

OSCILLATION OF A CLASS OF THIRD ORDER GENERALIZED FUNCTIONAL DIFFERENCE EQUATION

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Abstract

The authors intend to establish new oscillation criteria for a class of generalized third order functional difference equation of the form

$$\Delta_{\ell} \left(a_2(n) \left[\Delta_{\ell} \left(a_1(n) [\Delta_{\ell} z(n)]^{\beta_1} \right) \right]^{\beta_2} \right) + q(n)f(x(g(n))) = 0, \quad n \geq n_0, \quad (1)$$

where $z(n) = x(n) + p(n)x(\tau(n))$. We also present sufficient conditions for the solutions to converges to zero. Suitable examples are presented to validate our main results.

Keywords: Generalized difference operator, Oscillation, Convergence.

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1. INTRODUCTION

Difference equations usually occur due to certain phenomena over time, and it play an important role in describing discrete dynamical systems [1]. Difference equation and their associated operators not only play a role in their own right as direct mathematical models of physical phenomena but also provide the field of numerical analysis with powerful tools. Difference equations also occur in combined form with differential equations, commonly called differential-difference equations yielding rich models, particularly in control theory. Difference equations are widely used in the theory of probability, biology, engineering, social and behavioral sciences.

Difference equations are evolved normally concerning the operator Δ and its higher orders defined as

$$\Delta x(n) = x(n+1) - x(n), \quad n \in \mathbb{Z} = \{0, 1, 2, 3, \dots\} \quad (2)$$

A few authors ([1], [10], [13]) also defined the operator Δ as

$$\Delta x(n) = x(n + \ell) - x(n), \quad (3)$$

ℓ is a positive integer. None has consider the definition of Δ given in (3) for further study on the theory of difference equations because of the difficulty involved when ℓ is different from unity. Recently, Adem Kilicman, Thandapani, Maria Susai Manuel and Britto Antony Xavier considered the definition of Δ as given in (3) and derived exiting results on number theory ([5]-[7]) by denoting the new operator as Δ_ℓ . Also new oscillation criteria and new asymptotic properties are obtained by considering difference equations involving Δ_ℓ .

Oscillation is one of the main topics of interest in the study of difference equations. Active research is on in the last few decades in analyzing the oscillatory behavior of the solution of difference equations involving Δ but the study of the same property of difference equations involving Δ_ℓ is rare. The present research is focused on establishing new oscillation criteria of the class of third order generalized difference equation given in (1). We also present sufficient conditions for the solution to converge to zero. For the theory related to the relevant topic, one can refer ([2], [4], [8], [9], [11], [12]).

In this paper, we deal with the oscillation and the asymptotic behavior of solutions of the third-order generalized functional difference equation of the form

$$\Delta_\ell \left(a_2(n) \left[\Delta_\ell \left(a_1(n) [\Delta_\ell z(n)]^{\beta_1} \right) \right]^{\beta_2} \right) + q(n)f(x(g(n))) = 0, \quad n \geq n_0,$$

where $z(n) = x(n) + p(n)x(\tau(n))$.

We make the following assumptions throughout this paper.

- (a) $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, $\mathbb{N}(a) = \{a, a + 1, a + 2, \dots\}$,
- (b) $\mathbb{N}_\ell(a) = \{a, a + \ell, a + 2\ell, \dots\}$.
- (c) $[x]$ = upper integer part of x and $\{x\}$ = integer part of x .
- (d) $j = n - n_i - \left[\frac{n - n_i}{\ell} \right] \ell, n_i \in [0, \infty)$.
- (e) $\{a_i(n)\}$ is a positive increasing sequence of real numbers for all $n \geq n_0$ and satisfies the condition $\sum_{s=n_0}^{\infty} \frac{1}{a_i^{1/\beta_i}(s)} = \infty, i = 1, 2$.
- (f) $\{p(n)\}$ and $\{q(n)\}$ with $0 \leq p(n) \leq p < 1, q(n) > 0$ are sequences of real numbers for all $n \geq n_0$.

- (g) $\{\tau(n)\}$ and $\{g(n)\}$ are sequences of integers with $\tau(n) \leq n$, $g(n) \leq n$, $\Delta_\ell g(n) > 0$,
 $\lim_{n \rightarrow \infty} \tau(n) = \infty$ and $\lim_{n \rightarrow \infty} g(n) = \infty$.
- (h) β_1 and β_2 are a quotient of odd positive integers with $\beta = \beta_1 \beta_2$.
- (i) f is a continuous real valued function such that $\frac{f(x)}{x^\beta} \geq k > 0$ for $x \neq 0$ and k is a constant.

2. PRELIMINARIES

In this section, we present some preliminaries which will be useful for future discussion.

Definition 2.1. [5] Let $x(n)$, $n \in [0, \infty)$ be a real or complex valued function and $\ell \in (0, \infty)$. We define, the generalized difference operators Δ_ℓ as

$$\Delta_\ell x(n) = x(n + \ell) - x(n). \quad (4)$$

Definition 2.2. [5] Let $x(n)$, $n \in [0, \infty)$ be a real or complex valued function and $\ell \in (0, \infty)$. Then, the inverse of Δ_ℓ denoted by Δ_ℓ^{-1} is defined as follows.

$$\text{If } \Delta_\ell y(n) = x(n), \text{ then } y(n) = \Delta_\ell^{-1} x(n) + c_j, \quad (5)$$

(or)

$$y(n) = y(n_0 + j) + \sum_{r=0}^{\left[\frac{n-n_0-j-\ell}{\ell}\right]} x(n_0 + j + r\ell), \quad (6)$$

where c_j is a constant for all $n \in \mathbb{N}_\ell(j)$, $j = n - \left[\frac{n}{\ell}\right] \ell$.

Definition 2.3. [5] For $\lambda \in \mathbb{N}(1)$, the generalized polynomial factorial is defined by

$$n_\ell^{(\lambda)} = n(n - \ell)(n - 2\ell) \cdots (n - (\lambda - 1)\ell). \quad (7)$$

Lemma 2.4. [5] Let $\ell \in [0, \infty)$. then $\Delta_\ell(n_\ell^{(\lambda)}) = (\lambda\ell)n_\ell^{(\lambda-1)}$

Lemma 2.5. Let $x(n)$ and $y(n)$ be any two real valued functions. Then

$$\Delta_\ell\{x(n)y(n)\} = x(n + \ell)\Delta_\ell y(n) + y(n)\Delta_\ell x(n) = y(n + \ell)\Delta_\ell x(n) + x(n)\Delta_\ell y(n).$$

Definition 2.6. The solution $x(n)$ of (1) is called oscillatory if for any $n_1 \in [a, \infty)$ there exists an $n_2 \in \mathbb{N}_\ell(n_1)$ such that $x(n_2)x(n_2 + \ell) \leq 0$. The difference equation itself is called oscillatory if all its solutions are oscillatory. If the solution $x(n)$ is not oscillatory, then it is said to be nonoscillatory (i.e. $x(n)x(n + \ell) > 0$ for all $n \in [n_1, \infty)$).

Lemma 2.7. [3] *If x and y are positive and unequal, then*

$$ry^{r-1}(x-y) < x^r - y^r < rx^{r-1}(x-y), \quad r < 0 \text{ or } r > 1 \quad (8)$$

$$rx^{r-1}(x-y) < x^r - y^r < ry^{r-1}(x-y), \quad 0 < r < 1 \quad (9)$$

There is obviously equality when $r = 0$, $r = 1$ or $x = y$.

3. MAIN RESULTS

We establish in this section, some new oscillation criteria for solutions of the equation (1). For the sake of convenience, we introduce the following notations.

$$E_0(n) = z(n), \quad E_i(n) = a_i(n) (\Delta_\ell E_{i-1}(n))^{\beta_i}, \quad i = 1, 2$$

$$R_N(n) = \frac{1}{a_1^{1/\beta_1}(n)} \left(\sum_{r=0}^{\frac{n-N-\ell-j}{\ell}} \frac{1}{[a_2(N+j+r\ell)]^{1/\beta_2}} \right)^{1/\beta_1} \quad \text{and}$$

$$\overline{R_N}(n) = \sum_{r=0}^{\frac{n-N-\ell-j}{\ell}} R_N(N+j+r\ell).$$

First, we state and prove some useful lemmas.

Lemma 3.1. *Let $\{x(n)\}$ be a positive solution of equation (1). Then $z(n)$ has only one of the following two properties eventually*

- (i) $z(n) > 0$, $\Delta_\ell z(n) > 0$ and $\Delta_\ell E_1(n) > 0$,
- (ii) $z(n) > 0$, $\Delta_\ell z(n) < 0$ and $\Delta_\ell E_1(n) > 0$.

Proof. Let $\{x(n)\}$ be a positive solution of equation (1). From (g), there exists an $n_1 \geq n_0$ such that $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n)) > 0$ for $n \geq n_1$. Then $z(n) > 0$ and equation (1) implies that

$$\Delta_\ell E_2(n) = -q(n)f(x(g(n))) \leq 0.$$

Hence, $E_2(n)$ is a non-increasing function and of one sign. We claim that $E_2(n) > 0$ for $n \geq n_1$. Suppose that $E_2 < 0$ for $n \geq n_2 \geq n_1$, then there exists an $n_3 \geq n_2$ and a constant $K_1 > 0$ such that

$$\Delta_\ell E_1(n) < -K_1 [a_2(n)]^{-1/\beta_2} < 0, \text{ for } n \geq n_3.$$

Hence, by equation (6)

$$E_1(n) \leq E_1(n_3 + j) - K_1 \sum_{r=0}^{\frac{n-n_3-\ell-j}{\ell}} \frac{1}{[a_2(n_3 + j + r\ell)]^{1/\beta_2}}.$$

Letting $n \rightarrow \infty$, from (e) we have $\lim_{n \rightarrow \infty} E_1(n) = -\infty$. Then there exists a $n_4 \geq n_3$ and a constant $K_2 > 0$ such that

$$\Delta_\ell z(n) < -K_2 [a_1(n)]^{-1/\beta_1}, \text{ for } n \geq n_4.$$

Hence, by equation (6) and using (e), we get $\lim_{n \rightarrow \infty} z(n) = -\infty$, which contradicts $z(n) > 0$. Now we have $E_2(n) > 0$ for $n \geq n_1$. Therefore, $E_1(n)$ is an increasing function. Thus property (i) or property (ii) holds for $z(n)$ eventually. \square

Lemma 3.2. Let $\{x(n)\}$ be a positive solution of equation (1), and $z(n)$ has the property (ii). Assume that

$$\sum_{t=0}^{\infty} \frac{1}{a_1^{1/\beta_1}(n_3 + j + t\ell)} \left[\sum_{s=0}^{\frac{t-n_2-\ell-j}{\ell}} \frac{1}{a_2^{1/\beta_2}(n_2 + j + s\ell)} \left[\sum_{r=0}^{\frac{s-n_1-\ell-j}{\ell}} q(n_1 + j + r\ell) \right]^{\frac{1}{\beta_2}} \right]^{\frac{1}{\beta_1}} = \infty. \quad (10)$$

Then, the solution $\{x(n)\}$ of equation (1) converges to zero as $n \rightarrow \infty$.

Proof. Let $\{x(n)\}$ be a positive solution of equation (1). Since $z(n)$ satisfies the property (ii), we get

$$\lim_{n \rightarrow \infty} z(n) = \gamma \geq 0.$$

Now, we shall prove that $\gamma = 0$. Let $\gamma > 0$, then we have $\gamma < z(n) < \gamma + \epsilon$ for all $\epsilon > 0$ and n sufficiently large. Choosing $\epsilon < \frac{1-p}{p}\gamma$, we obtain

$$x(n) = z(n) - p(n)x(\tau(n)) > \gamma - p(n)z(\tau(n)) > L(\gamma + \epsilon) > Lz(n),$$

where $L = \frac{\gamma - p(\gamma + \epsilon)}{\gamma + \epsilon} > 0$. Hence, from equation (1) and (i), we have

$$\Delta_\ell E_2(n) \leq -kq(n)x^\beta(g(n)) < -kL^\beta q(n)z^\beta(g(n)) < -kL^\beta \gamma^\beta q(n)$$

Therefore, by equation (6), summing this inequality from n_1 to ∞ , we get

$$\Delta_\ell E_1(n) > -k^{\frac{1}{\beta_2}} L^{\frac{\beta}{\beta_2}} \gamma^{\frac{\beta}{\beta_2}} \frac{1}{a_2^{1/\beta_2}(n)} \left(\sum_{r=0}^{\frac{n-n_1-\ell-j}{\ell}} q(n_1 + j + r\ell) \right)^{1/\beta_2}.$$

Summing again from n_2 to ∞ , we obtain

$$\Delta_\ell z(n) < \frac{-C}{a_1^{1/\beta_1}(n)} \left(\sum_{s=0}^{\frac{n-n_2-\ell-j}{\ell}} \frac{1}{a_2^{1/\beta_2}(n_2 + j + s\ell)} \left(\sum_{r=0}^{\frac{s-n_1-\ell-j}{\ell}} q(n_1 + j + r\ell) \right)^{1/\beta_2} \right)^{1/\beta_1},$$

where $C = k^{\frac{1}{\beta}} L \gamma$. Summing the last inequality from n_3 to ∞ , we have

$$z(n) > -C \sum_{t=0}^{\infty} \frac{1}{a_1^{1/\beta_1}(n_3 + j + t\ell)} \left[\sum_{s=0}^{\frac{t-n_2-\ell-j}{\ell}} \frac{1}{a_2^{1/\beta_2}(n_2 + j + s\ell)} \left[\sum_{r=0}^{\frac{s-n_1-\ell-j}{\ell}} q(n_1 + j + r\ell) \right]^{\frac{1}{\beta_2}} \right]^{\frac{1}{\beta_1}}.$$

This contradicts condition (10). Hence, $\lim_{n \rightarrow \infty} z(n) = 0$, which implies that $\lim_{n \rightarrow \infty} x(n) = 0$. \square

Lemma 3.3. Let $\{x(n)\}$ be a positive solution of equation (1), and $z(n)$ has the property (i). Then we have

$$\Delta_\ell E_2(n) \leq -kq(n)(1 - p(g(n)))^\beta z^\beta(g(n)), \quad (11)$$

$$\Delta_\ell z(g(n)) \geq E_2^{1/\beta}(n) R_{n_0}(g(n)) \quad (12)$$

and

$$\overline{R}_{n_0}^\beta(g(n)) \frac{E_2(n)}{z^\beta(g(n))} \leq 1. \quad (13)$$

Proof. Let $\{x(n)\}$ be a positive solution of equation (1). From (g), there exists an $n_1 \geq n_0$ such that $x(n) > 0$, $x(\tau(n)) > 0$ and $x(g(n)) > 0$ for $n \geq n_1$. Since $z(n)$ satisfies the property (i), we have

$$x(n) = z(n) - p(n)x(\tau(n)) \geq (1 - p(n))z(n).$$

Thus, by equation (1) and (i), we have

$$\Delta_\ell E_2(n) \leq -kq(n)x^\beta(g(n)) \leq -kq(n)(1-p(g(n)))^\beta z^\beta(g(n)) < 0.$$

Again, from property (i), there exists an $N \geq n_0$ such that

$$E_1(n) = E_1(N+j) + \sum_{r=0}^{\frac{n-N-\ell-j}{\ell}} \frac{E_2^{1/\beta_2}(N+j+r\ell)}{[a_2(N+j+r\ell)]^{1/\beta_2}}.$$

Since $\Delta_\ell E_2(n) < 0$, we obtain

$$E_1(n) \leq E_2^{1/\beta_2}(n) \sum_{r=0}^{\frac{n-N-\ell-j}{\ell}} \frac{1}{[a_2(N+j+r\ell)]^{1/\beta_2}}.$$

This implies that

$$\Delta_\ell z(n) \geq E_2^{1/\beta}(n) R_N(n). \quad (14)$$

Since $g(n) \leq n$, we have

$$\Delta_\ell z(g(n)) \geq E_2^{1/\beta}(n) R_N(g(n)).$$

By summing the inequality (14) from N to n and using $\Delta_\ell E_2(n) < 0$, we get

$$z(n) = z(N+j) + E_2^{1/\beta}(n) \sum_{r=0}^{\frac{n-N-\ell-j}{\ell}} R_N(N+j+r\ell).$$

This implies that

$$z(n) \leq E_2^{1/\beta}(n) \overline{R}_N(n).$$

Thus, we get

$$z(g(n)) \geq E_2^{1/\beta}(n) \overline{R}_N(g(n)),$$

and so

$$\overline{R}_N^\beta(g(n)) \frac{E_2(n)}{z^\beta(g(n))} \leq 1.$$

This completes the proof. □

Remark 3.4. For simplicity, we introduce the following notations.

$$P = \liminf_{n \rightarrow \infty} \overline{R}_{n_0}^\beta(g(n+j+\ell)) \sum_{s=n}^{\infty} \phi(n+j+s\ell)$$

and

$$Q = \limsup_{n \rightarrow \infty} \frac{1}{\overline{R}_{n_0}(g(n+j+\ell))} \sum_{s=0}^{\left[\frac{n-n_0-j-\ell}{\ell}\right]} \overline{R}_{n_0}^{\beta+1}(g(n_0+j+s\ell+\ell))\phi(n_0+j+s\ell),$$

where $\phi(n) = kq(n)(1-p(g(n)))^\beta$. Moreover for $z(n)$ satisfying property (i), we define

$$\omega(n) = \frac{E_2(n)}{z^\beta(g(n))} \quad (15)$$

and

$$\begin{aligned} l &= \liminf_{n \rightarrow \infty} \overline{R}_{n_0}^\beta(g(n+j+\ell))\omega(n+j+\ell), \\ U &= \limsup_{n \rightarrow \infty} \overline{R}_{n_0}^\beta(g(n+\ell))\omega(n). \end{aligned} \quad (16)$$

Lemma 3.5. *Let $\{x(n)\}$ be a positive solution of equation (1).*

(1) *Let $P < \infty$, $Q < \infty$ and $z(n)$ satisfies property (i). If*

$$\lim_{n \rightarrow \infty} \overline{R}_{n_0}(n) = \infty, \quad (17)$$

then

$$P \leq l - l^{\frac{1+\beta}{\beta}} \text{ and } P + Q \leq 1 \quad (18)$$

(2) *If $P = \infty$ or $Q = \infty$, then $z(n)$ does not have property (i).*

Proof. Part(1); Let $\{x(n)\}$ be a positive solution of equation (1) and $z(n)$ satisfies property (i). By Lemma 3.3, we have that (11), (12) and (13) hold. From the definition of $\omega(n)$, we see that $\omega(n)$ is positive and satisfies

$$\Delta_\ell \omega(n) = \frac{\Delta_\ell E_2(n)}{z^\beta(g(n))} - \frac{E_2(n+\ell)\Delta_\ell z^\beta(g(n))}{z^\beta(g(n))z^\beta(g(n+\ell))}. \quad (19)$$

Now, by using equation (8), we find that

$$\Delta_\ell z^\beta(g(n)) = z^\beta(g(n+\ell)) - z^\beta(g(n)) < \beta z^{\beta-1}(g(n+\ell))\Delta_\ell z(g(n)). \quad (20)$$

The equation (19) can be rewritten as

$$\Delta_\ell \omega(n) = \frac{\Delta_\ell E_2(n)}{z^\beta(g(n))} - \frac{\beta E_2(n+\ell)\Delta_\ell z(g(n))}{z^\beta(g(n))z(g(n+\ell))}.$$

Thus, from (11) and (12), there exists an $N \geq n_0$ such that

$$\Delta_\ell \omega(n) \leq -kq(n)(1 - p(g(n)))^\beta - \frac{\beta E_2^{\frac{1+\beta}{\beta}}(n + \ell) R_N(g(n + \ell))}{z^{1+\beta}(g(n + \ell))}.$$

for $n \geq N$. This implies that

$$\Delta_\ell \omega(n) \leq -\phi(n) - \beta R_N(g(n + \ell)) \omega^{\frac{1+\beta}{\beta}}(n + \ell). \quad (21)$$

From (13), we get

$$\overline{R}_N^\beta(g(n)) \omega(n) \leq 1,$$

which with (17) gives

$$\lim_{n \rightarrow \infty} \omega(n) = 0. \quad (22)$$

On the other hand, from the definition of $\omega(n)$, l and U , we see that

$$0 \leq l \leq U \leq 1. \quad (23)$$

Now, we prove that the first inequality in (18) holds. Let $\epsilon > 0$, then from the definitions of P and l , we can choose $n_2 \geq N$ sufficiently large that

$$\overline{R}_N^\beta(g(n + j + \ell)) \sum_{s=0}^{\infty} \phi(n + j + s\ell) \geq P - \epsilon \quad (24)$$

$$\text{and } \overline{R}_N^\beta(g(n + j + \ell)) \omega(n + j + \ell) \geq l - \epsilon \quad \text{for } n \geq n_2. \quad (25)$$

By summing (21) from n to ∞ and using (22), we have

$$\omega(n + j) \geq \sum_{s=0}^{\infty} \phi(n + j + s\ell) + \beta \sum_{s=0}^{\infty} R_N(g(n + j + s\ell + \ell)) \omega^{\frac{1+\beta}{\beta}}(n + j + s\ell + \ell). \quad (26)$$

Multiplying the above inequality by $\overline{R}_N^\beta(g(n + j + \ell))$, we obtain

$$\begin{aligned} \overline{R}_N^\beta(g(n + j + \ell)) \omega(n + j) &\geq \omega(n + j) \geq \overline{R}_N^\beta(g(n + j + \ell)) \sum_{s=0}^{\infty} \phi(n + j + s\ell) \\ &+ \beta \overline{R}_N^\beta(g(n + j + \ell)) \sum_{s=0}^{\infty} \frac{R_N(g(n + j + s\ell + \ell))}{\overline{R}_N^{1+\beta}(g(n + j + s\ell + \ell))} \left(\overline{R}_N^\beta(g(n + j + s\ell + \ell)) \omega(n + j + s\ell + \ell) \right)^{\frac{1+\beta}{\beta}}. \\ &\geq (P - \epsilon) + (l - \epsilon)^{\frac{1+\beta}{\beta}} \beta \overline{R}_N^\beta(g(n + j + \ell)) \sum_{s=0}^{\infty} \frac{R_N(g(n + j + s\ell + \ell))}{\overline{R}_N^{1+\beta}(g(n + j + s\ell + \ell))}. \\ &\geq (P - \epsilon) + (l - \epsilon)^{\frac{1+\beta}{\beta}}. \end{aligned}$$

Taking limit inferior on both sides as $n \rightarrow \infty$, we get

$$l \geq (P - \epsilon) + (l - \epsilon)^{\frac{1+\beta}{\beta}}. \quad (27)$$

Since $\epsilon > 0$ is arbitrary, we obtain the desired result

$$P \leq l - l^{\frac{1+\beta}{\beta}}. \quad (28)$$

Next, we prove the second inequality in part (1). Multiplying (21) by $\bar{R}_N^{\beta+1}(g(n + \ell))$ and summing it from n_2 to $n - \ell$, we obtain

$$\begin{aligned} & \sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}} \bar{R}_N^{\beta+1}(g(n_2 + j + s\ell + \ell)) \Delta_\ell \omega(n_2 + j + s\ell) \\ & \leq - \sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}} \bar{R}_N^{\beta+1}(g(n_2 + j + s\ell + \ell)) \phi(n_2 + j + s\ell) \\ & \quad - \beta \sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}} R_N(g(n_2 + j + s\ell + \ell)) \left(\bar{R}_N^\beta(g(n_2 + j + s\ell + \ell)) \omega(n_2 + j + s\ell + \ell) \right)^{\frac{1+\beta}{\beta}}. \end{aligned}$$

By Summation by parts, we obtain

$$\begin{aligned} & \bar{R}_N^{\beta+1}(g(n + \ell)) \omega(n) \leq \bar{R}_N^{\beta+1}(g(n_2 + j + \ell)) \omega(n_2 + j) \\ & \quad + \sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}} \omega(n_2 + j + s\ell + \ell) \Delta_\ell \bar{R}_N^{\beta+1}(g(n_2 + j + s\ell + \ell)) \\ & \quad - \sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}} \bar{R}_N^{\beta+1}(g(n_2 + j + s\ell + \ell)) \phi(n_2 + j + s\ell) \\ & \quad - \beta \sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}} R_N(g(n_2 + j + s\ell + \ell)) \left(\bar{R}_N^\beta(g(n_2 + j + s\ell + \ell)) \omega(n_2 + j + s\ell + \ell) \right)^{\frac{1+\beta}{\beta}}. \end{aligned}$$

Hence,

$$\begin{aligned}\overline{R}_N^{\beta+1}(g(n+\ell))\omega(n) &\leq \overline{R}_N^{\beta+1}(g(n_2+j+\ell))\omega(n_2+j) \\ &\quad - \sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}} \overline{R}_N^{\beta+1}(g(n_2+j+s\ell+\ell))\phi(n_2+j+s\ell) \\ &\quad + \sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}} R_N(g(n_2+j+s\ell+\ell)) \left((\beta+1)M - \beta M^{\frac{1+\beta}{\beta}} \right),\end{aligned}$$

where $M = \overline{R}_N^{\beta}(g(n_2+j+s\ell+\ell))\omega(n_2+j+s\ell+\ell)$.

Using the inequality

$$Au - Bu^{\frac{1+\beta}{\beta}} \leq \frac{\beta^{\beta}}{(1+\beta)^{1+\beta}} \frac{A^{1+\beta}}{B^{\beta}} \quad (29)$$

with $u = M$, $A = (1+\beta)$ and $B = \beta$, we obtain

$$\begin{aligned}\overline{R}_N^{\beta+1}(g(n+\ell))\omega(n) &\leq \overline{R}_N^{\beta+1}(g(n_2+j+\ell))\omega(n_2+j) \\ &\quad - \sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}} \overline{R}_N^{\beta+1}(g(n_2+j+s\ell+\ell))\phi(n_2+j+s\ell) + \overline{R}_N(g(n+j+\ell)).\end{aligned}$$

It follows that

$$\begin{aligned}\overline{R}_N^{\beta}(g(n+\ell))\omega(n) &\leq \frac{\overline{R}_N^{\beta+1}(g(n_2+j+\ell))\omega(n_2+j)}{\overline{R}_N(g(n+j+\ell))} \\ &\quad - \frac{1}{\overline{R}_N(g(n+j+\ell))} \sum_{s=0}^{\frac{n-n_2-j-\ell}{\ell}} \overline{R}_N^{\beta+1}(g(n_2+j+s\ell+\ell))\phi(n_2+j+s\ell) + 1.\end{aligned}$$

Taking limit superior on both sides as $n \rightarrow \infty$ and using (17) we get

$$U \leq 1 - Q.$$

Thus, from (23), we have

$$P \leq l - l^{\frac{1+\beta}{\beta}} \leq l \leq U \leq 1 - Q, \quad (30)$$

which completes the proof of Part (1).

Part (2); Assume that $\{x(n)\}$ is a positive solution of equation (1). We shall prove that $z(n)$

does not have property (i). On the contrary, we assume that $P = \infty$. Then, from (26), we get

$$\overline{R}_N^\beta(g(n+\ell))\omega(n) \geq \overline{R}_N^\beta(g(n+\ell)) \sum_{s=n}^{\infty} \phi(n+j+s\ell)$$

Taking limit inferior on both sides as $n \rightarrow \infty$, we get in view of (23) that

$$1 \geq l \geq P = \infty.$$

This is a contradiction. Now we admit that $Q = \infty$. Then by (30), $U = -\infty$, which contradicts (23). The proof is now complete. \square

Theorem 3.6. Assume that (10) and (17) hold. If

$$P = \liminf_{n \rightarrow \infty} \overline{R}_{n_0}^\beta(g(n+j+\ell)) \sum_{s=n}^{\infty} \phi(n+j+s\ell) > \frac{\beta^\beta}{(\beta+1)^{\beta+1}}, \quad (31)$$

then every solution of Equation (1) is either oscillatory or tends to zero as $n \rightarrow \infty$.

Proof. Let $\{x(n)\}$ be a non-oscillatory solution of equation (1). Without loss of generality we may assume that $x(n) > 0$. If $P = \infty$, then by Lemma 3.5, $z(n)$ does not have property (i). That is, $z(n)$ satisfies property (ii). Therefore, from Lemma 3.2, we have $\lim_{n \rightarrow \infty} x(n) = 0$.

Now, Let $P < \infty$. By Lemma 3.1, we have that $z(n)$ has either property (i) or property (ii). If $z(n)$ has the property (ii), from Lemma 2.2, we obtain $\lim_{n \rightarrow \infty} x(n) = 0$.

Next, we assume that $z(n)$ holds property (i). Let ω and l be defined by (15) and (16), respectively. Then from Lemma 3.5, we have $P \leq l - l^{\frac{\beta+1}{\beta}}$. Using inequality (29) with $u = l$ and $A = B = 1$, we get that

$$P \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}},$$

which contradicts (31). This completes the proof. \square

Example 3.7. Consider the third-order neutral delay difference equation

$$\Delta_\ell \left(\frac{1}{n} \Delta_\ell \left(\frac{1}{n} \Delta_\ell \left(x(n) + \frac{1}{2} x(n-2\ell) \right) \right) \right) + \frac{3(4n^3 + 10n^2\ell + 7n\ell^2 + 2\ell^3)}{n^2(n+\ell)^2(n+2\ell)} x(n-2\ell) = 0. \quad (32)$$

We note that $\beta = 1$ and $f(x) = x$. Hence, it is easy to see that (10) and (17) hold and by Theorem 3.6, we see that every solution of equation (32) is either oscillatory or converges to zero as $n \rightarrow \infty$. In fact, $\{x(n)\} = \left\{ (-1)^{\left[\frac{n}{\ell}\right]} \right\}$ is one such solution of equation (32).

Example 3.8. Consider the third-order neutral delay difference equation

$$\Delta_\ell \left(n \Delta_\ell \left(n(n+\ell) \Delta_\ell \left(x(n) + \frac{(n-\ell)^2}{n^2} x(n-\ell) \right)^3 \right) \right) + 8\ell^7(n-2\ell)^2 \left(\frac{n(2n+\ell)^3}{n^5(n+\ell)^5} - \frac{(2n+3\ell)^3(2n+\ell)}{(n+\ell)^5(n+2\ell)^5} + \frac{(n+\ell)(2n+5\ell)^3}{(n+2\ell)^5(n+3\ell)^5} \right) x(n-2\ell) = 0. \quad (33)$$

We note that $\beta = 3$ and $f(x) = x$. Hence, it is easy to see that (10) and (17) hold and by Theorem 3.6, we see that every solution of equation (42) is either oscillatory or converges to zero as $n \rightarrow \infty$. In fact, $\{x(n)\} = \left\{ \frac{\ell^2}{n^2} \right\}$ is one such solution of equation (42).

Theorem 3.9. Assume that (10) and (17) hold. If

$$P + Q > 1, \quad (34)$$

then every solution of Equation (1) is either oscillatory or tends to zero as $n \rightarrow \infty$.

Proof. Let $x(n)$ be a non-oscillatory solution of equation (1). Let us assume that $x(n) > 0$. If $P = \infty$ or $Q = \infty$, then by Lemma 3.5, $z(n)$ does not have property (i). That is, $z(n)$ satisfies property (ii). Then from Lemma 3.2, we get $\lim_{n \rightarrow \infty} x(n) = 0$.

Next, Let $P < \infty$ and $Q < \infty$. By Lemma 3.1, we have $z(n)$ satisfies property (i) or property (ii). If $z(n)$ satisfies property (ii), from Lemma 3.2, we obtain $\lim_{n \rightarrow \infty} x(n) = 0$. If for $z(n)$ property (i) holds, then continuing as above and by lemma 3.2 we obtain $\lim_{n \rightarrow \infty} x(n) = 0$. Now, we assume that $z(n)$ satisfies property (i). Then from Lemma 3.5, we have $P + Q \leq 1$ which contradicts (31) and hence this completes the proof. \square

Example 3.10. Consider the third-order neutral delay difference equation

$$\Delta_\ell \left(n \Delta_\ell \left(\Delta_\ell \left(x(n) + \frac{n-\ell}{2n} x(n-\ell) \right) \right)^3 \right) + \frac{27\ell^7(8n^2 + 27n\ell + 27\ell^2)(n-\ell)^3}{n^2((n+\ell)(n+2\ell)(n+3\ell))^3} x^3(n-\ell) = 0. \quad (35)$$

It is easy to see that all conditions of Theorem 3.9 are satisfied and hence every solution of equation (37) is either oscillatory or converges to zero as $n \rightarrow \infty$. In fact, $\{x(n)\} = \left\{ \frac{\ell}{n} \right\}$ is one such solution of equation (37).

Example 3.11. Consider the third-order neutral delay difference equation

$$\Delta_\ell \left(n^6 \Delta_\ell \left(n^2 \Delta_\ell \left(x(n) + \frac{1}{n} x(n-\ell) \right) \right)^3 \right) + \ell^2 \left(\frac{n^6 (2\ell^3 + \ell^2(7n+1) + 7\ell n^2 + 2n^3)}{(n-3\ell)(\ell+n)(2\ell+n)} - \frac{(\ell+n)^7 (18\ell^3 + \ell^2(27n+1) + 13\ell n^2 + 2n^3)}{(n-3\ell)(n+\ell)(n+2\ell)(n+3\ell)} \right) x(n-3\ell) = 0. \quad (36)$$

We note that $\beta = 3$ and $f(x) = x$. It is easy to see that all conditions of Theorem 3.9 are not satisfied and hence every solution of equation (36) tends to infinity as $n \rightarrow \infty$.

Corollary 3.12. Assume that (10) and (17) hold. If

$$Q = \limsup_{n \rightarrow \infty} \frac{1}{R_{n_0}(g(n+j+\ell))} \sum_{s=0}^{\frac{n-n_0-j-\ell}{\ell}} \bar{R}_{n_0}^{\beta+1}(g(n_0+j+s\ell+\ell)) \phi(n_0+j+s\ell) > 1,$$

then every solution of Equation (1) is either oscillatory or tends to zero as $n \rightarrow \infty$,

Example 3.13. Consider the third-order neutral delay difference equation

$$\Delta_\ell \left(\Delta_\ell \left(\Delta_\ell \left(x(n) + \frac{1}{3} x(n-\ell) \right) \right)^3 \right) + q(n) f(x(g(n))) = 0, n > \ell. \quad (37)$$

Here $q(n) = \frac{32\ell^7 (16n^4 + 40n^3\ell + 13n^2\ell^2 - 30n\ell + 9\ell^4)}{3n^2 ((n+\ell)(n+2\ell)(n+3\ell))^3}$, $f(x) = x^3$ and $g(n) = n - \ell$. It is easy to see that all conditions of Corollary 3.12 are satisfied and hence every solution of equation (37) is either oscillatory or converges to zero as $n \rightarrow \infty$. In fact, $\{x(n)\} = \{\frac{\ell}{n}\}$ is one such solution of equation (37).

Theorem 3.14. Let (10) holds. Assume that there exists a positive function $\rho(n)$ such that

$$\limsup_{n \rightarrow \infty} \sum_{s=0}^{\frac{n-n_0-j-\ell}{\ell}} (\rho(n_0+j+s\ell) \phi(n_0+j+s\ell) - \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \left(\frac{\Delta_\ell \rho(n_0+j+s\ell)}{\rho(n_0+j+s\ell)} \right)^{\beta+1} \psi(n_0+j+s\ell)) = \infty. \quad (38)$$

where $\psi(n) = \rho(n)(\beta R_{n_0}(g(n+\ell)))^{-\beta}$. Then, every solution of equation (1) is either oscillatory or tends to zero as $n \rightarrow \infty$.

Proof. Let $x(n)$ be a non-oscillatory solution of equation (1). Without loss of generality we may assume that $x(n) > 0$. By Lemma 3.1, we have that $z(n)$ has the property (i) or property (ii). If $z(n)$ possess property (ii), from Lemma 3.2, we obtain $\lim_{n \rightarrow \infty} x(n) = 0$. Next, let $z(n)$ satisfies the property (i). By Lemma 3.3, we have that (11) and (12) hold. Now, we define

$$\omega_1(n) = \rho(n) \frac{E_2(n)}{z^\beta(g(n))}$$

By applying Δ_ℓ and using (11) and (12), we have

$$\Delta_\ell \omega_1(n) \leq -\rho(n)\phi(n) + \frac{\Delta_\ell \rho(n)}{\rho(n+\ell)} \omega_1(n+\ell) - \psi^{-\frac{1}{\beta}}(n) w_1^{\frac{\beta+1}{\beta}}(n+\ell). \quad (39)$$

Using inequality (29) with $u = \omega_1(n+\ell)$, $A = \frac{\Delta_\ell \rho(n)}{\rho(n+\ell)}$ and $B = \psi^{-\frac{1}{\beta}}(n)$, we obtain

$$\frac{\Delta_\ell \rho(n)}{\rho(n+\ell)} \omega_1(n+\ell) - \psi(n)^{-\frac{1}{\beta}} w_1^{\frac{\beta+1}{\beta}}(n+\ell) \leq \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \left(\frac{\Delta_\ell \rho(n)}{\rho(n+\ell)} \right)^{\beta+1} \psi(n) \quad (40)$$

Therefore, we get

$$\Delta_\ell \omega_1(n) \leq -\rho(n)\phi(n) + \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \left(\frac{\Delta_\ell \rho(n)}{\rho(n+\ell)} \right)^{\beta+1} \psi(n). \quad (41)$$

By summing the above inequality from n_0 to $n - \ell$ we have

$$\begin{aligned} \omega_1(n) \leq \omega_1(n_2 + j) - \sum_{s=0}^{n-n_0-j-\ell\ell} (\rho(n_0 + j + s\ell)\phi(n_0 + j + s\ell) \\ - \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \left(\frac{\Delta_\ell \rho(n_0 + j + s\ell)}{\rho(n_0 + j + s\ell + \ell)} \right)^{\beta+1} \psi(n_0 + j + s\ell)) \end{aligned}$$

Taking limit superior as $n \rightarrow \infty$ and using (38), we have $\omega_1(n) \rightarrow -\infty$, which contradicts that $\omega_1(n) > 0$. This completes the proof of the theorem. \square

Example 3.15. Consider the third-order neutral delay difference equation

$$\Delta_\ell \left(n \Delta_\ell \left(\frac{1}{n} \Delta_\ell \left(x(n) + \frac{1}{3} x(n-3\ell) \right)^5 \right) \right) + \frac{4n^2 + 10n\ell + 5\ell^2}{(n+\ell)(n+2\ell)} x^5(n-4\ell) = 0. \quad (42)$$

We have $\beta = 5$ and $f(x) = x^5$. It is easy to see that condition (10) holds. Hence, by Theorem 3.6, we see that every solution of equation (42) is either oscillatory or converges to zero as $n \rightarrow \infty$. In fact, $\{x(n)\} = \left\{ (-1)^{\left[\frac{n}{\ell} \right]} \right\}$ is one such solution of equation (32).

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