ON CONFORMABLE DOUBLE LAPLACE TRANSFORM AND
ONE DIMENSIONAL FRACTIONAL COUPLED BURGERS’
equation

HASSAN ELTAYEB1, IMED BACHAR2 AND ADEM KILICMAN3

Abstract. This article deals with the conformable double Laplace transforms and
their some properties with examples and also the existence Condition for
the conformable double Laplace transform is studied. Finally, in order to
obtain the solution of nonlinear fractional problems, we present a modified
conformable double Laplace that we call conformable double Laplace decom-
position methods (CDLDM). Then, we apply it to solve, Regular and singular
conformable fractional coupled burgers equation illustrate the effectiveness of
our method some examples are given.

1. Introduction

The fractional partial differential equations play a crucial role in fields of physics,
chemistry and engineering. Cheng and Yao, in [8] studied the solution of some
time-fractional partial differential equations by simplest equation method. In this
work, we deal with burgers equation, this equations appear in the area of applied
sciences such as fluid mechanics, mathematic model. The Burgers Equation was
first proposed by Bateman [1] who found its steady solutions, descriptive of certain
viscous flows and modified by J. M. Burgers (1895-1981) then it is widely named
as Burgers’ Equation [2]. Many researchers are Concentrated to studying the ex-
act and numerical solutions of this equation. The conformable double Laplace
transform method was introduced by ¨Ozkan and Kurt [3] in the study of fractional
to establish the exact solutions for time-fractional Burgers’ equation. Jincun and
Guolin in [12] is applied the generalized two-dimensional differential transform
method (DTM) to obtain the solution of the coupled Burgers equations with space-
fractional Laplace transform to solve the coupled system of conformable fractional
differential equation. The aim of this article is to propose an analytic solution of
the one dimensional Regular and singular conformable fractional coupled burgers
equation by using a conformable double Laplace decomposition method (CDLDM).
However, conformable double Laplace decomposition method can be used to ap-
proximate the solutions of the nonlinear differential equations with the linearization

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gular one dimensional coupled Burgers equations.
of non-linear terms by using Adomian polynomials. A new type of Burger’s equation was proposed in further work named as time-space fractional order coupled Burger’s equations [7], which has the form

\[
\frac{\partial^\beta u}{\partial t^\beta} - \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \eta \frac{\partial^\alpha u}{\partial x^\alpha} + \xi \frac{\partial^\alpha}{\partial x^\alpha} (uv) = f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right)
\]

\[
\frac{\partial^\beta v}{\partial t^\beta} - \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}} + \eta \frac{\partial^\alpha v}{\partial x^\alpha} + \mu \frac{\partial^\alpha}{\partial x^\alpha} (uv) = g \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right). \tag{1.1}
\]

Conformable fractional derivatives were studied by Khalil et al. [4] and extended by Abdeljawad [5]. First of all, we start to recall the definition of the conformable fractional derivatives, which are used in this article.

**Definition 1.** Given a function \( f : (0, \infty) \to R \), then the conformable fractional derivative of \( f \) of order \( \beta \) is defined by

\[
\frac{d^\beta}{dt^\beta} f(t) = \lim_{\epsilon \to 0} \frac{f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} + \epsilon^{1-\beta} \right) - f(t)}{\epsilon}, \quad t > 0, \ 0 < \beta \leq 1,
\]

see [4, 9, 10].

**Conformable Partial Derivatives:**

**Definition 2.** Given a function \( f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) : R \times (0, \infty) \to R \). Then, the conformable space fractional partial derivative of order \( \alpha \) a function \( f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \) is defined as:

\[
\frac{\partial^\alpha}{\partial x^\alpha} f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = \lim_{\epsilon \to 0} \frac{f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} + \epsilon x^{1-\alpha} \right) - f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right)}{\epsilon}, \quad x, t > 0, \ 0 < \alpha, \beta \leq 1,
\]

see [6].

**Definition 3.** Given a function \( f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) : R \times (0, \infty) \to R \). Then, the conformable time fractional partial derivative of order \( \beta \) a function \( f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \) is defined as:

\[
\frac{\partial^\beta}{\partial t^\beta} f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = \lim_{\sigma \to 0} \frac{f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} + \sigma^{1-\beta} \right) - f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right)}{\sigma}, \quad x, t > 0, \ 0 < \alpha, \beta \leq 1,
\]

see [6].

**Conformable fractional derivative of certain functions:**

**Example 1.**

\[
\frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = \frac{t^\beta}{\beta}, \quad \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \right)^n \frac{t^\beta}{\beta} = n \left( \frac{x^\alpha}{\alpha} \right)^{n-1} \frac{t^\beta}{\beta}
\]

\[
\frac{\partial^\beta}{\partial t^\beta} \left( \frac{x^\alpha}{\alpha} \right)^n \frac{t^\beta}{\beta} = \frac{x^\alpha}{\alpha}, \quad \frac{\partial^\beta}{\partial t^\beta} \left( \frac{x^\alpha}{\alpha} \right)^n \left( \frac{t^\beta}{\beta} \right)^m = m \left( \frac{x^\alpha}{\alpha} \right)^n \left( \frac{t^\beta}{\beta} \right)^{m-1}
\]

\[
\frac{\partial^\beta}{\partial t^\beta} \left( \sin \left( \frac{x^\alpha}{\alpha} \right) \sin \left( \frac{t^\beta}{\beta} \right) \right) = \sin \left( \frac{x^\alpha}{\alpha} \cos \left( \frac{t^\beta}{\beta} \right) \right)
\]

\[
\frac{\partial^\alpha}{\partial x^\alpha} \left( \sin \left( \frac{x^\alpha}{\alpha} \right) \sin \left( \frac{t^\beta}{\beta} \right) \right) = \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \sin \left( \frac{t^\beta}{\beta} \right) \right)
\]

\[
\frac{\partial^\alpha}{\partial x^\alpha} \left( e^{\lambda \frac{x^\alpha}{\alpha} + \frac{t^\beta}{\beta}} \right) = \lambda e^{\lambda \frac{x^\alpha}{\alpha} + \frac{t^\beta}{\beta}}, \quad \frac{\partial^\beta}{\partial t^\beta} \left( e^{\lambda \frac{x^\alpha}{\alpha} + \frac{t^\beta}{\beta}} \right) = \tau e^{\lambda \frac{x^\alpha}{\alpha} + \frac{t^\beta}{\beta}}.
\]
Conformal Laplace transform:

Definition 4. Let \( f : [0, \infty) \to \mathbb{R} \) be a real valued function. The conformal Laplace transform of \( f \) is defined by

\[
L_\alpha^\beta \left( f \left( \frac{t^\beta}{\beta} \right) \right) = \int_0^\infty e^{-st^\beta} f \left( \frac{t^\beta}{\beta} \right) t^{\beta-1} dt
\]

for all values of \( s \), the integral is correct, see [?].

Definition 5. Let \( u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \) be a piecewise continuous function on the interval \([0, \infty) \times [0, \infty)\) of exponential order. Consider for some \( a, b \in \mathbb{R} \) \(\sup_{\alpha, \beta} \frac{\alpha}{\beta} > 0\), \(\frac{u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right)}{\frac{x^\alpha}{\alpha} + \frac{t^\beta}{\beta}}\). Under these conditions conformal double Laplace transform is defined by

\[
L_\alpha^\beta L_i^\beta \left( u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \right) = U \left( p, s \right) = \int_0^\infty \int_0^\infty e^{-p t^\beta} e^{-s \frac{t^\beta}{\beta}} u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) t^{\beta-1} x^{\alpha-1} dt dx
\]

where \( p, s \in \mathbb{C} \), \( 0 < \alpha, \beta \leq 1 \) and the integrals are by means of conformal fractional integral with respect to \( \frac{x^\alpha}{\alpha} \) and \( \frac{t^\beta}{\beta} \) respectively.

Example 2. This example we calculate the double fractional Laplace for certain functions

1. \( L_\alpha^\beta L_i^\beta \left( \left( \frac{x^\alpha}{\alpha} \right)^n \left( \frac{t^\beta}{\beta} \right)^m \right) = L_x L_i \left[ (x)^n (t)^m \right] = \frac{n! m!}{p^{n+1} s^{m+1}} \).

2. \( L_\alpha^\beta L_i^\beta \left( e^{\lambda \frac{x^\alpha}{\alpha} + \frac{t^\beta}{\beta}} \right) = L_x L_i \left[ e^{x\lambda + \lambda t} \right] = e^{(p-\lambda) (s-\lambda)} \).

3. \( L_\alpha^\beta L_i^\beta \left( \sin \left( \frac{x^\alpha}{\alpha} \right) \sin \left( \frac{t^\beta}{\beta} \right) \right) = L_x L_i \left[ \sin (x) \sin (t) \right] = \frac{1}{p^2 + s^2} \).

4. If \( a > -1 \) and \( b > -1 \) are real numbers, then double fractional Laplace of the function \( f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = \left( \frac{x^\alpha}{\alpha} \right)^a \left( \frac{t^\beta}{\beta} \right)^b \) is given by

\[
L_x L_i \left( \frac{x^\alpha}{\alpha} \right)^a \left( \frac{t^\beta}{\beta} \right)^b = \frac{\Gamma (a+1) \Gamma (b+1)}{p^{a+1} s^{b+1}}.
\]

Theorem 1. Let \( 0 < \alpha, \beta \leq 1 \) and \( m, n \in \mathbb{N} \) such that \( u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \in C^l \left( \mathbb{R}^+ \times \mathbb{R}^+ \right) \), \( l = \max (m, n) \). Also let the conformal Laplace transforms of the functions \( u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \) and \( \frac{\partial^m u}{\partial x^m} \), \( i = 1, \ldots, m \), \( j = 1, \ldots, n \) exist. Then

\[
L_x \left( \frac{\partial^m u}{\partial x^m} \right) \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = \frac{\partial^m u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right)}{\partial x^m} \left( 0, 0 \right)
\]

\[
L_x \left( \frac{\partial^m u}{\partial x^m} \right) \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = \frac{\partial^m u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right)}{\partial x^m} \left( 0, 0 \right)
\]

where \( \frac{\partial^m u}{\partial x^m} \) and \( \frac{\partial^m u}{\partial x^m} \) denotes \( m \) \( m \) times conformable fractional derivatives of function \( u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \) with order \( b \) and \( a \) respectively. For more details see [3]

In the following theorem, we study double Laplace transform of the function \( \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right)^n \) as follows:
Theorem 2. If conformable double Laplace transform of the partial derivatives \( \frac{\partial^n}{\partial t^p} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \) is given by Eqs. (2.21), then double Laplace transform of \( f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \) and \( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \) are given by

\[
(-1)^n \frac{d}{dp} \left( L_x^\alpha L_t^\beta \left[ \frac{\partial^n}{\partial t^p} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right] \right) = L_x^\alpha L_t^\beta \left[ \left( \frac{x^\alpha}{\alpha} \right)^n \frac{\partial^n}{\partial t^n} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right] \tag{1.2}
\]

and

\[
(-1)^n \frac{d}{dp} \left( L_x^\alpha L_t^\beta \left[ g\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right] \right) = L_x^\alpha L_t^\beta \left[ \left( \frac{x^\alpha}{\alpha} \right)^n g\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right]. \tag{1.3}
\]

where \( n = 1, 2, 3, \ldots \)

Proof. Using the definition of double Laplace transform of the fractional partial derivatives one gets

\[
L_x^\alpha L_t^\beta \left[ \frac{\partial^n}{\partial t^p} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right] = \int_0^\infty \int_0^\infty e^{-\frac{x^\alpha}{\alpha} - \frac{t^\beta}{\beta}} \left( \frac{\partial^n}{\partial t^p} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right) t^{\beta - 1} x^{\alpha - 1} dt dx,
\]

by taking the \( n \)th derivative with respect to \( p \) for both sides of Eq. (1.2), we have

\[
\frac{d^n}{dp^n} \left( L_x^\alpha L_t^\beta \left[ \frac{\partial^n}{\partial t^p} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right] \right) = \int_0^\infty \int_0^\infty \frac{d^n}{dp^n} \left( e^{-\frac{x^\alpha}{\alpha} - \frac{t^\beta}{\beta}} \frac{\partial^n}{\partial t^p} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right) t^{\beta - 1} x^{\alpha - 1} dt dx
\]

\[
= (-1)^n \int_0^\infty \int_0^\infty \left( \frac{x^\alpha}{\alpha} \right)^n e^{-\frac{x^\alpha}{\alpha} - \frac{t^\beta}{\beta}} t^{\beta - 1} x^{\alpha - 1} \frac{\partial^n}{\partial t^p} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) dt dx
\]

\[
= (-1)^n L_x^\alpha L_t^\beta \left[ \left( \frac{x^\alpha}{\alpha} \right)^n \frac{\partial^n}{\partial t^p} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right].
\]

we obtain

\[
(-1)^n \frac{d}{dp} \left( L_x^\alpha L_t^\beta \left[ \frac{\partial^n}{\partial t^p} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right] \right) = L_x^\alpha L_t^\beta \left[ \left( \frac{x^\alpha}{\alpha} \right)^n \frac{\partial^n}{\partial t^p} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right].
\]

Similarly, we can prove Eq. (1.3).

Existence Condition for the conformable double Laplace transform:

If \( f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \) is an exponential order \( a \) and \( b \) as \( \frac{x^\alpha}{\alpha} \to \infty, \frac{t^\beta}{\beta} \to \infty \), if there exists a positive constant \( K \) such that for all \( x > X \) and \( t > T \)

\[
\left| f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right| \leq Ke^{\frac{a}{\alpha} + \frac{b}{\beta}} x^\frac{\alpha}{\alpha} t^\frac{\beta}{\beta}, \tag{1.5}
\]

it is easy to get,

\[
f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = O(e^{\frac{a}{\alpha} + \frac{b}{\beta}}) \text{ as } \frac{x^\alpha}{\alpha} \to \infty, \frac{t^\beta}{\beta} \to \infty.
\]

Or, equivalently,

\[
\lim_{\frac{x^\alpha}{\alpha} \to \infty} e^{-\frac{a}{\alpha} - \frac{b}{\beta}} \left| f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right| = K \lim_{\frac{x^\alpha}{\alpha} \to \infty} e^{-\frac{a}{\alpha} - \frac{b}{\beta}} = 0,
\]

\[
\lim_{\frac{t^\beta}{\beta} \to \infty} e^{-\frac{a}{\alpha} - \frac{b}{\beta}} \left| f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \right| = K \lim_{\frac{t^\beta}{\beta} \to \infty} e^{-\frac{a}{\alpha} - \frac{b}{\beta}} = 0.
\]
CONFORMABLE LAPLACE TRANSFORM

where \( \mu > a \) and \( \eta > a \). The function \( f(\alpha, \beta) \) is called an exponential order as \( \frac{x^\alpha}{\alpha} \to \infty, \frac{x^\beta}{\beta} \to \infty \), and clearly, it does not grow faster than \( Ke^{\frac{x^\alpha}{\alpha} + \frac{x^\beta}{\beta}} \) as \( \frac{x^\alpha}{\alpha} \to \infty, \frac{x^\beta}{\beta} \to \infty \).

**Theorem 3.** If a function \( f(\frac{x^\alpha}{\alpha}, \frac{x^\beta}{\beta}) \) is a continuous function in every finite intervals \((0, X)\) and \((0, T)\) and of exponential order \( e^{\frac{x^\alpha}{\alpha} + \frac{x^\beta}{\beta}} \), then the conformable double Laplace transform of \( f(\frac{x^\alpha}{\alpha}, \frac{x^\beta}{\beta}) \) exists for all \( \Re p > \mu, \Re s > \eta \).

**Proof.** From the definition of the conformable double Laplace transform of \( f(\frac{x^\alpha}{\alpha}, \frac{x^\beta}{\beta}) \), we have

\[
|U(p, s)| = \left| \int_0^\infty \int_0^\infty e^{-p \frac{x^\alpha}{\alpha} - s \frac{x^\beta}{\beta}} f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) t^{\beta-1} x^{\alpha-1} dt \, dx \right|
\leq K \left| \int_0^\infty \int_0^\infty e^{-(p-a) \frac{x^\alpha}{\alpha} -(s-b) \frac{x^\beta}{\beta}} t^{\beta-1} x^{\alpha-1} dt \, dx \right|
= \frac{1}{(p-a)(s-b)}. \tag{1.6}
\]

For \( \Re p > \mu, \Re s > \eta \), from Eq.(1.6), we have

\[
\lim_{p \to \infty \atop s \to \infty} |U(p, s)| = 0 \text{ or } \lim_{p \to \infty \atop s \to \infty} U(p, s) = 0.
\]

- **2. ONE DIMENSIONAL FRACTIONAL COUPLED BURGERS’ EQUATION**

In this Section we discuss the solution of regular One dimensional conformable fractional coupled burgers equation and singular one dimensional conformable fractional coupled burgers equation by using conformable double Laplace decomposition methods (CDLDM).

**The first problem:** One dimensional conformable fractional coupled burgers equation is given by

\[
\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \eta u \frac{\partial^\alpha}{\partial x^\alpha} u + \zeta \frac{\partial^\alpha}{\partial x^\alpha} (uv) = f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right),
\]

\[
\frac{\partial^\beta v}{\partial t^\beta} - \frac{\partial^{2\beta} v}{\partial x^{2\beta}} + \eta v \frac{\partial^\beta}{\partial x^\beta} v + \mu \frac{\partial^\beta}{\partial x^\beta} (uv) = g\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right), \tag{2.1}
\]

subject to

\[
u \left(\frac{x^\alpha}{\alpha}, 0\right) = f_1 \left(\frac{x^\alpha}{\alpha}\right), \quad v \left(\frac{x^\alpha}{\alpha}, 0\right) = g_1 \left(\frac{x^\alpha}{\alpha}\right). \tag{2.2}
\]

for \( t > 0 \). Here, \( f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right), g\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right), f_1 (x) \) and \( g_1 (x) \) are given functions, \( \eta, \zeta \) and \( \mu \) are arbitrary constants depending on the system parameters such as Peclet number, Stokes velocity of particles due to gravity and Brownian diffusivity [14]. By taking conformable double Laplace transform for both sides of Eq(2.1) and conformable single Laplace transform for Eq(2.2), we have

\[
U(p, s) = \frac{F_1(p)}{s} + \frac{F(p, s)}{s} + \frac{1}{s} \int_0^\infty t^\beta \left[ \eta u \frac{\partial^\alpha}{\partial x^\alpha} u - \zeta \frac{\partial^\alpha}{\partial x^\alpha} (uv) \right] dt, \tag{2.3}
\]

where \( \mu > a \) and \( \eta > a \). The function \( f\left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) \) is called an exponential order as \( \frac{x^\alpha}{\alpha} \to \infty, \frac{x^\beta}{\beta} \to \infty \), and clearly, it does not grow faster than \( Ke^{\frac{x^\alpha}{\alpha} + \frac{x^\beta}{\beta}} \) as \( \frac{x^\alpha}{\alpha} \to \infty, \frac{x^\beta}{\beta} \to \infty \).
The Adomian polynomials for the nonlinear terms

\[ V(p, s) = \frac{G_1(p)}{s} + \frac{G(p, s)}{s} + \frac{1}{s} L_2^{\alpha} L_1^{\beta} \left[ \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} - \eta v \frac{\partial^{\alpha} v}{\partial x^{\alpha}} v - \mu \frac{\partial^{\alpha} v}{\partial x^{\alpha}} (uv) \right]. \]  

(2.4)

The conformable double Laplace decomposition methods (CDLDM) defines the solution of one dimensional conformable fractional coupled burgers equation as \[ u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \] and \[ v \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \] by the infinite series

\[ u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = \sum_{n=0}^{\infty} u_n \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right), \quad v \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = \sum_{n=0}^{\infty} v_n \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right). \]  

(2.5)

We can give Adomian’s polynomials \( A_n, B_n \) and \( C_n \) respectively as follows

\[ A_n = \sum_{n=0}^{\infty} u_n u_{2n}, \quad B_n = \sum_{n=0}^{\infty} v_n v_{2n}, \quad C_n = \sum_{n=0}^{\infty} u_n v_n. \]  

(2.6)

The Adomian polynomials for the nonlinear terms \( uu_x \), \( vv_x \) and \( uv \) are given by

\[ \begin{align*}
A_0 &= u_0 u_0 x, \\
A_1 &= u_0 u_1 x + u_1 u_0 x, \\
A_2 &= u_0 u_2 x + u_1 u_1 x + u_2 u_0 x, \\
A_3 &= u_0 u_3 x + u_1 u_2 x + u_2 u_1 x + u_3 u_0 x, \\
A_4 &= u_0 u_4 x + u_1 u_3 x + u_2 u_2 x + u_3 u_1 x + u_4 u_0 x, \\
B_0 &= v_0 v_0 x, \\
B_1 &= v_0 v_1 x + v_1 v_0 x, \\
B_2 &= v_0 v_2 x + v_1 v_1 x + v_2 v_0 x, \\
B_3 &= v_0 v_3 x + v_1 v_2 x + v_2 v_1 x + v_3 v_0 x, \\
B_4 &= v_0 v_4 x + v_1 v_3 x + v_2 v_2 x + v_3 v_1 x + v_4 v_0 x.
\end{align*} \]  

(2.7)

and

\[ \begin{align*}
C_0 &= u_0 v_0, \\
C_1 &= u_0 v_1 + u_1 v_0, \\
C_2 &= u_0 v_2 + u_1 v_1 + u_2 v_0, \\
C_3 &= u_0 v_3 + u_1 v_2 + u_2 v_1 + u_3 v_0, \\
C_4 &= u_0 v_4 + u_1 v_3 + u_2 v_2 + u_3 v_1 + u_4 v_0.
\end{align*} \]  

(2.8)

By applying inverse conformable double Laplace transform on both sides of Eq.(2.3) and Eq.(2.4), making use of Eq.(2.6), we have

\[ \begin{align*}
\sum_{n=0}^{\infty} u_n \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) &= f_1 (x) + L_p^{-1} L_s^{-1} \left[ \frac{F(p, s)}{s} \right] \\
&\quad + L_p^{-1} L_s^{-1} \left[ \frac{1}{s} L_2^{\alpha} L_1^{\beta} \left[ \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} \right] \right] \\
&\quad - L_p^{-1} L_s^{-1} \left[ \frac{1}{s} L_2^{\alpha} L_1^{\beta} \left[ \eta A_n \right] \right] \\
&\quad - L_p^{-1} L_s^{-1} \left[ \frac{1}{s} L_2^{\alpha} L_1^{\beta} \left[ \zeta (C_n) x \right] \right].
\end{align*} \]  

(2.10)
and
\[ \sum_{n=0}^{\infty} v_n \left( \frac{x^\alpha t^\beta}{\alpha \beta} \right) = g_1(x) + L_p^{-1} L_s^{-1} \left[ \frac{G(p,s)}{s} \right] \]
\[ + L_p^{-1} L_s^{-1} \left[ \frac{L_2^\alpha L_t^\beta}{s} \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}} \right] \]
\[ - L_p^{-1} L_s^{-1} \left[ \frac{L_2^\alpha L_t^\beta}{s} [\eta B_n] \right] \]
\[ - L_p^{-1} L_s^{-1} \left[ \frac{L_2^\alpha L_t^\beta}{s} [\mu (C_n)_{2\alpha}] \right]. \]  
(2.11)

On comparing both sides of the Eq.(2.10) and Eq.(2.11) we have
\[ u_0 = f_1(x) + L_p^{-1} L_s^{-1} \left[ \frac{F(p,s)}{s} \right], \]
\[ v_0 = g_1(x) + L_p^{-1} L_s^{-1} \left[ \frac{G(p,s)}{s} \right]. \]  
(2.12)

In general, the recursive relation is given by
\[ u_{n+1} = L_p^{-1} L_s^{-1} \left[ \frac{1}{s} L_2^\alpha L_t^\beta \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} \right] \]
\[ - L_p^{-1} L_s^{-1} \left[ \frac{1}{s} L_2^\alpha L_t^\beta [\eta A_n] \right] \]
\[ - L_p^{-1} L_s^{-1} \left[ \frac{1}{s} L_2^\alpha L_t^\beta [\zeta (C_n)_{2\alpha}] \right]. \]  
(2.13)

and
\[ v_{n+1} = L_p^{-1} L_s^{-1} \left[ \frac{1}{s} L_2^\alpha L_t^\beta \frac{\partial^{2\alpha} v}{\partial x^{2\alpha} v_n} \right] \]
\[ - L_p^{-1} L_s^{-1} \left[ \frac{1}{s} L_2^\alpha L_t^\beta [\eta B_n] \right] \]
\[ - L_p^{-1} L_s^{-1} \left[ \frac{1}{s} L_2^\alpha L_t^\beta [\mu (C_n)_{2\alpha}] \right]. \]  
(2.14)

Here, we provide double inverse laplace transform with respect to \( p \) and \( s \) exist for each terms in the right hand side of above equations. To illustrate this method for one dimensional conformable fractional coupled burgers equation we take the following example:

**Example 3.** Consider the following homogeneous form of a one dimensional conformable fractional coupled burgers equation
\[ \frac{\partial^\beta u}{\partial t^\beta} - \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} - 2u \frac{\partial^\alpha}{\partial x^\alpha} \left( u \right) = 0, \]
\[ \frac{\partial^\beta v}{\partial t^\beta} - \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}} - 2v \frac{\partial^\alpha}{\partial x^\alpha} \left( v \right) = 0, \]  
(2.15)

with initial condition
\[ u \left( \frac{x^\alpha}{\alpha}, 0 \right) = \sin \frac{x^\alpha}{\alpha}, \quad v \left( \frac{x^\alpha}{\alpha}, 0 \right) = \sin \frac{x^\alpha}{\alpha}. \]  
(2.16)
By using Eqs. (2.12), Eq. (2.13) and Eq. (2.14) we have

\[
\begin{align*}
    u_0 &= \sin \frac{x^2}{\alpha},
    v_0 &= \sin \frac{x^2}{\alpha}, \\
    u_1 &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + 2u_{0x}D^{(1)}_\alpha u_0 - \frac{\partial^\alpha}{\partial x^\alpha} (u_0v_0) \right) \right] \\
    &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( - \sin \frac{x^2}{\alpha} \right) \right] = L_p^{-1}L_s^{-1} \left[ \frac{1}{s^2 (p^2 + 1)} \right] = -\frac{t^3}{\beta} \sin \frac{x^2}{\alpha}, \\
    v_1 &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}} + 2v_{0x}D^{(1)}_\alpha v_0 - \frac{\partial^\alpha}{\partial x^\alpha} (u_0v_0) \right) \right] \\
    &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( - \sin \frac{x^2}{\alpha} \right) \right] = L_p^{-1}L_s^{-1} \left[ \frac{1}{s^2 (p^2 + 1)} \right] = -\frac{t^3}{\beta} \sin \frac{x^2}{\alpha}, \\
    u_2 &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + 2 \left( u_{0x}D^{(1)}_\alpha u_1 + u_{1x}D^{(1)}_\alpha x_0 \right) - \frac{\partial^\alpha}{\partial x^\alpha} (u_0v_1 + u_1v_0) \right) \right] \\
    &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( \frac{t^3}{\beta} \sin \frac{x^2}{\alpha} \right) \right] = L_p^{-1}L_s^{-1} \left[ \frac{1}{s^3 (p^2 + 1)} \right] = -\frac{(t^3)^2}{2} \sin \frac{x^2}{\alpha}, \\
    v_2 &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}} + 2 \left( v_{0x}D^{(1)}_\alpha v_1 + v_{1x}D^{(1)}_\alpha v_0 \right) - \frac{\partial^\alpha}{\partial x^\alpha} (u_0v_1 + u_1v_0) \right) \right] \\
    &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( \frac{t^3}{\beta} \sin \frac{x^2}{\alpha} \right) \right] = L_p^{-1}L_s^{-1} \left[ \frac{1}{s^3 (p^2 + 1)} \right] = -\frac{(t^3)^2}{2} \sin \frac{x^2}{\alpha}, \\
    \text{and} \\
    u_3 &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + 2 \left( u_{0x}D^{(1)}_\alpha u_2 + u_{1x}D^{(1)}_\alpha u_1 + u_{2x}D^{(1)}_\alpha u_0 \right) \right) \right] \\
    &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( \frac{t^3}{\beta} \sin \frac{x^2}{\alpha} \right) \right] = L_p^{-1}L_s^{-1} \left[ \frac{1}{s^3 (p^2 + 1)} \right] = -\frac{(t^3)^2}{6} \sin \frac{x^2}{\alpha}, \\
    v_3 &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}} + 2 \left( v_{0x}D^{(1)}_\alpha v_2 + v_{1x}D^{(1)}_\alpha v_1 + v_{2x}D^{(1)}_\alpha v_0 \right) \right) \right] \\
    &= L_p^{-1}L_s^{-1} \left[ \frac{1}{s} L_x^\alpha L_t^\beta \left( \frac{t^3}{\beta} \sin \frac{x^2}{\alpha} \right) \right] = L_p^{-1}L_s^{-1} \left[ \frac{1}{s^3 (p^2 + 1)} \right] = -\frac{(t^3)^2}{6} \sin \frac{x^2}{\alpha}.
\end{align*}
\]
and so on for other components. Using Eq.(2.5), the series solutions are therefore given by

\[ u \left( \frac{x^\alpha}{\alpha} \frac{t^\beta}{\beta} \right) = u_0 + u_2 + u_3 + \ldots = \left( 1 - \left( \frac{t^\beta}{\beta} \right) + \left( \frac{t^\beta}{\beta} \right)^2 - \frac{t^\beta}{\beta} \right)^3 + \ldots \sin \frac{x^\alpha}{\alpha} \]

\[ v \left( \frac{x^\alpha}{\alpha} \frac{t^\beta}{\beta} \right) = v_0 + v_2 + v_3 + \ldots = \left( 1 - \left( \frac{t^\beta}{\beta} \right) + \left( \frac{t^\beta}{\beta} \right)^2 - \frac{t^\beta}{\beta} \right)^3 + \ldots \sin \frac{x^\alpha}{\alpha} \]

and hence the exact solutions become

\[ u \left( \frac{x^\alpha}{\alpha} \frac{t^\beta}{\beta} \right) = e^{-\frac{\beta}{\alpha}} \sin \frac{x^\alpha}{\alpha}, \quad v \left( \frac{x^\alpha}{\alpha} \frac{t^\beta}{\beta} \right) = e^{-\frac{\beta}{\alpha}} \sin \frac{x^\alpha}{\alpha}. \]

By taking \( \alpha = 1 \) and \( \beta = 1 \), the fractional solution become

\[ u \left( \frac{x^\alpha}{\alpha} \frac{t^\beta}{\beta} \right) = e^{-t} \sin x, \quad v \left( \frac{x^\alpha}{\alpha} \frac{t^\beta}{\beta} \right) = e^{-t} \sin x. \]

The second problem: Singular one dimensional conformable fractional coupled burgers equation with Bessel operator are given by

\[
\frac{\partial^\beta u}{\partial t^\beta} - \frac{\alpha}{x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} u \right) + \eta u \frac{\partial^\alpha}{\partial x^\alpha} u + \zeta \frac{\partial^\alpha}{\partial x^\alpha} (uv) = f \left( \frac{x^\alpha}{\alpha} \frac{t^\beta}{\beta} \right) \\
\frac{\partial^\beta v}{\partial t^\beta} - \frac{\alpha}{x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} v \right) + \eta v \frac{\partial^\alpha}{\partial x^\alpha} v + \mu \frac{\partial^\alpha}{\partial x^\alpha} (uv) = g \left( \frac{x^\alpha}{\alpha} \frac{t^\beta}{\beta} \right) 
\]

with initial conditions

\[ u \left( \frac{x^\alpha}{\alpha}, 0 \right) = f_1 \left( \frac{x^\alpha}{\alpha} \right), \quad v \left( \frac{x^\alpha}{\alpha}, 0 \right) = g_1 \left( \frac{x^\alpha}{\alpha} \right), \quad (2.18) \]

where the linear terms \( \frac{\alpha}{x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \right) \) is the called conformable Bessel operator where \( \zeta, \mu \) and \( \eta \) are real constants. In order to obtain the solution of Eq.(2.17),

First: multiply both sides of Eq.(2.17) by \( \frac{x^\alpha}{\alpha} \) we have

\[
\frac{x^\alpha}{\alpha} \frac{\partial^\beta u}{\partial t^\beta} - \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} u \right) + \eta \frac{\partial^\alpha}{\partial x^\alpha} u \frac{\partial^\alpha}{\partial x^\alpha} u + \zeta \frac{\partial^\alpha}{\partial x^\alpha} (uv) = \frac{x^\alpha}{\alpha} f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \\
\frac{x^\alpha}{\alpha} \frac{\partial^\beta v}{\partial t^\beta} - \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} v \right) + \eta \frac{\partial^\alpha}{\partial x^\alpha} v \frac{\partial^\alpha}{\partial x^\alpha} v + \mu \frac{\partial^\alpha}{\partial x^\alpha} (uv) = \frac{x^\alpha}{\alpha} g \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right). 
\]

\[ (2.19) \]

Second: we apply conformable double Laplace transform on both sides of Eq.(2.19) and single conformable Laplace transform for initial condition, we get

\[
L_x^\alpha L_t^\beta \left[ \frac{x^\alpha}{\alpha} \frac{\partial^\beta u}{\partial t^\beta} \right] = L_x^\alpha L_t^\beta \left[ \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} u \right) - \eta \frac{\partial^\alpha}{\partial x^\alpha} u \frac{\partial^\alpha}{\partial x^\alpha} u - \zeta \frac{\partial^\alpha}{\partial x^\alpha} (uv) + \frac{x^\alpha}{\alpha} f \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \right], \\
L_x^\alpha L_t^\beta \left[ \frac{x^\alpha}{\alpha} \frac{\partial^\beta v}{\partial t^\beta} \right] = L_x^\alpha L_t^\beta \left[ \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} v \right) - \eta \frac{\partial^\alpha}{\partial x^\alpha} v \frac{\partial^\alpha}{\partial x^\alpha} v - \mu \frac{\partial^\alpha}{\partial x^\alpha} (uv) + \frac{x^\alpha}{\alpha} g \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \right]. 
\]

\[ (2.20) \]
The nonlinear operators can be defined as

\[ \left[ \frac{d}{dp} + L_x \right] f_1(x) = \frac{d}{dp} L_x L_t^\beta \left[ \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha \partial^\alpha}{\alpha \partial x^\alpha} u \right) \right] - \eta x^{\alpha u} \frac{\partial^\alpha}{\partial x^\alpha} u - \zeta x^{\alpha \partial^\alpha} (uv) \]

\[ \left[ \frac{d}{dp} L_x L_t^\beta \right] \left[ f \left( \frac{x^\alpha \partial^\alpha}{\alpha \partial x^\alpha} u \right) \right], \]

simplifying Eq.(2.21), we obtain

\[ \frac{d}{dp} U(p, s) = \frac{1}{s} \int_0^p \left( \frac{d}{dp} L_x L_t^\beta \left[ \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha \partial^\alpha}{\alpha \partial x^\alpha} u \right) \right] - \eta x^{\alpha u} \frac{\partial^\alpha}{\partial x^\alpha} u - \zeta x^{\alpha \partial^\alpha} (uv) \right) dp \]

Third: by integrating both sides of Eq.(2.22) from 0 to \( p \) with respect to \( p \), we have

\[ U(p, s) = \frac{1}{s} \int_0^p \left( \frac{d}{dp} L_x L_t^\beta \left[ \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha \partial^\alpha}{\alpha \partial x^\alpha} u \right) \right] - \eta x^{\alpha u} \frac{\partial^\alpha}{\partial x^\alpha} u - \zeta x^{\alpha \partial^\alpha} (uv) \right) dp \]

Using conformable double Laplace decomposition method to defines the solution of the system as \( u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \) and \( v \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) \) by the infinite series

\[ u \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = \sum_{n=0}^{\infty} u_n \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right), \quad v \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = \sum_{n=0}^{\infty} v_n \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right). \]
The first few components can be written as

\[\sum_{n=0}^{\infty} u_n \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = f_1(x) + L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p dF (p, s) \right] + L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \frac{\partial ^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial ^\alpha}{\partial x^\alpha} \left( \sum_{n=0}^{\infty} u_n \right) \right) \right) \right] dp \]

\[+ L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \eta \frac{x^\alpha}{\alpha} \sum_{n=0}^{\infty} A_n \right) \right) dp \right] + L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \zeta \frac{x^\alpha}{\alpha} \sum_{n=0}^{\infty} C_n \right) \right) dp \right], \quad (2.26)\]

and

\[\sum_{n=0}^{\infty} v_n \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = g_1(x) + L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p dG (p, s) \right] + L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \frac{\partial ^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial ^\alpha}{\partial x^\alpha} \left( \sum_{n=0}^{\infty} v_n \right) \right) \right) \right] dp \]

\[+ L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \eta \frac{x^\alpha}{\alpha} \sum_{n=0}^{\infty} B_n \right) \right) dp \right] + L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \zeta \frac{x^\alpha}{\alpha} \sum_{n=0}^{\infty} C_n \right) \right) dp \right], \quad (2.27)\]

The first few components can be written as

\[u_0 = f_1(x) + L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p dF (p, s) \right] \],

\[v_0 = g_1(x) + L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p dG (p, s) \right] \], \quad (2.28)\]

and

\[u_{n+1} \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = -L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \frac{\partial ^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial ^\alpha}{\partial x^\alpha} \left( \sum_{n=0}^{\infty} u_n \right) \right) \right) \right] dp \]

\[+ L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \eta \frac{x^\alpha}{\alpha} \sum_{n=0}^{\infty} A_n \right) \right) dp \right] + L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \zeta \frac{x^\alpha}{\alpha} \sum_{n=0}^{\infty} C_n \right) \right) dp \right], \quad (2.29)\]

and

\[v_{n+1} \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) = -L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \frac{\partial ^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial ^\alpha}{\partial x^\alpha} \left( \sum_{n=0}^{\infty} v_n \right) \right) \right) \right] dp \]

\[+ L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \eta \frac{x^\alpha}{\alpha} \sum_{n=0}^{\infty} B_n \right) \right) dp \right] + L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_x^\alpha L_t^\beta \left( \zeta \frac{x^\alpha}{\alpha} \sum_{n=0}^{\infty} C_n \right) \right) dp \right]. \quad (2.30)\]
Here we provide double inverse Laplace transform with respect to \( p \) and \( s \) exist for each terms in the right hand side of of Eqs. (2.28), (2.29) and (2.30).

**Example 4.** Singular one dimensional conformable fractional coupled burgers equation

\[
\begin{align*}
\frac{\partial^\beta u}{\partial t^\beta} &- \frac{\alpha}{x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} u \right) - 2u \frac{\partial^\alpha}{\partial x^\alpha} u + \frac{\partial^\alpha}{\partial x^\alpha} (uv) = \left( \frac{x^\alpha}{\alpha} \right)^2 e^{\frac{\beta}{\alpha}} - 4e^{\frac{\beta}{\alpha}} \\
\frac{\partial^\beta v}{\partial t^\beta} &- \frac{\alpha}{x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} v \right) - 2v \frac{\partial^\alpha}{\partial x^\alpha} v + \frac{\partial^\alpha}{\partial x^\alpha} (uv) = \left( \frac{x^\alpha}{\alpha} \right)^2 e^{\frac{\beta}{\alpha}} - 4e^{\frac{\beta}{\alpha}}, \\
\end{align*}
\]

(2.31)

subject to

\[
\begin{align*}
u(x, 0) &= \left( \frac{x^\alpha}{\alpha} \right)^2, & v(x, 0) &= \left( \frac{x^\alpha}{\alpha} \right)^2. \\
\end{align*}
\]

(2.32)

By applying the above steps, we obtain

\[
\begin{align*}
\sum_{n=0}^{\infty} u_n \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) &= \left( \frac{x^\alpha}{\alpha} \right)^2 e^{\frac{\beta}{\alpha}} - 4e^{\frac{\beta}{\alpha}} + 4 \\
&- L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_{\frac{\alpha}{\alpha}}^\beta L_{\frac{\beta}{\beta}}^\alpha \left[ \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \left( \sum_{n=0}^{\infty} u_n \right) \right) \right] \right) dp \\
&- L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_{\frac{\alpha}{\alpha}}^\beta L_{\frac{\beta}{\beta}}^\alpha \left[ 2 \frac{x^\alpha}{\alpha} \sum_{n=0}^{\infty} A_n \right] \right) dp \\
&+ L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_{\frac{\alpha}{\alpha}}^\beta L_{\frac{\beta}{\beta}}^\alpha \left[ \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \left( \sum_{n=0}^{\infty} C_n \right) \right] \right) dp \right], \quad (2.33)
\end{align*}
\]

and

\[
\begin{align*}
\sum_{n=0}^{\infty} v_n \left( \frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta} \right) &= \left( \frac{x^\alpha}{\alpha} \right)^2 e^{\frac{\beta}{\alpha}} - 4e^{\frac{\beta}{\alpha}} + 4 \\
&- L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_{\frac{\alpha}{\alpha}}^\beta L_{\frac{\beta}{\beta}}^\alpha \left[ \frac{\partial^\alpha}{\partial x^\alpha} \left( \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \left( \sum_{n=0}^{\infty} u_n \right) \right) \right] \right) dp \\
&- L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_{\frac{\alpha}{\alpha}}^\beta L_{\frac{\beta}{\beta}}^\alpha \left[ 2 \frac{x^\alpha}{\alpha} \sum_{n=0}^{\infty} B_n \right] \right) dp \\
&+ L_p^{-1} L_s^{-1} \left[ \frac{1}{s} \int_0^p \left( L_{\frac{\alpha}{\alpha}}^\beta L_{\frac{\beta}{\beta}}^\alpha \left[ \frac{x^\alpha}{\alpha} \sum_{n=0}^{\infty} C_n \right] \right) dp \right], \quad (2.34)
\end{align*}
\]
where \( A_n, B_n \) and \( C_n \) are defined in Eqs. (2.7), (2.8) and (2.9) respectively. On using Eqs. (2.28), (2.29) and (2.30) the components are given by

\[
\begin{align*}
u_0 &= \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} - 4e^\frac{t^\beta}{\beta} + 4, \\
u_1 &= -L_p^{-1}L_s^{-1}\left[1 + \int_0^p L_x^\alpha L_t^\beta \left(\frac{\partial^\alpha}{\partial x^\alpha} \left(\frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} u_0\right) + 2\frac{x^\alpha}{\alpha} u_0 \frac{\partial^\alpha}{\partial x^\alpha} u_0 - \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} (u_0 v_0)\right) dp \right] \\
u_2 &= -L_p^{-1}L_s^{-1}\left[1 + \int_0^p L_x^\alpha L_t^\beta \left(\frac{\partial^\alpha}{\partial x^\alpha} \left(\frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} u_0\right) + 2\frac{x^\alpha}{\alpha} u_0 \frac{\partial^\alpha}{\partial x^\alpha} u_0 - \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} (u_0 v_0)\right) dp \right]
\end{align*}
\]

In the same manner, we obtain that

\[
\begin{align*}
u_2 &= -L_p^{-1}L_s^{-1}\left[1 + \int_0^p L_x^\alpha L_t^\beta \left(\frac{\partial^\alpha}{\partial x^\alpha} \left(\frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} u_0\right) + 2\frac{x^\alpha}{\alpha} u_0 \frac{\partial^\alpha}{\partial x^\alpha} u_0 - \frac{x^\alpha}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} (u_0 v_0)\right) dp \right] \\
u_2 &= 0 \\
u_2 &= 0,
\end{align*}
\]

It is obvious that the self-cancelling some terms appear between various components and the connected by coming terms, then we have,

\[
u \left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = u_0 + u_1 + u_2 + ... , \quad v \left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = v_0 + v_1 + v_2 + ...
\]

Therefore, the exact solution is given by

\[
u \left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}} \quad \text{and} \quad v \left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = \left(\frac{x^\alpha}{\alpha}\right)^2 e^{\frac{t^\beta}{\beta}}
\]

By taking \( \alpha = 1 \) and \( \beta = 1 \), the fractional solution becomes

\[
u \left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = x^2 e^t, \quad \quad v \left(\frac{x^\alpha}{\alpha}, \frac{t^\beta}{\beta}\right) = x^2 e^t
\]

**Conclusion 1.** In this work, we give the solution of the one dimensional Regular and singular conformable fractional coupled Burgers equation by using conformable double Laplace decomposition method. Moreover, two examples were given to validate our method. This method can be apply to solve some nonlinear time-fractional differential equations with conformable derivative.
HASSAN ELTAYEB¹, IMED BACHAR² AND ADEM KILICMAN³

REFERENCES


¹,² Mathematics Department, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia and ³ Department of Mathematics, University Putra Malaysia, 43400 Serdang, Selangor, Malaysia

E-mail address: hgadain@ksu.edu.sa, abachar@ksu.edu.sa, akilic@upm.edu.my