Article

A NEW SET THEORY FOR ANALYSIS

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Abstract: We present the real number system as a generalization of the natural numbers. First, we prove the co-finite topology, $Cof(\mathbb{N})$, is isomorphic to the natural numbers. Then we prove the power set $2^{\mathbb{Z}}$ contains a subset isomorphic to the non-negative real numbers, with all its defining structure of operations and order. Finally, we provide two different constructions of the entire real number line, and we see that the power set $2^{\mathbb{N}}$ can be given the defining structure of $\mathbb{R}$. We give simple rules for calculating addition, multiplication, subtraction, division, powers and rational powers of real numbers, and logarithms. The supremum and infimum are explicitly constructed by means of a well defined algorithm that ends in denumerable steps. In section 5 we give evidence our construction of $\mathbb{N}$ and $\mathbb{R}$ are canonical; these constructions are as natural as possible. In the same section, we propose a new axiomatic basis for analysis. This answers Benacerraf's identification problem by giving a canonical representation of numbers as sets. In the last section we provide a series of graphic representations and physical models that can be used to represent the real number system. We conclude that the system of real numbers is completely defined by the order structure of $\mathbb{N}$.

Keywords: General Topology, Axiomatic Set Theory, Real Analysis, Continuum, Graph Theory, Benacerraf's Identification Problem, Mathematical Structuralism

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Introduction

In building the continuum we make use of properties of integers and sets. Apart from this, we assume the basic concepts of category theory. Mainly, the concept of isomorphism between categories. Background knowledge on previous axiomatic constructions of the real numbers will be of help. The more modern constructions of the real number system can be found in the references [1-5]. It is notable that the real number system has been studied in detail through the generations, and still new insights and more useful constructions are sought. The mathematical objects that have previously been denominated as real numbers, are objects of complex and illusive structure. The mathematician has always had to recur to advanced tools and objects in building the real number structure. In the words of Dr. K. Knopp [6](p. 4)

> "...these preliminary investigations are tedious and troublesome, and have actually, it must be confessed, not yet reached any entirely satisfactory conclusion at all."

The construction provided here, stands out in the fact that we do not have to build a new structure. We prove the real number system is isomorphic in terms of structure (not only cardinality) to $2^{\mathbb{N}}$. One can find many bijective functions from the set of real numbers to the power set of integers, but it is completely different to prove they have the same structure. Of course, the structure of $2^{\mathbb{N}}$ can be defined artificially in terms of any bijection, but that does not give us any new information or computational advantages. Our work is to define a simple structure for $2^{\mathbb{N}}$ and prove this structure is equivalent to the real number system as a totally ordered field with the property of the supremum element (completeness). In other wrods, we provide a set of rules that define order and operations $<, \oplus, \otimes$ in $2^{\mathbb{N}}$, in such a way that $\mathcal{F}a < \mathcal{F}b$ if and only if $a < b$, and $\mathcal{F}(a+b) = \mathcal{F}a \oplus \mathcal{F}b$ and $\mathcal{F}(a \cdot b) = \mathcal{F}a \odot \mathcal{F}b$, for a bijection $\mathcal{F} : \mathbb{R} \to 2^{\mathbb{N}}$.

Real numbers are usually presented axiomatically as a set that satisfies certain rules. In undergraduate school, constructions of this system are rarely taught, even in advanced courses

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of analysis. This leads to a certain gap in the learning of the student. This problem is one of major motivations of the present work. We first provide an isomorphic structure to $\mathbb{N}$, by defining order and operations for the closed sets of the cofinite topology $Cof(\mathbb{N})$. A similar construction is given for the discrete topology $2^{\mathbb{N}}$; the continuum $[0,1]$ is isomorphic in structure to the collection of all subsets of $\mathbb{N}$. We generalize these constructions to obtain a structure isomorphic to $\mathbb{R}^+$. We define order and operation for the closed sets of $\bar{\mathbb{Z}}$, where $\bar{\mathbb{Z}}$ is the topology of $\mathbb{Z}$ whose closed sets are the subsets of $\mathbb{Z}$ that are bounded above. In other words, non-negative real numbers are represented by subsets of $\mathbb{Z}$ that are bounded above. We define order, addition and product for these objects.

1. Motivation

Every positive integer has unique representation as sum of natural powers of 2. This expression is usually treated as a sequence of 0s and 1s. A 0 in the n-th place indicates that 2^n is not a summand in the expression. A digit 1 in that same place, would indicate that power is indeed a summand. This is the usual binary expression of natural numbers. We will refer to finite subsets of $\mathbb{N}$ as *set numbers*. Given the binary representation of a positive integer N, we can naturally assign a set number to it. The elements of the set number are the natural numbers that correspond to spaces in the sequence with a digit 1. Where a 0 would appear we say that integer does not belong to the set number. For example, the number $5 = (\ldots, 0, 0, 0, 1, 0, 1)$ is assigned the set number $\{0, 2\}$ because the 0-th and 2-nd space are occupied by the digit 1. The number $13 = (\ldots, 0, 0, 0, 1, 1, 0, 1)$ will be mapped to the set number $\{0, 2, 3\}$ and the number $17 = (\ldots, 0, 0, 0, 1, 0, 0, 0, 1)$ would be given the set number $\{0, 4\}$. Given set numbers a, b, we say $a < b$ if $\max(a \triangle b) \in b$, where $a \triangle b = (a \cup b) - (a \cap b)$ is the symmetric difference of the two sets. This order is isomorphic to the order of natural numbers $\mathbb{N}_<$.

Next, we define a sum operation for set numbers, in the form of a recursive formula that ends in finite steps. We express the sum of two set numbers a, b as the sum of two new sets: the symmetric difference and the intersection, with a slight change. We add 1 to the elements of the intersection to form a new set, $s(a \cap b)$. In other words, to add two numbers, we add the powers of 2 that are not repeated, then we add the powers of 2 that appear repeated. In terms of set numbers, this means we carry out the set number addition between the symmetric difference, and the intersection (with the elements of the intersection increased by 1). We add 1 to the elements of $a \cap b$, because in terms of natural numbers it is equivalent to multiplication by 2. Thus, our formula $a \oplus b = (a \triangle b) \oplus s(a \cap b)$. The function $s : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ adds 1 to the elements of the argument; in specific $s\emptyset = \emptyset$ and $s\{0\} = \{1\}$. The set number sum $13 + 5 = \{0, 2, 3\} \oplus \{0, 2\}$ is equal to $(2^0 + 2^2 + 2^3) + (2^0 + 2^2) = [2^3] + 2 * [2^0 + 2^2] = [2^3] + [2^1 + 2^3] = (2^1) + 2 * (2^3) = (2^1) + (2^4) = 18$. In terms of set numbers, $\{0, 2, 3\} \oplus \{0, 2\} = \{3\} \oplus \{0 + 1, 2 + 1\} = \{3\} \oplus \{1, 3\} = \{1\} \oplus \{3 + 1\} = \{1\} \oplus \{4\} = \{1, 4\} \oplus \emptyset = \{1, 4\}$. In general, the process ends in finite steps. The sum of two sets, a, b is equal to the sum of the sets $a' = a \triangle b$ and $b' = s(a \cap b)$. The sum of these two is in turn equal to the sum of $a'' = a' \triangle b'$ and $b'' = s(a' \cap b')$, etc.... This process ends when $b^{(n)}$ becomes the empty set (after a finite number of iterations). We give another example, $15 + 23 = \{0, 1, 2, 3\} \oplus \{0, 1, 2, 4\}$. If $a = 15, b = 23$, then $15 + 23 = \{3, 4\} \oplus s\{0, 1, 2\} = \{3, 4\} \oplus \{1, 2, 3\} = \{1, 2, 4\} \oplus \{4\} = \{1, 2\} \oplus \{5\} = \{1, 2, 5\} \oplus \emptyset = \{1, 2, 5\} = 38$.

This process can be modeled with particles occupying energy levels. A set number will be represented by a column with numbered energy levels, and any given arrangement of particles occupying levels (with at most one particle on each level). To perform addition of two columns, we give one rule: two particles in the same level are replaced by a single particle, one level up. Let us describe this in detail. Given two columns A, B, we call the ordered pair $\mathbf{S} = (A, B)$ a *state*. Thus, a state is determined by two columns with occupied levels. Call the levels of the left column A_i and the levels of the right column B_j, so that A_4 means *the fourth level of the left column* and B_0 means *the lowest (or 0-th) level of the right column.*

We give a system that evolves discretely in time, with our rule for simplifying columns. To go from a state t_n to t_{n+1}, we form two new columns. The left column of state t_{n+1} is occupied in the energy levels that were not repeated in the previous state t_n. The right column of state t_{n+1} is occupied

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in the energy level B_{j+1} if the energy levels A_j and B_j where both occupied during the state t_n. To find the set number sum of two set numbers, we merge the two columns into a single column, by applying our sum formula. The observation is that the right column becomes the empty set in finite steps. We define $x \oplus \emptyset = \emptyset \oplus x = x$. See Figure 1.

1.png Fig 1.png

It can be proven the state stabilizes in a finite number of steps. That is to say, there exists a state $\mathbf{S}(t_n)$ such that $\mathbf{S}(t_k) = \mathbf{S}(t_n)$ for all $k \geq n$. Say A_i^n represents the level A_i at time t_n. Then stability means for all $i, j \geq 0$ and for all $n \geq k$, it is true $A_i^n = A_i^k$ and $B_j^n = 0$. Notice, the same diagram is valid under vertical displacements. In figure 2 we illustrate the fact that the same system of states from Figure 1 holds under displacement of the energy levels (Figure 2).

2.png Fig 2.png

Real numbers in the unit interval can be expressed as the sum of negative powers of 2. We use the same rules, now with arbitrary sets of negative integers. For example, $\frac{1}{2} = 2^{-1} = \{-1\}$, and $\frac{3}{2} = \{-1, 0\}$ because $\frac{3}{2} = 2^{-1} + 2^0$. Adding these, $\frac{1}{2} + \frac{3}{2} = \{-1\} \oplus \{-1, 0\} = \{0\} \oplus s\{-1\} = \{0\} \oplus \{0\} = \emptyset \oplus s\{0\} = s\{0\} = \{1\} = 2^1$ (Figure 3).

3.png Fig 3.png

If we rearrange the levels so that $a = \{0\}$ and $b = \{0, 1\}$, then Figure 3 is a graphic representation of $1 + 3 = 4$. We have denumerable representations of sums, actually. Observe that $\sum_{i<n} 2^i = 2^n$, for every integer n. This is equivalent to defining $\{i\}_{i=-\infty}^{n-1} = \{n\}$. It comes from an iteration of Figure 3, and we will see why this is consistent (Figure 4).

4.png Fig 4.png

A subset $A \subseteq \mathbb{Z}$, with maximum in $\mathbb{Z}$, is a positive real number where the integer part is given by the (finite) intersection $\mathbb{N} \cap A$ and the fractional part is given by $-\mathbb{N} \cap A$.

2. $\mathbb{N} \cong Cof(\mathbb{N})$

Let $A \subset \mathbb{N}$ be a closed set of the cofinite topology of $\mathbb{N}$. Equivalently, A has a maximum and we say that A is a *set number*.

2.1. Order

Given two set numbers $A \neq B$, we define an order relation $<$ between them. We say $A < B$ if $\max(A \triangle B) \in B$. In the contrary case, $\max(A \triangle B) \in A$ and we define $B < A$. Every two set numbers

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are comparable because the symmetric difference is non-empty and has maximum. It is not difficult to see that we have a well ordering on the family of set numbers. We give three examples in Figure 5.

Fig 5.png

5.png

2.2. Addition

We define the sum of two set numbers as a recursive formula that ends in finite steps. When adding powers of two, we have the following simple rule $2^{n+1} = 2^n + 2^n$. So, we ask the sum obey $\{x\} \oplus \{x\} = \{x+1\}$, for every $x \in \mathbb{N}$. Given any set number $A = \{x\}_{x \in A} \subset \mathbb{N}$, we define $sA = \{x+1\}_{x \in A}$. Then, define the operation as $A \oplus B = (A \triangle B) \oplus s(A \cap B)$, where $A \triangle B = (A \cup B) - (A \cap B)$ is the usual symmetric difference. Successive applications yield

$$\begin{aligned}
A \oplus B &= (A \triangle B) \oplus s(A \cap B) \\
&= [(A \triangle B) \triangle s(A \cap B)] \oplus s[(A \triangle B) \cap s(A \cap B)] \\
&= [((A \triangle B) \triangle s(A \cap B)) \triangle s((A \triangle B) \cap s(A \cap B))] \\
&\quad \oplus s[((A \triangle B) \triangle s(A \cap B)) \cap s((A \triangle B) \cap s(A \cap B))].
\end{aligned}$$

Before complicating things more, we stop here to see what is happening. Let $C_{n+1} = C_n \triangle D_n$ and $D_{n+1} = s(C_n \cap D_n)$, where $C_1 = A \triangle B$ and $D_1 = s(A \cap B)$. Then the previous equalities can be rewritten as

$$\begin{aligned}
A \oplus B &= (C_1 \triangle D_1) \oplus s(C_1 \cap D_1) \\
&= C_2 \oplus D_2 \\
&= (C_2 \triangle D_2) \oplus s(C_2 \cap D_2) \\
&= C_3 \oplus D_3 \\
&= C_k \oplus D_k \qquad \forall k \in \mathbb{Z}.
\end{aligned}$$

We are calculating $C_k \oplus D_k$ as a sum $C_{k+1} \oplus D_{k+1}$, where the term D_{k+1} is dependent on the intersection. We call the term D_{k+1} the remainder, and although it is not getting smaller in value (as natural number), the cardinality (as set number) does go to 0. However, it can be the case that the cardinality of the remainder does not get smaller. It may stay constant for finite iterations. It is guaranteed that in a finite number of steps, the remainder becomes the empty set, and our result is now obvious because we have iterated the formula until we reach $A \oplus B = C_{k+1} \oplus D_{k+1}$, where $D_{k+1} = \emptyset$. The system stabilizes when $D_{k+1} = \emptyset$; we get $A \oplus B = C_{k+1} = C_{k+2} = \cdots = C_k \cup D_k$.

It is left to the reader to prove 1) $D_k = \emptyset$ in at most $\max(A \cup B) + 1$ steps. 2) $X \oplus Y = X \oplus \{y_1\} \oplus \{y_2\} \oplus \cdots \oplus \{y_n\}$, for any set numbers X, Y where the y_i are the elements of Y. The proofs of the algebraic properties are not all trivial. It is not trivial to prove associativity for $\oplus$. The best thing to do to avoid direct proofs of these properties, is to prove $\mathcal{F}(a+b) = \mathcal{F}a \oplus \mathcal{F}b$ and $\mathcal{F}^{-1}(A \oplus B) = \mathcal{F}^{-1}A + \mathcal{F}^{-1}B$, where $\mathcal{F}$ is the isomorphism between natural numbers and set numbers.

2.3. Product

In defining the product, consider that multiplying by 2 is equivalent to adding +1 to the elements of the set number. For example, $2 * 14 = \{1\} \odot \{1,2,3\} = \{1+1, 2+1, 3+1\} = \{2,3,4\} = 28$. In general, $2 \cdot x = s(X)$, where X is the set number corresponding to $x \in \mathbb{N}$. Recall $s(X) = \{a+1\}_{a \in X}$. Multiplication by 4, is equivalent to adding +2 to all the elements of X. In general, multiplication by

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2^n is equivalent to adding n, to the elements of X. That is to say, $2^n \cdot x = \{a+n\}_{a\in X}$. The unit of this operation is the set $\{0\}$. To multiply numbers that are not powers of 2, we define using distributivity. Let us first give an example. To multiply $7 \cdot 9 = \{0,1,2\} \odot \{0,3\}$, one of the two numbers is going to be distributed over the other in a sense that will now be made clear. We form three new sets (one for each element of 7). These are the sets $\{0+0, 3+0\}, \{0+1, 3+1\}, \{0+2, 3+2\}$. Adding these set numbers, $\{0,3\} \oplus \{1,4\} \oplus \{2,5\} = (\{0,3\} \oplus \{1,4\}) \oplus \{2,5\} = \{0,1,3,4\} \oplus \{2,5\} = \{0,1,2,3,4,5\} = 63$. This example is illustrated in Figure 6.

6.png Fig 6.png

To define the product of two set numbers $A \odot B$, we will refer to A as the *pivot* and B as the *base*. The product is the set number sum, of the set numbers obtained by displacing the base; the displacements of the base correspond to elements of the pivot. So to give another example, if we wish to multiply 32 and 12, and take 32 as pivot and 12 as base, we proceed as follows. We know $32 = \{5\}$ has one element so that we add 5 to the elements of $\{2,3\}$ to obtain $\{7,8\}$ which is indeed 384. If we chose 32 to be the base and 12 to be the pivot, then we would have to add the set numbers $\{5+2\} \oplus \{5+3\} = \{7\} \oplus \{8\} = \{7,8\}$. The product is then expressed by:

$$A \odot B = \bigoplus_{a\in A} \{a+b\}_{b\in B}$$

Let $a_1 < a_2 < \cdots < a_n$ be the elements of A, and $b_1 < b_2 < \cdots < b_m$ the elements of B. Then the formula is the set number sum $\{a_1+b\}_{b\in B} \oplus \{a_2+b\}_{b\in B} \oplus \cdots \oplus \{a_n+b\}_{b\in B}$, where the set $\{a_i+b\}_{b\in B} = \{a_i+b_1, a_i+b_2, \ldots, a_i+b_m\}$. Developing the expression we get

$$A \odot B = \{a_1+b_1, a_1+b_2, \ldots, a_1+b_m\} \oplus \{a_2+b_1, a_2+b_2, \ldots, a_2+b_m\} \oplus \cdots \oplus \{a_n+b_1, a_n+b_2, \ldots, a_n+b_m\}.$$

Figure 7 illustrates that the product formula is also valid for negative integers.

7.png Fig 7.png

The reader can prove $\mathcal{F}(a \cdot b) = \mathcal{F}a \odot \mathcal{F}b$. which will be useful in proving commutativity and associativity of product $A \odot B = B \odot A$. Commutativity can be rewritten as

$$\begin{aligned} \bigoplus_{a\in A} \{a+b\}_{b\in B} &= \{a_1+b_1, \ldots, a_n+b_1\} \oplus \{a_1+b_2, \ldots, a_n+b_2\} \oplus \cdots \oplus \{a_1+b_m, \ldots, a_n+b_m\} \\ &= \{a_1+b_1, \ldots, a_1+b_m\} \oplus \{a_2+b_1, \ldots, a_2+b_m\} \oplus \cdots \oplus \{a_n+b_1, \ldots, a_n+b_m\} \\ &= \bigoplus_{b\in B} \{a+b\}_{a\in A} \end{aligned}$$

Direct proofs (in terms of sets) of the set sum are difficult, but direct proofs of the product for set numbers seems to be almost impossible. For instance, in proving commutativity, we have to prove that the set number sum of n sets (each with cardinal number m), is equal to a set number sum of m sets (each with cardinal number n). Again, the only easy way out of this is to use the isomorphism $\mathcal{F}$.

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2.4. Subtraction

We wish to give forms of inverse operating elements, namely we wish to provide well defined algorithms for finding the difference between two object $A \ominus B$ and their quotient $A \otimes B$ in such a way that the corresponding operation in real numbers is equal. In this section we will concentrate on finding an algorithm (and, ultimately, a well defined formula) for subtraction.

Before defining the general case, let us begin with two set numbers subject to the relation $A \subset B$. Then the subtraction is defined by $B \ominus A = B - A$, where $B - A$ is the usual set difference. So that if we put the two columns side by side, the result $B - A$ is obtained by taking away the particles in the column of B that also appear in A. For example, $45 = \{0,2,3,5\}$ and $9 = \{0,3\}$ is a subset of 45, so that we can easily find $45 - 9 = \{0,2,3,5\} - \{0,3\} = \{2,5\} = 36$.

Next step to generality is to consider two set numbers $A < B$ such that $\max A < \max B$ in the strict sense. We must use our basic rule of addition, now in reverse order, so that we can take away particles from any level of the column B. We have described our basic rule as $\{x+1\} = \{x\} + \{x\}$, for every $k = 1,2,3,\ldots$. It can be rewritten as $\{x\} = \{x-1\} \oplus \{x-1\}$. Applying the same rule to $\{x-1\}$, we obtain $\{x\} = \{x-1\} \oplus (\{x-2\} \oplus \{x-2\})$. Continuing in this manner, the result is

$$\{x\} = \{x-1\} \oplus \{x-2\} \oplus \cdots \oplus \{1\} \oplus \{0\} \oplus \{0\}$$

Therefore,

$$\begin{aligned} X &= (X - \{k\}) \oplus \{k\} \\ &= (X - \{k\}) \oplus (\{k-1\} \oplus \{k-2\} \oplus \cdots \oplus \{1\} \oplus \{0\} \oplus \{0\}). \end{aligned} \quad (1)$$

for any set number X and any $k \in X$. Now, we wish to find a set number X such that $A \oplus X = B$, and to do this we rewrite B:

$$\begin{aligned} X &= B \ominus A \\ &= [(B - \{N\}) \oplus (\{N-1\} \oplus \{N-2\} \oplus \cdots \oplus \{1\} \oplus \{0\} \oplus \{0\})] \ominus A \end{aligned} \quad (2)$$

where $N = \max B$. We know $A \subseteq \{x\}_{x=0}^{N-1} = \{0,1,2,\ldots,N-1\}$. Thus,

$$B \ominus A = (B - \{N\}) \oplus \{n_1, n_2, \cdots, n_k\} \oplus \{0\},$$

where $\{n_1, n_2, \ldots, n_k\} = \{0,1,2,\ldots,N-1\} - A$. Finally,

$$B \ominus A = (B - \{N\}) \oplus C \oplus \{0\},$$

where $C(N,A) = \{n_i\}_{i=1}^{k} = \{0,1,2,\ldots,N-1\} - A$.

To find $A \ominus B$ with $B = 42 = \{1,3,5\}$ and $A = 21 = \{0,2,4\}$.

$$\begin{aligned} B \ominus A &= (\{1,3,5\} - \{5\}) \oplus (\{0,1,2,3,4\} - A) \oplus \{0\} \\ &= \{1,3\} \oplus \{1,3\} \oplus \{0\} \\ &= \{0,2,4\}. \end{aligned}$$

This is represented in Figure 8.

8.png Fig 8.png

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Figures 9 and 10 are also examples of subtraction.

9.png Fig 9.png

10.png Fig 10.png

Let $A < B$ two set numbers, and let $N_0 = \max(A \triangle B) \in B$. We observe $B \ominus A = B' \ominus A'$ where $A' = A \cap \{0, 1, \ldots, N_0\}$ and $B' = B \cap \{0, 1, \ldots, N_0\}$. The subtraction $B \ominus A$ is treated as in the last paragraph, $\max A' < N_0 = \max B'$. This defines subtraction in the most general case (Figures 11-13).

11.png Fig 11.png

12.png Fig 12.png

13.png Fig 13.png

3. Continuum

We extend the family of set numbers to include sets that represent reciprocals of natural numbers. The unit interval continuum $[0, 1]$ is the power set $2^{-\mathbb{N}}$. First, we prove that every subset of $2^{-\mathbb{N}}$ has supremum and infimum. In particular $\sup(2^{-\mathbb{N}}) = -\mathbb{N} < \{0\}$, and $\{0\}$ is the smallest set number larger than $-\mathbb{N}$. We define $\{0\} = -\mathbb{N}$ without fear of contradiction. This translates to $\sum_{i \in \mathbb{N}} 2^{-i} = 2^0 = 1$. In general, $\sum_{i<n} 2^i = 2^n$, or equivalently, $\{n\} = \{i\}_{i<n}$, for any integer n. This means a bounded (above and below) subset of $\mathbb{Z}$ has two representations. The inverse of $N \subset \mathbb{N}$ is $\frac{1}{N} \subset -\mathbb{N}$ such that $\frac{1}{N} \odot N = -\mathbb{N} = \{0\}$. After describing the unit interval, we prove $\bar{\mathbb{Z}} \cong \mathbb{R}^+$. The closed sets of the topology $\bar{\mathbb{Z}}$ are $A \subset \mathbb{Z}$ that are bounded above. The integer part of $A \in \mathbb{R}^+$ is given by $A \cap \mathbb{N}$ while its fractional part is $A \cap -\mathbb{N}$.

3.1. $[0, 1] \cong 2^{-\mathbb{N}}$

Supremum and Infimum. We define a limiting process that defines a supremum function in the continuum $[0, 1]$. The order is defined as before, two sets relate $A < B$ if and only if $\max(A \triangle B) \in B$. The only change is that A, B are arbitrary subsets of $-\mathbb{N}$, instead of finite sets of $\mathbb{N}$. We must prove that the supremum of any family of subsets of $-\mathbb{N}$ exists.

First, we give the intuitive idea for the more general case of finding the supremum of a bounded above set X, of positive real numbers $A_i \subset \mathbb{Z}$. The family of sets X is bounded above in the order of set numbers and every set number is also bounded above in the order of integers. Therefore $x_1 = \max \bigcup X$ exists. The supremum of X is among the sets A_j that contain the integer x_1. We look to see which of those set numbers contains the second largest integer appearing in the family. Let $x_2 = \max \bigcup_j A_j$ where the index j include only the elements of X such that $x_1 \in A_j$. Now we know the supremum is among the sets that contain both x_1 and x_2. This procedure continues for denumerable steps and generates a denumerable set of integers $x_1 > x_2 > x_3 > \cdots$. The set of these integers is $\sup X$.

Now we formally discuss this procedure for an arbitrary set of real numbers in the unit interval. Let $X \subseteq 2^{-\mathbb{N}}$. This means $A \subseteq -\mathbb{N}$ for every $A \in X$. Because of the well ordering principle, we know $x_1 := \max \bigcup X$ exists. Define the sub family $Y_1 := \{A \in X | x_1 \in A\}$, of set numbers containing x_1, and $X_1 := (\bigcup Y_1) - \{x_1\}$, then $x_2 := \max X_1 < x_1$. Now make $Y_2 := \{A \in Y_1 | x_2 \in A\}$, and $X_2 := (\bigcup Y_2) - \{x_1, x_2\}$, and $x_3 := \max X_2$. Continue in this manner, so that $x_{n+1} := \max X_n$, where $X_n := \bigcup Y_n - \{x_i\}_{i=1}^{n}$ and $Y_n := \{A \in Y_{n-1} | x_n \in A\}$. The supremum is defined as $\sup X = \{x_i\}_i$, which is by construction greater than every set in X, and it is the least subset of $\mathbb{Z}$ that is greater than all the sets of X. In a countable number of steps we have determined a unique set number (Figure 14).

Fig 14.png

14.png

$2^{2^{-\mathbb{N}}}$ represents the power set, of the power set of $-\mathbb{N}$. The supremum function is a function of the form $\sup : 2^{2^{-\mathbb{N}}} \to 2^{-\mathbb{N}}$. If $\sup(X) \in X$, we say $\sup(X) = \max(X)$. In particular, this is true if X is finite. There is also an infimum function of the form $\inf : 2^{2^{-\mathbb{N}}} \to 2^{-\mathbb{N}}$, where the image of the family is its greatest lower bound. Given a family $X \subseteq 2^{-\mathbb{N}}$, define X_* as the family of set numbers that are smaller than all the elements of X. In other words a set number $A \subseteq -\mathbb{N}$ is in X_* if and only if $A < Y$, for all $Y \in X$. Define $\inf(X) = \sup(X_*)$. If $X_* = \emptyset$, then $\emptyset \in X$, and $\inf(X) = \emptyset$. In particular, $\inf[0,1] = 0$. To calculate the infimum, look for the largest integer x_1 such that $x_1 < A$, for all $A \in X$. If there is no such integer (the elements of X are arbitrarily small) define $\inf(X) = \emptyset$. If x_1 exists and $\{x_1\} \in X$, then $\inf(X) = \{x_1\}$. If x_1 exists and $\{x_1\} \notin X$, then we compare $\{x_1, x_1 - 1\}$ with the elements of X. If $\{x_1, x_1 - 1\} \in X$, then $\inf(X) = \{x_1, x_1 - 1\}$. If $\{x_1, x_1 - 1\} < A$ for every $A \in X$, then $x_2 := x_1 - 1 \in \inf(X)$. The last possible case is $\{x_1, x_1 - 1\} > A$ for some $A \in X$, so that $x_1 - 1 \notin \inf(X)$, and we would then have to verify if $x_1 - 2 \in \inf(X)$. The process is ends in denumerable steps, with a set $\inf(X) = \{x_i\}_i \subseteq -\mathbb{N}$.

Reciprocals. If we want to multiply $A \subset \mathbb{N}$ by $B \subseteq -\mathbb{N}$, we use distributivity. Define the product as a natural generalization of the last definition of product. The index of the operation is now a subset of $-\mathbb{N}$. Refer to Figure 7. The unit of product, $\{0\}$, is bounded and therefore has a second representation, $-\mathbb{N}$. This is the representation used in providing a well-defined algorithm for finding the reciprocal of a natural number. We illustrate our motivation with an example. To find the inverse (under set product) of the set number $137 = \{0,3,7\}$. We seek out a set number $\frac{1}{N} = \{x_1, x_2, \ldots\} \subset -\mathbb{N}$ such that

$$-\mathbb{N} \quad = \quad \{x_1 + 0, x_2 + 0, \ldots\} \oplus \{x_1 + 3, x_2 + 3, \ldots\} \oplus \{x_1 + 7, x_2 + 7, \ldots\}.$$

If we propose $\frac{1}{N} = \{-7\}$ we find we *"go over"* because:

$$\begin{aligned} \{0,3,7\} \odot \{-7\} &= \{-7+0\} \oplus \{-7+3\} \oplus \{-7+7\} \\ &= \{-7, -4, 0\} \\ &> -\mathbb{N}. \end{aligned}$$

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Now we try $\frac{1}{N} = \{-8\}$ and we see that we do not go over:

$$\begin{aligned}
\{0,3,7\} \odot \{-8\} &= \{-8+0\} \oplus \{-8+3\} \oplus \{-8+7\} \\
&= \{-8,-5,-1\} \\
&< -\mathbb{N}.
\end{aligned}$$

Naturally, proceed to approximate $\frac{1}{N} = \{-8,-9\}$ and find

$$\begin{aligned}
\{0,3,7\} \odot \{-8,-9\} &= \{-8+0,-9+0\} \oplus \{-8+3,-9+3\} \oplus \{-8+7,-9+7\} \\
&= \{-1,-2,-5,-6,-8,-9\} \\
&< -\mathbb{N}
\end{aligned}$$

The reader can easily verify $\{0,3,7\} \odot \{-8,-9,-10\} = \{-1,-2,-3,-5,-6,-7,-8,-9,-10\}$. It is equally easy to prove $\{0,3,7\} \odot \{-8,-9,-10,-11\} > -\mathbb{N}$. Continue and verify Figure 15

$$\begin{aligned}
\{0,3,7\} \odot \{-8,-9,-10,-12\} &= \{-8+0,-9+0,-10+0,-12+0\} \oplus \{-8+3,-9+3,-10+3,-12+3\} \\
&\quad \oplus \{-8+7,-9+7,-10+7,-12+7\} \\
&= \{-1,-2,-3,-4,-5,-10,-12\} \\
&< -\mathbb{N}.
\end{aligned}$$

Fig 15.png

15.png

Let us describe the procedure for finding the reciprocal of a set number $N = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{N}$, with $x_1 > x_2 > \cdots > x_n$. We wish to find a set number $\frac{1}{N} \subset -\mathbb{N}$ such that $N \odot \frac{1}{N} = -\mathbb{N}$. The reciprocal $\frac{1}{N} = \{y_1, y_2, \ldots\} \subset -\mathbb{N}$ will be found in denumerable steps, as follows. First, make $y_1 = -(x_1+1)$ so that

$$N \odot \{-(x_1+1)\} = \{x_1-(x_1+1), x_2-(x_1+1), \ldots, x_n-(x_1+1)\} = \{-1, x_2-(x_1+1), \ldots, x_n-(x_1+1)\}.$$

Next find the largest negative integer y_2 such that

$$\{-1, x_2-(x_1+1), \ldots, x_n-(x_1+1)\} \oplus \{x_1+y_2, x_2+y_2, \ldots, x_n+y_2\} < -\mathbb{N}.$$

We know there is at least one negative integer that satisifes this inequality. By the well ordering principle, we can find the maximum of such a set of solutions. This maximum is our integer y_2. The finitude of $\#(N)$ plays a fundamental role in allowing us to guarantee such y_2 exists. Now, to find y_3, we look for the largest negative integer such that

$$(\{-1, x_2-(x_1+1), \ldots, x_n-(x_1+1)\} \oplus \{x_1+y_2, x_2+y_2, \ldots, x_n+y_2\}) \oplus \{x_1+y_3, x_2+y_3, \ldots, x_n+y_3\} < -\mathbb{N}.$$

If we continue in this manner we have well defined $\frac{1}{N}$.

3.2. $\mathbb{R}^+ \cong \bar{\mathbb{Z}}$

Topology of Bounded Subsets of $\mathbb{Z}$. A positive real number is a subset of $\mathbb{Z}$, bounded above. The intersection with the negative integers represents the fractional part, and the finite intersection with the positive integers represents the integer part. We use the continuum $[0,1]$, built in the last section, to produce a series of copies, squeezed in between consecutive natural numbers. We do

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not need to make any modifications to the basic rules and relations of order and operation already used. We extend the same relations to the closed sets of the topology $\bar{\mathbb{Z}}$. For example, given a binary representation 110101.00101_1, the set number is $\{5,4,2,0,-3,-5\}$. We have a natural bijection between binary sequences and closed sets of $\bar{\mathbb{Z}}$. The bijection is $A \mapsto \sum_{i\in A} 2^i$ for every closed set A of $\bar{\mathbb{Z}}$.

The positive real line is constructed by piecing together copies of $[0,1]$. Let $A \subset -\mathbb{N}$ be the corresponding set number to $x \in [0,1)$, so that $x = \sum_{i\in A} 2^i$. Then $A \cup \{0\}$ is the set number corresponding to $1+x = 2^0 + \sum_{i\in A} 2^i$. More generally, let $N \subset \mathbb{N}$ be the set number corresponding to a natural number $n \in \mathbb{N}$; this means $n = \sum_{i\in \mathbb{N}} 2^i$. Now we have $n + x = \sum_{i\in N} 2^i + \sum_{i\in A} 2^i = \sum_{i\in (A\cup N)} 2^i$. We can summarize our work as follows:

i. $\mathbb{N} \cong Cof(\mathbb{N})$ where the cofinite topology uses the order and operations of set numbers.
ii. $[0,1] \cong 2^{-\mathbb{N}}$, with $\emptyset = 0$ and $-\mathbb{N} = 1$, and the same definitions for set order and operations.
iii. The continuum of non-negative real numbers is built as a natural generalization of both $Cof(\mathbb{N})$ and $2^{-\mathbb{N}}$. We piece together $[0,1], [1,2], [2,3], \ldots$, into a single continuum $[0,\infty)$. This is done by considering the upper bounded sets of $\mathbb{Z}$ and a proper extension of the set number relations.

We generalize the previous methods into a single structure isomorphic to $\mathbb{R}^+$. The main difference between our construction of the real numbers is that we do not build a complex space which we will have great difficulty in understanding. Our approach here is to prove the simplest set with cardinality of the continuum can be given the structure of positive real numbers, in a natural manner. In fact, gives reason to speculate it is a canonical construction of $\mathbb{R}$.

Supremum. In the previous section we provided a well defined algorithm for finding the supremum of a family $X \subseteq [0,1]$, where the elements of X are arbitrary subsets of $-\mathbb{N}$. Now, we generalize this process to define the supremum of a bounded set of positive real numbers.

Observe $\max X$ exists if and only if X is a bounded family, of bounded above subsets of $\mathbb{Z}$. That is to say, $\sup(X)$ exists if and only if there exists $n \in \mathbb{Z}$ such that for all $A \in X$ it is true $A < \{n\}$. Notice it is not the same as saying *"X is a set of bounded above subsets of* $\mathbb{Z}$*"*; this would be written as *"...for all* $A \in X$ *there exists* $n \in \mathbb{Z}$ *such that* $A < \{n\}$*"*. We need to guarantee the existence of $\max \bigcup X$ in order to find the supremum of X. This is done by asking that X be a bounded above subset of $\bar{\mathbb{Z}}$ (each of the elements of the set are bounded above subsets of $\mathbb{Z}$). For example, there is no $\max \bigcup X$ for

$$X = \{\{1,0,-1,-2,\ldots\},\{2,1,0,-1,\ldots\},\{3,2,1,0,\ldots\},\{4,3,2,1,\ldots\},\ldots\},$$

although every element of X has a maximum.

Let us find the supremum of a finite list of set numbers, obviously the result should be the maximum of the list. These are $A = \{4,2,1,-1,-3,-5,-7,\ldots\}$, $B = \{4,3,-1,-3,-5,-7,\ldots\}$, $C = \{4,2,1,0,-2,-4,-6,\ldots\}$, $D = \{4,3,0,-2,-4,-6,\ldots\}$, $E = \{4,3,0,-1,-2,-4,-6,\ldots\}$. We wish to find $\sup(X)$, where $X = \{A,B,C,D,E\}$. First we find the maximum of $\bigcup X$; the maximum integer that appears in our set numbers is $x_1 = 4$. Then, we define the set $Y_1 \subseteq X$ of those set numbers that have 4 as an element. In this case A,B,C,D,E all have 4 as element so that $Y_1 = X$. Now we find the maximum of $(\bigcup Y_1) - \{x_1\}$; the second largest number that appears in the family Y_1 is $x_2 = 3$. Now $Y_2 = \{B,D,E\}$ because these are the only set numbers of Y_1 that contain $x_2 = 3$. The maximum of $(\bigcup Y_2) - \{x_1,x_2\}$ is the third largest number that appears in the family Y_2. It is $x_3 = 0$, and the only elements of Y_2 that contain 0 are D,E so that $Y_3 = \{D,E\}$. We find $x_4 = \max((\bigcup Y_3) - \{4,3,0\}) = -1$ and -1 is not an element of D but it is an element of E, then $Y_4 = \{E\}$. We find $x_5 = -2$, $x_6 = -4$, $x_7 = -6, \ldots$. We conclude $\sup(X) = \max(X) = E$.

Division. We now describe division and rational numbers. Let us find the rational representation of the set number $\{5,0,-2,-4,-5\} = 33.34375$. Add 5 to the elements; the result is $\{10,5,3,1,0\} = 1067$. The action of adding 5 to the elements of a set number is actually multiplication by 2^5, so that $\{5,0,-2,-4,-5\} = \frac{1067}{32}$. This fraction is said to be irreducible because $32 = \{5\}$ is the smallest natural number that we can multiply with $\{5,0,-2,-4,-5\}$ and obtain a subset of $\mathbb{N}$, which is the numerator $1067 = \{10,5,3,1,0\}$. Given a finite subset of integers, we give an infinite collection of ordered pairs of

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natural numbers, which will be called *fractions*. In other words, given a finite set number $A \subset \mathbb{Z}$, we can give infinite, but equivalent, representations of a single irreducible fraction $\frac{m}{n}$. The set numbers N, M, corresponding to n, m are easy to find; $N \subset \mathbb{N}$ is the smallest set number such that $N \odot A \subset \mathbb{N}$. Then, M is the result of $N \odot A$. More generally, any set number $B \subset \mathbb{N}$ such that $B \odot A \subset \mathbb{N}$, allows the construction of a fraction $\frac{\alpha}{\beta}$, where $\alpha = \sum_{i\in(A\odot B)} 2^i$ and $\beta = \sum_{i\in B} 2^i$.

Consider the inverse problem of finding the set number corresponding to an ordered pair $\frac{m}{n}$. We use our algorithm for finding the inverse, $1/N$, and multiply that with M. We will approximate $\frac{51}{137}$, by multiplying $\frac{1}{137} \approx \{-8,-9,-10,-12\}$ with $51 = \{5,4,1,0\}$, to obtain $0.361083984375 = \{-2,-4,-5,-6,-10,-11,-12\}$. For more precision, we must give a better approximation to the number $\frac{1}{137}$; for example $\frac{1}{137} \approx \{-8,-9,-10,-12,-13,-14,-17\}$.

Now, we know how to provide an irreducible fraction, given a set number. We can also approximate the set number corresponding to any given fraction of natural numbers. Now let us make our first consideration as far as infinite subsets of $\mathbb{Z}$ are concerned. Let $A \subset \mathbb{Z}$ be a set number with $A \cap -\mathbb{N}$ infinite. Then we are unable to give a set number $B \subset \mathbb{N}$ such that $A \odot B \subset \mathbb{N}$, with one crucial exception. If the set $A \cap -\mathbb{N}$ is infinite and periodic, then we can find a finite set $B \subset \mathbb{N}$ such that $A \odot B \subset \mathbb{N}$. In other words, infinite but periodic set numbers have rational representation as an ordered pair of natural numbers. Any set number that can be identified with two natural numbers, as just described, is called a *rational number*. Approximate $\frac{137}{51}$ by two methods. First, finding $1/51$ and multiplying that by $\{7,2,0\}$. Secondly, approximating B such that $\{-2,-4,-5,-6,-10,-11,-12\} \odot B = -\mathbb{N}$. The set number associated with the form $\frac{a}{b} = a \cdot \frac{1}{b}$ has inverse, under product, $\frac{b}{a} = b \cdot \frac{1}{a}$.

Sum and Product of Infinite Sets. We must define the set number sum of two infinite subsets of $\mathbb{Z}$. Let them be A, the set of integers $a_1 > a_2 > a_3 > \cdots$, and B the set of integers $b_1 > b_2 > b_3 > \cdots$. Define $A_n = \{a_i\}_{i=1}^{n}$, and in a similar manner B_m. The sum is defined as $A \oplus B := \sup_{n,m}(A_n \oplus B_m)$ for all $n, m \in N$. The reader can define the multiplication of two infinite set numbers.

Powers. To take powers of set numbers A^B, we start by defining A^B, where $A, B \subset \mathbb{N}$. In this case, $X = A^B$ is the result of carrying out successive products of set numbers. The empty set is representing the integer 0, so we define the power $A^{\emptyset} = \{0\}$. We define the power $A^{\{0\}} = A$ because $\{0\} = 1$. With this, we are able to give a recursive formula $A^B = A \odot A^{B\ominus\{0\}}$. To find power of $127^4 = \{0,1,2,3,4,5,6\}^{\{2\}}$ we first reduce the expression. We know $A^{\{2\}} = A \odot A^{\{2\}\ominus\{0\}}$. We know, from the subtraction of set numbers, that $\{2\} \ominus \{0\} = \{0,1\}$, then $A^{\{2\}} = A \odot A^{\{0,1\}}$. Then we find $A^{\{0,1\}} = A \odot A^{\{0,1\}\ominus\{0\}} = A \odot A^{\{1\}} = A \odot (A \odot A^{\{1\}\ominus\{0\}}) = A \odot (A \odot A)$. Finally, $A^{\{2\}} = A \odot (A \odot (A \odot A))$, as we expect since $\{2\} = 4$.

$$
\begin{aligned}
127 \cdot 127 &= \{0,1,2,3,4,5,6\} \odot \{0,1,2,3,4,5,6\} \\
&= \bigoplus_{a\in 127} \{a+b\}_{b\in 127} \\
&= \{0,1,2,3,4,5,6\} \oplus \{1,2,3,4,5,6,7\} \oplus \{2,3,4,5,6,7,8\} \oplus \{3,4,5,6,7,8,9\} \\
&\quad \oplus \{4,5,6,7,8,9,10\} \oplus \{5,6,7,8,9,10,11\} \oplus \{6,7,8,9,10,11,12\} \\
&= \{0,8,9,10,11,12,13\}
\end{aligned}
$$

Now we take $\{0,8,9,10,11,12,13\}$ as base and use $\{0,1,2,3,4,5,6\}$ as pivot so that

$$
\begin{aligned}
127^3 &= \{0,1,2,3,4,5,6\} \odot \{0,8,9,10,11,12,13\} \\
&= \bigoplus_{a\in 127} \{a+b\}_{b\in 16,129} \\
&= \{0,1,2,3,4,5,6,8,14,16,17,18,19,20\}
\end{aligned}
$$

The reader can verify the final result, carrying out the following addition of set numbers.

$$\begin{aligned} 127^4 &= \{0,1,2,3,4,5,6\} \odot \{0,1,2,3,4,5,6,8,14,16,17,18,19,20\} \\ &= \bigoplus_{a\in 127} \{a+b\}_{b\in 127^3} \end{aligned}$$

If the set number $A \subset \mathbb{Z}$ has infinite elements $a_1 > a_2 > a_3 > \cdots$, we define the power $A^B = \sup_m A_m^B$, where $A_n = \{a_i\}_{i=1}^n$. For now, taking negative powers is only notation for symbolizing the reciprocal of another number, so we have nothing to do in that respect because A^{-3} is simply $1/A^3 = 1/A^{\{0,1\}}$. When we define negative real numbers, this will make sense as we will extend our definition. When taking powers, we notice two different behavior: 1) if $A < \{0\} = 1$ then A^X is a decreasing function 2) if $A > \{0\} = 1$ then A^X is an increasing function. Proving A^X is decreasing (or increasing, depending on the case) is not difficult but requires some labor. Before proving it for arbitrary set numbers X, it has to be proven for $X \subset \mathbb{N}$ and $1/X$.

Roots. The process of finding a root is to take a reciprocal power, that is to say $A^{\frac{1}{B}}$, for some finite set $B \subset \mathbb{N}$. To find $X = A^{\frac{1}{B}}$ we must find a set $X \subset \mathbb{Z}$ such that $X^B = A$. Consider the set $P(A, B)$ of all set numbers X such that $X^B < A$, then this set is bounded above. We define $A^{\frac{1}{B}} = \sup P$. Notice we have three cases: 1) $A > \{0\}$, 2) $A = \{0\}$, 3) $A < \{0\}$. In the first case, $A^{\frac{1}{B}} < A$, while in the last case $A^{\frac{1}{B}} > A$. Of course, if $A = \{0\}$, any power of A is equal to A, so that $\{0\}^{\frac{1}{B}} = \{0\}$, regardless of B. We illustrate this with an example to find $3^{1/4} = \{0,1\}^{1/\{2\}}$. If we elevate $\{0,-1\}$ to the fourth power, the result is greater than $3 = \{0,1\}$. But, we find that $\{0,-2\}^4 < \{0,1\}$. In the next step we find $\{0,-2,-3\} > \{0,1\}$. Later we find $\{0,-2,-4\} < \{0,1\}$. We continue in this manner, with trial and error to elevate sets to fourth power all the time using larger numbers but making sure the result does not go over 3. If we are to take a rational power A^B, in the sense that B is a periodic set as mentioned above, we find its irreducible fraction $B = \frac{m}{n}$. Now we can say $A^B = (A^m)^{\frac{1}{n}}$ is well defined, since it can be proven $(A^m)^{\frac{1}{n}} = (A^{\frac{1}{n}})^m$. Consider next the more general case where B is not rational. Consider $A, B \subset \mathbb{Z}$ to be arbitrary set numbers, and the elements of B are $b_1 > b_2 > b_3 \cdots$. Let $B_k = \{b_i\}_{i=1}^k$, then to every B_k there corresponds an irreducible fraction $Q_k = \frac{m_k}{n_k}$. We define $A^B = \sup_k A^{Q_k}$. Of course, for this definition to be justified, we have to prove the set $\{A^{Q_k}\}_{k=1}^\infty$ is bounded above. Hint: prove the power function A^X is increasing with X, then it suffices to show $\{B_k\}_{k=1}^\infty$ is bounded above by $\{b_1 + 1\} > B_k$ for all k.

Logarithms. In the last section, we only extended the definition of powers. Now, we explore the inverse function. To find $\log_B A$ we find a set number X such that $B^X = A$. It is not difficult to show we have two defining cases for the logarithm. Consider $B > \{0\} = 1$, and $A > \{0\}$. Then $X > 0$ because there is a positive real number X such that $B^X = A$. If $A < \{0\}$, then X is negative, because the negative powers of B map to numbers $< \{0\} = 1$. For the case $B < \{0\}$ we have the contrary arrangement; if $A > \{0\}$ then $X < 0$, and if $A < \{0\}$ then $X > 0$. Let us calculate $log_{2.5}3.125$ which is the *logarithm base* 2.5 *of* 3.125; the numerical value is ≈ 1.24353. We know $B = \{-1,1\}$ and $A = \{-3,0,1\}$. We wish to find a set number $X = \frac{n}{m}$ such that $(B^n)^{\frac{1}{m}} = A$. We begin by calculating 2.5^2, to see if we go over 3.125 or not. Multiplying $\{-1,1\}$ by itself is equal to the set sum $\{-2,0\} \oplus \{0,2\}$ which is $\{-2,1,2\}$. Since the result is a set number larger than $\{-3,0,1\}$, we next try Y smaller than $\{1\}$ and bigger than {0} because $2.5^{\{0\}} = 2.5$. We try with $\{0,-1\} = \frac{3}{2}$, so that first we find the third power of 2.5.

$$\begin{aligned} 2.5^3 &= 2.5^2 \cdot 2.5 \\ &= \{-2,1,2\} \odot \{-1,1\} \\ &= \{-3,-1,0,1,2,3\}. \end{aligned}$$

Next we find the square root of $\{-3,-1,0,1,2,3\}$. It is not difficult to see $\{-1,0,1\} \subset \{-3,-1,0,1,2,3\}^{\frac{1}{2}}$, so that $2.5^{\frac{3}{2}} > \{-3,0,1\}$. Our next candidate for Y is $\{0,-2,\} = \frac{5}{4}$. The

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fifth power of 2.5 is equal to $\{-1,1\}^5 = \{-5,-3-1,0,5,6\}$. Searching the fourth root of this last set number gives $\{-6,-3,0,1\} \subset \{-5,-3-1,0,5,6\}^{\frac{1}{4}}$. Our next approximation for Y is $\{-3,0\} = \frac{9}{8}$. We find $\{-1,1\}^{\{-3,0\}} < \{-3,0,1\}$. Then we find $\{-1,1\}^{\{-4,-3,0\}} < \{-3,0,1\}$ and $\{-1,1\}^{\{-5,-4,-3,0\}} < \{-3,0,1\}$ so that we approximate $\log_{2.5} 3.125 \approx 1.21875$. Taking another step gives $\log_{2.5} 3.125 \approx \{-6,-5,-4,-3,0\} = 1.234375$.

Properties of Operation. The axiomatic properties of the field of real numbers hold, taking into account that we have not yet described negative real numbers. The unit for set sum is $\emptyset$, while the unit for product is $\{0\}$. Commutativity of set number addition is trivial because of the commutative properties of $\triangle$ and $\cap$. It is not easy to give a direct proof of associativity for set number addition. We first have to show $\{n\} \oplus (A \oplus \{m\}) = (\{n\} \oplus A) \oplus \{m\}$, for any singletons $\{n\}, \{m\} \subset \mathbb{Z}$. Let $N = \{x_1, x_2, \dots, x_n\}$, $M = \{y,1_2, \dots, y_m\}$ and $A = \{a_1, a_2, \dots, a_p\}$ three finite subsets of $\mathbb{Z}$. The sum of these can be written

$$\begin{aligned} N \oplus (A \oplus M) &= (\{x_1\} \oplus \{x_2\} \cdots \{x_n\}) \oplus [A \oplus (\{y_1\} \oplus \{y_2\} \cdots \{y_m\})] \\ &= \{x_1\} \oplus (\{x_2\} \oplus \cdots \oplus (\{x_n\} \oplus (((A \oplus \{y_1\}) \oplus \{y_2\}) \oplus \cdots \oplus \{y_m\}))) \end{aligned}$$

From this it is possible to prove $N \oplus (A \oplus M) = (N \oplus A) \oplus M$. The commutativity and associativity of product is much more difficult to prove, and is left for future work. The same can be said of distributivity; the proof does not seem to be trivial. It is easy, however, to prove that 2^n commutes with any set number X, under product. In virtue of this, if we prove distributivity then we have proven commutativity of product for two finite set numbers.

4. Construction of $\mathbb{R}$

We provide two constructions for the real number system. The first method consists of fitting all the real numbers into the unit interval; we define an explicit isomorphism between the set of real numbers and the power set of $-\mathbb{N}$. The extreme values, 0 and 1, of the unit interval correspond to $-\infty, +\infty$, respectively. The number $0 \in \mathbb{R}$ is identified with $1/2$ in the unit interval. The real numbers of the interval $[0,1]$ are put in bijection with the set numbers of the interval $[1/2, 3/4]$. Real numbers in the interval $[1,2]$ are put into bijection with $[3/4, 7/8] \subset -\mathbb{N}$. The interval $[-1,0]$ is put into bijection with $[1/4, 1/2] \subset -\mathbb{N}$, and $[-2,-1]$ is bijective with $[1/8, 1/4] \subset -\mathbb{N}$. We continue in this manner with the rest of the intervals (Figure 16). Our second method consists of building the real numbers as a space of functions; the positive real numbers are functions $+x$ so that $+x(a) = a + x$. The negative real numbers are their inverse functions.

Unit Interval. In our first method, we fit the entire real number line into the unit interval. We give an explicit bijection $2^{-\mathbb{N}} \cong [0,1] \to \mathbb{R}$ such that the corresponding set numbers can be operated. Graphically, we will associate $-\infty$ to the set number $\emptyset = 0$, and $+\infty$ will be associated to the set number $-\mathbb{N} = \{0\} = 1$. The real number $0 \in \mathbb{R}$ corresponds to the set number $\{-1\} = \frac{1}{2}$.

16.png Fig 16.png

The real numbers in $I_1 = [0,1) \subset \mathbb{R}$ are set numbers of the form $\{-1, x_1, x_2, x_3, \dots\}$, with $x_i \leq -3$. Next, we make the numbers in the interval $I_2 = [1,2) \subset \mathbb{R}$ bijective with the collection of set numbers of the form $\{-1,-2, x_1, x_2, x_3, \dots\}$, with $x_i \leq -4$. The interval $I_3 = [2,3) \subset \mathbb{R}$ is bijective to the collection of set numbers $\{-1,-2,-3, x_1, x_2, x_3, \dots\}$, with $x_i \leq -5$. The negative interval $-I_1 = [-1,0) \subset \mathbb{R}$ is bijective with the collection of set numbers of the form $\{-2, x_1, x_2, x_3, \dots\}$ with $x_i \leq -3$. In specific, the real number $-1 \in \mathbb{R}$ is identified with $\frac{1}{4} = \{-2\}$. Also, $-1 \in \mathbb{R}$ corresponds to the set number $\{-2\}$, and $-\frac{1}{2} \in \mathbb{R}$ is the set number $\{-2,-3\}$.

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To summarize, there are two types of set numbers: 1) If $-1 \in X$ then we say X is a positive set number 2) If $-1 \notin X$ then X is a negative set number. Positive set numbers are of the form $\{-1,-2,-3,\ldots,-n,a_i\}$ with $-2 \geq -n-1 > a_1 > a_2 > a_3 > \cdots$, while negative numbers are sets of the form $\{-n,a_1,a_2,a_3,\ldots\}$ with $-1 > -n > a_1 > a_2 > a_3 > \cdots$. Let $X \in \bar{\mathbb{Z}}$ a set number with $\{a_1,a_2,a_3,\ldots\} = X \cap -\mathbb{N}$ and a finite set of natural numbers $N = X \cap \mathbb{N}$. Define $n = \sum_{i \in N} 2^i$, then $X' = \{-1,-2,-3,\ldots,-n,a_1-(n+1),a_2-(n+1),a_3-(n+1),\ldots\}$ is the new representation of the set number X. The set number X can also be identified with another set we will call the negative of X', and it is $-X' = \{-(n+1),a_1-(n+1),a_2-(n+1),a_3-(n+1),\ldots\}$. To give another example, consider the number $X = 7.28125 = \{-5,-2,0,1,2\}$. Its new representation is

$$X' = \{-1,-2,-3,-4,-5,-6,-7,-8,-2-(8+1),-5-(8+1)\} = \{-1,-2,-3,-4,-5,-6,-7,-8,-11,-14\}$$

because the integer part is 7 and therefore $n = 8$. The negative is $-X' = \{-9,-2-9,-5-9\} = \{-9,-11,-14\}$. It is left as an exercise to prove every subset of $-\mathbb{N}$ corresponds to a unique real number, and vice-versa. We can obviously identify every real number with a unique subset of $\mathbb{N}$, now that we can identify it with a unique subset of $-\mathbb{N}$. The main idea behind this construction is that we use the first n natural numbers of a set number to determine the sign and the integer part.

This construction of real numbers has been given mostly for its role in giving graphic representations of the real numbers. In this paragraph we have proven that there is a way of defining an order relation for $2^{\mathbb{N}}$, and that this order is isomorphic to the order of the extender real number line $\mathbb{R}$. The operations can be meticulously defined case by case, but we will not unnecessarily extend our discussion. These have a simple interpretation in our graphic representations. The operations of real numbers are developed in our next construction.

Function Space. Our second construction, of negative real numbers, involves inverse functions. Every positive real number $x \in \bar{\mathbb{Z}}$ can be associated a unique isomorphism of the form $\bar{\mathbb{Z}} \to \mathbb{R}_x$ where $\mathbb{R}_x \subset \bar{\mathbb{Z}}$ is the collection of positive real numbers that are greater than or equal to x. In other words, an isomorphism $\bar{\mathbb{Z}} \to \{+x\}$, between the positive real numbers and a collection of functions $\{+x\}$ is given; the function $+x : \bar{\mathbb{Z}} \to \mathbb{R}_x$ is the bijection that acts by $+x(a) = a + x$. Let $\mathbb{R}^* = \{+x\} \cup \{(+x)^{-1}\}$ the set of functions that consists of the functions $+x$ and their respective inverse function $(+x)^{-1} : \mathbb{R}_x \to \bar{\mathbb{Z}}$, plus the identity function.

We will define an operation on the elements of $\mathbb{R}^*$. Let $x, y \in \bar{\mathbb{Z}}$, and their functions $+x, +y \in \mathbb{R}^*$, respectively. We define the operation in $\mathbb{R}^*$ as the function $+x \circ +y : \bar{\mathbb{Z}} \to \mathbb{R}_{x+y}$. We have two classes of functions in $\mathbb{R}^*$, as far as domain and range; the objects we call negative real numbers, as well as the positive real numbers. But, so far we have not defined an operation for all of these objects. Now we must find a suitable definitions for $+x \circ (+y)^{-1}$, and for the sum of two inverse functions in $\mathbb{R}^*$.

First, let us find the sum of two negative elements in $\mathbb{R}^*$. The sum of two positive elements in $\mathbb{R}^*$ is the composition, so given two negative real numbers $(+x)^{-1}, (+y)^{-1}$, it is natural to set $(+x)^{-1} \circ (+y)^{-1} := (+x \circ +y)^{-1} \in \mathbb{R}^*$. If $y < x$, then we can find the number $x - y > 0$ and a unique function $\bar{\mathbb{Z}} \to \mathbb{R}_{x-y}$ in $\mathbb{R}^*$; this function is defined to be the result of $+y^{-1} \circ +x = +x \circ +y^{-1}$. Now, define $+y^{-1} \circ +x = +x \circ +y^{-1} = (+y \circ +x^{-1})^{-1} \in \mathbb{R}^*$ if $x < y$. The operation we have defined for $\mathbb{R}^*$ is isomorphic to addition of real numbers.

Dual Function Space. The elements of $\mathbb{R}^*$ will be represented with bold letters such as **x**, **y**, **-x**, and **y+x** $= +x \circ +y$. We can build an isomorphism $+ : \mathbb{R}^* \to \mathbb{R}^{**}$, where $\mathbb{R}^{**}$ is a collection of bijections of the form $\mathbb{R}^* \to \mathbb{R}^*$. Specifically, the elements of $\mathbb{R}^{**}$ are the bijections $+(\mathbf{x}) : \mathbb{R}^* \to \mathbb{R}^*$ such that $\mathbf{y} \mapsto_{+(\mathbf{x})}$ **y+x**. Our real numbers, the elements of $\mathbb{R}^*$, are functions on positive numbers. The space $\mathbb{R}^{**}$ is a collection of bijective functions of the form $\mathbb{R}^* \to \mathbb{R}^*$, and it is isomorphic to $\mathbb{R}^*$ under addition. In conclusion $\mathbb{R} \cong \mathbb{R}^* \cong \mathbb{R}^{**}$, and $\mathbb{R}^{**}$ gives a complete description of addition for real numbers.

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5. Universe of Finite Sets

Previous expositions of axiomatic set theory for $\mathbb{R}$, begin describing natural numbers in two main forms [7](pp.21-22). These have become to be known as Zermelo ordinals, and Neumann ordinals. The first is the set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots\}$. That is to say, natural numbers have been characterized as $0 = \emptyset$, $1 = \{0\}$, $2 = \{1\}$, and in general $n+1 := \{n\}$ (Zermelo, 1908). The second way, due to von Neumann, is $\mathbb{N} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots\}$ and in this case we have that every natural number is the set of all natural numbers smaller than it (the set of its predecessors) so $x + 1 := x \cup \{x\}$. These constructions assign natural numbers to a certain kind of sets that are obtainable from the empty set. However, in both cases we do not have a surjective function because not all sets obtainable from are assigned a natural number. For example, the sets $\{\emptyset, \{\{\emptyset\}\}\}$ and $\{\emptyset, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$ is not a natural number in either ordinal family. Part of the problem in Benacerraf's identification problem, is that the identifications are not using all of the sets of $\mathcal{U}_0$.

One of our results is that the set $\mathcal{U}_0$, of objects constructed from the empty set, with finite steps, is equivalent to $\mathbb{N}$. In this sense, we don't leave out any sets of $\mathcal{U}_0$ when identifying them with natural numbers. Indeed, all elements of $\mathcal{U}_0$ represent a unique natural number. Furthermore, the power set of $\mathcal{U}_0$ is equivalent to the real number system. Going back to our original definition of set numbers, that represent natural numbers, we defined:

$$\begin{aligned}
0 &= \emptyset \\
1 &= \{0\} = \{\emptyset\} \\
2 &= \{1\} = \{\{\emptyset\}\} \\
3 &= \{0,1\} = \{\emptyset, \{\emptyset\}\} \\
4 &= \{2\} = \{\{\{\emptyset\}\}\} \\
5 &= \{0,2\} = \{\emptyset, \{\{\emptyset\}\}\} \\
6 &= \{1,2\} = \{\{\emptyset\}, \{\{\emptyset\}\}\} \\
7 &= \{0,1,2\} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\} \\
8 &= \{3\} = \{\{\emptyset, \{\emptyset\}\}\} \\
9 &= \{0,3\} = \{\emptyset, \{\emptyset, \{\emptyset\}\}\}.
\end{aligned}$$

In the following we give our proposal of definitions and axioms for analysis.

Definition 1. *Define a universe of sets* $\mathcal{U}_0$.

1. $\emptyset \in \mathcal{U}_0$
2. $x_1, x_2, x_3, \ldots, x_n \in \mathcal{U}_0$, *then* $\{x_1, x_2, x_3, \ldots, x_n\} \in \mathcal{U}_0$
3. $\mathcal{U}_0$ *is the set of objects that satisfy 1. or 2.*

Definition 2. *For any* $\emptyset \neq x \in \mathcal{U}_0$, *define* $x \oplus \{\emptyset\} = (x \triangle \{\emptyset\}) \oplus R$, *where* $R = \{y \oplus \{\emptyset\}\}_{y \in (x \cap \{\emptyset\})}$. *For the empty set, we define* $\emptyset \oplus \{\emptyset\} = \{\emptyset\} \oplus \emptyset = \{\emptyset\}$. *For any two* $A, B \in \mathcal{U}_0$ *we define the operation* $A \oplus B = (A \triangle B) \oplus \{s(y)\}_{y \in (A \cap B)}$. *In particular, define* $A \oplus \emptyset = \emptyset \oplus A = A$.

Let us use the notation $\mathcal{U}_0 - \{\emptyset\}$ to represent the set that contains all the elements of $\mathcal{U}_0$, except $\emptyset$.

Axiom 1. *Definition 2. Provides a bijection* $\oplus\{\emptyset\} : \mathcal{U}_0 \to (\mathcal{U}_0 - \{\emptyset\})$ *that defines a recursion. This recursion is a successor function;* $s(x) = \oplus\{\emptyset\}(x)$.

Axiom 2. *The definition of* $A \oplus B$ *defines the operation of addition for natural numbers. We can always find the set* $A \oplus B$ *in finite steps, and this operation is isomorphic to the usual addition of natural numbers.*

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It is left as an exercise for the reader to prove $s(x) = x + 1$ for every $0 \leq x \leq 8$, and compare the results to the equalities above. We do the first few.

$$\begin{aligned} \{\emptyset\} \oplus \{\emptyset\} &= (\{\emptyset\} \triangle \{\emptyset\}) \oplus \{s(y)\}_{y \in (\{\emptyset\} \cap \{\emptyset\})} \\ &= \emptyset \oplus \{s(y)\}_{y \in (\{\emptyset\} \cap \{\emptyset\})} \\ &= \{s(y)\}_{y \in \{\emptyset\}} \\ &= \{s(\emptyset)\} \\ &= \{\{\emptyset\}\} \end{aligned}$$

Next we find the sum $2 + 1 = 3$

$$\begin{aligned} \{\{\emptyset\}\} \oplus \{\emptyset\} &= (\{\{\emptyset\}\} \triangle \{\emptyset\}) \oplus \{s(y)\}_{y \in (\{\{\emptyset\}\} \cap \{\emptyset\})} \\ &= \{\emptyset, \{\emptyset\}\} \oplus \{s(y)\}_{y \in \emptyset} \\ &= \{\emptyset, \{\emptyset\}\} \oplus \emptyset \\ &= \{\emptyset, \{\emptyset\}\} \end{aligned}$$

Our next example is $3 + 1 = 4$

$$\begin{aligned} \{\emptyset, \{\emptyset\}\} \oplus \{\emptyset\} &= (\{\emptyset, \{\emptyset\}\} \triangle \{\emptyset\}) \oplus \{s(y)\}_{y \in (\{\emptyset, \{\emptyset\}\} \cap \{\emptyset\})} \\ &= \{\{\emptyset\}\} \oplus \{s(y)\}_{y \in \{\emptyset\}} \\ &= \{\{\emptyset\}\} \oplus \{s(\emptyset)\} \\ &= \{\{\emptyset\}\} \oplus \{\{\emptyset\}\} \\ &= (\{\{\emptyset\}\} \triangle \{\{\emptyset\}\}) \oplus \{s(y)\}_{y \in \{\{\emptyset\}\}} \\ &= \emptyset \oplus \{s(y)\}_{y \in \{\{\emptyset\}\}} \\ &= \{s(\{\emptyset\})\} \\ &= \{\{\{\emptyset\}\}\}. \end{aligned}$$

Lastly, we show $4 + 1 = 5$

$$\begin{aligned} \{\{\{\emptyset\}\}\} \oplus \{\emptyset\} &= (\{\{\{\emptyset\}\}\} \triangle \{\emptyset\}) \oplus \{s(y)\}_{y \in (\{\{\{\emptyset\}\}\} \cap \{\emptyset\})} \\ &= \{\emptyset, \{\{\emptyset\}\}\} \oplus \{s(y)\}_{y \in \emptyset} \\ &= \{\emptyset, \{\{\emptyset\}\}\} \oplus \emptyset \\ &= \{\emptyset, \{\{\emptyset\}\}\}. \end{aligned}$$

Now that we have defined the natural numbers, we can construct the integers. We do this using the method we used in defining negative real numbers. That means, every element $x \in \mathcal{U}_0$ is associated with a bijective function $+x : \mathcal{U}_0 \to \mathcal{U}_0^x$, where $\mathcal{U}_0^x$ is the set of elements of $\mathcal{U}_0$ that are greater than x. In particular, $\emptyset$ is the identity function, and $\{\emptyset\}$ is associated to the successor function $\oplus\{\emptyset\}$. If x is associated a function $+x$, then $s(x)$ is associated the function $+x \circ \oplus\{\emptyset\}$. We have inductively defined the functions as compositions. With this we can build a relation order isomorphic to the integers. The objects of this order are the functions just defined (the powers of composition for the successor function) and their respective inverse functions. The order of these objects is defined naturally, and now we consider subsets of this space of ordered functions to be positive real numbers. Of course, we can already have said that the extended real number line is the set of all subsets of $\mathcal{U}_0$, if we consider the first construction we gave of real numbers.

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Theorem 1. *The set of real numbers is $\mathcal{U}_1 = 2^{\mathcal{U}_0}$, the power set of $\mathcal{U}_0$. The suitable functions of addition, product and order exist and are well defined in this universe. The system of real numbers is an extension to infinity, of the natural numbers.*

In a sense, we have answered a more general question than Benacerraf's identification problem, since we have provided a canonical set theory for natural numbers, which can be naturally extended to obtain a canonical identification of $2^{\mathcal{U}_0}$ with the continuum $[0, 1]$, and later with the extended real number line. Our axioms are stating that every set in $\mathcal{U}_0 - \{\emptyset\}$ is successor of a set in $\mathcal{U}_0$, and that every set in $\mathcal{U}_0$ has a successor in $\mathcal{U}_0 - \{\emptyset\}$. Furthermore, there is a general procedure for finding the successor of any set in $\mathcal{U}_0$; the successor of x is $(x \triangle \{\emptyset\}) \oplus \{y \oplus \{\emptyset\}\}_{y \in (x \cap \{\emptyset\})}$. Our axioms assure us every set in $\mathcal{U}_0$ is of the form $(\oplus\{\emptyset\} \circ \oplus\{\emptyset\} \circ \cdots \circ \oplus\{\emptyset\})(\emptyset)$, and that every composition $(\oplus\{\emptyset\} \circ \oplus\{\emptyset\} \circ \cdots \circ \oplus\{\emptyset\})(\emptyset)$ is an element of $\mathcal{U}_0$. We have an identification between $2^{-\mathbb{N}}$ and the continuum $[0, 1]$ of the unit interval. Therefore we have an identification of $[0, 1]$ with $2^{\mathcal{U}_0}$. Since we can again identify each number of the unit interval (refer back to the first construction of the real number line) with the extended real number line, we conclude there is a natural identification of the extended real number line with the set $2^{\mathcal{U}_0}$.

6. Graphic Representations

We can carry out our constructions into a series of real-world representations. That is to say, we can give simple rules for manipulating physical systems, and these model numerical systems.

Collections of Arrows. In developing General Theory of Systems, we have to classify a system by its objects and its relations. In category theory, we focus on one special type of relations: binary relations. Of course, binary relations are really collections of arrows. Every number $0 \leq a \leq 31$ is the collection of arrows $\{x \to a\}$, for all $0 \leq x < a$. For example, $23 = \{0 \to 23, 1 \to 23, 2 \to 23, 4 \to 23\}$. This means we give an isomorphism that sends $a \in \mathbb{N}$ to the set of arrows $\{x \to a\}$, for all x element of the set number a. For example, 6 is represented by the collection of arrows $\{1 \to 6, 2 \to 6\}$ and 13 is the collection of arrows $\{0 \to 13, 2 \to 13, 3 \to 13\}$ (Figure 17).

17.png Fig 17.png

We provide a diagram of objects and arrows to describe the structure of the natural numbers. The objects are the natural numbers $0, 1, 2, 3 \ldots$, and the arrows are $0 \to 1$, $0 \to 3$, $0 \to 5, \ldots$; $1 \to 2$, $1 \to 3$, $1 \to 6$, $1 \to 7, \ldots$; $2 \to 4$, $2 \to 5$, $2 \to 6$, $2 \to 7$, $2 \to 12, \ldots$; $3 \to 8$, $3 \to 9$, $3 \to 10$, $3 \to 11$, $3 \to 12, \ldots$;... (see Figure 18). The pattern that these relations follow is obvious $0 \in \{1 + 2i\}_{i \in \mathbb{N}}$, $1 \in \{2 + 4i, 3 + 4i\}_{i \in \mathbb{N}}$, $2 \in \{4 + 8i, 5 + 8i, 6 + 8i, 7 + 8i\}_{i \in \mathbb{N}}$, $3 \in \{8 + 16i, 9 + 16i, 10 + 16i, 11 + 16i, 12 + 16i, 13 + 16i, 14 + 16i, 15 + 16i\}_{i \in \mathbb{N}}$, etc. In Figure 18, we represent the natural numbers from 0 to 31. We arrange the objects along a circumference, and start adding the arrows to obtain Figure 18:

18.png Fig 18.png

Let us transform the real number line into a circumference. The integers correspond to the discrete points (in red) of Figure 19. These have been determined by successive bisections of the circumference. Given an arbitrary point (blue) on the continuum of the circumference, we have a

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unique number $x \in \mathbb{R}$. At the same time, $|x|$ is a set of integers. Every real number is identified with a subset of the red points. We draw arrows from the red points, corresponding to the elements of $|x|$, into the blue point that corresponds to x. We can give a graphic representation of real numbers. Following the same procedure as in Figure 18, we can represent multiple (approximations) of real numbers, in a single diagram.

Fig 19.png

19.png

Trees. A tree is a graph of nodes and edges such that 1) We can identify a *trunk*: a principle edge with a number of *branches* attached to one of the nodes. 2) Each branch on the tree is a tree. 3) A single edge is a tree; we call it the 0-tree. 4) The successor of a tree is obtained by adding a single edge to the trunk. Adding an edge to the trunk of the 0-tree gives its successor, the 1-tree, which is two edges joined together at one node. Adding an edge to the 1-tree, we find its successor, the 2-tree. If two branches are repeated on the same trunk, we substitute the two repeated branches with a single branch; the successor of these. This is called *reduction*. If one tree can be reduced to obtain another tree they are in the same equivalence class. An irreducible tree is said to be in canonical form. Reducing the 2-tree, we obtain find the canonical form (Figure 20). Adding a single edge to that, we obtain the canonical form of the 3-tree. If we add an edge to the 3-tree we have to reduce and obtain the canonical form of the 4-tree, etc. Continuing in this manner, we find all trees in sequence.

Fig 20.png

20.png

Rings. A ring, R, is a circumference passing through denumerable number of rings R_i; the center point of every R_i is a point on R; a central circumference passing through the center of a denumerable set of circumferences of degree 1. The central circumference is said to have degree 0. The circumferences R_i are rings themselves; each R_i is a circumference passing through the center of a denumerable set $R_i^{j_i}$ of circumferences of degree 2, and so on. A natural number n, with set number N, is represented by an equivalence class of rings. To build the canonical ring corresponding to a natural number, we draw an R_i, for each element of the set number $N = \{a_1, a_2, a_3, \ldots\}$. That is to say, the central ring is a circumference going through $\#(N)$ circumferences; each of these a ring R_i. Then, R_i is a circumference going through the center of $\#(a_i)$ circumferences. We apply this recursively, until we *bottom out*. The equivalence relation is defined analagous to trees. If we have a ring with two identical rings, we substitute these both with a single ring; the successor ring of the repeated ring. The successor of R is found by adding a single 0-ring to the central circumference of R, and reducing.

Fig 21.png

21.png

Consider the ring of the number 27={4,3,1,0}. Then R is a circumference passing through the center of 4 circumferences R_1, R_2, R_3, R_4; each one representing a number of the set $\{4, 3, 1, 0\}$.

22.png Fig 22.png

Giving a degree of freedom to the rings of degree 0 and 1, we are able to represent the set of real numbers. If the number is negative, we paint the degree 0 ring red; if it is positive the degree 0 ring is blue. The degree 1 rings are allowed to be red or blue in order to represent negative integers; a red degree 1 ring means we have a negative power in the binary representation. A 0 ring of degree 1 is neither red nor blue because $2^0 = 1$ is its own reciprocal.

23.png Fig 23.png

7. Conclusions

Even though these methods and definitions are elementary, and the presentation may seem trivial and in plain sight, these constructions and proofs are not apparent. The real number system has been revisited on many occasions but has never had simple solution. The initial purpose was to find a system theoretic treatment of numbers, as opposed to the set theoretic foundations of the classical axiomatic systems. It was our aim to propose new models of the relations of numerical systems, and finding their true representation in the universe of sets. This work comes after an initial attempt was made in [8]. Our construction of real numbers has provided 1) New algorithms for calculating operations of real numbers 2) Graphic representations of real numbers. We can associate numbers to certain classes of physical models. 3) We have provided a canonical set theory for arithmetic of natural numbers, and for analysis; one set being the power set of the other. We have answered Benacerraf's identification problem [9], by giving these canonical set representations of numbers, thus proving there is an intrinsic connection between the universe of sets and arithmetic. An extended version of these results and methods is under work. Several topics are treated in the same spirit. The topics include a calculus defined in terms of the order of $\mathbb{N}$. A number of applications are also being prepared. A computing device that operates using radio frequency signals emitted back and forth between two stations, is tempting. Station A emits two sets of signals to station B, which then emits two signal back to station A. This process continues until the signal stabilizes (we can use this physical process to model addition, using the interpretation of energy levels). Other applications may be explored based on the graphic representations of the last section.

8. References

[1] Norbert A'Campo, *A Natural Construction for the Real Numbers.* arXiv:math.GN/0301015 v1 (3 Jan 2003).

[2] R.D. Arthan, *The Eudoxus Real Numbers*. (preprint, September 2003).

[3] N. Metropolis, Gian-Carlo Rota, And S. Tanny. *Significance Arithmetic: The Carrying Algorithm*. Journal of Combinatorial Theory (A) 14, 386-421 (1973).

[4] N. G. De Bruijn, *Definig Reals Without the Use of Rationals.* Koninkl. Nederl. Akademie Van Wetenschappen-Amsterdam. Reprinted from Proceedings, Series A, 79, No. 2 and Indag. Math., 38, No.2 1976.

[5] Arnold Knopfmacher and John Knopfmacher, *Two Concrete New Constructions of the Real Numbers.* Rocky Mountain Journal of Mathematics Volume 18, Number 4, Fall 1988.

[6] Konrad Knopp. *Theory and Application of Infinite Series.* Dover, New York (1990).

[7] Paul Bernays. *Axiomatic Set Theory.* Dover, New York (1991).

 doi:10.20944/preprints201901.0089.v1

[8] Juan Pablo Ramírez, *Systems and Categories.* arXiv:1509.03649v5 [math.CT](11 Sep 2015)
[9] P. Benacerraf (1965), "What Numbers Could Not Be", Philosophical Review Vol. 74, pp. 47–73.