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Article

Minimizing an Insurer's Ultimate Ruin Probability by Noncheap Proportional Reinsurance Arrangements and Investments

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Abstract: In this paper, we work with a diffusion-perturbed risk model comprising a surplus generating process and an investment return process. The investment return process is of standard Black-Scholes type, that is, it comprises a single risk-free asset that earns interest at a constant rate and a single risky asset whose price process is modelled by a geometric Brownian motion. Additionally, the company is allowed to purchase noncheap proportional reinsurance priced via the expected value principle. Using the Hamilton-Jacobi-Bellman approach, we derive a second-order Volterra integrodifferential equation which we transform into a linear Volterra integral equation of the second kind. We proceed to solve this integral equation numerically using the block-by-block method for the optimal reinsurance retention level that minimizes the ultimate ruin probability. The numerical results based on light- and heavy-tailed distributions show that proportional reinsurance and investments play a vital role in enhancing the survival of insurance companies. But the ruin probability exhibits sensitivity to the volatility of the stock price.

Keywords: ruin probability; jump-diffusion; HJB equation; Volterra equation; block-by-block method; proportional reinsurance; investments

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1. Introduction

The problem of minimizing the ruin probability, when the insurance company is allowed to invest part of its surplus in the money and stock markets and to reduce its risk by entering into proportional reinsurance treaties, has been extensively studied in different forms since the ground-breaking work of Bachelier [1]. Liang and Guo [2] found that the minimal ruin probability maximizes the adjustment coefficient γ under proportional reinsurance and that it satisfies the Lundberg inequality $\psi(u) \leq Ce^{-\gamma u}$, where C is a constant. Wang [3] considered the case of multiple risky assets in an optimal investment problem for an insurer whose surplus evolves according to a jump-diffusion process, while Liang and Guo [4] considered the optimal reinsurance problem by combining quota-share and excess-of-loss reinsurance. The authors derived explicit expressions for the value function and the optimal strategies.

Kasozi et al. [5] studied the problem of controlling ultimate ruin probability by quota-share reinsurance arrangements for an insurer that is allowed to invest part of the surplus in a risk-free and risky asset. They found that, for chosen parameter values, the optimal quota-share retention lies in the interval $(0.2, 0.4)$, i.e., the company should cede between 60 and 80% of its risks to a reinsurer. This study also found that the ruin probabilities increase when stock prices become more volatile. However, while [5] assumed cheap reinsurance, in this paper we use noncheap reinsurance. Zhou et al. [6] investigated the optimal proportional reinsurance and investment problem for a jump-diffusion surplus process in a constant elasticity of variance (CEV) stock market.

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Liu and Yang [7] revisited the model in Hipp and Plum [8] by incorporating a risk-free interest rate. Since they could not obtain closed-form solutions in this case, they provided numerical results for optimal strategies for maximizing the survival probability under different claim-size distribution assumptions. Schmidli [9] proved the existence and uniqueness of a solution of the ruin probability minimization problem in a model compounded by investment and dynamic proportional reinsurance for the case $\lambda > 0$ and $\sigma = 0$, i.e., when there is no diffusion and when F has a bounded density. But while [9] uses proportional reinsurance in minimizing ruin probabilities in the Cramér-Lundberg model, this paper considers proportional reinsurance and investments of Black-Scholes type in the diffusion-perturbed model.

With the objective of determining the optimal investment and reinsurance strategies, Liang and Young [10] studied the problem of minimizing the probability of ruin in the presence of per-loss reinsurance for an insurance company whose risk process follows a compound Poisson process or its diffusion approximation. Assuming that the financial market in which the company invests follows the Black-Scholes model, and under minimal assumptions regarding admissible reinsurance forms, [10] showed that the optimal per-loss reinsurance policy is excess-of-loss reinsurance. They found that for cheap reinsurance under both models full reinsurance is never optimal, a result consistent with Mossin [11]. While under the compound Poisson model it is optimal not to buy reinsurance when the surplus is sufficiently low, for the diffusion approximation model the insurer always buys some amount of reinsurance but the optimal retention is inversely proportional to the surplus. This is also true of the optimal investment level as it decreases with an increase in the surplus. However, [10] concerned itself with excess-of-loss reinsurance while this paper explores optimality of noncheap proportional reinsurance and employs different numerical methods from those of [10].

Zhu et al. [12] studied the optimal proportional reinsurance and investment problem in a general jump-diffusion financial market. With the objective of maximizing the expected exponential utility of terminal wealth, they added a general jump to the price of the risky asset, so that the financial market follows a general jump-diffusion model. They also incorporated a reasonable constraint on the proportional reinsurance strategy, thus making the model more reasonable and realistic, and derived closed-form expressions for the value function and optimal strategy. Glineur and Walhin [13] revisited de Finetti's retention problem for proportional reinsurance by applying the *convex optimization method*. The authors extended the result to variable quota-share and surplus reinsurance with table of lines and showed, by means of a numerical example, that neither variable quota share reinsurance nor surplus reinsurance with table of lines may be considered as optimal reinsurance structures. They were able to determine the optimal quota-share and surplus reinsurance strategies. However, the numerical example also led them to the conclusion that there exists no general rule asserting superiority of either quota-share-type or surplus-type reinsurance above the other.

An insurance company is said to have experienced ruin when its surplus becomes negative, thus making it impossible for the company to meet its financial obligations (e.g., claims). The time of ruin is the first time that the cedent's surplus process enters $(-\infty, 0)$ and the associated probability is referred to as the ultimate ruin probability. Ruin is a technical term which does not necessarily mean that the company is bankrupt but rather that bankruptcy is at hand and that the company should therefore be prompted to take action to improve its solvency status. Thus, insurance companies customarily take precautions to avoid ruin. These precautions are referred to as *control variables* and include investments, capital injections or refinancing, portfolio selection, volume control through the setting of premiums and reinsurance arrangements, to mention but a few. This study focuses on reinsurance as a risk control mechanism for a company that also invests part of its surplus in risk-free and risky assets.

According to Jang and Kim [14], insurance companies generally face two sources of risk, viz., an insolvency risk that arises from unexpectedly large insurance claims, and a market risk that arises from

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risky investments in financial markets. Reinsurance can help mitigate the insolvency risk, while investing in some risk-free assets such as short-term bonds and money market funds could reduce the market risk. Reinsurance is the transfer of risk from a direct insurer (the cedent) to a second insurance carrier (the reinsurer). It serves the purpose of offering protection to cedents against very large individual claims or fluctuations in their aggregate portfolio of risks, as well as diversifying the financial losses caused by it. Reinsurance therefore allows the cedent to pass on some of its risk to the reinsurer but at the expense of a portion of the aggregate premiums receivable from the policyholders [15].

Mikosch [16] has pointed out that reinsurance treaties are of two types: *Random walk* type reinsurance which includes proportional, excess-of-loss and stop-loss reinsurance, and *extreme value* type reinsurance which includes largest claims and ECOMOR reinsurance (excédent du coût moyen relatif or 'excess of the average cost'). Proportional, or pro rata, reinsurance is a common form of reinsurance for claims of 'moderate' size, and requires the reinsurer to cover a fraction of each claim equal to the fraction of total premiums ceded to the reinsurer. Proportional reinsurance treaties are traditionally subdivided into two forms: quota-share and surplus reinsurance. Quota-share reinsurance is a common type of proportional reinsurance in which the cedent and the reinsurer agree to share claims and premiums in the same proportion which remains constant throughout the portfolio [17]. With surplus reinsurance the reinsurer agrees to accept an individual risk with sum insured in excess of the direct retention limit set by the cedent [18].

It has been noted in [19] that proportional reinsurance is the easiest way of covering an insurance portfolio. This paper focuses on quota-share (QS) proportional reinsurance due to its simplicity, but other forms of reinsurance could also be used. In addition, the reinsurer pays a 'ceding commission' to the cedent to compensate for the costs of underwriting the ceded business. This commission is ignored in this study. Thus, if a cedent enters into a quota-share reinsurance treaty with a reinsurer, then they will share claims and premiums according to a retention level $k \in [0, 1]$. For every claim X that occurs at the time where the surplus prior to the claim payment is u, the cedent pays kX while the reinsurer pays $(1-k)X$. Similarly, for every premium amount c received by the insurer, $c^R = (1-k)c$ is paid to the reinsurer and $c^k = c - c^R$ is retained by the cedent. Since the factor $(1-k)$ represents the proportion of claims or premiums ceded to the reinsurer, it is called the *cession level*. It should be noted that for cheap reinsurance, $c^k = kc$.

It has been argued in the literature that the Cramér-Lundberg model is somewhat inadequate for modelling real-world insurance processes in that it does not account for interest earned on the reserve and for long tail business with claims that are settled long after occurrence. Furthermore, it does not include time-dependence or randomness of premium income or of the size of the portfolio. For these reasons, we make generalisations to the well known Cramér-Lundberg model by adding a diffusion term and also allowing the company to invest in the financial markets with returns of Black-Scholes type. Thus, this paper focuses on ultimate ruin and considers proportional reinsurance coupled with investments as mechanisms for reducing the insurer's ultimate ruin probability. Reinsurance can protect insurers against potentially large losses, while investment of insurance premiums enables insurers to achieve certain management objectives, some of the most common of which are the minimization of the ruin probability, maximization of expected utility and mean-variance criteria. Li et al. [20] have pointed out that insurance companies commonly employ integrated reinsurance and investment strategies to increase their underwriting capacity, stabilize underwriting results, protect themselves against catastrophic losses and achieve financial growth.

The remainder of the paper is organized as follows. In Section 2, we present the models to be studied and the underlying assumptions. In Section 3, we give the Hamilton-Jacobi-Bellman (HJB) equation and verification theorems for the ruin probabilities under proportional reinsurance, as well as the corresponding Volterra integrodifferential and integral equations. Section 4 gives a brief outline of the numerical method used for solving the Volterra integral equation. In Section 5, we present numerical results and examples based on light- and heavy-tailed distributions. Finally, in Section 6 we give some concluding remarks and possible extensions to this work.

2. The Models

To give a rigorous mathematical formulation of the problem, we assume that all stochastic quantities are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_\sqcup\}_{\sqcup \in \mathbb{R}^+}, \mathbb{P})$ satisfying the usual conditions, i.e. the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$, which represents the information available at time t and forms the basis for all decision-making, is right-continuous and $\mathbb{P}$-complete. Right-continuity is necessary for ensuring that the ruin time defined later in this section is a stopping time. The risk process considered in this paper is made up of two important processes: the insurance process and the investment-generating process. In the absence of reinsurance, the insurance process $\{P_t\}_{t\in\mathbb{R}^+}$ is given by the diffusion-perturbed model

$$P_t = ct + \sigma_1 W_{1,t} - \underline{S}, \quad t \geq 0, \tag{1}$$

where the process $\underline{S} = \{S_t\}_{t\in\mathbb{R}^+}$ defined as

$$S_t = \begin{cases} \sum_{i=1}^{N_t} X_i & \text{if } N_t > 0 \\ 0 & \text{if } N_t = 0 \end{cases}$$

is a compound Poisson process representing the aggregate claims made by policyholders. Here, the premiums are assumed to be calculated according to the expected value premium principle and to be collected continuously over time at a constant rate $c = (1+\eta)\lambda\mu > 0$, where $\eta > 0$ is the relative safety loading of the insurer. W_1 is a one-dimensional standard Brownian motion independent of the compound Poisson process S_t, $\{N_t\}$ is a homogeneous Poisson process with constant intensity λ and the claim sizes $\{X_i\}_{i\in\mathbb{N}}$ are a sequence of strictly positive i.i.d. random variables. We assume that the processes $\{X_i\}_{i\in\mathbb{N}}$, $\{N_t\}_{t\in\mathbb{R}^+}$ and $\{W_{1,t}\}_{t\in\mathbb{R}^+}$ are mutually independent. We denote by F the distribution function of X_i, by $\mu = \mathbb{E}[X_i]$ its first moment and by $M_X(t) = \mathbb{E}\left[e^{tX_i}\right]$ its moment-generating function. We will assume that $F(0) = 0$ and that at least one of σ_1 or λ is non-zero.

The diffusion term $\sigma_1 W_1$ in the basic model (1) has been interpreted in a two-fold manner in the literature. On the one hand, $\sigma_1 W_1$ could be understood as standing for the uncertainty or random fluctuations associated with the insurance process at time t (the *U-S* case). This means that the aggregate claims up to time t are given by the compound Poisson process S_t. On the other hand, $\sigma_1 W_1$ could represent the additional small claims which account for uncertainty associated with the insurance market or the economic environment (the *A-C* case), so that the aggregate claims process is $\hat{S}_t = S_t - \sigma_1 W_{1,t}$ (see, e.g., [6]). It should be noted that, given an initial surplus u, when there is no volatility in the surplus and claim amounts (i.e., when $\sigma_1 = 0$), Equation (1) becomes the well-known classical risk process (or the Cramér-Lundberg model).

Given that the insurer controls its insurance risk by taking QS proportional reinsurance at a retention level $k \in [0,1]$, the insurance process in the presence of QS reinsurance is now

$$P_t^k = c^k t + k\sigma_1 W_{1,t} - k\underline{S} \tag{2}$$

with dynamics

$$dP_t^k = c^k dt + k\sigma_1 dW_{1,t} - kd\underline{S}. \tag{3}$$

If $k = 0$ then there is full reinsurance, i.e., the entire portfolio of risks is ceded to the reinsurer, whereas if $k = 1$ then there is no reinsurance. The case $k = 1$ is precisely the model considered in [21,22]. In this study, we assume *noncheap* reinsurance, meaning that the reinsurer uses a higher safety loading than the insurer. Otherwise, the insurance company can take full reinsurance and receive a positive return without any risk, which is undesirable from the reinsurer's standpoint, as was demonstrated in [23]. Thus, if $c^R = (1-k)(1+\theta)\lambda\mu$ is the reinsurance premium to be paid for the QS reinsurance, then the insurance premium rate is $c^k = c - c^R = [k(1+\theta) - (\theta - \eta)]\lambda\mu$, where $\theta \in (\eta, \infty)$ is the reinsurer's safety loading. In order for the net profit condition (NPC) to be fulfilled, that is,

$$[k(1+\theta) - (\theta-\eta)]\lambda\mu - k\lambda\mu > 0,$$

we need

$$k > \underline{k} = 1 - \frac{\eta}{\theta}, \tag{4}$$

otherwise ruin is certain for any initial capital $u > 0$. Note that in noncheap reinsurance the fraction of the premiums diverted to the reinsurer is larger than that of each claim covered by the reinsurer. The classical risk process with noncheap reinsurance was also studied by, among others, Ma et al. [24] who obtained the minimal probability of ruin as well as the optimal proportional reinsurance strategy using the dynamic programming approach, while cheap reinsurance (i.e., $\theta = \eta$) was considered in Schmidli [25] who allowed for investment in a risky asset and obtained, by means of an HJB equation, the optimal reinsurance and investment strategies for minimizing the ultimate ruin probability.

Suppose the insurer invests part of its surplus, into say, a risk-free asset (a bond) and a risky asset (stocks) as in [7]. Let the return on investments process be:

$$R_t = rt + \sigma_2 W_{2,t}, \quad t \geq 0,\ R_0 = 0, \tag{5}$$

where r is the risk-free interest rate, so that $R_t = rt$ implies that one unit invested now will be worth e^{rt} at time t; W_2 is another one-dimensional Brownian motion independent of the surplus-generating process P and σ_2 is the volatility of the stock price, so that the diffusion term $\sigma_2 W_2$ accounts for random fluctuations in the investment returns. Equation (5) is actually the famous Black-Scholes option pricing formula according to which the price of a stock is assumed to follow the stochastic differential equation

$$Y_t = Y_0 + \int_0^t Y_s dR_s, \tag{6}$$

where Y_0 is the stock price at $t = 0$. The process Y is a geometric Brownian motion. The solution to (6) is the value of the stock at time t and is given by $Y_t = Y_0 \exp\{(r - \frac{1}{2}\sigma_2^2)t + \sigma_2 W_{2,t}\}$.

The risk process is therefore made up of a combination of the surplus-generating process compounded by proportional reinsurance (2) and the investment-generating process (5). Thus, the insurance portfolio is represented by the risk process $U^k = \{U_t^k\}_{t\in\mathbb{R}^+}$ which has dynamics

$$dU_t^k = dP_t^k + U_{t-}^k dR_t. \tag{7}$$

A reinsurance strategy k is said to be *admissible* if it is $\mathcal{F}_t$-progressively measurable and takes values from the set $\mathcal{R} = [0,1]$. Thus, given an admissible reinsurance strategy $k \in \mathcal{R}$, and assuming that the mutually independent processes P and R belong to the rather general class of semimartingales, then under some weak additional assumptions the risk process U^k is mathematically the solution of the linear SDE

$$U_t^k = u + P_t^k + \int_0^t U_{s-}^k dR_s, \tag{8}$$

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where $U_0^k = u > 0$ is the initial surplus of the insurance company, P_t^k is the basic insurance (or surplus-generating) process in equation (2), R_t the investment-generating process in equation (5) and U_{t-}^k denotes the insurer's surplus (incorporating both proportional reinsurance and investments) just prior to time t. Paulsen [26], gave the solution of (8) as

$$U_t^k = \overline{R}_t \left(u + \int_0^t \overline{R}_s^{-1} dP_s^k \right), \tag{9}$$

where

$$\overline{R}_t = \exp\left\{ \left(r - \tfrac{1}{2}\sigma_2^2 \right) t + \sigma_2 W_{2,t} \right\}, \qquad t \geq 0$$

is the geometric Brownian motion so extensively used in mathematical finance and is the solution of the SDE $d\overline{R}_t \overset{It\hat{o}}{=} r\overline{R}_t dt + \sigma_2 \overline{R}_t dW_{2,t}$, with $\overline{R}_0 = 1$.

Since both P and R have stationary independent increments, U_t is a homogeneous strong Markov process. We define the value function of this optimization problem as

$$\psi^k(u) = \mathbb{P}(U_t^k \leq 0 \text{ for some } t \geq 0 | U_0^k = u) = \mathbb{P}(\tau^k < \infty | U_0^k = u),$$

where $\psi^k(u)$ is the ultimate ruin probability under the reinsurance policy k when the initial surplus is u and $\tau^k = \inf\{t > 0 | U_t^k < 0\}$ is the time of ruin, with $\tau^k = \infty$ if U_t^k remains positive. Then the objective is to find the optimal value function, i.e., the minimal ruin probability

$$\psi(u) = \inf_{k \in \mathcal{R}} \psi^k(u) \tag{10}$$

and the optimal policy k^* such that $\psi^{k^*}(u) = \psi(u)$, considered optimal if k^* minimizes the ruin probability. Since the ultimate survival probability $\phi^k(u) = \mathbb{P}(\tau^k = \infty | U_0^k = u) = 1 - \psi^k(u)$, we may alternatively find the value of k^* which maximizes $\phi(u)$, so that the optimal value function becomes

$$\phi(u) = \sup_{k \in \mathcal{R}} \phi^k(u). \tag{11}$$

3. HJB, Integrodifferential and Integral Equations

In this section, we derive the HJB equation for the problem and the corresponding integrodifferential and integral equations. Since the investment-generating process R_t follows (5), it follows that under weak assumptions the ruin probability $\psi(u)$ is twice continuously differentiable on $(0, \infty)$ and is a solution to the equation (see [27])

$$\mathcal{A}\psi(u) = -\lambda \overline{F}(u), \tag{12}$$

where $\overline{F}(u) = 1 - F(u)$, with boundary conditions $\lim_{u\to\infty} \psi(u) = 0$ and $\psi(u) = 1$ if $\sigma_1 > 0$ (see Theorem 1 below). Here $\mathcal{A}$ is the integrodifferential operator

$$\mathcal{A}g(u) = \frac{1}{2}\left(\sigma_2^2 u^2 + k^2\sigma_1^2\right) g''(u) + (ru + c^k)g'(u) + \lambda \int_0^\infty (g(u - kx) - g(u))\, dF(x) \tag{13}$$

Sometimes it is more convenient, as we do in this paper, to work with the survival probability $\phi(u) = 1 - \psi(u)$, in which case (12) becomes

$$\mathcal{A}\phi(u) = 0.$$

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The integrodifferential operator (13), which is the infinitesimal generator for the process U_t^k, does not easily give rise to closed-form solutions, hence the need for the use of numerical methods. The following theorem is proved in [27].

Theorem 1. *Let $\tau^k = \inf\{t > 0 | U_t^k < 0\}$ be the ruin time, with $\tau^k = \infty$ if $U_t^k \geq 0 \; \forall \; t$. Assume that the equation $\mathcal{A}\phi(u) = 0$ has a bounded, twice continuously differentiable solution (once continuously differentiable if $\sigma_1 = \sigma_2 = 0$) that satisfies the boundary conditions*

$$\begin{aligned} \phi(u) &= 0 \text{ on } u < 0, \\ \phi(0) &= 0 \text{ if } \sigma_1^2 > 0, \\ \lim_{u \to \infty} \phi(u) &= 1 \end{aligned} \tag{14}$$

Then $\phi(u) = 1 - \psi(u)$ is the survival probability.

We now present the HJB equation for this optimization problem.

Theorem 2. *Assume that the survival probability $\phi(u)$ defined by (11) is twice continuously differentiable on $(0, \infty)$. Then $\phi(u)$ satisfies the HJB equation*

$$\sup_{k \in \mathcal{R}} \left\{ \frac{1}{2} \left(\sigma_2^2 u^2 + k^2 \sigma_1^2 \right) \phi^{''}(u) + (ru + c^k)\phi^{'}(u) + \lambda \int_0^u (\phi(u - kx) - \phi(u)) \, dF(x) \right\} = 0 \tag{15}$$

for $u > 0$, where $\mathcal{R} = [0, 1]$.

Proof. See [28]. □

The function $\phi(u)$ will satisfy the HJB equation (15) only if it is strictly increasing, strictly concave, twice continuously differentiable and satisfies $\phi(u) \to 1$ for $u \to \infty$ [8]. In the following, therefore, $\phi(u)$ will be assumed to be strictly increasing. This is consistent with the smoothness assumption and the intuition that the more wealth there is (through investment), the higher the probability of survival of the insurance company. It will also be assumed that $\phi(u)$ is concave. To ensure smoothness and concavity, the claim density function must be locally-bounded [7].

The following verification theorem, whose proof is similar to that of Theorem 2 in Kasumo et al. [23], is essential for solving the associated control problem as it leads to the integrodifferential equation for the problem.

Theorem 3. *Suppose $\Phi \in C^2$ is an increasing strictly concave function satisfying the HJB Equation (15) subject to the boundary conditions*

$$\begin{aligned} \Phi(u) &= 0 \text{ on } u < 0 \\ \Phi(0) &= 0 \text{ if } \sigma^2 > 0 \\ \lim_{u \to \infty} \Phi(u) &= 1 \end{aligned}$$

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for $0 < u \leq \infty$. *Then the maximal survival probability* $\phi(u)$ *given by* (11) *coincides with* Φ. *Furthermore, if* k^* *satisfies*

$$\frac{1}{2}\left(\sigma_2^2 u^2 + k^{*2}\sigma_1^2\right)\Phi''(u) + (ru + c^{k^*})\Phi'(u) + \lambda\left[\int_0^u \Phi\left(u - k^* x\right) dF(x) - \Phi(u)\right] = 0 \tag{16}$$

when $0 \leq u < \infty$, *where* $c^{k^*} = [k^*(1+\theta) - (\theta - \eta)]\lambda\mu$, *then the policy* k^* *is an optimal policy, that is,* $\Phi(u) = \phi(u) = \phi^{k^*}(u)$.

The integrodifferential equation for the survival probability $\phi(u)$, which follows immediately from Theorem 3, is of the form $\mathcal{A}\phi(u) = 0$ (since, by (14), $\phi(u) = 0$ for $u < 0$), where $\mathcal{A}$ is the infinitesimal generator (13) of the underlying risk process, that is,

$$\frac{1}{2}\left(\sigma_2^2 u^2 + k^2\sigma_1^2\right)\phi''(u,k) + (ru + c^k)\phi'(u,k) + \lambda\int_0^u \phi(u - kx, k)dF(x) - \lambda\phi(u,k) = 0, \tag{17}$$

for $0 < u \leq \infty$. (17) can be rewritten as

$$\frac{1}{2}\left(\sigma_2^2 u^2 + k^2\sigma_1^2\right)\phi''(u) + (ru + c^k)\phi'(u) + \lambda\int_0^u \phi(u - kx)dF(x) - \lambda\phi(u) = 0, \tag{18}$$

for $0 < u \leq \infty$. Equation (18) is a second-order Volterra integrodifferential equation (VIDE) which is easily transformed, using successive integration by parts, into a linear Volterra integral equation of the second kind (VIE-2) to be used in this study. This leads to the following theorem which is our main result.

Theorem 4. *The integrodifferential equation* (18) *can be represented as a VIE-2*

$$\phi(u) + \int_0^u K(u,x)\phi(x)dx = \alpha(u) \tag{19}$$

with $u \in [0,\infty)$, *where* $K : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *and* $\alpha : [0,\infty) \to \mathbb{R}$ *are two known continuous functions,* $\phi : [0,\infty) \to \mathbb{R}$ *is the unknown function to be determined, and*

1. *For the case without diffusion (i.e., when* $\sigma_1^2 = \sigma_2^2 = 0$*), the kernel and forcing function are given, respectively, by*

$$\begin{aligned} K(u,x) &= -\frac{r + \lambda\overline{F}(u - kx)}{ru + c^k}, \\ \alpha(u) &= \frac{c^k}{ru + c^k}\phi(0), \end{aligned} \tag{20}$$

with $\overline{F}(x) = 1 - F(x)$.

2. *For the case with diffusion (i.e., when* $\sigma_1^2 + \sigma_2^2 > 0$*), the kernel and forcing function are, respectively,*

$$\begin{aligned} K(u,x) &= 2\frac{(2r - 3\sigma_2^2 + \lambda)kx + c^k + \lambda G(u - kx) - (r - \sigma_2^2 + \lambda)u}{\sigma_2^2 u^2 + k^2\sigma_1^2}, \\ \alpha(u) &= \begin{cases} \frac{2c^k}{\sigma_2^2 u}\phi(0) & \text{if } \sigma_1^2 = 0, \\ \frac{\sigma_1^2 u}{\sigma_2^2 u^2 + k^2\sigma_1^2}\phi'(0) & \text{if } \sigma_1^2 > 0, \end{cases} \end{aligned} \tag{21}$$

with $G(x) = \int_0^x F(v)dv$

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Setting $k = 1$ in both of the above cases gives the VIE-2 for the case without reinsurance, while setting $\sigma_2^2 = r = 0$ leads to the VIE-2 for the case without investments.

Proof. Integrating Equation (18) by parts with respect to u on $[0, z]$ gives

$$\begin{aligned}
0 &= \frac{1}{2}\int_0^z \left(\sigma_2^2 u^2 + k^2\sigma_1^2\right)\phi''(u)du + \int_0^z (ru + c^k)\phi'(u)du - \lambda\int_0^z \phi(u)du \\
&\quad +\lambda\int_0^z\int_0^u \phi(u - kx)dF(x)du \\
&= \frac{1}{2}\left(\sigma_R^2 z^2 + \sigma_P^2\right)\phi'(z) - \frac{1}{2}\sigma_P^2\phi'(0) + \int_0^z [(r - \sigma_R^2)u + c^k]\phi'(u)du - \lambda\int_0^z \phi(u)du \\
&\quad +\lambda\int_0^z\int_0^u \phi(v)f(u - v)dvdu \ (v := u - kx)
\end{aligned} \tag{22}$$

Evaluating the third term in (22) by integrating by parts yields

$$\begin{aligned}
0 &= \frac{1}{2}\left(\sigma_2^2 z^2 + k^2\sigma_1^2\right)\phi'(z) - \frac{1}{2}\sigma_1^2\phi'(0) + [(r - \sigma_2^2)z + c^k]\phi(z) - c^k\phi(0) \\
&\quad -(r - \sigma_2^2 + \lambda)\int_0^z \phi(v)dv + \lambda\int_0^z F(z - v)\phi(v)dv
\end{aligned}$$

Integrating (23) by parts over $[0, u]$ with respect to z gives

$$\begin{aligned}
0 &= \int_0^u \frac{1}{2}\left(\sigma_2^2 z^2 + k^2\sigma_1^2\right)\phi'(z)dz + \int_0^u [(r - \sigma_2^2)z + c^k]\phi(z)dz - \left(\frac{1}{2}\sigma_1^2\phi'(0) + c^k\phi(0)\right)u \\
&\quad -(r - \sigma_2^2 + \lambda)\int_0^u\int_0^z \phi(v)dvdz + \lambda\int_0^u\int_0^z F(z - v)\phi(v)dvdz \\
&= \frac{1}{2}\left(\sigma_2^2 z^2 + k^2\sigma_1^2\right)\phi'(z)\Big|_0^u - \sigma_2^2\int_0^u z\phi(z)du + \int_0^u [(r - \sigma_2^2)z + c^k]\phi(z)dz \\
&\quad -\left(\frac{1}{2}\sigma_1^2\phi'(0) + c^k\phi(0)\right)u - (r - \sigma_2^2 + \lambda)\int_0^u\int_v^u dz\phi(v)dv + \lambda\int_0^u\int_v^u F(z - v)dz\phi(v)dv
\end{aligned} \tag{23}$$

The above is obtained by simplifying the double integrals in the last two terms by using integration by parts again and switching the order of integration using Fubini's Theorem [29]. Recall that $F(0) = 0$ and $F(x^-) = F(x)$ for $x \in \mathbb{R}$, F being absolutely continuous with respect to Lebesgue measure. Thus, further simplification yields

$$\begin{aligned}
0 &= \frac{1}{2}\left(\sigma_2^2 u^2 + k^2\sigma_1^2\right)\phi(u) - \frac{1}{2}\sigma_1^2(\phi(0) + u\phi'(0)) - c^k u\phi(0) \\
&\quad + \int_0^u \left[\left(2r - 3\sigma_2^2 + \lambda\right)z + c^k + \lambda G(u - z) - \left(r - \sigma_2^2 + \lambda\right)u\right]\phi(z)dz,
\end{aligned} \tag{24}$$

where $G(x) = \int_0^x F(v)dv$. Replacing z with x gives

$$\begin{aligned}
0 &= \frac{1}{2}\left(\sigma_2^2 u^2 + k^2\sigma_1^2\right)\phi(u) - \frac{1}{2}\sigma_1^2(\phi(0) + u\phi'(0)) - c^k u\phi(0) \\
&\quad + \int_0^u \left[\left(2r - 3\sigma_2^2 + \lambda\right)x + c^k + \lambda G(u - x) - \left(r - \sigma_2^2 + \lambda\right)u\right]\phi(x)dx.
\end{aligned} \tag{25}$$

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Equation (25) can be written as

$$\begin{aligned} \phi(u) &+ 2\int_0^u \frac{\left(2r-3\sigma_2^2+\lambda\right)x+c^k+\lambda G(u-x)-\left(r-\sigma_2^2+\lambda\right)u}{\sigma_2^2u^2+k^2\sigma_1^2}\phi(x)dx \\ &= \frac{\sigma_1^2(\phi(0)+u\phi'(0))+2c^ku\phi(0)}{\sigma_2^2u^2+k^2\sigma_1^2} \end{aligned} \tag{26}$$

which is a VIE-2. Replacing x with kx gives the kernel and forcing function as

$$\begin{aligned} K(u,x) &= 2\frac{(2r-3\sigma_2^2+\lambda)kx+c^k+\lambda G(u-kx)-(r-\sigma_2^2+\lambda)u}{\sigma_2^2u^2+k^2\sigma_1^2}, \\ \alpha(u) &= \begin{cases} \frac{2c^k}{\sigma_2^2u}\phi(0) & \text{if } \sigma_1^2=0, \\ \frac{\sigma_1^2u}{\sigma_2^2u^2+k^2\sigma_1^2}\phi'(0) & \text{if } \sigma_1^2>0, \end{cases} \end{aligned} \tag{27}$$

This is simply Equations (19) and (21) (the diffusion case). The case without diffusion is really the Cramér-Lundberg model with a reinsurance retention and a constant force of interest, that is, the IDE is

$$(ru+c^k)\phi'(u)+\lambda\int_0^u[\phi(u-kx)-\phi(u)]dF(x)=0 \tag{28}$$

It is known that $\phi(u)=0$ for $u<0$, and that $\lim_{u\to\infty}\phi(u)=1$. Integrating (28) by parts on $[0,z]$ with respect to u and replacing x with kx transforms the IDE into a VIE of the second kind with kernel and forcing function given, respectively, by

$$\begin{aligned} K(u,x) &= -\frac{r+\lambda\overline{F}(u-kx)}{ru+c^k}, \\ \alpha(u) &= \frac{c^k}{ru+c^k}\phi(0) \end{aligned} \tag{29}$$

which is the case without diffusion (that is, when $\sigma_1^2=\sigma_2^2=0$), as given by Equations (19) and (20) above. □

The following theorem has been proved in [30] for $k=1$.

Theorem 5. *Let $\phi(u)$ be the survival probability and assume that $c^k>0$, $\lambda>0$ and $r>0$. Then $\phi(0)>0\,\forall\, u>0$ iff $r>\frac{1}{2}\sigma_2^2$, and in this case $\phi(\infty)=1$. When $r\leq\frac{1}{2}\sigma_2^2$, $\phi(u)=0\,\forall\, u$.*

4. Numerical Methods

This section discusses the numerical method to be applied in finding numerical solutions of the survival probability $\phi(u)$ using a fixed grid $u=0,h,2h,\ldots$. The assumptions of Theorem 1 are assumed to hold throughout. For this to happen, by Theorem 4, it is necessary that Theorem 5 also holds. The numerical solution of the general linear VIE-2

$$\phi(u)+\int_0^u K(u,x)\phi(x)dx=\alpha(u), \tag{30}$$

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where the kernel $K(u,x)$ and the forcing function $\beta(u)$ are known functions and $\phi(u)$ is the unknown function to be determined, is of the form

$$\phi_n + h\sum_{i=1}^{n} w_i K_{n,i}\phi_i = \alpha_n, \tag{31}$$

where ϕ_i is the numerical approximation to $\phi(ih)$, $K_{n,i} = K(nh,ih)$, $\phi_n = \phi(nh)$ and $\alpha_n = \alpha(nh)$. The w_i are the integration weights. Here, the block-by-block method will be used in conjunction with Simpson's Rule of integration, known to have an error of order 4, to obtain solutions in blocks of two values (see Theorem 6 and Remark 1). A comprehensive description of the block-by-block method can be found in [22,28,31,32].

Definition 1. ***(Convergence)*** *Let $\phi_0(h), \phi_1(h), \ldots$ denote the approximation obtained by a given method using step-size h. Then a method is said to be convergent iff*

$$|\phi_i(h) - \phi(u_i)| \to 0, \ for\ i = 0,1,2,\ldots,N$$

as $h \to 0$, $N \to \infty$, s.t. $Nh = a$.

Definition 2. ***(Order of convergence)*** *A method is said to be of order q if q is the largest number for which there exists a finite constant C s.t.*

$$|\phi_i(h) - \phi(u_i)| \le Ch^q, \quad i = 0,1,2,\ldots, \text{ for all } h > 0$$

We need to show that the method we use converges and also establish its order of convergence. The following lemma given by [31] is required as it forms the basis for the theorem that follows.

Lemma 1. *If $|\xi_n| \le A\sum_{i=0}^{n-1}|\xi_i| + B, \quad A > 0, B > 0$ then $|\xi_n| \le B(1+A)^n$*

The proof follows immediately by induction. As a corollary we have that, if $A = hK$ and $u = nh$, then

$$|\xi_n| \le Be^{Ku} \tag{32}$$

Theorem 6. *The block-by-block method is convergent and its order of convergence is four.*

The proof of Theorem 6 is given by Linz [31].

Remark 1. By Theorem 3.1 in [22] and from results in Chapter 7 of Linz [32], it follows that for a fixed u so that $nh = u$, the solution satisfies

$$|\phi_n - \phi(u)| = O(h^4), \tag{33}$$

provided that g is four times continuously differentiable as is the case here by Theorem 2.4 in [22]. On the other hand, for the block-by-block method $|\phi_{2m+2} - \phi_{2m+1}| = O(h^4)$ as well.

5. Numerical Results

We now present some numerical results and study the impact of the volatility of stock prices on the ruin probability. We assume that the small claim sizes are exponentially distributed and the large ones are Pareto distributed. The merits of using these two distributions for modelling insurance claims are briefly

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well articulated in [33]. The VIE (19) was solved using the fourth-order block-by-block method described in Section 4.

Exp(β) refers to the exponential density $f(x) = \beta e^{-\beta x}$. The exponential distribution has distribution function $F(x) = 1 - e^{-\beta x}$ from which the tail distribution is $\overline{F}(x) = 1 - F(x) = e^{-\beta x}$. Its mean excess function is $e_F(x) = \frac{1}{\mu}$, so that $G(x) = x - \frac{1}{\beta}F(x)$. The Pareto distribution is commonly used for modelling large claims. The probability density function of the Pareto distribution is $f(x) = \frac{\alpha\kappa^\alpha}{(\kappa+x)^{\alpha+1}}$ where $\alpha > 0$, $\kappa = \alpha - 1 > 0$ and the distribution function $F(x) = 1 - \left(\frac{\kappa}{\kappa+x}\right)^\alpha$. Hence the tail distribution is $\overline{F}(x) = \left(\frac{\kappa}{\kappa+x}\right)^\alpha$. Also, $G(x) = x - 1 + \left(\frac{\kappa}{\kappa+x}\right)^\kappa$. Note also that the Pareto distribution has a mean excess function $e_F(x) = \frac{\kappa+x}{\alpha-1}$ (or $1 + \frac{x}{\kappa}$), meaning that $G(x)$ can alternatively be written as $x - \left(1 + \frac{x}{\kappa}\right) F(x)$.

A step-size of $h = 0.01$ was used throughout. All numerical simulations in this section were performed using a Samsung Series 3 PC with an Intel Celeron CPU 847 at 1.10GHz and 6.0GB internal memory. The block-by-block method was implemented using the FORTRAN programming language and taking advantage of its Double Precision feature to obtain satisfactory accuracy. Slower programs such as R, MATLAB, Maple or Mathematica could, of course, have been used but at the expense of considerably longer computing time. Although Theorem 4 deals with the survival probability $\phi(u)$ as the value function, the programs have been adjusted to output infinite ruin probabilities (since $\psi(u) = 1 - \phi(u)$). Since the block-by-block method does not require special starting procedures, it can be initiated using any value of $\phi(0)$. The values stabilize at $g(\infty)$ which is used for scaling the probabilities. For $\phi(\overline{u} - 999h)$ to be virtually equal to 1, the corresponding upper bound $\overline{u}$ should be sufficiently large. Without reinsurance, the results for ruin probabilities have been published widely (see, e.g., [22] and the references therein). The graphs were constructed using MATLAB R2016a. Five cases will now be presented by way of illustration. Without loss of generality, we use the parameter values shown in Table 1 in the numerical examples that follow.

Table 1. Model parameter values.

Parameter	Value	Source/Reference
θ	0.8	Cheng and Zhao [34]
η	0.5	Cheng and Zhao [34]
σ_1	0.001	Kasozi et al. [5]
σ_2	0.001	Kasozi et al. [5]
r	0.05	Kasozi et al. [5]
λ	2	Kasumo et al. [23]
μ	1.5	Estimated

From the net profit condition (4), we must use QS retention values k in the set $(\underline{k}, 1]$, where $\underline{k} = 1 - \frac{\eta}{\theta} = 1 - \frac{0.5}{0.8} = 0.375$. In addition, we will take $\beta = 0.5$ as the parameter of the exponential distribution, and $\alpha = 3, \kappa = 2$ as the parameters of the Pareto distribution.

5.1. Proportional Reinsurance in the Cramér-Lundberg Model

When $\sigma_1^2 = \sigma_2^2 = r = 0$ and $0 \le k \le 1$, then the SDE (8) takes the form of the classical risk process compounded by proportional reinsurance

$$U_t^k = u + c^k t - \sum_{i=1}^{N_t} kX_i$$

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By Itô's formula, the infinitesimal generator for the process U^k is given by

$$\mathcal{A}g(u) = c^k g'(u) + \lambda \int_0^u [g(u - kx) - g(u)]dF(x)$$

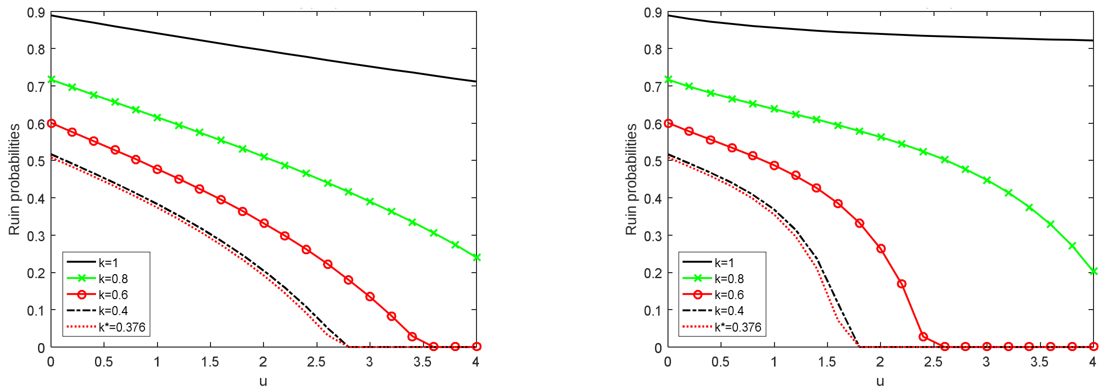
from which the VIDE corresponding to the survival probability $\phi(u)$ follows as

$$c^k \phi'(u) + \lambda \int_0^u [\phi(u - kx) - \phi(u)]dF(x) = 0 \tag{34}$$

This VIDE reduces to an ordinary VIE of the second kind with kernel $K(u, x) = -\frac{\lambda \overline{F}(u-kx)}{c^k}$, where $\overline{F}(x) = 1 - F(x)$, and forcing function $\alpha(u) = \phi(0)$. This is simply (19) and (20) with $r = 0$.

Example 1. *Exponential distribution with* $\lambda = 2,\ \beta = 0.5,\ \mu = 1.5,\ \theta = 0.8,\ \eta = 0.5.$

Since the ruin probability is a function of the initial surpus u, we observe from Figure 1a that the ruin probability reduces as the initial surplus increases. We also note that the higher the cession level $1 - k$ for QS reinsurance, the lower the ruin probability. From the results presented in Figure 1, we see that the lowest value of k that satisfies the NPC (4) and at the same time gives the minimal ultimate ruin probability is 0.376. Thus, the optimal retention for QS reinsurance is $k^* = 0.376$. This means that the company should cede about 62.4% of its risks to a reinsurer.



(a) **(b)**

Figure 1. Ultimate ruin probabilities for the CLM compounded by proportional reinsurance. (**a**) CLM with QS reinsurance: Exp(0.5) claims. (**b**) CLM with QS reinsurance: Par(3,2) claims.

Example 2. *Pareto distribution with* $\lambda = 2,\ \alpha = 3,\ \kappa = 2,\ \theta = 0.8,\ \eta = 0.5.$

The ultimate ruin probabilities for large claims reduce more when QS reinsurance is applied to the portfolio of risks as shown in Figure 1b. As for the small claim case, the optimal QS retention level in the large claim case is $k^* = 0.376$.

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5.2. Proportional Reinsurance in the Cramér-Lundberg Model under Interest Force

Here we consider the case when $\sigma_1^2 = \sigma_2^2 = 0$, $r > 0$ and $0 \leq k \leq 1$ which leads to the CLM compounded by proportional reinsurance and a constant force of interest

$$U_t^k = u + c^k t - \sum_{i=1}^{N_t} kX_i + r\int_0^t U_s ds$$

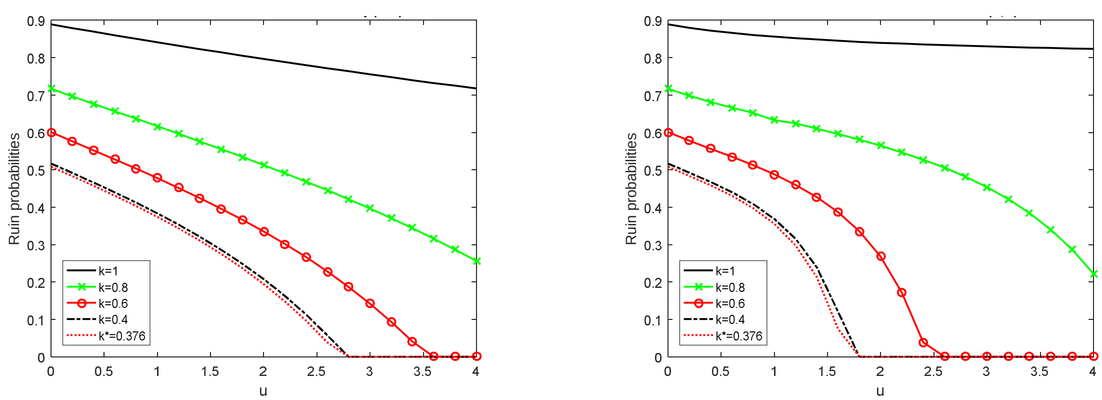
The survival probability satisfies the VIDE

$$(ru + c^k)\phi'(u) + \lambda \int_0^u [\phi(u - kx) - \phi(u)]dF(x) = 0 \tag{35}$$

which reduces to a linear VIE of the second kind with kernel and forcing function given in (20).

Example 3. *Exponential distribution with $\lambda = 2$, $\beta = 0.5$, $\mu = 1.5$, $\theta = 0.8$, $\eta = 0.5$, $r = 0.05$.*

The comments made under Example 1 apply here as well and the optimal QS reinsurance policy in this case is again $k^* = 0.376$ (see Figure 2a).



(a) (b)

Figure 2. Ultimate ruin probabilities for the CLM compounded by proportional reinsurance and a constant force of interest. (**a**) CLM with interest force: Exp(0.5) claims. (**b**) CLM with interest force: Par(3,2) claims.

Example 4. *Pareto distribution with $\lambda = 2$, $\alpha = 3$, $\kappa = 2$, $\mu = 1.5$, $\theta = 0.8$, $\eta = 0.5$, $r = 0.05$.*

The comments made under Example 2 apply to this case also. Again, the optimal QS reinsurance policy is $k^* = 0.376$ as shown in Figure 2b.

5.3. Proportional Reinsurance in the Diffusion-Perturbed Model

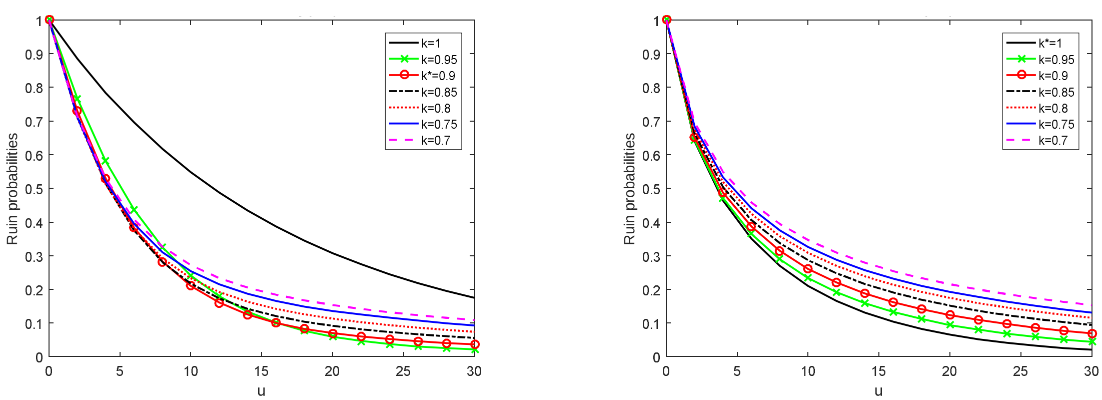
When $\sigma_1^2 > 0$, $\sigma_2^2 = r = 0$ and $0 \le k \le 1$, then we have the diffusion-perturbed model (DPM) compounded by proportional reinsurance

$$U_t^k = u + c^k t + k\sigma_1 W_{1,t} - \sum_{i=1}^{N_t} kX_i$$

In this case, the associated VIE has kernel and forcing function given, respectively, by $K(u,x) = \frac{2[c^k - \lambda(u-kx) + \lambda G(u-kx)]}{k^2\sigma_1^2}$ and $\alpha(u) = \frac{u}{k^2}\phi'(0)$. This is simply (21) with $\sigma_2^2 = r = 0$.

Example 5. *Exponential distribution with* $\lambda = 2,\ \beta = 0.5,\ \mu = 1.5,\ \theta = 0.8,\ \eta = 0.5,\ r = \sigma_2 = 0,\ \sigma_1 = 0.001.$

It can be seen from Figure 3a that $k^* \approx 0.9$ for $u \in [0, 15]$ and $k^* = 0.95$ for $u > 15$. It is expected that when u is sufficiently large, it is optimal for the company not to reinsure, i.e., $k^* = 1$.



(a) **(b)**

Figure 3. Ultimate ruin probabilities for the DPM compounded by proportional reinsurance. (**a**) DPM with QS reinsurance: Exp(0.5) claims. (**b**) DPM with QS reinsurance: Par(3,2) claims.

Example 6. *Pareto distribution with* $\lambda = 2,\ \alpha = 3,\ \kappa = 2,\ \theta = 0.8,\ \eta = 0.5,\ r = \sigma_2 = 0,\ \sigma_1 = 0.001.$

For the large claim cases in the DPM, the ruin probabilities increase instead of reducing with the application of proportional reinsurance, as can be seen from Figure 3b. We can therefore conclude that it is optimal not to reinsure, i.e., $k^* = 1$.

5.4. Proportional Reinsurance in the Perturbed Model under Interest Force

This is the case when $\sigma_1^2 > 0$, $\sigma_2^2 = 0$, $r > 0$ and $0 \le k \le 1$, then we have the DPM compounded by proportional reinsurance and a constant force of interest

$$U_t^k = u + c^k t + k\sigma_1 W_{1,t} - \sum_{i=1}^{N_t} kX_i + r\int_0^t U_s ds$$

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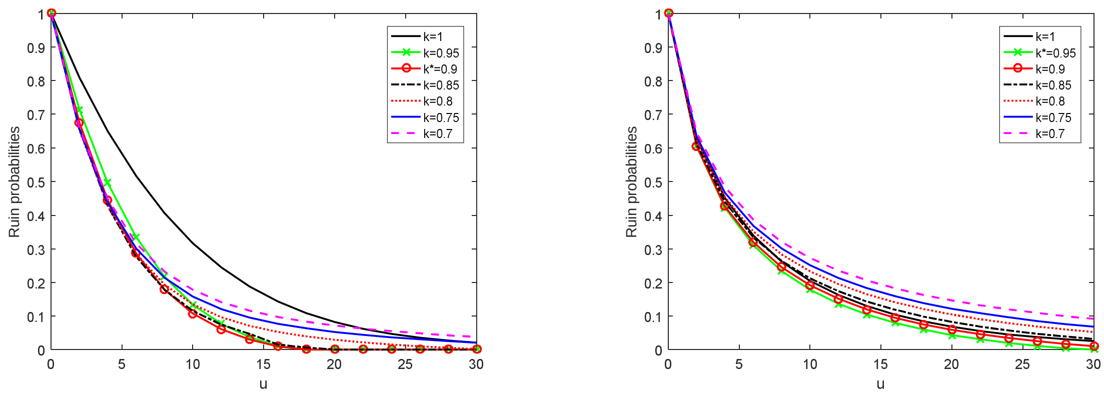
The corresponding VIE has kernel and forcing function given in (21) with $\sigma_2^2 = 0$.

Example 7. *Exponential distribution with* $\lambda = 2,\ \beta = 0.5,\ \mu = 1.5,\ \theta = 0.8,\ \eta = 0.5,\ r = 0.05,\ \sigma_1 = 0.001,\ \sigma_2 = 0.$

For the DPM under interest force, it is evident from Figure 4a that for exponentially distributed claim sizes the optimal QS reinsurance retention $k^* \in (0.85, 0.9)$ since the graph for $k = 0.85$ is slightly higher for the first time than that for $k = 0.9$. Thus, the optimal policy is to reinsure 10% of the risks, i.e., $k^* \approx 0.9$.



(a) **(b)**

Figure 4. Ultimate ruin probabilities for the DPM compounded by proportional reinsurance and a constant force of interest. (**a**) DPM with interest force: Exp(0.5) claims. (**b**) DPM with interest force: Par(3,2) claims.

Example 8. *Pareto distribution with* $\lambda = 2,\ \alpha = 3,\ \kappa = 2,\ \theta = 0.8,\ \eta = 0.5,\ r = 0.05,\ \sigma_1 = 0.001,\ \sigma_2 = 0.$

For the large claim case in the DPM with interest force, Figure 4b shows that the optimal QS retention $k^* \in (0.9, 0.95)$ since the graph for $k = 0.9$ is higher for the first time than that for $k = 0.95$. In this case, the company should cede only about 5% of its risks to a reinsurer since $k^* \approx 0.95$.

5.5. Proportional Reinsurance with Investments of Black-Scholes Type

When we have stochastic return on investments, the model takes the form

$$U_t^k = u + \int_0^t \left(rU_s^k + c^k \right) ds + \int_0^t \sqrt{\sigma_1^2 + \sigma_2^2 \left(U_s^k \right)^2} dW_s - \underline{S},\ U_0^k = u > 0$$

Theorem 2, together with the integrodifferential operator (13), gives the corresponding integrodifferential equation for the survival probability $\phi(u)$ as

$$\frac{1}{2}(\sigma_2^2 u^2 + k^2 \sigma_1^2)\phi''(u) + (ru + c^k)\phi'(u) + \lambda \int_0^u \phi(u - kx)dF(x) - \lambda\phi(u) = 0 \quad (36)$$

for $0 \leq u \leq \infty$, which is a second-order Volterra integrodifferential equation (VIDE). Repeated integration by parts transforms this into a VIE of the second kind with kernel and forcing function as given in (21).

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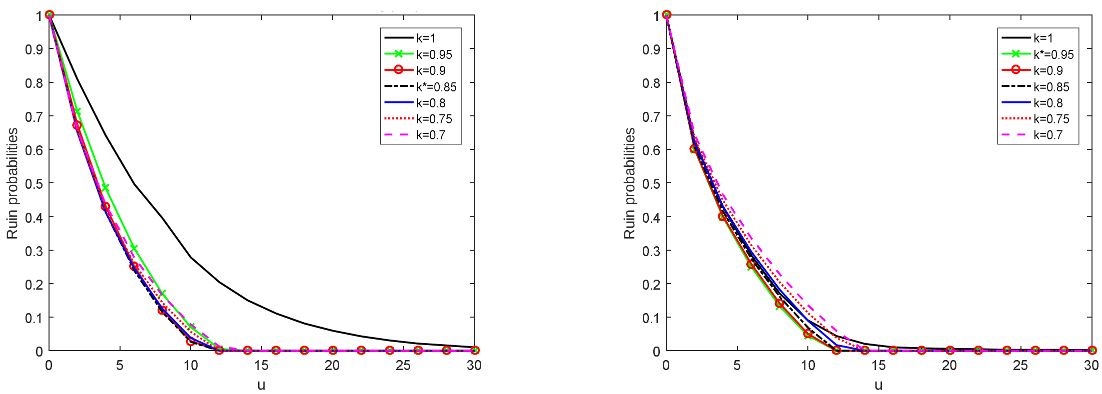
Example 9. *Exponential distribution with* $\lambda = 2,\ \beta = 0.5,\ \mu = 1.5,\ \theta = 0.8,\ \eta = 0.5,\ r = 0.05,\ \sigma_1 = \sigma_2 = 0.001$.

This is the small claim case assuming that, in addition to purchasing noncheap proportional reinsurance, the insurance company invests part of its surplus in risk-free and risky assets according to the Black-Scholes options pricing formula. As shown in Figure 5a, the optimal QS retention $k^* \in (0.8, 0.85)$. From the graph, we see that $k^* \approx 0.85$, meaning that the company should reinsure about 15% of its risks.



(a) **(b)**

Figure 5. Ultimate ruin probabilities for the DPM compounded by proportional reinsurance and investments of Black-Scholes type. (**a**) DPM with stochastic interest: Exp(0.5) claims. (**b**) DPM with stochastic interest: Par(3,2) claims.

Example 10. *Pareto distribution with* $\lambda = 2,\ \alpha = 3,\ \kappa = 2,\ \theta = 0.8,\ \eta = 0.5,\ r = 0.05,\ \sigma_1 = \sigma_2 = 0.001$.

For the large claim case in the model involving investments of Black-Scholes type, $k^* \in (0.9, 0.95)$ as shown in Figure 5b. In fact, $k^* \approx 0.95$.

5.6. Sensitivity of Ruin Probability to Volatility of Stock Prices

Figure 6 shows the effect of volatility of stock prices on the ultimate ruin probability. Evidently, as stock prices become more volatile (that is, as σ_2 increases), the ruin probability also increases, and vice versa. Volatility is actually a measure of the riskiness of a stock. If the volatility of the stock price increases but the expected rate of return of the stock stays the same, then the insurer will find the reward for accepting the risk unattractive and would rather invest less in stocks and more in bonds. Conversely, a decrease in the volatility of the stock price enables the insurer to receive the same return but with a lower risk. For this reason, the company will find that it makes economic sense to invest in the stock. This applies to both the exponential and Pareto distributions as Figure 6 makes abundantly clear.

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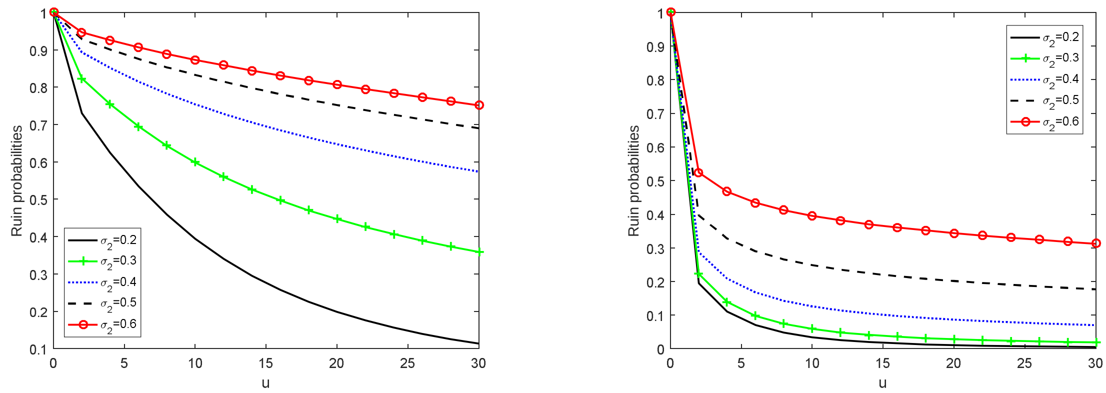

(a) **(b)**

Figure 6. Effects of volatility of stock prices on the ultimate ruin probability in the small and large claim cases. (**a**) Effect of volatility coefficient on $\psi(u)$: Exp(0.5) claims. (**b**) Effect of volatility coefficient on $\psi(u)$: Par(3,2) claims.

However, we also observe from Figure 6 that the ruin probabilities for large claims are much lower than those for small claims.

6. Conclusion

It is evident from the research findings that in the CLM, the ruin probabilities keep reducing as k reduces up to the smallest k that satisfies the NPC, so that the optimal QS retention level for both small and large claim cases in the CLM with and without a constant force of interest is $k^* = 0.376$. However, for the DPM the ruin probabilities keep reducing up to a given retention level, after which they begin to increase. This is true for both small and large claim cases. This means that the optimal retention level for proportional reinsurance lies somewhere around the point at which the ruin probabilities begin to rise again after consistently falling with a reduction in k. This is in line with our expectation that the ruin probabilities should keep reducing as the quota-share retention level reduces and then start rising again after a certain k, giving an indication of where the optimal retention k^* lies. The results from the previous section indicate that proportional reinsurance does have a positive impact on the survival of insurance companies as it minimizes their ultimate ruin probabilities.

Overall, the results for the DPM show that in the small claim case the optimal policy is $k^* \geq 0.85$, while in the large claim case it is $k^* \geq 0.95$. This means that an insurance company should reinsure up to about 15% of its portfolio in the small claim case and only up to about 5% of its risks in the large claim case. The reason for this difference is that since large claims are also extremal and therefore rare the company can afford to retain more of its large-scale risks.

The results presented in this paper indicate that investment of the surplus plays an important role in the survival of insurance companies as it significantly drives down the ultimate ruin probabilities. Noncheap proportional reinsurance also has an impact on the minimization of the ultimate ruin probabilities of insurance companies, thus enhancing their chances of survival in the market. Possible extensions of this work include the use of other forms of reinsurance arrangements (e.g., surplus, excess-of-loss or stop-loss) as well as inclusion of jumps in the investment process.

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Abbreviations

The following abbreviations are used in this paper:

CLM	Cramér-Lundberg Model
DPM	Diffusion-Perturbed Model
NPC	Net Profit Condition
SDE	stochastic differential equation
HJB	Hamilton-Jacobi-Bellman
VIDE	Volterra integrodifferential equation
VIE	Volterra integral equation
VIE-2	Volterra integral equation of the second kind
QS	quota-share

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