
$$= (-D_x)^m \phi(x)|_{x=x_0} \forall\, x_0 \,\epsilon\, \mathbb{R}^n \quad [6].$$

This is called point wise convergence of the wavelet inversion formula.

Lemma 3b: Let $\phi \in S(\mathbb{R}^n)$ and ψ be a wavelet belonging to $S(\mathbb{R}^n)$, then

$$\frac{1}{C_\psi} \int\limits_{a\,\epsilon\,\mathbb{R}^n} \int\limits_{b\,\epsilon\,\mathbb{R}^n} \int\limits_{t\epsilon\,\mathbb{R}^n} (-D_t)^m \phi(t) \bar{\psi}\left(\frac{t-b}{a}\right) \psi\left(\frac{x-b}{a}\right) \frac{dt\, db\, da}{a^2|a|}$$

converges to $(-D_x^m)\ \phi(x)$ uniformly $\forall\ x \in\ \mathbb{R}^n$.

Proof: Let $F(\wedge) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} (-D_t)^m \phi(t)\ e^{-i\wedge.t}\ dt$, be the Fourier transform of $(-D_t)^m \phi(t)$ then it follows that in the sense of $L^2(\mathbb{R}^n)$ convergence

$$\frac{1}{C_\psi} \int\limits_{a\,\epsilon\,\mathbb{R}^n} \int\limits_{b\,\epsilon\,\mathbb{R}^n} \int\limits_{c\,\epsilon\,\mathbb{R}^n} (-D_t)^m \phi(t) \bar{\psi}\left(\frac{t-b}{a}\right) \psi\left(\frac{x-b}{a}\right) \frac{dt\, db\, da}{a^2|a|}$$
$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\wedge^n} F(\wedge) e^{i\wedge.x} d\wedge = (-D_x)^m \phi(x) \quad [9].$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 January 2019 doi:10.20944/preprints201901.0131.v1

Peer-reviewed version available at *Symmetry* **2019**, *11*, 235; doi:10.3390/sym11020235

This convergence is also uniform by Weierstrass M-test because

$$\left| \frac{1}{(2\pi)^{\frac{n}{2}}} \int\limits_{\mathbb{R}^n} F(\wedge)\, e^{i\wedge . x}\, d\wedge \right| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int |F(\wedge)| d\wedge < \infty$$

$$F(\wedge) \in S(\mathbb{R}^n).$$

Theorem 2. *Let $f \in S'(\mathbb{R}^n)$ and $W_f(a,b)$ be its wavelet transform defined by*

$$W_f(a,b) = \left\langle f(x), \ \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) \right\rangle.$$

Then

$$\left\langle \frac{1}{C_\psi} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} W_f(a,b) \psi\left(\frac{t-b}{a}\right) \frac{db\, da}{\sqrt{|a|}a^2}, \phi(t) \right\rangle = \langle f, \phi \rangle \tag{12}$$

$$\forall \ f \in S'(\mathbb{R}^n), \phi \in S(\mathbb{R}^n).$$

64 **Proof:** Using the structure formula for f we have

$$\begin{aligned} W_f(a,b) &= \left\langle D_x^m g(x), \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) \right\rangle \\ &= \left\langle g(x), (-D_x)^m \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) \right\rangle \end{aligned}$$

by distributional differentiation

65 Here,

$$(-D_x) = (-D_{x_1})(-D_{x_2}) \ldots (-D_{x_n})$$

$$D_{x_i} \equiv \frac{\partial}{\partial x_i}, \ i = 1, 2, \ldots n.$$

66 So,

$$\begin{aligned} W_f(a,b) &= \left\langle g(x), (D_b)^m \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) \right\rangle \\ D_b &= \frac{\partial}{\partial b_1} \frac{\partial}{\partial b_2} \cdots \frac{\partial}{\partial b_n}. \end{aligned}$$

67 So, the L.H.S. expression in (12) can be written as

$$\frac{1}{C_\psi} \int\limits_{t\epsilon\mathbb{R}^n} \int\limits_{a\epsilon\mathbb{R}^n} \int\limits_{b\epsilon\mathbb{R}^n} \int\limits_{x\epsilon\mathbb{R}^n} g(x) D_b^m \frac{1}{\sqrt{|a|}} \bar{\psi}\left(\frac{x-b}{a}\right) \psi\left(\frac{t-b}{a}\right) \bar{\phi}(t) dx\, db\, da\, dt$$

$$= \frac{1}{C_\psi} \int\limits_{t\epsilon\mathbb{R}^n} \int\limits_{a\epsilon\mathbb{R}^n} \int\limits_{x\epsilon\mathbb{R}^n} g(x) \left[\int\limits_{b\epsilon\mathbb{R}^n} \left\{ D_b^m \bar{\psi}\left(\frac{x-b}{a}\right) \right\} \psi\left(\frac{t-b}{a}\right) db \right] \frac{\bar{\phi}(t)\, dx\, da\, dt}{a^2 |a|}. \tag{13}$$

68 We now integrate the integral in the big bracket by parts to get (13)

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 14 January 2019 doi:10.20944/preprints201901.0131.v1

Peer-reviewed version available at Symmetry 2019, 11, 235; doi:10.3390/sym11020235

$$= \frac{1}{C_\psi} \int\limits_{t\epsilon\mathbb{R}^n} \int\limits_{a\epsilon\mathbb{R}^n} \int\limits_{x\epsilon\mathbb{R}^n} g(x) \left[\int\limits_{b\epsilon\mathbb{R}^n} \bar{\psi}\left(\frac{x-b}{a}\right) (-D_b)^m \psi\left(\frac{t-b}{a}\right) db \right] \frac{\bar{\phi}(t)\, dx\, da\, dt}{a^2|a|}$$

$$= \frac{1}{C_\psi} \int\limits_{t\epsilon\mathbb{R}^n} \int\limits_{a\epsilon\mathbb{R}^n} \int\limits_{x\epsilon\mathbb{R}^n} g(x) \left[\int\limits_{b\epsilon\mathbb{R}^n} \bar{\psi}\left(\frac{x-b}{a}\right) (+D_t)^m \psi\left(\frac{t-b}{a}\right) db \right] \frac{\bar{\phi}(t) da\, dt}{a^2|a|}$$

By switching the order of integration in a and t we have (13)

$$= \frac{1}{C_\psi} \int\limits_{a\epsilon\mathbb{R}^n} \int\limits_{t\epsilon\mathbb{R}^n} \int\limits_{b\epsilon\mathbb{R}^n} \int\limits_{x\epsilon\mathbb{R}^n} g(x)\bar{\psi}\left(\frac{x-b}{a}\right) dx\, D_t^m\, \psi\left(\frac{t-b}{a}\right) db\bar{\phi}(t) \frac{dt\, da}{|a|^2|a|} \tag{14}$$

$$= \frac{1}{C_\psi} \int\limits_{a\epsilon\mathbb{R}^n} \int\limits_{b\epsilon\mathbb{R}^n} \int\limits_{x\epsilon\mathbb{R}^n} g(x)\bar{\psi}\left(\frac{x-b}{a}\right) dx \int\limits_{t\epsilon\mathbb{R}^n} \psi\left(\frac{t-b}{a}\right) db\, (-D_t)^m\, \bar{\phi}(t) \frac{dt\, da}{|a|^2|a|}$$

To justify the switch in the order of integration with respect to a and t we perform the integration in the region $[(a,t) : |a| > \epsilon, a, t \in \mathbb{R}^n]$ and then switch the order of integration and then let $\epsilon \to 0$. This existence of the n-triple integral in terms of b, a and t in (14) is proved by using Plancherel theorem with respect to the variable b and, using $C_\psi = (2\pi)^n \int\limits_{\wedge^n} \frac{|\hat{\psi}(\wedge)|^2}{|\wedge|} d\wedge$ we notice that the variable a disappears from the denominator and every calculation goes on smoothly. Since the functions ϕ and ψ are elements of $S(\mathbb{R}^n)$, the Fubini's theorem can be applied to justify the above switches in order of integration.

Now, (14) can be written as

$$\left\langle g(x), \frac{1}{C_\psi} \int\limits_{a\epsilon\mathbb{R}^n} \int\limits_{b\epsilon\mathbb{R}^n} \int\limits_{t\epsilon\mathbb{R}^n} (-D_t)^m\, \phi(t)\bar{\psi}\left(\frac{t-b}{a}\right) dt\, \psi\left(\frac{x-b}{a}\right) \frac{db\, da}{|a|^2|a|} \right\rangle \tag{15}$$

$$= \langle g(x), (-D_x)^m\, \phi(x) \rangle . \tag{16}$$

By the wavelet inversion formula in $\mathbb{R}^n$ [8] and Lemma 3b.

Note that the triple integral in (15) converges uniformly to $(-D_x)^m\, \phi(x) \forall\ x \in\ \mathbb{R}^n$. So (15) becomes (16).

$$= \langle (D_x)^m\, g(x),\ \phi(x) \rangle$$
$$= \langle f(x),\ \ \phi(x) \rangle .$$

References

1. Bremmermann, H., *Distributions, Complex Variables and Fourier Transforms*; Addison-Wesley Publishing Company Inc., 1965.
2. Daubechies, Ingrid., *Ten lectures on wavelets*; Mathematics Department, University of Lowell, Massachusetts, 1990.
3. Holschneider, M., *Wavelets an Analysis tool*; Oxford Science Publications, Clarendon Press, 1995.
4. Kantorovich, L.V. and Akilov, G.P., *Functional Analysis in Normed Spaces*; Macmillan, New York, 1963.
5. Meyer, Yves, Wavelets and Operators, *Cambridge Studies in Advanced Mathematics* **1992**, *37*.

6. Pandey, J.N., Jha, N.K. and Singh, O.P., The continuous wavelet transform in n-dimensions, *International J. Wavelets, Multiwavelets, Multiresolution and Information Processing* **2016**, *14*, 13 pages.
7. Pandey, J.N., *The Hilbert transform of Schwartz Distributions and Applications*; John Wiley and Sons Inc., 1996.
8. Pandey, J.N. and Upadhyay, S.K., Continuous Wavelet transform and window functions, *Proceeding of American Math. Soc.* **2015**, *143*, 4750-4773.
9. Pathak, R.S., *The Wavelet Transform*; Atlantic Press, World Scientific, 2009.
10. Postnikov, Eugene B., Lebedeva Elena A., Lavrova Anastasia I., Computational implementation of the inverse continuous wavelet transform without a requirement of the admissibility condition, *Applied Mathematics and Computation* **2016**, *282*, 128-136.
11. Schwartz, Laurent, *Theorie des distributions*; Hermann, Paris, 1966.
12. Shilov, George E., *Generalized functions and partial differential equations*; Gordon and Breach Publishing Co.
13. Treves, François, *Topological vector spaces, distributions and kernels*; Academic Press, 1997.
14. Vladimirov, V.S., *Generalized functions in Mathematical Physics, translated from Russian by G. Yankovasky*.
15. Walter, Gilbert G. and Shen, Xiaoping, *Wavelets and other orthogonal systems, second edition*; Chapman and Hall/CRC, 2009.
16. Weisz F., Convergence of the inverse continuous wavelet transform in Wiener amalgam spaces, *Analysis* **2015**, *35*, 33-46.
17. Weisz F., Inversion Formulas for the Continuous Wavelet Transform, *Acta Math. Hungar.* **2013**, *138*, 237-258.
18. Yosida, Kosaku, *Functional Analysis*; Springer Verlag, 1995.
19. Zemanian, A.H., *Distribution Theory and Transform Analysis*; McGraw-Hill Book Company, 1965.
20. Zemanian, A.H., *Generalized Integral Transformation*; Inter Science Publishers, John Wiley and Sons Inc., 1968.