Linear statistical models, least-squares estimators, and classification analysis to reverse-order laws for generalized inverses of matrix products

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Abstract. Reverse-order laws for generalized inverses of matrix products is a classic object of study in the theory of generalized inverses. One of the well-known reverse-order laws for a matrix product AB is $(AB)^{(i,\dots,j)} = B^{(i,\dots,j)}A^{(i,\dots,j)}$, where $(\cdot)^{(i,\dots,j)}$ denotes an $\{i,\dots,j\}$-generalized inverse of matrix. Because $\{i,\dots,j\}$-generalized inverse of a general matrix is not necessarily unique, the relationships between both sides of the reverse-order law can be divided into four situations for consideration. In this paper, we first introduce a linear mixed model $y = AB\beta + A\gamma + \epsilon$, present two least-squares methodologies to estimate the fixed parameter vector β in the model, and describe the connections between the two least-squares estimators and the reverse-order laws for generalized inverses of the matrix product AB. We then prepare some valued matrix analysis tools, including a general theory on linear or nonlinear matrix identities, a group of expansion formulas for calculating ranks of block matrices, two groups of explicit formulas for calculating the maximum and minimum ranks of $B^{(i,\dots,j)}A^{(i,\dots,j)}$, as well as necessary and sufficient conditions for $B^{(i,\dots,j)}A^{(i,\dots,j)}$ to be invariant with respect to the choice of $A^{(i,\dots,j)}$ and $B^{(i,\dots,j)}$. We then present a unified approach to the 512 set inclusion problems $\{(AB)^{(i,\dots,j)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$ for the eight commonly-used types of generalized inverses of A, B, and AB using the block matrix representation method (BMRM), the matrix equation method (MEM), and the matrix rank method (MRM), where $\{(\cdot)^{(i,\dots,j)}\}$ denotes the collection of all $\{i,\dots,j\}$-generalized inverses of a matrix.

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1 Introduction

Throughout this paper, we use the following notations: the symbol $\mathbb{C}^{m\times n}$ stands for the collection of all $m \times n$ complex matrices; $r(A)$ and $\mathscr{R}(A)$, and $\mathscr{N}(A)$ stand for the rank, the range (column space), and the null space of a matrix $A \in \mathbb{C}^{m\times n}$, respectively; I_m denotes the identity matrix of order m; $[\,A,\, B\,]$ denotes a row block matrix consisting of A and B. We next introduce the definition and notation of generalized inverses of a matrix. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four Penrose equations

$$\text{(i) } AXA = A, \quad \text{(ii) } XAX = X, \quad \text{(iii) } (AX)^{*} = AX, \quad \text{(iv) } (XA)^{*} = XA. \tag{1.1}$$

A matrix X is called an $\{i,\dots,j\}$-generalized inverse of A, denoted by $A^{(i,\dots,j)}$, if it satisfies the ith,…,jth equations in (1.1). The collection of all $\{i,\dots,j\}$-generalized inverses of A is denoted by $\{A^{(i,\dots,j)}\}$. The generalized inverses of A including the first equation in (1.1) are given by

$$A^{\dagger}, \quad A^{(1,3,4)}, \quad A^{(1,2,4)}, \quad A^{(1,2,3)}, \quad A^{(1,4)}, \quad A^{(1,3)}, \quad A^{(1,2)}, \quad A^{(1)},$$

which are usually called the eight commonly-used types of generalized inverses of A in the literature; see e.g., [3, 4, 21]. In addition, we denote $E_A = I_m - AA^{\dagger}$ and $F_A = I_n - A^{\dagger}A$.

Parametric regression analysis is perhaps the most commonly employed tool of statistical model building in data analysis and inference. When using parametric regression models to fit given data, unknown parameters in the models may not necessarily be assumed to be fixed, instead, to vary at more than one level, or to be given in nested forms. Multilevel statistical modeling is such an analytical technique for estimating regression and related models with data that have a hierarchical structure. Because of the occurrence of parameters at more than one level, inference of multilevel statistical models involve various nested calculations of given matrices and vectors in the models. In fact, many problems in statistics and applications involve analyzing and manipulating this kind of nested structured data and models; see a number of books including [5, 6, 9, 15, 17, 23, 25, 45]. In this paper, we consider a two-level linear regression model defined by

$$\mathscr{M} : \begin{cases} y = A\alpha + \epsilon, \qquad \alpha = B\beta + \gamma, \\ E(\epsilon) = 0, \quad E(\gamma) = 0, \quad Cov(\epsilon) = \sigma^2 I_n, \quad Cov(\gamma) = \tau^2 I_p, \quad Cov(\epsilon, \gamma) = 0, \end{cases} \tag{1.2}$$

where in the first-level model, $y \in \mathbb{R}^{n\times 1}$ is a vector of observable response variables, $A \in \mathbb{R}^{n\times p}$ is a known matrix of arbitrary rank, $\alpha \in \mathbb{R}^{p\times 1}$ is a vector of unobservable random variables, $\epsilon \in \mathbb{R}^{n\times 1}$ is a vector of randomly distributed error terms, σ^2 is an arbitrary positive scaling factor; in the second-level model, $B \in \mathbb{R}^{p\times k}$ is a

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known matrix of arbitrary rank, $\beta \in \mathbb{R}^{k\times 1}$ is a vector of fixed but unknown parameters, $\gamma \in \mathbb{R}^{p\times 1}$ is a vector of unobservable random variables, τ^2 is an arbitrary positive scaling factor. This kind of models have different names in statistical analysis according to their origination, such as, random-effect models, hierarchical models, nested models, etc. Substituting the second equation in (1.2) into the first equation leads to the following linear mixed model

$$\mathscr{N}: \quad y = AB\beta + A\gamma + \epsilon, \quad E(y) = E(AB\beta) = AB\beta, \quad Cov(y) = Cov(A\gamma + \epsilon) = \sigma^2 I_n + \tau^2 AA^T. \tag{1.3}$$

The most common technique used to estimate the unknown parameters of linear regression models is the method of least-squares. In this paper, we shall present two kinds of the ordinary least-squares estimator (OLSE) of unknown parameter spaces under $\mathscr{M}$ and $\mathscr{N}$ using generalized inverses of matrices, and establish some connections between the formulas for calculating the OLSEs and reverse-order laws for generalized inverses of matrix products. To illustrate the importance of these connections, the purpose of this paper is to classify various reverse-order laws for generalized inverses of a matrix product, and to establish necessary and sufficient conditions for these classified reverse-order laws to hold using various matrix analysis tools, including matrix rank optimization methodologies.

Since there are two alternative forms in (1.2) and (1.3), respectively, we are able to adopt different procedures to calculate the OLSEs of the unknown parameter vector β and the mean vector $AB\beta$ in (1.2) and (1.3).

(I) The standard method is to

$$\text{minimize } (y - AB\beta)^T(y - AB\beta) \tag{1.4}$$

in the context of (1.3). It is easy to verify that the norm $(y-AB\beta)^T(y-AB\beta)$ in (1.4) can be decomposed as

$$(\, y - AB\beta\,)^T(\, y - AB\beta\,) = y^T E_{AB} y + (\, P_{AB}y - AB\beta\,)^T(\, P_{AB}y - AB\beta\,),$$

where the two terms on the right-hand side satisfy $y^T E_{AB} y \geqslant 0$ and $(\, P_{AB}y - AB\beta\,)^T(\, P_{AB}y - AB\beta\,) \geqslant 0$. Hence,

$$\min_{\beta\in\mathbb{R}^{p\times 1}} (\, y - AB\beta\,)^T(\, y - AB\beta\,) = y^T E_{AB} y + \min_{\beta\in\mathbb{R}^{p\times 1}} (\, P_{AB}y - AB\beta\,)^T(\, P_{AB}y - AB\beta\,) = y^T E_{AB} y,$$

where the equation $AB\beta = AB(AB)^\dagger y$, which is equivalent to the normal equation $(AB)^T AB\beta = AB^T y$ by pre-multiplying $(AB)^T$, is always consistent; see e.g., [10, p. 114] and [24, pp. 164–165]. Solving the equation gives the well-known OLSEs of β and $AB\beta$ under $\mathscr{N}$:

$$\mathrm{OLSE}_{\mathscr{N}}(\beta) = [\,(AB)^\dagger + F_{AB}U_1\,]y = (AB)^{(1,3)}y, \tag{1.5}$$
$$\mathrm{OLSE}_{\mathscr{N}}(AB\beta) = AB\mathrm{OLSE}_{\mathscr{N}}(\beta) = AB(AB)^{(1,3)}y. \tag{1.6}$$

Furthermore, the expectations and the covariance matrices of $\mathrm{OLSE}_{\mathscr{N}}(\beta)$ and $\mathrm{OLSE}_{\mathscr{N}}(AB\beta)$ are given by

$$E[\mathrm{OLSE}_{\mathscr{N}}(\beta)] = (AB)^\dagger AB\beta, \tag{1.7}$$
$$Cov[\mathrm{OLSE}_{\mathscr{N}}(\beta)] = (AB)^\dagger(\sigma^2 I_n + \tau^2 AA^T)[(AB)^\dagger]^T, \tag{1.8}$$
$$E[\mathrm{OLSE}_{\mathscr{N}}(AB\beta)] = AB\beta, \tag{1.9}$$
$$Cov[\mathrm{OLSE}_{\mathscr{N}}(AB\beta)] = AB(AB)^\dagger(\sigma^2 I_n + \tau^2 AA^T)AB(AB)^\dagger. \tag{1.10}$$

(II) On the other hand, we may first solve the least-squares problem $(\, y - A\alpha\,)^T(\, y - A\alpha\,) = \min$ under (1.2) and obtain the OLSE of α as follows

$$\mathrm{OLSE}_{\mathscr{M}}(\alpha) = (\, A^\dagger + F_A U_1\,)y = A^{(1,3)}y. \tag{1.11}$$

Substituting this formula into the second equation in (1.2) yields

$$A^{(1,3)}y = B\beta + \gamma. \tag{1.12}$$

In this case, solving $\|A^{(1,3)}y - B\beta\|^2 = \min$ under (1.12) leads to

$$\mathrm{OLSE}_{\mathscr{M}}(\beta) = (B^\dagger + F_B U_2)A^{(1,3)}y = B^{(1,3)}A^{(1,3)}y, \tag{1.13}$$
$$\mathrm{OLSE}_{\mathscr{M}}(AB\beta) = ABB^{(1,3)}A^{(1,3)}y, \tag{1.14}$$

where U_2 is an arbitrary matrix. In the case of Moore–Penrose inverses, the expectations and the covariance matrices of these estimators are given by given by

$$E[\mathrm{OLSE}_{\mathscr{M}}(\beta)] = B^\dagger A^\dagger AB\beta, \tag{1.15}$$
$$Cov[\mathrm{OLSE}_{\mathscr{M}}(\beta)] = B^\dagger A^\dagger(\sigma^2 I_n + \tau^2 AA^T)(B^\dagger A^\dagger)^T, \tag{1.16}$$
$$E[\mathrm{OLSE}_{\mathscr{M}}(AB\beta)] = ABB^\dagger A^\dagger AB\beta, \tag{1.17}$$
$$Cov[\mathrm{OLSE}_{\mathscr{M}}(AB\beta)] = ABB^\dagger A^\dagger(\sigma^2 I_n + \tau^2 AA^T)(ABB^\dagger A^\dagger)^T. \tag{1.18}$$

Note from (1.5)–(1.10) and (1.13)–(1.18) that the OLSEs under $\mathscr{N}$ and $\mathscr{M}$ are given in different formulas. Thus they have different performance, and it would be of interest to describe the relationships between the OLSEs under $\mathscr{N}$ and $\mathscr{M}$, in particular, it is necessary to establish identifying conditions for the following four equalities for the OLSEs and their expectations

$$\text{OLSE}_{\mathscr{M}}(\beta) = \text{OLSE}_{\mathscr{N}}(\beta), \tag{1.19}$$
$$\text{OLSE}_{\mathscr{M}}(AB\beta) = \text{OLSE}_{\mathscr{N}}(AB\beta), \tag{1.20}$$
$$E[\text{OLSE}_{\mathscr{M}}(\beta)] = E[\text{OLSE}_{\mathscr{N}}(\beta)], \tag{1.21}$$
$$E[\text{OLSE}_{\mathscr{M}}(AB\beta)] = E[\text{OLSE}_{\mathscr{N}}(AB\beta)] \tag{1.22}$$

to hold, respectively. It is clear that we need to compare the coefficient matrices of y in (1.5) and (1.13), the expectations in (1.7), (1.9), (1.15), and (1.17) in order to examine the four equalities, and obtain the following facts.

Lemma 1.1. *Let the OLSEs of β and $AB\beta$ in $\mathscr{M}$ and $\mathscr{N}$ be as given in* (1.33), (1.34), (1.7), (1.9), (1.15), *and* (1.17), *respectively. Then the following equivalent facts hold*

$$\text{OLSE}_{\mathscr{M}}(\beta) = \text{OLSE}_{\mathscr{N}}(\beta) \Longleftrightarrow (AB)^{\dagger} = B^{\dagger}A^{\dagger}, \tag{1.23}$$
$$\text{OLSE}_{\mathscr{M}}(AB\beta) = \text{OLSE}_{\mathscr{N}}(AB\beta) \Longleftrightarrow AB(AB)^{\dagger} = ABB^{\dagger}A^{\dagger}, \tag{1.24}$$
$$E[\text{OLSE}_{\mathscr{M}}(\beta)] = E[\text{OLSE}_{\mathscr{N}}(\beta)] \Longleftrightarrow (AB)^{\dagger}AB = B^{\dagger}A^{\dagger}AB, \tag{1.25}$$
$$E[\text{OLSE}_{\mathscr{M}}(AB\beta)] = E[\text{OLSE}_{\mathscr{N}}(AB\beta)] \Longleftrightarrow AB = ABB^{\dagger}A^{\dagger}AB. \tag{1.26}$$

The matrix equality in (1.23) is the well-known reverse-order law for the Moore–Penrose generalized inverses of the product AB, while the three matrix equalities in (1.24)–(1.26) are obtained by pre- and post-multiplying the equality in (1.23) with AB, respectively. It is obvious that the four matrix equalities in (1.23)–(1.26) are algebraic issues in matrix mathematics. The equivalent statements in (1.23)–(1.26), however, show that the four matrix equalities in (1.23)–(1.26) can be used to describe and solve some fundamental problems on performance of OLSEs in statistical analysis of regression models, and therefore can be taken as remarkable motivation and valuable explanation for approaching various matrix equalities that involve generalized inverses. in mathematics and applications. It is should be pointed out that the four matrix equalities in (1.23)–(1.26) do not necessarily hold for two general matrices A and B. Thus it is imperative to establish necessary and sufficient conditions for the four matrix equalities in (1.23)–(1.26) to hold in order to interpret and use the four statistical statements in (1.23)–(1.26).

Since generalized inverses of a matrix are not necessarily unique, the four matrix equalities in (1.23)–(1.26) can be regarded as the special cases of the following matrix equalities

$$(AB)^{(i,\dots,j)} = B^{(i,\dots,j)}A^{(i,\dots,j)}, \tag{1.27}$$
$$AB(AB)^{(i,\dots,j)} = ABB^{(i,\dots,j)}A^{(i,\dots,j)}, \tag{1.28}$$
$$(AB)^{(i,\dots,j)}AB = B^{(i,\dots,j)}A^{(i,\dots,j)}AB, \tag{1.29}$$
$$AB = AB(AB)^{(i,\dots,j)}AB, \tag{1.30}$$

where (1.27) is also called a reverse-order law for generalized inverses of the product AB. Since $\{i,\dots,j\}$-generalized inverses of a matrix are not necessarily unique, both sides of (1.27) can be regarded as two matrix sets. In this situation, it is natural to divide the link between both sides of (1.27) into the following four reasonable relationships for the two matrix sets

$$\{(AB)^{(i,\dots,j)}\} \cap \{B^{(i,\dots,j)}A^{(i,\dots,j)}\} \neq \{\emptyset\}, \tag{1.31}$$
$$\{(AB)^{(i,\dots,j)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}, \tag{1.32}$$
$$\{(AB)^{(i,\dots,j)}\} \subseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}, \tag{1.33}$$
$$\{(AB)^{(i,\dots,j)}\} = \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}. \tag{1.34}$$

Because multiplication of matrices is non-commutative, and also because $AA^{(i,\dots,j)} = I_m$, $A^{(i,\dots,j)}A = I_n$, $BB^{(i,\dots,j)} = I_n$, and $B^{(i,\dots,j)}B = I_p$ do not necessarily hold for the two singular matrices A and B, the product $B^{(i,\dots,j)}A^{(i,\dots,j)}$ on the right-hand side of (1.27) does not necessarily satisfy the matrix equations defined for $(AB)^{(i,\dots,j)}$. Thus it is a primary task to derive necessary and sufficient conditions for (1.27) to hold before using it to deal with matrix computations that involve generalized inverses of matrix products. It is obvious that the four types of relation for the two matrix sets include $4\times 8^3 = 2,048$ situations for the eight commonly-used types of generalized inverses of A, B, and AB. So that it is a tremendous work to characterize all these relations. Eq. (1.27) and its extensions for generalized inverses of matrix products have been a classic objects of study in the theory of generalized inverses and applications, and have attracted considerable attention since 1960s. Literature

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on reverse-order product of generalized inverses of matrix products is abundant, while many cases of (1.31)–(1.34) were investigated by use of various matrix analysis tools, including the three principal methodologies: the block matrix representation methodology (BMRM), the matrix equation methodology (MEM), and the matrix rank methodology (MRM).

Recall a basic fact about matrix that $A = 0$ if and only if $r(A) = 0$. Thus, two matrices X and Y of the same size are equal, namely, $X = Y$, if and only if $r(X - Y) = 0$. Furthermore, assume that $\mathcal{S}$ and $\mathcal{T}$ are two sets consisting of matrices of the same size. Then, the following two assertions hold

$$\mathcal{S} \cap \mathcal{T} \neq \emptyset \Leftrightarrow \min_{X\in\mathcal{S},\, Y\in\mathcal{T}} r(X - Y) = 0; \tag{1.35}$$

$$\mathcal{S} \subseteq \mathcal{T} \Leftrightarrow \max_{X\in\mathcal{S}} \min_{Y\in\mathcal{T}} r(X - Y) = 0. \tag{1.36}$$

These equivalent facts provide a highly flexible framework for characterizing equalities of matrices via ranks of matrices. If certain formulas for calculating the rank of $X - Y$ are derived, we can use the formulas to characterize relationships between two matrices A and B and to obtain many valuable results on relationships between two matrix sets. This method, called the matrix rank method, is available for studying various matrix expressions involving generalized inverses of matrices. Perhaps, no methods in linear algebra and matrix theory, as described above, is more elementary than the rank method in characterizing equalities of matrices. Applying (1.35) and (1.36) to (1.31)–(1.33), we see that

$$\{(AB)^{(i,\dots,j)}\} \cap \{B^{(i,\dots,j)}A^{(i,\dots,j)}\} \neq \{\emptyset\} \Leftrightarrow \min_{(AB)^{(i,\dots,j)},\, A^{(i,\dots,j)},\, B^{(i,\dots,j)}} r\Big[(AB)^{(i,\dots,j)} - B^{(i,\dots,j)}A^{(i,\dots,j)}\Big] = 0, \tag{1.37}$$

$$\{(AB)^{(i,\dots,j)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\} \Leftrightarrow \max_{A^{(i,\dots,j)},\, B^{(i,\dots,j)}} \min_{(AB)^{(i,\dots,j)}} r\Big[(AB)^{(i,\dots,j)} - B^{(i,\dots,j)}A^{(i,\dots,j)}\Big] = 0, \tag{1.38}$$

$$\{(AB)^{(i,\dots,j)}\} \subseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\} \Leftrightarrow \max_{(AB)^{(i,\dots,j)}} \min_{A^{(i,\dots,j)},\, B^{(i,\dots,j)}} r\Big[(AB)^{(i,\dots,j)} - B^{(i,\dots,j)}A^{(i,\dots,j)}\Big] = 0. \tag{1.39}$$

Thus, if certain formulas for calculating the max-min ranks are given, we can derive necessary and sufficient conditions for (1.31)–(1.33) to hold from the rank formulas, respectively.A seminal work on applications of rank formulas in the theory of generalized inverses of matrices was presented in [18]. In particular, the present author introduced the matrix rank formulas in the investigation of reverse-order laws for the generalized inverses of products of matrices; see e.g., [28, 29, 32, 33, 37, 38, 41].

As an integral part of the research on the reverse-order law in (1.27), the present author considers in this paper the $8^3 = 512$ set inclusions in (1.32) for the eight commonly-used types of generalized inverses of A, B, and AB. This paper is organized as follows. In Section 2, I introduce a group of results on linear and nonlinear matrix identities that involve one or more separated variable matrices. In Section 3, I introduce some fundamental properties of generalized inverses of matrices and prepare some well-known expansion formulas for calculating the ranks of block matrices. In Section 4, I establish various matrix-valued functions associated with the products $B^{(i,\dots,j)}A^{(i,\dots,j)}$, $MB^{(i,\dots,j)}A^{(i,\dots,j)}M$, $M^*MB^{(i,\dots,j)}A^{(i,\dots,j)}$, and $B^{(i,\dots,j)}A^{(i,\dots,j)}MM^*$, present 126 known analytical formulas for calculating the maximum and minimum ranks of $B^{(i,\dots,j)}A^{(i,\dots,j)}$ subject to the choice of the generalized inverses, and give several groups of conclusions on the invariance properties of the previous matrix products with respect to the choice of the generalized inverses. The main results and their derivations are presented in Sections 5, 6, and 7.

2 The theory of linear or nonlinear matrix identities with variable entries

Matrix equations occupy a central place in the development of matrix theory. Because generalized inverses of a matrix are defined to be solutions of some/all of the four Penrose equations, (1.27) can be regarded as a nonlinear matrix equation of the form $X = YZ$ subject to the restrictions $X \in \{(AB)^{(i,\dots,j)}\}$, $Y \in \{B^{(i,\dots,j)}\}$, and $Z \in \{A^{(i,\dots,j)}\}$. For a general matrix equation $f(X_1, \dots, X_k) = 0$, it is a fundamental and challenging problem to establish necessary and sufficient conditions for the equality to hold for all the variable matrices $X_1, \dots, X_k$ due to the noncommutativity of matrix algebra. For the two simplest matrix equations $AX = 0$ and $AXB = 0$, it is well known that $AX = 0$ holds for all X if and only if $A = 0$; $AXB = 0$ holds for all X if and only if either $A = 0$ or $B = 0$; see e.g., [1]. As matrix equations are given in general forms, the derivations and representations of identifying conditions become increasingly difficult for the matrix equations to always hold for all the variable matrices in them. In a recent paper [14], Jiang and Tian provided a short and readily comprehensible procedure of establishing various types of linear and multilinear matrix identities that involve separated variable matrices using the block matrix representation method (BMRM).

In this section, we present a group of known results on linear and multilinear matrix equations to hold for all the unknown matrices in them.

Lemma 2.1 ([19]). *Let*

$$BX = A \tag{2.1}$$

be a given linear matrix equation, where $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{m\times p}$ *are known matrices, and* $X \in \mathbb{C}^{p\times n}$ *is an unknown matrix. Then*

$$(2.1) \text{ is solvable for } X \Leftrightarrow \mathscr{R}(A) \subseteq \mathscr{R}(B) \Leftrightarrow BB^{\dagger}A = A. \tag{2.2}$$

In this situation, the general solution of (2.1) *can be written in the following parametric form*

$$X = B^{\dagger}A + F_B V, \tag{2.3}$$

where $V \in \mathbb{C}^{p\times n}$ *is arbitrary. In particular,* (2.1) *holds for all matrices* $X \in \mathbb{C}^{p\times n}$ *if and only if both* $A = 0$ *and* $B = 0$*, or equivalently,* $[A,\, B\,] = 0$.

Lemma 2.2 ([19]). *Let*

$$BXC = A \tag{2.4}$$

be a given linear matrix equation, where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$, *and* $C \in \mathbb{C}^{q\times n}$ *are known matrices. Then the following statements are equivalent:*

(a) *Eq.* (2.4) *is solvable for* $X \in \mathbb{C}^{p\times q}$.

(b) *Both* $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ *and* $\mathscr{R}(A^{*}) \subseteq \mathscr{R}(C^{*})$.

(c) $r[\,A,\, B\,] = r(B)$ *and* $r\begin{bmatrix} A \\ C \end{bmatrix} = r(C)$.

(d) $BB^{\dagger}A = A$ *and* $AC^{\dagger}C = A$.

(e) $BB^{\dagger}AC^{\dagger}C = A$.

In this situation, the general solution of (2.4) *can be written in the following parametric form*

$$X = A^{\dagger}CB^{\dagger} + F_A V + W E_B, \tag{2.5}$$

where $V,\, W \in \mathbb{C}^{p\times q}$ *are arbitrary. In particular,* (2.4) *holds for all matrices* $X \in \mathbb{C}^{p\times q}$ *if and only if*

$$\text{either} \quad [A,\, B\,] = 0 \quad \text{or} \quad \begin{bmatrix} A \\ C \end{bmatrix} = 0. \tag{2.6}$$

Lemma 2.3 ([14]). *Let*

$$B_1X_1C_1 + B_2X_2C_2 = A \tag{2.7}$$

be a given linear matrix equation, where $A \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{m\times p_1}$, $B_2 \in \mathbb{C}^{m\times p_2}$, $C_1 \in \mathbb{C}^{q_1\times n}$, *and* $C_2 \in \mathbb{C}^{q_2\times n}$ *are known matrices. Then the following results hold.*

(a) *Eq.* (2.7) *holds for all matrices* $X_1 \in \mathbb{C}^{p_1\times q_1}$ *and* $X_2 \in \mathbb{C}^{p_2\times q_2}$ *if and only if the 5 given matrices satisfy one of the following 4 block matrix equalities:*

$$\text{(i) } [\,A,\, B_1,\, B_2\,] = 0. \quad \text{(ii) } \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} = 0. \quad \text{(iii) } \begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} = 0. \quad \text{(iv) } \begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} = 0.$$

(b) *Under the assumptions that* $\mathscr{R}(B_1) \subseteq \mathscr{R}(B_2)$ *and* $\mathscr{R}(C_1^{*}) \supseteq \mathscr{R}(C_2^{*})$, (2.7) *holds for all matrices* X_1 *and* X_2 *if and only if one of the following 2 block matrix equalities holds:*

$$\text{(i) } [\,A,\, B_2\,] = 0. \quad \text{(ii) } \begin{bmatrix} A \\ C_1 \end{bmatrix} = 0. \quad \text{(iii) } \begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} = 0.$$

Lemma 2.4 ([14]). *Let*

$$(\,A_1 + B_1X_1C_1\,)(\,A_2 + B_2X_2C_2\,) = A \tag{2.8}$$

be a given multilinear matrix equation, where $A \in \mathbb{C}^{m\times n}$, $A_1 \in \mathbb{C}^{m\times s}$, $B_1 \in \mathbb{C}^{m\times p_1}$, $C_1 \in \mathbb{C}^{q_1\times s}$, $A_2 \in \mathbb{C}^{s\times n}$, $B_2 \in \mathbb{C}^{s\times p_2}$, *and* $C_2 \in \mathbb{C}^{q_2\times n}$ *are known matrices. Then the following results hold.*

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(a) *Eq.* (2.8) *holds for all matrices* $X_1 \in \mathbb{C}^{p_1\times q_1}$ *and* $X_2 \in \mathbb{C}^{p_2\times q_2}$ *if and only if one of the following 4 block matrix equalities holds:*

$$\text{(i)}\ \ [\, A_1A_2 - A,\ A_1B_2,\ B_1\,] = 0. \quad \text{(ii)}\ \ \begin{bmatrix} A_1A_2 - A & B_1 \\ C_2 & 0 \end{bmatrix} = 0.$$

$$\text{(iii)}\ \ \begin{bmatrix} A_1A_2 - A & A_1B_2 \\ C_1A_2 & C_1B_2 \end{bmatrix} = 0. \quad \text{(iv)}\ \ \begin{bmatrix} A_1A_2 - A \\ C_1A_2 \\ C_2 \end{bmatrix} = 0.$$

(b) *Under the assumption* $\mathscr{R}(A_1) \subseteq \mathscr{R}(B_1)$ *and* $\mathscr{R}(A_2) \subseteq \mathscr{R}(B_2)$, (2.8) *holds for all* $X_1 \in \mathbb{C}^{p_1\times q_1}$ *and* $X_2 \in \mathbb{C}^{p_2\times q_2}$ *if and only if one of the following 3 block matrix equalities holds:*

$$\text{(i)}\ \ [\, A,\ B_1\,] = 0. \quad \text{(ii)}\ \ \begin{bmatrix} A & A_1B_2 \\ 0 & C_1B_2 \end{bmatrix} = 0. \quad \text{(iii)}\ \ \begin{bmatrix} A_1A_2 - A \\ C_1A_2 \\ C_2 \end{bmatrix} = 0.$$

(c) *Under the assumption* $\mathscr{R}(A_1^*) \subseteq \mathscr{R}(C_1^*)$ *and* $\mathscr{R}(A_2^*) \subseteq \mathscr{R}(C_2^*)$, (2.8) *holds for all* $X_1 \in \mathbb{C}^{p_1\times q_1}$ *and* $X_2 \in \mathbb{C}^{p_2\times q_2}$ *if and only if one of the following 3 block matrix equalities holds:*

$$\text{(i)}\ \ [\, A_1A_2 - A,\ A_1B_2,\ B_1\,] = 0. \quad \text{(ii)}\ \ \begin{bmatrix} A & 0 \\ C_1A_2 & C_1B_2 \end{bmatrix} = 0. \quad \text{(iii)}\ \ \begin{bmatrix} A \\ C_2 \end{bmatrix} = 0.$$

(d) *Under the assumption* $\mathscr{R}(A_1) \subseteq \mathscr{R}(B_1)$ *and* $\mathscr{R}(A_2^*) \subseteq \mathscr{R}(C_2^*)$, (2.8) *holds for all* $X_1 \in \mathbb{C}^{p_1\times q_1}$ *and* $X_2 \in \mathbb{C}^{p_2\times q_2}$ *if and only if one of the following 3 block matrix equalities holds:*

$$\text{(i)}\ \ [\, A,\ B_1\,] = 0. \quad \text{(ii)}\ \ \begin{bmatrix} A \\ C_2 \end{bmatrix} = 0. \quad \text{(iii)}\ \ \begin{bmatrix} A_1A_2 - A & A_1B_2 \\ C_1A_2 & C_1B_2 \end{bmatrix} = 0.$$

(e) *Under the assumption* $\mathscr{R}(A_1^*) \subseteq \mathscr{R}(C_1^*)$ *and* $\mathscr{R}(A_2) \subseteq \mathscr{R}(B_2)$, (2.8) *holds for all* $X_1 \in \mathbb{C}^{p_1\times q_1}$ *and* $X_2 \in \mathbb{C}^{p_2\times q_2}$ *if and only if one of the following 4 block matrix equalities holds:*

$$\text{(i)}\ \ [\, A,\ A_1B_2,\ B_1\,] = 0. \quad \text{(ii)}\ \ \begin{bmatrix} A_1A_2 - A & B_1 \\ C_2 & 0 \end{bmatrix} = 0.$$

$$\text{(iii)}\ \ \begin{bmatrix} A & 0 \\ 0 & C_1B_2 \end{bmatrix} = 0. \quad \text{(iv)}\ \ \begin{bmatrix} A \\ C_1A_2 \\ C_2 \end{bmatrix} = 0.$$

Lemma 2.5 ([14]). *Let*

$$(\, A_1 + B_1X_1C_1\,)(\, A_2 + B_2X_2C_2\,)(\, A_3 + B_3X_3C_3\,) = A \tag{2.9}$$

be a given multilinear matrix equation, where A, A_i, B_i, *and* C_i *are known matrices of appropriate sizes,* $i = 1, 2, 3$. *Then the following results hold.*

(a) *Eq.* (2.9) *holds for all matrices* X_1, X_2, *and* X_3 *if and only if one of the following 8 block matrix equalities holds:*

$$\text{(i)}\ \ [\, A_1A_2A_3 - A,\ A_1A_2B_3,\ A_1B_2,\ B_1\,] = 0. \quad \text{(ii)}\ \ \begin{bmatrix} A_1A_2A_3 - A & A_1A_2B_3 & A_1B_2 \\ C_1A_2A_3 & C_1A_2B_3 & C_1B_2 \end{bmatrix} = 0.$$

$$\text{(iii)}\ \ \begin{bmatrix} A_1A_2A_3 - A & A_1A_2B_3 & B_1 \\ C_2A_3 & C_2B_3 & 0 \end{bmatrix} = 0. \quad \text{(iv)}\ \ \begin{bmatrix} A_1A_2A_3 - A & A_1B_2 & B_1 \\ C_3 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{(v)}\ \ \begin{bmatrix} A_1A_2A_3 - A & A_1A_2B_3 \\ C_1A_2A_3 & C_1A_2B_3 \\ C_2A_3 & C_2B_3 \end{bmatrix} = 0. \quad \text{(vi)}\ \ \begin{bmatrix} A_1A_2A_3 - A & A_1B_2 \\ C_1A_2A_3 & C_1B_2 \\ C_3 & 0 \end{bmatrix} = 0.$$

$$\text{(vii)}\ \ \begin{bmatrix} A_1A_2A_3 - A & B_1 \\ C_2A_3 & 0 \\ C_3 & 0 \end{bmatrix} = 0. \quad \text{(viii)}\ \ \begin{bmatrix} A_1A_2A_3 - A \\ C_1A_2A_3 \\ C_2A_3 \\ C_3 \end{bmatrix} = 0.$$

(b) *Under the assumption* $\mathscr{R}(A_1) \subseteq \mathscr{R}(B_1)$ *and* $\mathscr{R}(A_3) \subseteq \mathscr{R}(B_3)$, (2.9) *holds for all matrices* X_1, X_2, *and* X_3 *if and only if one of the following 5 block matrix equalities holds*

$$\text{(i)}\ \ [\, A,\ B_1\,] = 0. \quad \text{(ii)}\ \ \begin{bmatrix} A & A_1A_2B_3 & A_1B_2 \\ 0 & C_1A_2B_3 & C_1B_2 \end{bmatrix} = 0.$$

$$\text{(iii)}\ \ \begin{bmatrix} A & A_1A_2B_3 \\ 0 & C_1A_2B_3 \\ 0 & C_2B_3 \end{bmatrix} = 0. \quad \text{(iv)}\ \ \begin{bmatrix} A_1A_2A_3 - A & A_1B_2 \\ C_1A_2A_3 & C_1B_2 \\ C_3 & 0 \end{bmatrix} = 0. \quad \text{(v)}\ \ \begin{bmatrix} A_1A_2A_3 - A \\ C_1A_2A_3 \\ C_2A_3 \\ C_3 \end{bmatrix} = 0.$$

(c) *Under the assumption* $\mathscr{R}(A_1^*) \subseteq \mathscr{R}(C_1^*)$ *and* $\mathscr{R}(A_3^*) \subseteq \mathscr{R}(C_3^*)$, (2.9) *holds for all matrices* X_1, X_2, *and* X_3 *if and only if one of the following 5 block matrix equalities holds*

(i) $[\,A_1A_2A_3 - A,\ A_1A_2B_3,\ A_1B_2,\ B_1\,] = 0.$ (ii) $\begin{bmatrix} A & 0 & 0 \\ C_1A_2A_3 & C_1A_2B_3 & C_1B_2 \end{bmatrix} = 0.$

(iii) $\begin{bmatrix} A_1A_2A_3 - A & A_1A_2B_3 & B_1 \\ C_2A_3 & C_2B_3 & 0 \end{bmatrix} = 0.$ (iv) $\begin{bmatrix} A & 0 \\ C_1A_2A_3 & C_1A_2B_3 \\ C_2A_3 & C_2B_3 \end{bmatrix} = 0.$ (v) $\begin{bmatrix} A \\ C_3 \end{bmatrix} = 0.$

(d) *Under the assumption* $\mathscr{R}(A_1) \subseteq \mathscr{R}(B_1)$ *and* $\mathscr{R}(A_3^*) \subseteq \mathscr{R}(C_3^*)$, (2.9) *holds for all matrices* X_1, X_2, *and* X_3 *if and only if one of the following 4 block matrix equalities holds*

(i) $[\,A,\ B_1\,] = 0.$ (ii) $\begin{bmatrix} A \\ C_3 \end{bmatrix} = 0.$

(iii) $\begin{bmatrix} A_1A_2A_3 - A & A_1A_2B_3 & A_1B_2 \\ C_1A_2A_3 & C_1A_2B_3 & C_1B_2 \end{bmatrix} = 0.$ (vi) $\begin{bmatrix} A_1A_2A_3 - A & A_1A_2B_3 \\ C_1A_2A_3 & C_1A_2B_3 \\ C_2A_3 & C_2B_3 \end{bmatrix} = 0.$

(e) *Under the assumption* $\mathscr{R}(A_1^*) \subseteq \mathscr{R}(C_1^*)$ *and* $\mathscr{R}(A_3) \subseteq \mathscr{R}(B_3)$, (2.9) *holds fo all matrices* X_1, X_2, *and* X_3 *if and only if one of the following 8 block matrix equalities holds*

(i) $[\,A,\ A_1A_2B_3,\ A_1B_2,\ B_1\,] = 0.$ (ii) $\begin{bmatrix} A & 0 & 0 \\ 0 & C_1A_2B_3 & C_1B_2 \end{bmatrix} = 0.$

(iii) $\begin{bmatrix} A & A_1A_2B_3 & B_1 \\ 0 & C_2B_3 & 0 \end{bmatrix} = 0.$ (iv) $\begin{bmatrix} A_1A_2A_3 - A & A_1B_2 & B_1 \\ C_3 & 0 & 0 \end{bmatrix} = 0.$

(v) $\begin{bmatrix} A & 0 \\ 0 & C_1A_2B_3 \\ 0 & C_2B_3 \end{bmatrix} = 0.$ (vi) $\begin{bmatrix} A & 0 \\ C_1A_2A_3 & C_1B_2 \\ C_3 & 0 \end{bmatrix} = 0.$

(vii) $\begin{bmatrix} A_1A_2A_3 - A & B_1 \\ C_2A_3 & 0 \\ C_3 & 0 \end{bmatrix} = 0.$ (viii) $\begin{bmatrix} A \\ C_1A_2A_3 \\ C_2A_3 \\ C_3 \end{bmatrix} = 0.$

Lemma 2.6 ([14]). *The matrix equation*

$$(\,A_1 + B_1X_1C_1\,)(A_2 + B_2X_2 + Y_2C_2\,)(\,A_3 + B_3X_3C_3\,) = A \tag{2.10}$$

holds for all matrices X_1, X_2, X_3, *and* Y_2 *if and only if one of the following 8 block matrix equalities holds*:

(i) $[\,A,\ A_1,\ B_1\,] = 0.$ (ii) $\begin{bmatrix} A & A_1 \\ 0 & C_1 \end{bmatrix} = 0.$ (iii) $\begin{bmatrix} A & 0 \\ A_3 & B_3 \end{bmatrix} = 0.$ (iv) $\begin{bmatrix} A \\ A_3 \\ C_3 \end{bmatrix} = 0.$

(v) $\begin{bmatrix} A_1A_2A_3 - A & A_1A_2B_3 & A_1B_2 & B_1 \\ C_2A_3 & C_2B_3 & 0 & 0 \end{bmatrix} = 0.$ (vi) $\begin{bmatrix} A_1A_2A_3 - A & A_1B_2 & B_1 \\ C_2A_3 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix} = 0.$

(vii) $\begin{bmatrix} A_1A_2A_3 - A & A_1A_2B_3 & A_1B_2 \\ C_1A_2A_3 & C_1A_2B_3 & C_1B_2 \\ C_2A_3 & C_2B_3 & 0 \end{bmatrix} = 0.$ (viii) $\begin{bmatrix} A_1A_2A_3 - A & A_1B_2 \\ C_1A_2A_3 & C_1B_2 \\ C_2A_3 & 0 \\ C_3 & 0 \end{bmatrix} = 0.$

Under the assumption $\mathscr{R}(A_1) \subseteq \mathscr{R}(B_1)$ *and* $\mathscr{R}(A_3^*) \subseteq \mathscr{R}(C_3^*)$, (2.10) *holds for all matrices* X_1, X_2, X_3, *and* Y_2 *if and only if one of the following 5 block matrix equalities holds*:

(i) $[\,A,\ B_1\,] = 0.$ (ii) $\begin{bmatrix} A \\ C_3 \end{bmatrix} = 0.$ (iii) $\begin{bmatrix} A & A_1 \\ 0 & C_1 \end{bmatrix} = 0.$ (iv) $\begin{bmatrix} A & 0 \\ A_3 & B_3 \end{bmatrix} = 0.$

(v) $\begin{bmatrix} A_1A_2A_3 - A & A_1A_2B_3 & A_1B_2 \\ C_1A_2A_3 & C_1A_2B_3 & C_1B_2 \\ C_2A_3 & C_2B_3 & 0 \end{bmatrix} = 0.$

Lemma 2.7 ([14]). *Let*

$$(\,A_1 + B_1X_1C_1 + D_1Y_1E_1\,)(A_2 + B_2X_2C_2 + D_2Y_2E_2\,) = A \tag{2.11}$$

be a given multilinear matrix equation, where A, A_i, B_i, C_i, D_i, *and* E_i *are known matrices of appropriate sizes,* $i = 1, 2$. *Then the following results hold.*

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(a) *Eq.* (2.11) *holds for all matrices* X_1, X_2, Y_1, *and* Y_2 *if and only if one of the following 16 block matrix equalities holds*:

$$\text{(i)}\quad [\,A_1A_2 - A,\ A_1B_2,\ A_1D_2,\ B_1,\ D_1\,] = 0. \qquad \text{(ii)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 & A_1D_2 & B_1 \\ E_1A_2 & E_1B_2 & E_1D_2 & 0 \end{bmatrix} = 0.$$

$$\text{(iii)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 & A_1D_2 & D_1 \\ C_1A_2 & C_1B_2 & C_1D_2 & 0 \end{bmatrix} = 0. \qquad \text{(iv)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 & B_1 & D_1 \\ E_2 & 0 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{(v)}\quad \begin{bmatrix} A_1A_2 - A & A_1D_2 & B_1 & D_1 \\ C_2 & 0 & 0 & 0 \end{bmatrix} = 0. \qquad \text{(vi)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 & A_1D_2 \\ C_1A_2 & C_1B_2 & C_1D_2 \\ E_1A_2 & E_1B_2 & E_1D_2 \end{bmatrix} = 0.$$

$$\text{(vii)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 & B_1 \\ E_1A_2 & E_1B_2 & 0 \\ E_2 & 0 & 0 \end{bmatrix} = 0. \qquad \text{(viii)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 & D_1 \\ C_1A_2 & C_1B_2 & 0 \\ E_2 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{(ix)}\quad \begin{bmatrix} A_1A_2 - A & A_1D_2 & B_1 \\ E_1A_2 & E_1D_2 & 0 \\ C_2 & 0 & 0 \end{bmatrix} = 0. \qquad \text{(x)}\quad \begin{bmatrix} A_1A_2 - A & A_1D_2 & D_1 \\ C_1A_2 & C_1D_2 & 0 \\ C_2 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{(xi)}\quad \begin{bmatrix} A_1A_2 - A & B_1 & D_1 \\ C_2 & 0 & 0 \\ E_2 & 0 & 0 \end{bmatrix} = 0. \qquad \text{(xii)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 \\ C_1A_2 & C_1B_2 \\ E_1A_2 & E_1B_2 \\ E_2 & 0 \end{bmatrix} = 0.$$

$$\text{(xiii)}\quad \begin{bmatrix} A_1A_2 - A & A_1D_2 \\ C_1A_2 & C_1D_2 \\ E_1A_2 & E_1D_2 \\ C_2 & 0 \end{bmatrix} = 0. \qquad \text{(xiv)}\quad \begin{bmatrix} A_1A_2 - A & B_1 \\ E_1A_2 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} = 0.$$

$$\text{(xv)}\quad \begin{bmatrix} A_1A_2 - A & D_1 \\ C_1A_2 & 0 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} = 0. \qquad \text{(xvi)}\quad \begin{bmatrix} A_1A_2 - A \\ C_1A_2 \\ E_1A_2 \\ C_2 \\ E_2 \end{bmatrix} = 0.$$

(b) *The matrix equation*

$$(\,A_1 + B_1X_1 + Y_1E_1\,)(A_2 + B_2X_2 + Y_2E_2\,) = A \tag{2.12}$$

holds for all matrices X_1, X_2, Y_1, *and* Y_2 *if and only if one of the following 3 block matrix equalities holds*:

$$\text{(i)}\quad \begin{bmatrix} A & A_1 & B_1 \\ 0 & E_1 & 0 \end{bmatrix} = 0. \quad \text{(ii)}\quad \begin{bmatrix} A & 0 \\ A_2 & B_2 \\ E_2 & 0 \end{bmatrix} = 0. \quad \text{(ii)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 & B_1 \\ E_1A_2 & E_1B_2 & 0 \\ E_2 & 0 & 0 \end{bmatrix} = 0.$$

(c) *The matrix equation*

$$(\,A_1 + B_1X_1C_1\,)(A_2 + B_2X_2C_2 + D_2Y_2E_2\,) = A \tag{2.13}$$

holds for all matrices X_1, X_2, *and* Y_2 *if and only if one of the following 8 block matrix equalities holds:*

$$\text{(i)}\quad [\,A_1A_2 - A,\ A_1B_2,\ A_1D_2,\ B_1\,] = 0. \qquad \text{(ii)}\quad \begin{bmatrix} A_1A_2 - A & A_1D_2 & B_1 \\ C_2 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{(iii)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 & B_1 \\ E_2 & 0 & 0 \end{bmatrix} = 0. \qquad \text{(iv)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 & A_1D_2 \\ C_1A_2 & C_1B_2 & C_1D_2 \end{bmatrix} = 0.$$

$$\text{(v)}\quad \begin{bmatrix} A_1A_2 - A & B_1 \\ C_2 & 0 \\ E_2 & 0 \end{bmatrix} = 0. \qquad \text{(vi)}\quad \begin{bmatrix} A_1A_2 - A & A_1D_2 \\ C_1A_2 & C_1D_2 \\ C_2 & 0 \end{bmatrix} = 0.$$

$$\text{(vii)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 \\ C_1A_2 & C_1B_2 \\ E_2 & 0 \end{bmatrix} = 0. \qquad \text{(viii)}\quad \begin{bmatrix} A_1A_2 - A \\ C_1A_2 \\ C_2 \\ E_2 \end{bmatrix} = 0.$$

(d) *The matrix equation*

$$(\,A_1 + B_1X_1C_1 + D_1Y_1E_1\,)(A_2 + B_2X_2C_2\,) = A \tag{2.14}$$

holds for all matrices X_1, Y_1, and X_2 if and only if one of the following 8 block matrix equalities holds:

$$\text{(i)}\quad [\, A_1A_2 - A,\ A_1B_2,\ B_1,\ D_1 \,] = 0. \qquad \text{(ii)}\quad \begin{bmatrix} A_1A_2 - A & B_1 & D_1 \\ C_2 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{(iii)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 & B_1 \\ E_1A_2 & E_1B_2 & 0 \end{bmatrix} = 0. \qquad \text{(iv)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 & D_1 \\ C_1A_2 & C_1B_2 & 0 \end{bmatrix} = 0.$$

$$\text{(v)}\quad \begin{bmatrix} A_1A_2 - A & B_1 \\ E_1A_2 & 0 \\ C_2 & 0 \end{bmatrix} = 0. \qquad \text{(vi)}\quad \begin{bmatrix} A_1A_2 - A & D_1 \\ C_1A_2 & 0 \\ C_2 & 0 \end{bmatrix} = 0.$$

$$\text{(vii)}\quad \begin{bmatrix} A_1A_2 - A & A_1B_2 \\ C_1A_2 & C_1B_2 \\ E_1A_2 & E_1B_2 \end{bmatrix} = 0. \qquad \text{(viii)}\quad \begin{bmatrix} A_1A_2 - A \\ C_1A_2 \\ E_1A_2 \\ C_2 \end{bmatrix} = 0.$$

Lemma 2.8 ([14]). *Let*

$$(\, A_1 + B_1X_1C_1 \,)(\, A_2 + B_2X_2C_2 \,)(\, A_3 + B_3X_3C_3 \,)(\, A_4 + B_4X_4C_4 \,) = A \tag{2.15}$$

be a given multilinear matrix equation, where A, A_i, B_i, and C_i are known matrices of appropriate sizes, $i = 1, 2, 3, 4$. Then the following results hold.

(a) *Eq.* (2.15) *holds for all matrices X_1, X_2, X_3, and X_4 if and only if one of the following 16 block matrix equalities holds*:

$$\text{(i)}\quad [\, A_1A_2A_3A_4 - A,\ A_1A_2A_3B_4,\ A_1A_2B_3,\ A_1B_2,\ B_1 \,] = 0.$$

$$\text{(ii)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2B_3 & A_1B_2 & B_1 \\ C_4 & 0 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{(iii)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 & A_1B_2 & B_1 \\ C_3A_4 & C_3B_4 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{(iv)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 & A_1A_2B_3 & B_1 \\ C_2A_3A_4 & C_2A_3B_4 & C_2B_3 & 0 \end{bmatrix} = 0.$$

$$\text{(v)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 & A_1A_2B_3 & A_1B_2 \\ C_1A_2A_3A_4 & C_1A_2A_3B_4 & C_1A_2B_3 & C_1B_2 \end{bmatrix} = 0.$$

$$\text{(vi)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1B_2 & B_1 \\ C_3A_4 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{(vii)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2B_3 & B_1 \\ C_2A_3A_4 & C_2B_3 & 0 \\ C_4 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{(viii)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2B_3 & A_1B_2 \\ C_1A_2A_3A_4 & C_1A_2B_3 & C_1B_2 \\ C_4 & 0 & 0 \end{bmatrix} = 0.$$

$$\text{(ix)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 & B_1 \\ C_2A_3A_4 & C_2A_3B_4 & 0 \\ C_3A_4 & C_3B_4 & 0 \end{bmatrix} = 0.$$

$$\begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 & A_1B_2 \\ C_1A_2A_3A_4 & C_1A_2A_3B_4 & C_1B_2 \\ C_3A_4 & C_3B_4 & 0 \end{bmatrix} = 0.$$

$$\text{(xi)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 & A_1A_2B_3 \\ C_1A_2A_3A_4 & C_1A_2A_3B_4 & C_1A_2B_3 \\ C_2A_3A_4 & C_2A_3B_4 & C_2B_3 \end{bmatrix} = 0.$$

$$\text{(xii)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & B_1 \\ C_2A_3A_4 & 0 \\ C_3A_4 & 0 \\ C_4 & 0 \end{bmatrix} = 0.$$

$$\text{(xiii)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1B_2 \\ C_1A_2A_3A_4 & C_1B_2 \\ C_3A_4 & 0 \\ C_4 & 0 \end{bmatrix} = 0.$$

$$\text{(xiv)}\quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2B_3 \\ C_1A_2A_3A_4 & C_1A_2B_3 \\ C_2A_3A_4 & C_2B_3 \\ C_4 & 0 \end{bmatrix} = 0.$$

$$\text{(xv)} \quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 \\ C_1A_2A_3A_4 & C_1A_2A_3B_4 \\ C_2A_3A_4 & C_2A_3B_4 \\ C_3A_4 & C_3B_4 \end{bmatrix} = 0. \qquad \text{(xvi)} \quad \begin{bmatrix} A_1A_2A_3A_4 - A \\ C_1A_2A_3A_4 \\ C_2A_3A_4 \\ C_3A_4 \\ C_4 \end{bmatrix} = 0.$$

(b) *Under the assumptions* $\mathscr{R}(A_1) \subseteq \mathscr{R}(B_1)$ *and* $\mathscr{R}(A_4^*) \subseteq \mathscr{R}(C_4^*)$, (2.15) *holds for all matrices* X_1, X_2, X_3, *and* X_4 *if and only if one of the following 6 block matrix equalities holds*:

$$\text{(i)} \quad [\, A, \; B_1 \,] = 0. \qquad \text{(ii)} \quad \begin{bmatrix} A \\ C_4 \end{bmatrix} = 0.$$

$$\text{(iii)} \quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 & A_1A_2B_3 & A_1B_2 \\ C_1A_2A_3A_4 & C_1A_2A_3B_4 & C_1A_2B_3 & C_1B_2 \end{bmatrix} = 0.$$

$$\text{(iv)} \quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 & A_1B_2 \\ C_1A_2A_3A_4 & C_1A_2A_3B_4 & C_1B_2 \\ C_3A_4 & C_3B_4 & 0 \end{bmatrix} = 0.$$

$$\text{(v)} \quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 & A_1A_2B_3 \\ C_1A_2A_3A_4 & C_1A_2A_3B_4 & C_1A_2B_3 \\ C_2A_3A_4 & C_2A_3B_4 & C_2B_3 \end{bmatrix} = 0.$$

$$\text{(vi)} \quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 \\ C_1A_2A_3A_4 & C_1A_2A_3B_4 \\ C_2A_3A_4 & C_2A_3B_4 \\ C_3A_4 & C_3B_4 \end{bmatrix} = 0.$$

(c) *Under the assumption* $\mathscr{R}(A_1) \subseteq \mathscr{R}(B_1)$, $\mathscr{R}(A_2^*) \subseteq \mathscr{R}(C_2^*)$, $\mathscr{R}(A_3) \subseteq \mathscr{R}(B_3)$, *and* $\mathscr{R}(A_4^*) \subseteq \mathscr{R}(C_4^*)$, (2.15) *holds for all matrices* X_1, X_2, X_3, *and* X_4 *if and only if one of the following 6 block matrix equalities holds*:

$$\text{(i)} \quad [\, A, \; B_1 \,] = 0. \quad \text{(ii)} \quad \begin{bmatrix} A \\ C_4 \end{bmatrix} = 0. \quad \text{(iii)} \quad \begin{bmatrix} A & 0 \\ 0 & C_2B_3 \end{bmatrix} = 0. \quad \text{(iv)} \quad \begin{bmatrix} A & A_1A_2B_3 & A_1B_2 \\ 0 & C_1A_2B_3 & C_1B_2 \end{bmatrix} = 0.$$

$$\text{(v)} \quad \begin{bmatrix} A & 0 \\ C_2A_3A_4 & C_2A_3B_4 \\ C_3A_4 & C_3B_4 \end{bmatrix} = 0. \qquad \text{(vi)} \quad \begin{bmatrix} A_1A_2A_3A_4 - A & A_1A_2A_3B_4 & A_1B_2 \\ C_1A_2A_3A_4 & C_1A_2A_3B_4 & C_1B_2 \\ C_3A_4 & C_3B_4 & 0 \end{bmatrix} = 0.$$

Obviously, the block matrix equalities given in Lemmas 2.2–2.8 reveal some essential connections among the products of the given matrices in the linear and multilinear matrix equations, which therefore show many beautiful facets of linear and multilinear matrix identities that involve separated variable matrices. These lemmas in fact provide a set of highly efficient methods to establish and verify many types of matrix equalities, and can be used to characterize the relationships between matrix sets composed by matrix-valued functions.

3 Fundamental properties of generalized inverses of matrices and some matrix rank formulas

In this section, we first describe some fundamental properties of generalized inverses of matrices, and then prepare some well-known expansion formulas for ranks of block matrices, which will serve to clarify the plan of the present paper.

Note from the definitions of generalized inverses of a matrix that they are in fact defined to be (common) solutions of some matrix equations. Thus analytical expressions of generalized inverses of matrices can be written as certain matrix-valued functions with one or more variable matrices. In fact, analytical formulas of generalized inverses of matrices and their functions are important issues and tools in matrix analysis. For instance, the basic formulas in the following lemma can be found, e.g., in [3, 4, 21].

Lemma 3.1. *Let* $A \in \mathbb{C}^{m\times n}$. *Then the following results hold.*

(a) *The general expressions of the seven commonly-used types of generalized inverses* $A^{(1,3,4)}$, $A^{(1,2,4)}$, $A^{(1,2,3)}$,

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$A^{(1,4)}$, $A^{(1,3)}$, $A^{(1,2)}$, and $A^{(1)}$ of A can be written in the following 7 matrix-valued functions

$$A^{(1,3,4)} = A^{\dagger} + F_AUE_A, \tag{3.1}$$
$$A^{(1,2,4)} = A^{\dagger} + A^{\dagger}AUE_A, \tag{3.2}$$
$$A^{(1,2,3)} = A^{\dagger} + F_AUAA^{\dagger}, \tag{3.3}$$
$$A^{(1,4)} = A^{\dagger} + UE_A, \tag{3.4}$$
$$A^{(1,3)} = A^{\dagger} + F_AU, \tag{3.5}$$
$$A^{(1,2)} = (\,A^{\dagger} + F_AU_1\,)A(\,A^{\dagger} + U_2E_A\,), \tag{3.6}$$
$$A^{(1)} = A^{\dagger} + F_AU_1 + U_2E_A, \tag{3.7}$$

where U, U_1, $U_2 \in \mathbb{C}^{n\times m}$ are arbitrary. In particular,

$$A^{(1,3,4)} \text{ is unique} \Leftrightarrow \text{ either } r(A) = m \text{ or } r(A) = n, \tag{3.8}$$
$$A^{(1,2,4)} \text{ is unique} \Leftrightarrow \text{ either } A = 0 \text{ or } r(A) = m, \tag{3.9}$$
$$A^{(1,2,3)} \text{ is unique} \Leftrightarrow \text{ either } A = 0 \text{ or } r(A) = n, \tag{3.10}$$
$$A^{(1,4)} \text{ is unique} \Leftrightarrow r(A) = m, \tag{3.11}$$
$$A^{(1,3)} \text{ is unique} \Leftrightarrow r(A) = n, \tag{3.12}$$
$$A^{(1,2)} \text{ is unique} \Leftrightarrow \text{ either } A = 0 \text{ or } r(A) = m = n, \tag{3.13}$$
$$A^{(1)} \text{ is unique} \Leftrightarrow r(A) = m = n, \text{ namely, } A \text{ is nonsingular.} \tag{3.14}$$

(b) *The following matrix equalities hold*

$$AA^{(1,3,4)} = AA^{(1,2,3)} = AA^{(1,3)} = AA^{\dagger}, \tag{3.15}$$
$$AA^{(1,2,4)} = AA^{(1,4)} = AA^{(1,2)} = AA^{(1)} = AA^{\dagger} + AUE_A, \tag{3.16}$$
$$A^{(1,3,4)}A = A^{(1,2,4)}A = A^{(1,4)}A = A^{\dagger}A, \tag{3.17}$$
$$A^{(1,2,3)}A = A^{(1,3)}A = A^{(1,2)}A = A^{(1)}A = A^{\dagger}A + F_AUA, \tag{3.18}$$

where $U \in \mathbb{C}^{n\times m}$ is arbitrary. In particular,

$$AA^{(1,3,4)} = AA^{(1,2,3)} = AA^{(1,3)} \text{ is always unique}, \tag{3.19}$$
$$A^{(1,3,4)}A = A^{(1,2,4)}A = A^{(1,4)}A \text{ is always unique}, \tag{3.20}$$
$$AA^{(1,2,4)} = AA^{(1,4)} = AA^{(1,2)} = AA^{(1)} \text{ is unique} \Leftrightarrow \text{ either } A = 0 \text{ or } r(A) = m, \tag{3.21}$$
$$A^{(1,2,3)}A = A^{(1,3)}A = A^{(1,2)}A = A^{(1)}A \text{ is unique} \Leftrightarrow \text{ either } A = 0 \text{ or } r(A) = n. \tag{3.22}$$

(c) *The following set inclusions hold*

$$A^{\dagger} \in \{A^{(1,3,4)}\} \subseteq \{A^{(1,4)}\} \subseteq \{A^{(1)}\}, \tag{3.23}$$
$$A^{\dagger} \in \{A^{(1,3,4)}\} \subseteq \{A^{(1,3)}\} \subseteq \{A^{(1)}\}, \tag{3.24}$$
$$A^{\dagger} \in \{A^{(1,2,4)}\} \subseteq \{A^{(1,4)}\} \subseteq \{A^{(1)}\}, \tag{3.25}$$
$$A^{\dagger} \in \{A^{(1,2,4)}\} \subseteq \{A^{(1,2)}\} \subseteq \{A^{(1)}\}, \tag{3.26}$$
$$A^{\dagger} \in \{A^{(1,2,3)}\} \subseteq \{A^{(1,3)}\} \subseteq \{A^{(1)}\}, \tag{3.27}$$
$$A^{\dagger} \in \{A^{(1,2,3)}\} \subseteq \{A^{(1,2)}\} \subseteq \{A^{(1)}\}. \tag{3.28}$$

(d) *The following matrix set equalities hold*

$$\{(A^{(1,3,4)})^*\} = \{(A^*)^{(1,3,4)}\}, \quad \{(A^{(1,2,4)})^*\} = \{(A^*)^{(1,2,3)}\}, \tag{3.29}$$
$$\{(A^{(1,2,3)})^*\} = \{(A^*)^{(1,2,4)}\}, \quad \{(A^{(1,4)})^*\} = \{(A^*)^{(1,3)}\}, \tag{3.30}$$
$$\{(A^{(1,3)})^*\} = \{(A^*)^{(1,4)}\}, \quad \{(A^{(1,2)})^*\} = \{(A^*)^{(1,2)}\}, \tag{3.31}$$
$$\{(A^{(1)})^*\} = \{(A^*)^{(1)}\}. \tag{3.32}$$

The fact in the following lemma is obvious.

Lemma 3.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be two matrix sets consisting of matrices of the same size, and let P and Q be two matrices of appropriate sizes. Then*

$$\mathcal{S} \supseteq \mathcal{T} \Rightarrow P\mathcal{S}Q \supseteq P\mathcal{T}Q. \tag{3.33}$$

It has been realized in the past several decades that the rank of matrix is a fundamental and valued tool for discovering and investigating the relationships between two matrix expressions. This tool is based on establishing various algebraic expansion formulas for calculating ranks of matrix expressions. So that it is also called the matrix rank method (MRM) for convenience of statement. The MRM has essential applications in the theory of generalized inverses of matrices.

In order to establish and simplify various matrix equalities, we need to use a family of rank and dimensional formulas for matrices and their ranges. The following two lemmas on ranks, dimensions, and ranges are well known in linear algebra.

Lemma 3.3. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$. Then*

$$r[\,A,\ B\,] = r(A) + r(B) - \dim[\mathscr{R}(A) \cap \mathscr{R}(B)]. \tag{3.34}$$

Lemma 3.4. *Let $P_1 \in \mathbb{C}^{m\times p_1}$, $P_2 \in \mathbb{C}^{m\times p_2}$, $Q_1 \in \mathbb{C}^{m\times q_1}$, and $Q_2 \in \mathbb{C}^{m\times q_2}$. Then*

$$\text{both } \mathscr{R}(P_1) = \mathscr{R}(Q_1) \text{ and } \mathscr{R}(P_2) = \mathscr{R}(Q_2) \Rightarrow r[\,P_1,\ P_2\,] = r[\,Q_1,\ Q_2\,]. \tag{3.35}$$

We also need to use the following formulas for matrices and their generalized inverses.

Lemma 3.5 ([18]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$. Then*

$$r[\,A,\ B\,] = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{3.36}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{3.37}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_BAF_C). \tag{3.38}$$

If $\mathscr{R}(B) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$, then

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(\,D - CA^{\dagger}B\,). \tag{3.39}$$

Furthermore, the following results hold.

(a) $r[\,A,\ B\,] = r(A) \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A) \Leftrightarrow AA^{\dagger}B = B \Leftrightarrow E_AB = 0.$

(b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) \Leftrightarrow \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*) \Leftrightarrow CA^{\dagger}A = C \Leftrightarrow CF_A = 0.$

(c) $r[\,A,\ B\,] = r(A) + r(B) \Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\} \Leftrightarrow \mathscr{R}[(E_AB)^*] = \mathscr{R}(B^*) \Leftrightarrow \mathscr{R}[(E_BA)^*] = \mathscr{R}(A^*).$

(d) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C) \Leftrightarrow \mathscr{R}(A^*) \cap \mathscr{R}(C^*) = \{0\} \Leftrightarrow \mathscr{R}(CF_A) = \mathscr{R}(C) \Leftrightarrow \mathscr{R}(AF_C) = \mathscr{R}(A).$

(e) $r(\,A + B\,) = r(A) + r(B) \Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$ *and* $\mathscr{R}(A^*) \cap \mathscr{R}(B^*) = \{0\}$ *for* $A,\ B \in \mathbb{C}^{m\times n}$.

Lemma 3.6 ([28, 29]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$. Then*

$$r(\,D - CA^{\dagger}B\,) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A). \tag{3.40}$$

In particular,

$$r(\,D - CAA^{\dagger}B\,) = r\begin{bmatrix} A^*A & A^*B \\ CA & D \end{bmatrix} - r(A), \tag{3.41}$$

$$r(\,D - CA^{\dagger}AB\,) = r\begin{bmatrix} AA^* & AB \\ CA^* & D \end{bmatrix} - r(A), \tag{3.42}$$

$$r(\,A^* - A^{\dagger}\,) = r(\,AA^*A - A\,). \tag{3.43}$$

The two formulas in following lemma are best known in elementary linear algebra.

Lemma 3.7. *Let $A \in \mathbb{C}^{m\times m}$. Then*

$$r(\,A - A^2\,) = r(\,I_m - A\,) + r(A) - m, \tag{3.44}$$

$$r(\,A - A^3\,) = r(\,I_m + A\,) + r(\,I_m - A\,) + r(A) - 2m. \tag{3.45}$$

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Lemma 3.8 ([30]). *Let $A \in \mathbb{C}^{m\times n}$ and assume that $X_1,\, X_2 \in \{A^{(2)}\}$. Then*

$$r(\, X_1 - X_2\,) = r\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + r[\, X_1,\, X_2\,] - r(X_1) - r(X_2). \tag{3.46}$$

Lemma 3.9 ([43]). *Let $P,\, Q \in \mathbb{C}^{m\times m}$ be two orthogonal projectors. Then*

$$r(\, P + Q\,) = r[\, P,\, Q\,], \tag{3.47}$$
$$r(\, P - Q\,) = 2r[\, P,\, Q\,] - r(P) - r(Q), \tag{3.48}$$
$$r(\, PQ - QP\,) = 2r[\, P,\, Q\,] + 2r(PQ) - 2r(P) - 2r(Q). \tag{3.49}$$

Lemma 3.10 ([43]). *Let $A \in \mathbb{C}^{m\times n}$ be given, and let $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be two idempotent matrices. Then the difference $PA - AQ$ satisfies the two rank equalities*

$$r(\, PA - AQ\,) = r\begin{bmatrix} PA \\ Q \end{bmatrix} + r[\, AQ,\, P\,] - r(P) - r(Q) = r(\, PA - PAQ\,) + r(\, PAQ - AQ\,), \tag{3.50}$$

$$r(\, A - PAQ\,) = r\begin{bmatrix} A & AQ & P \\ PA & 0 & 0 \\ Q & 0 & 0 \end{bmatrix} - r(P) - r(Q) = r\begin{bmatrix} (\,I - P\,)A(\,I - Q\,) & (\,I - P\,)AQ \\ PA(\,I - Q\,) & 0 \end{bmatrix}. \tag{3.51}$$

Lemma 3.11 ([31, 40]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, and $C \in \mathbb{C}^{l\times n}$. Then*

$$\max_{X\in\mathbb{C}^{k\times l}} r(\, A - BXC\,) = \min\left\{ r[\, A,\, B\,],\;\; r\begin{bmatrix} A \\ C \end{bmatrix} \right\}, \tag{3.52}$$

$$\min_{X\in\mathbb{C}^{k\times l}} r(\, A - BXC\,) = r[\, A,\, B\,] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{3.53}$$

Lemma 3.12. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and denote*

$$t_1 = \min\{r(A) + r(B),\;\; n\},\;\; t_2 = r[\, A^*,\, B\,],\;\; t_3 = r[\, A^*AB,\, B\,],$$
$$t_4 = r[\, AA^*AB,\, AB\,],\qquad t_5 = r(A) + r[\, A^*AB,\, B\,] - r[\, A^*,\, B\,],$$
$$t_6 = r(AB),\;\; t_7 = r(A) + r(B) - r[\, A^*,\, B\,],\;\; t_8 = \max\{0,\;\; r(A) + r(B) - n\}.$$

Then the following inequalities

$$t_1 \geq t_2 \geq t_3 \geq t_4 \geq t_5 \geq t_6 \geq t_7 \geq t_8 \tag{3.54}$$

hold.

Proof. The first inequality in (3.54) follows from the two well-known inequalities $r[\, A^*,\, B\,] \leq r(A) + r(B)$ and $r[\, A^*,\, B\,] \leq n$. The second inequality in (3.54) follows directly the matrix product $[\, A^*AB,\, B\,] = [\, A^*,\, B\,]\begin{bmatrix} AB & 0 \\ 0 & I_p \end{bmatrix}$. The third inequality in (3.54) follows directly the matrix product $[\, AA^*AB,\, AB\,] = A[\, A^*AB,\, B\,]$. Furthermore, rewrite $[\, AA^*AB,\, AB\,]$ as a triple matrix product $[\, AA^*AB,\, AB\,] = A[\, A^*A,\, B\,]\begin{bmatrix} B & 0 \\ 0 & I_p \end{bmatrix}$, and applying the well-known Frobenius' inequality $r(XYZ) \geq r(XY) + r(YZ) - r(Y)$ to this triple product yields

$$r[\, AA^*AB,\, AB\,] \geq r(A[\, A^*A,\, B\,]) + r\left([\, A^*A,\, B\,]\begin{bmatrix} B & 0 \\ 0 & I_p \end{bmatrix}\right) - r[\, A^*A,\, B\,] = r(A) + r[\, A^*AB,\, B\,] - r[\, A^*,\, B\,],$$

establishing the fourth inequality in (3.54). Applying the Frobenius' inequality to $[\, A^*AB,\, B\,] = [\, A^*,\, I_n\,]\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & I_p \end{bmatrix}$ yields

$$r[\, A^*AB,\, B\,] \geq r\left([\, A^*,\, I_n\,]\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) + r\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & I_p \end{bmatrix}\right) - r\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = r[\, A^*,\, B\,] + r(AB) - r(A),$$

establishing the fifth inequality in (3.54). Applying (3.36) and inequality $r(X - Y) \geq r(X) - r(Y)$ to $[\, A^*,\, B\,]$ yields

$$r[\, A^*,\, B\,] = r(A^*) + r(B - A^{\dagger}AB) \geq r(A) + r(B) - r(A^{\dagger}AB) = r(A) + r(B) - r(AB),$$

thus establishing the sixth inequality in (3.54). The last inequality in (3.54) is equivalent to the first inequality. □

Lemma 3.13. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and denote*

$$M = \begin{bmatrix} AB(AB)^*AB & ABB^*B \\ AA^*AB & AB \end{bmatrix} = \begin{bmatrix} ABB^* \\ A \end{bmatrix} [\, A^*AB,\, B\,],$$

$$s_1 = r\begin{bmatrix} ABB^* \\ A \end{bmatrix} + r[\, A^*AB,\, B\,] - r[A^*,\, B\,],$$

$$s_2 = r[A^*,\, B\,] + 2r(AB) - r(A) - r(B),$$

$$s_3 = r\begin{bmatrix} ABB^*B \\ AB \end{bmatrix} + r[\, AA^*AB,\, AB\,] - r(AB),$$

$$s_4 = r\begin{bmatrix} ABB^* \\ A \end{bmatrix} + r[\, A^*AB,\, B\,] + r(A) + r(B) - 2r[\, A^*,\, B\,] - r(AB).$$

Then

$$r(M) \geq s_1 \geq s_2 \geq r(AB), \tag{3.55}$$

$$r(M) \geq s_3 \geq s_4 \geq r(AB). \tag{3.56}$$

Proof. Follows from Lemma 3.12. □

4 Fundamental properties of the 63 products $B^{(i,\dots,j)}A^{(i,\dots,j)}$

We begin with a group of obvious results and facts on the set inclusions in (1.32).

Lemma 4.1. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$, *and denote* $M = AB$. *Then the following results hold.*

(a) *The following 5 statements are equivalent:*

(i) $\{M^{(1)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(ii) $\{MM^{(1)}\} = \{MM^{(1,2)}\} = \{MM^{(1,4)}\} = \{MM^{(1,2,4)}\} \supseteq \{MB^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(iii) $\{M^{(1)}M\} = \{M^{(1,2)}M\} = \{M^{(1,3)}M\} = \{M^{(1,2,3)}M\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}M\}$.

(iv) $\{BM^{(1)}A\} \supseteq \{BB^{(i,\dots,j)}A^{(i,\dots,j)}A\}$.

(v) $MB^{(i,\dots,j)}A^{(i,\dots,j)}M \equiv M$ *for all* $A^{(i,\dots,j)}$ *and* $B^{(i,\dots,j)}$.

(b) *The following 3 statements are equivalent:*

(i) $\{M^{(1,2)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(ii) $\{M^{(1)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$ *and* $r(B^{(i,\dots,j)}A^{(i,\dots,j)}) = r(M)$ *for all* $A^{(i,\dots,j)}$ *and* $B^{(i,\dots,j)}$.

(iii) $MB^{(i,\dots,j)}A^{(i,\dots,j)}M \equiv M$ *and* $r(B^{(i,\dots,j)}A^{(i,\dots,j)}) = r(M)$ *for all* $A^{(i,\dots,j)}$ *and* $B^{(i,\dots,j)}$.

(c) *The following 2 statements are equivalent:*

(i) $\{M^{(1,3)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(ii) $M^*MB^{(i,\dots,j)}A^{(i,\dots,j)} \equiv M^*$ *for all* $A^{(i,\dots,j)}$ *and* $B^{(i,\dots,j)}$.

(d) *The following 2 statements are equivalent:*

(i) $\{M^{(1,4)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(ii) $B^{(i,\dots,j)}A^{(i,\dots,j)}MM^* \equiv M^*$ *for all* $A^{(i,\dots,j)}$ *and* $B^{(i,\dots,j)}$.

(e) *The following 3 statements are equivalent:*

(i) $\{M^{(1,2,3)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(ii) $\{M^{(1,2)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$ *and* $\{M^{(1,3)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(iii) $M^*MB^{(i,\dots,j)}A^{(i,\dots,j)} \equiv M^*$ *and* $r(B^{(i,\dots,j)}A^{(i,\dots,j)}) = r(M)$ *for all* $A^{(i,\dots,j)}$ *and* $B^{(i,\dots,j)}$.

(f) *The following 3 statements are equivalent:*

(i) $\{M^{(1,2,4)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(ii) $\{M^{(1,2)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(iii) $B^{(i,\dots,j)}A^{(i,\dots,j)}MM^* \equiv M^*$ *and* $r(B^{(i,\dots,j)}A^{(i,\dots,j)}) = r(M)$ *for all* $A^{(i,\dots,j)}$ *and* $B^{(i,\dots,j)}$.

(g) *The following 3 statements are equivalent:*

(i) $\{M^{(1,3,4)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(ii) *Both* $\{M^{(1,3)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(iii) *Both* $M^*MB^{(i,\dots,j)}A^{(i,\dots,j)} \equiv M^*$ *and* $B^{(i,\dots,j)}A^{(i,\dots,j)}MM^* \equiv M^*$ *for all* $A^{(i,\dots,j)}$ *and* $B^{(i,\dots,j)}$.

(h) *The following 3 statements are equivalent:*

(i) *Both* $M^\dagger = B^{(i,\dots,j)}A^{(i,\dots,j)}$ *holds for all* $A^{(i,\dots,j)}$ *and* $B^{(i,\dots,j)}$.

(ii) *Both* $\{M^{(1,2,3)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$ *and* $\{M^{(1,2,4)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\}$.

(iii) $B^{(i,\dots,j)}A^{(i,\dots,j)}$ *does not change with respect to* $B^{(i,\dots,j)}A^{(i,\dots,j)}$ *and* $M^\dagger = B^\dagger A^\dagger$.

Proof. The equivalence of (i) and (v) in (a) follows directly from the definition of $\{1\}$-generalized inverse of AB. Pre- and post-multiplying both matrix sets in (i) with M respectively lead to the two matrix set inclusions in (ii) and (iii); pre- and post-multiplying both matrix sets in (i) with B and A respectively lead to the two matrix set inclusion in (iv). Furthermore, post-multiplying both matrix sets in (ii) with M, pre-multiplying both matrix sets in (iii) with M, and pre- and post-multiplying both matrix sets in (iv) with A and B leads to the assertion in (v).

It is easy to verify by elementary block matrix operations that

$$r\begin{bmatrix} M & MX \\ XM & X \end{bmatrix} = r\begin{bmatrix} M & 0 \\ 0 & X - MXM \end{bmatrix} = r(\, M - MXM\,) + r(X), \tag{4.1}$$

$$r\begin{bmatrix} M & MX \\ XM & X \end{bmatrix} = r\begin{bmatrix} M - XMX & 0 \\ 0 & X \end{bmatrix} = r(\, X - XMX\,) + r(M). \tag{4.2}$$

Combining (4.1) and (4.2) gives the well-known rank formula

$$r(\, M - MXM\,) + r(X) = r(\, X - XMX\,) + r(M). \tag{4.3}$$

A direct consequence of (4.3) is

$$\begin{aligned} & MXM = M \text{ and } X = XMX, \text{ i.e. } X \in \{M^{(1,2)}\} \\ & \Leftrightarrow MXM = M \text{ and } r(X) = r(M) \\ & \Leftrightarrow XMX = X \text{ and } r(X) = r(M). \end{aligned} \tag{4.4}$$

Applying (4.4) to (i) in (b) leads to the equivalence of (i) and (ii) in (b). The equivalence of (ii) and (iii) in (b) follows from (i) and (ii) in (a).

The following two facts

$$X \in \{M^{(1,3)}\} \Leftrightarrow MX = MM^\dagger \Leftrightarrow M^*MX = M^*, \tag{4.5}$$

$$X \in \{M^{(1,4)}\} \Leftrightarrow XM = M^\dagger M \Leftrightarrow XMM^* = M^* \tag{4.6}$$

are well known and can be easily verified by the definitions of $\{1,3\}$- and $\{1,4\}$-generalized inverses of a matrix; see [27, Propositions 3.5 and 3.6]. Applying (4.5) and (4.6) to (i) in (c) and (d) leads to the equivalences of (i) and (ii) in (c) and (d).

Results (e)–(h) follow from the definitions of $\{1,2,3\}$-, $\{1,2,4\}$-, $\{1,3,4\}$-generalized inverses and the facts in (b), (c), and (d). □

In order to characterize the set inclusions in (1.32), we first need to know and use some fundamental properties of the products $B^{(i,\dots,j)}A^{(i,\dots,j)}$. Note that there are 64 products

$$B^{(i,\dots,j)}A^{(i,\dots,j)} \tag{4.7}$$

for the eight commonly-used types of generalized inverses $B^{(i,\dots,j)}$ and $A^{(i,\dots,j)}$, respectively. From (3.1)–(3.7), the 63 matrix-valued functions generated from the products in (4.7) except $B^\dagger A^\dagger$ are given by

$$B^\dagger A^{(1,3,4)} = B^\dagger A^\dagger + B^\dagger F_A U E_A, \tag{4.8}$$

$$B^\dagger A^{(1,2,4)} = B^\dagger A^\dagger + B^\dagger A^\dagger A U E_A, \tag{4.9}$$

$$B^\dagger A^{(1,2,3)} = B^\dagger A^\dagger + B^\dagger F_A U A A^\dagger, \tag{4.10}$$

$$B^\dagger A^{(1,4)} = B^\dagger A^\dagger + B^\dagger U E_A, \tag{4.11}$$

$$B^\dagger A^{(1,3)} = B^\dagger A^\dagger + B^\dagger F_A U, \tag{4.12}$$

$$B^\dagger A^{(1,2)} = (\, B^\dagger A^\dagger + B^\dagger F_A U_1\,) A (\, A^\dagger + U_2 E_A\,), \tag{4.13}$$

$$B^{\dagger}A^{(1)} = B^{\dagger}A^{\dagger} + B^{\dagger}F_AU_1 + B^{\dagger}U_2E_A, \tag{4.14}$$
$$B^{(1,3,4)}A^{\dagger} = B^{\dagger}A^{\dagger} + F_BVE_BA^{\dagger}, \tag{4.15}$$
$$B^{(1,3,4)}A^{(1,3,4)} = (B^{\dagger} + F_BVE_B)(A^{\dagger} + F_AUE_A), \tag{4.16}$$
$$B^{(1,3,4)}A^{(1,2,4)} = (B^{\dagger} + F_BVE_B)(A^{\dagger} + A^{\dagger}AUE_A), \tag{4.17}$$
$$B^{(1,3,4)}A^{(1,2,3)} = (B^{\dagger} + F_BVE_B)(A^{\dagger} + F_AUAA^{\dagger}), \tag{4.18}$$
$$B^{(1,3,4)}A^{(1,4)} = (B^{\dagger} + F_BVE_B)(A^{\dagger} + UE_A), \tag{4.19}$$
$$B^{(1,3,4)}A^{(1,3)} = (B^{\dagger} + F_BVE_B)(A^{\dagger} + F_AU), \tag{4.20}$$
$$B^{(1,3,4)}A^{(1,2)} = (B^{\dagger} + F_BVE_B)(A^{\dagger} + F_AU_1)A(A^{\dagger} + U_2E_A), \tag{4.21}$$
$$B^{(1,3,4)}A^{(1)} = (B^{\dagger} + F_BVE_B)(A^{\dagger} + F_AU_1 + U_2E_A), \tag{4.22}$$
$$B^{(1,2,4)}A^{\dagger} = B^{\dagger}A^{\dagger} + B^{\dagger}BVE_BA^{\dagger}, \tag{4.23}$$
$$B^{(1,2,4)}A^{(1,3,4)} = (B^{\dagger} + B^{\dagger}BVE_B)(A^{\dagger} + F_AUE_A), \tag{4.24}$$
$$B^{(1,2,4)}A^{(1,2,4)} = (B^{\dagger} + B^{\dagger}BVE_B)(A^{\dagger} + A^{\dagger}AUE_A), \tag{4.25}$$
$$B^{(1,2,4)}A^{(1,2,3)} = (B^{\dagger} + B^{\dagger}BVE_B)(A^{\dagger} + F_AUAA^{\dagger}), \tag{4.26}$$
$$B^{(1,2,4)}A^{(1,4)} = (B^{\dagger} + B^{\dagger}BVE_B)(A^{\dagger} + UE_A), \tag{4.27}$$
$$B^{(1,2,4)}A^{(1,3)} = (B^{\dagger} + B^{\dagger}BVE_B)(A^{\dagger} + F_AU), \tag{4.28}$$
$$B^{(1,2,4)}A^{(1,2)} = (B^{\dagger} + B^{\dagger}BVE_B)(A^{\dagger} + F_AU_1)A(A^{\dagger} + U_2E_A), \tag{4.29}$$
$$B^{(1,2,4)}A^{(1)} = (B^{\dagger} + B^{\dagger}BVE_B)(A^{\dagger} + F_AU_1 + U_2E_A), \tag{4.30}$$
$$B^{(1,2,3)}A^{\dagger} = B^{\dagger}A^{\dagger} + F_BVBB^{\dagger}A^{\dagger}, \tag{4.31}$$
$$B^{(1,2,3)}A^{(1,3,4)} = (B^{\dagger} + F_BVBB^{\dagger})(A^{\dagger} + F_AUE_A), \tag{4.32}$$
$$B^{(1,2,3)}A^{(1,2,4)} = (B^{\dagger} + F_BVBB^{\dagger})(A^{\dagger} + A^{\dagger}AUE_A), \tag{4.33}$$
$$B^{(1,2,3)}A^{(1,2,3)} = (B^{\dagger} + F_BVBB^{\dagger})(A^{\dagger} + F_AUAA^{\dagger}), \tag{4.34}$$
$$B^{(1,2,3)}A^{(1,4)} = (B^{\dagger} + F_BVBB^{\dagger})(A^{\dagger} + UE_A), \tag{4.35}$$
$$B^{(1,2,3)}A^{(1,3)} = (B^{\dagger} + F_BVBB^{\dagger})(A^{\dagger} + F_AU), \tag{4.36}$$
$$B^{(1,2,3)}A^{(1,2)} = (B^{\dagger} + F_BVBB^{\dagger})(A^{\dagger} + F_AU_1)A(A^{\dagger} + U_2E_A), \tag{4.37}$$
$$B^{(1,2,3)}A^{(1)} = (B^{\dagger} + F_BVBB^{\dagger})(A^{\dagger} + F_AU_1 + U_2E_A), \tag{4.38}$$
$$B^{(1,4)}A^{\dagger} = B^{\dagger}A^{\dagger} + VE_BA^{\dagger}, \tag{4.39}$$
$$B^{(1,4)}A^{(1,3,4)} = (B^{\dagger} + VE_B)(A^{\dagger} + F_AUE_A), \tag{4.40}$$
$$B^{(1,4)}A^{(1,2,4)} = (B^{\dagger} + VE_B)(A^{\dagger} + A^{\dagger}AUE_A), \tag{4.41}$$
$$B^{(1,4)}A^{(1,2,3)} = (B^{\dagger} + VE_B)(A^{\dagger} + F_AUAA^{\dagger}), \tag{4.42}$$
$$B^{(1,4)}A^{(1,4)} = (B^{\dagger} + VE_B)(A^{\dagger} + UE_A), \tag{4.43}$$
$$B^{(1,4)}A^{(1,3)} = (B^{\dagger} + VE_B)(A^{\dagger} + F_AU), \tag{4.44}$$
$$B^{(1,4)}A^{(1,2)} = (B^{\dagger} + VE_B)(A^{\dagger} + F_AU_1)A(A^{\dagger} + U_2E_A), \tag{4.45}$$
$$B^{(1,4)}A^{(1)} = (B^{\dagger} + VE_B)(A^{\dagger} + F_AU_1 + U_2E_A), \tag{4.46}$$
$$B^{(1,3)}A^{\dagger} = B^{\dagger}A^{\dagger} + F_BVA^{\dagger}, \tag{4.47}$$
$$B^{(1,3)}A^{(1,3,4)} = (B^{\dagger} + F_BV)(A^{\dagger} + F_AUE_A), \tag{4.48}$$
$$B^{(1,3)}A^{(1,2,4)} = (B^{\dagger} + F_BV)(A^{\dagger} + A^{\dagger}AUE_A), \tag{4.49}$$
$$B^{(1,3)}A^{(1,2,3)} = (B^{\dagger} + F_BV)(A^{\dagger} + F_AUAA^{\dagger}), \tag{4.50}$$
$$B^{(1,3)}A^{(1,4)} = (B^{\dagger} + F_BV)(A^{\dagger} + UE_A), \tag{4.51}$$
$$B^{(1,3)}A^{(1,3)} = (B^{\dagger} + F_BV)(A^{\dagger} + F_AU), \tag{4.52}$$
$$B^{(1,3)}A^{(1,2)} = (B^{\dagger} + F_BV)(A^{\dagger} + F_AU_1)A(A^{\dagger} + U_2E_A), \tag{4.53}$$
$$B^{(1,3)}A^{(1)} = (B^{\dagger} + F_BV)(A^{\dagger} + F_AU_1 + U_2E_A), \tag{4.54}$$
$$B^{(1,2)}A^{\dagger} = (B^{\dagger} + F_BV_1)B(B^{\dagger}A^{\dagger} + V_2E_BA^{\dagger}), \tag{4.55}$$
$$B^{(1,2)}A^{(1,3,4)} = (B^{\dagger} + F_BV_1)B(B^{\dagger} + V_2E_B)(A^{\dagger} + F_AUE_A), \tag{4.56}$$
$$B^{(1,2)}A^{(1,2,4)} = (B^{\dagger} + F_BV_1)B(B^{\dagger} + V_2E_B)(A^{\dagger} + A^{\dagger}AUE_A), \tag{4.57}$$

$$B^{(1,2)}A^{(1,2,3)} = (\,B^{\dagger} + F_BV_1\,)B(\,B^{\dagger} + V_2E_B\,)(\,A^{\dagger} + F_AUAA^{\dagger}\,), \tag{4.58}$$

$$B^{(1,2)}A^{(1,4)} = (\,B^{\dagger} + F_BV_1\,)B(\,B^{\dagger} + V_2E_B\,)(\,A^{\dagger} + UE_A\,), \tag{4.59}$$

$$B^{(1,2)}A^{(1,3)} = (\,B^{\dagger} + F_BV_1\,)B(\,B^{\dagger} + V_2E_B\,)(\,A^{\dagger} + F_AU\,), \tag{4.60}$$

$$B^{(1,2)}A^{(1,2)} = (\,B^{\dagger} + F_BV_1\,)B(\,B^{\dagger} + V_2E_B\,)(\,A^{\dagger} + F_AU_1\,)A(\,A^{\dagger} + U_2E_A\,), \tag{4.61}$$

$$B^{(1,2)}A^{(1)} = (\,B^{\dagger} + F_BV_1\,)B(\,B^{\dagger} + V_2E_B\,)(\,A^{\dagger} + F_AU_1 + U_2E_A\,), \tag{4.62}$$

$$B^{(1)}A^{\dagger} = B^{\dagger}A^{\dagger} + F_BV_1A^{\dagger} + V_2E_BA^{\dagger}, \tag{4.63}$$

$$B^{(1)}A^{(1,3,4)} = (\,B^{\dagger} + F_BV_1 + V_2E_B\,)(\,A^{\dagger} + F_AUE_A\,), \tag{4.64}$$

$$B^{(1)}A^{(1,2,4)} = (\,B^{\dagger} + F_BV_1 + V_2E_B\,)(\,A^{\dagger} + A^{\dagger}AUE_A\,), \tag{4.65}$$

$$B^{(1)}A^{(1,2,3)} = (\,B^{\dagger} + F_BV_1 + V_2E_B\,)(\,A^{\dagger} + F_AUAA^{\dagger}\,), \tag{4.66}$$

$$B^{(1)}A^{(1,4)} = (\,B^{\dagger} + F_BV_1 + V_2E_B\,)(\,A^{\dagger} + UE_A\,), \tag{4.67}$$

$$B^{(1)}A^{(1,3)} = (\,B^{\dagger} + F_BV_1 + V_2E_B\,)(\,A^{\dagger} + F_AU\,), \tag{4.68}$$

$$B^{(1)}A^{(1,2)} = (\,B^{\dagger} + F_BV_1 + V_2E_B\,)(\,A^{\dagger} + F_AU_1\,)A(\,A^{\dagger} + U_2E_A\,), \tag{4.69}$$

$$B^{(1)}A^{(1)} = (\,B^{\dagger} + F_BV_1 + V_2E_B\,)(\,A^{\dagger} + F_AU_1 + U_2E_A\,), \tag{4.70}$$

where V, V_1, V_2, U, U_1, U_2 are arbitrary matrices of appropriate sizes. The 64 expressions of $MB^{(i,\ldots,j)}A^{(i,\ldots,j)}M$ for the eight commonly-used types of generalized inverses of A and B are divided into the following 4 groups:

$$\begin{aligned}
&MB^{\dagger}A^{(1,3,4)}M = MB^{\dagger}A^{(1,2,4)}M = MB^{\dagger}A^{(1,4)}M = MB^{(1,3,4)}A^{\dagger}M\\
&= MB^{(1,3,4)}A^{(1,3,4)}M = MB^{(1,3,4)}A^{(1,2,4)}M = MB^{(1,3,4)}A^{(1,4)}M\\
&= MB^{(1,2,3)}A^{\dagger}M = MB^{(1,2,3)}A^{(1,3,4)}M = MB^{(1,2,3)}A^{(1,2,4)}M\\
&= MB^{(1,2,3)}A^{(1,4)}M = MB^{(1,3)}A^{\dagger}M = MB^{(1,3)}A^{(1,3,4)}M\\
&= MB^{(1,3)}A^{(1,2,4)}M = MB^{(1,3)}A^{(1,4)}M = MB^{\dagger}A^{\dagger}M,
\end{aligned} \tag{4.71}$$

$$\begin{aligned}
&MB^{\dagger}A^{(1,2,3)}M = MB^{\dagger}A^{(1,3)}M = MB^{\dagger}A^{(1,2)}M = MB^{\dagger}A^{(1)}M\\
&= MB^{(1,3,4)}A^{(1,2,3)}M = MB^{(1,3,4)}A^{(1,3)}M = MB^{(1,3,4)}A^{(1,2)}M\\
&= MB^{(1,3,4)}A^{(1)}M = MB^{(1,2,3)}A^{(1,2,3)}M = MB^{(1,2,3)}A^{(1,3)}M\\
&= MB^{(1,2,3)}A^{(1,2)}M = MB^{(1,2,3)}A^{(1)}M = MB^{(1,3)}A^{(1,2,3)}M\\
&= MB^{(1,3)}A^{(1,3)}M = MB^{(1,3)}A^{(1,2)}M = MB^{(1,3)}A^{(1)}M\\
&= MB^{\dagger}A^{\dagger}M + MB^{\dagger}F_AUM,
\end{aligned} \tag{4.72}$$

$$\begin{aligned}
&MB^{(1,2,4)}A^{\dagger}M = MB^{(1,2,4)}A^{(1,3,4)}M = MB^{(1,2,4)}A^{(1,2,4)}M\\
&= MB^{(1,2,4)}A^{(1,4)}M = MB^{(1,4)}A^{\dagger}M = MB^{(1,4)}A^{(1,3,4)}M\\
&= MB^{(1,4)}A^{(1,2,4)}M = MB^{(1,4)}A^{(1,4)}M = MB^{(1,2)}A^{\dagger}M\\
&= MB^{(1,2)}A^{(1,3,4)}M = MB^{(1,2)}A^{(1,2,4)}M = MB^{(1,2)}A^{(1,4)}M\\
&= MB^{(1)}A^{\dagger}M = MB^{(1)}A^{(1,3,4)}M = MB^{(1)}A^{(1,2,4)}M\\
&= MB^{(1)}A^{(1,4)}M = MB^{\dagger}A^{\dagger}M + MVE_BA^{\dagger}M,
\end{aligned} \tag{4.73}$$

$$\begin{aligned}
&MB^{(1,2,4)}A^{(1,2,3)}M = MB^{(1,2,4)}A^{(1,3)}M = MB^{(1,2,4)}A^{(1,2)}M\\
&= MB^{(1,2,4)}A^{(1)}M = MB^{(1,4)}A^{(1,2,3)}M = MB^{(1,4)}A^{(1,3)}M\\
&= MB^{(1,4)}A^{(1,2)}M = MB^{(1,4)}A^{(1)}M = MB^{(1,2)}A^{(1,2,3)}M\\
&= MB^{(1,2)}A^{(1,3)}M = MB^{(1,2)}A^{(1,2)}M = MB^{(1,2)}A^{(1)}M\\
&= MB^{(1)}A^{(1,2,3)}M = MB^{(1)}A^{(1,3)}M = MB^{(1)}A^{(1,2)}M\\
&= MB^{(1)}A^{(1)}M = (\,MB^{\dagger} + MVE_B\,)(\,A^{\dagger}M + F_AUM\,),
\end{aligned} \tag{4.74}$$

where V and U are arbitrary matrices of appropriate sizes. The 64 products $M^*MB^{(i,\ldots,j)}A^{(i,\ldots,j)}$ for the eight commonly-used types of generalized inverses of A and B are divided into the following groups

$$M^*MB^{(1,3,4)}A^\dagger = M^*MB^{(1,2,3)}A^\dagger = M^*MB^{(1,3)}A^\dagger = M^*MB^\dagger A^\dagger, \tag{4.75}$$

$$M^*MB^\dagger A^{(1,3,4)} = M^*MB^{(1,3,4)}A^{(1,3,4)} = M^*MB^{(1,2,3)}A^{(1,3,4)} = M^*MB^{(1,3)}A^{(1,3,4)} = M^*MB^\dagger A^\dagger + M^*MB^\dagger F_A U E_A, \tag{4.76}$$

$$M^*MB^\dagger A^{(1,2,4)} = M^*MB^{(1,3,4)}A^{(1,2,4)} = M^*MB^{(1,2,3)}A^{(1,2,4)} = M^*MB^{(1,3)}A^{(1,2,4)} = M^*MB^\dagger A^\dagger + M^*MB^\dagger A^\dagger A U E_A, \tag{4.77}$$

$$M^*MB^\dagger A^{(1,2,3)} = M^*MB^{(1,3,4)}A^{(1,2,3)} = M^*MB^{(1,2,3)}A^{(1,2,3)} = M^*MB^{(1,3)}A^{(1,2,3)} = M^*MB^\dagger A^\dagger + M^*MB^\dagger F_A U A A^\dagger, \tag{4.78}$$

$$M^*MB^\dagger A^{(1,4)} = M^*MB^{(1,3,4)}A^{(1,4)} = M^*MB^{(1,2,3)}A^{(1,4)} = M^*MB^{(1,3)}A^{(1,4)} = M^*MB^\dagger A^\dagger + M^*MB^\dagger U E_A, \tag{4.79}$$

$$M^*MB^\dagger A^{(1,3)} = M^*MB^{(1,3,4)}A^{(1,3)} = M^*MB^{(1,2,3)}A^{(1,3)} = M^*MB^{(1,3)}A^{(1,3)} = M^*MB^\dagger A^\dagger + M^*MB^\dagger F_A U, \tag{4.80}$$

$$M^*MB^\dagger A^{(1,2)} = M^*MB^{(1,3,4)}A^{(1,2)} = M^*MB^{(1,2,3)}A^{(1,2)} = M^*MB^{(1,3)}A^{(1,2)} = M^*MB^\dagger(\, A^\dagger + F_A U_1)A(A^\dagger + U_2 E_A\,), \tag{4.81}$$

$$M^*MB^\dagger A^{(1)} = M^*MB^{(1,3,4)}A^{(1)} = M^*MB^{(1,2,3)}A^{(1)} = M^*MB^{(1,3)}A^{(1)} = M^*MB^\dagger(\, A^\dagger + F_A U_1 + U_2 E_A\,) \tag{4.82}$$

$$M^*MB^{(1,2,4)}A^\dagger = M^*MB^{(1,4)}A^\dagger = M^*MB^{(1,2)}A^\dagger = M^*MB^{(1)}A^\dagger = M^*MB^\dagger A^\dagger + M^*MVE_B A^\dagger, \tag{4.83}$$

$$M^*MB^{(1,2,4)}A^{(1,3,4)} = M^*MB^{(1,4)}A^{(1,3,4)} = M^*MB^{(1,2)}A^{(1,3,4)} = M^*MB^{(1)}A^{(1,3,4)} = M^*M(B^\dagger + VE_B)(A^\dagger + F_A U E_A), \tag{4.84}$$

$$M^*MB^{(1,2,4)}A^{(1,2,4)} = M^*MB^{(1,4)}A^{(1,2,4)} = M^*MB^{(1,2)}A^{(1,2,4)} = M^*MB^{(1)}A^{(1,2,4)} = M^*M(B^\dagger + VE_B)(A^\dagger + A^\dagger A U E_A), \tag{4.85}$$

$$M^*MB^{(1,2,4)}A^{(1,2,3)} = M^*MB^{(1,4)}A^{(1,2,3)} = M^*MB^{(1,2)}A^{(1,2,3)} = M^*MB^{(1)}A^{(1,2,3)} = M^*M(B^\dagger + VE_B)(A^\dagger + F_A U A A^\dagger), \tag{4.86}$$

$$M^*MB^{(1,2,4)}A^{(1,4)} = M^*MB^{(1,4)}A^{(1,4)} = M^*MB^{(1,2)}A^{(1,4)} = M^*MB^{(1)}A^{(1,4)} = M^*M(B^\dagger + VE_B)(A^\dagger + U E_A), \tag{4.87}$$

$$M^*MB^{(1,2,4)}A^{(1,3)} = M^*MB^{(1,4)}A^{(1,3)} = M^*MB^{(1,2)}A^{(1,3)} = M^*MB^{(1)}A^{(1,3)} = M^*M(B^\dagger + VE_B)(A^\dagger + F_A U), \tag{4.88}$$

$$M^*MB^{(1,2,4)}A^{(1,2)} = M^*MB^{(1,4)}A^{(1,2)} = M^*MB^{(1,2)}A^{(1,2)} = M^*MB^{(1)}A^{(1,2)} = M^*M(B^\dagger + VE_B)(A^\dagger + F_A U_1)A(A^\dagger + U_2 E_A), \tag{4.89}$$

$$M^*MB^{(1,2,4)}A^{(1)} = M^*MB^{(1,4)}A^{(1)} = M^*MB^{(1,2)}A^{(1)} = M^*MB^{(1)}A^{(1)} = M^*M(B^\dagger + VE_B)(A^\dagger + F_A U_1 + U_2 E_A), \tag{4.90}$$

where V, U, U_1, U_2 are arbitrary matrices of appropriate sizes. The 64 products $B^{(i,\ldots,j)}A^{(i,\ldots,j)}MM^*$ are classified as the following groups:

$$B^\dagger A^{(1,3,4)}MM^* = B^\dagger A^{(1,2,4)}MM^* = B^\dagger A^{(1,4)}MM^* = B^\dagger A^\dagger MM^*, \tag{4.91}$$

$$B^{(1,3,4)}A^\dagger MM^* = B^{(1,3,4)}A^{(1,3,4)}MM^* = B^{(1,3,4)}A^{(1,2,4)}MM^* = B^{(1,3,4)}A^{(1,4)}MM^* = B^\dagger A^\dagger MM^* + F_B V E_B A^\dagger MM^*, \tag{4.92}$$

$$B^{(1,2,4)}A^\dagger MM^* = B^{(1,2,4)}A^{(1,3,4)}MM^* = B^{(1,2,4)}A^{(1,2,4)}MM^* = B^{(1,2,4)}A^{(1,4)}MM^* = = B^\dagger A^\dagger MM^* + B^\dagger B V E_B A^\dagger MM^*, \tag{4.93}$$

$$B^{(1,2,3)}A^\dagger MM^* = B^{(1,2,3)}A^{(1,3,4)}MM^* = B^{(1,2,3)}A^{(1,2,4)}MM^* = B^{(1,2,3)}A^{(1,4)}MM^* = B^\dagger A^\dagger MM^* + F_B V B B^\dagger A^\dagger MM^*, \tag{4.94}$$

$$B^{(1,4)}A^{(1,3,4)}MM^* = B^{(1,4)}A^{(1,2,4)}MM^* = B^{(1,4)}A^{(1,4)}MM^* = B^{(1,4)}A^\dagger MM^* = B^\dagger A^\dagger MM^* + VE_B A^\dagger MM^*, \tag{4.95}$$

$$B^{(1,3)}A^{(1,3,4)}MM^* = B^{(1,3)}A^{(1,2,4)}MM^* = B^{(1,3)}A^{(1,4)}MM^* = B^{(1,3)}A^{\dagger}MM^* = B^{\dagger}A^{\dagger}MM^* + F_BVA^{\dagger}MM^*, \tag{4.96}$$

$$B^{(1,2)}A^{(1,3,4)}MM^* = B^{(1,2)}A^{(1,2,4)}MM^* = B^{(1,2)}A^{(1,4)}MM^* = B^{(1,2)}A^{\dagger}MM^* = (B^{\dagger} + F_BV_1)B(B^{\dagger} + V_2E_B)A^{\dagger}MM^*, \tag{4.97}$$

$$B^{(1)}A^{(1,3,4)}MM^* = B^{(1)}A^{(1,2,4)}MM^* = B^{(1)}A^{(1,4)}MM^* = B^{(1)}A^{\dagger}MM^* = (\,B^{\dagger} + F_BV_1 + V_2E_B\,)A^{\dagger}MM^*, \tag{4.98}$$

$$B^{\dagger}A^{(1,2,3)}MM^* = B^{\dagger}A^{(1,3)}MM^* = B^{\dagger}A^{(1,2)}MM^* = B^{\dagger}A^{(1)}MM^* = B^{\dagger}(A^{\dagger} + F_AU)MM^*, \tag{4.99}$$

$$B^{(1,3,4)}A^{(1,2,3)}MM^* = B^{(1,3,4)}A^{(1,3)}MM^* = B^{(1,3,4)}A^{(1,2)}MM^* = B^{(1,3,4)}A^{(1)}MM^* = (B^{\dagger} + F_BVE_B)(A^{\dagger} + F_AU)MM^*, \tag{4.100}$$

$$B^{(1,2,4)}A^{(1,2,3)}MM^* = B^{(1,2,4)}A^{(1,3)}MM^* = B^{(1,2,4)}A^{(1,2)}MM^* = B^{(1,2,4)}A^{(1)}MM^* = (B^{\dagger} + B^{\dagger}BVE_B)(A^{\dagger} + F_AU)MM^*, \tag{4.101}$$

$$B^{(1,2,3)}A^{(1,2,3)}MM^* = B^{(1,2,3)}A^{(1,3)}MM^* = B^{(1,2,3)}A^{(1,2)}MM^* = B^{(1,2,3)}A^{(1)}MM^* = (B^{\dagger} + F_BVBB^{\dagger})(A^{\dagger} + F_AU)MM^*, \tag{4.102}$$

$$B^{(1,4)}A^{(1,2,3)}MM^* = B^{(1,4)}A^{(1,3)}MM^* = B^{(1,4)}A^{(1,2)}MM^* = B^{(1,4)}A^{(1)}MM^* = (B^{\dagger} + VE_B)(A^{\dagger} + F_AU)MM^*, \tag{4.103}$$

$$B^{(1,3)}A^{(1,2,3)}MM^* = B^{(1,3)}A^{(1,3)}MM^* = B^{(1,2,3)}A^{(1,2)}MM^* = B^{(1,3)}A^{(1)}MM^* = (B^{\dagger} + F_BV)(A^{\dagger} + F_AU)MM^*, \tag{4.104}$$

$$B^{(1,2)}A^{(1,2,3)}MM^* = B^{(1,2)}A^{(1,3)}MM^* = B^{(1,2)}A^{(1,2)}MM^* = B^{(1,2)}A^{(1)}MM^* = (B^{\dagger} + F_BV_1)B(B^{\dagger} + V_2E_B)(A^{\dagger} + F_AU)MM^*, \tag{4.105}$$

$$B^{(1)}A^{(1,2,3)}MM^* = B^{(1)}A^{(1,3)}MM^* = B^{(1)}A^{(1,2)}MM^* = B^{(1)}A^{(1)}MM^* = (B^{\dagger} + F_BV_1 + V_2E_B)(A^{\dagger} + F_AU)MM^*, \tag{4.106}$$

where U, V, V_1, and V_2 are variable matrices of appropriate sizes. We shall use the above expressions in the characterization of the set inclusions in (1.32).

Lemma 4.2 ([29]). *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ be given. Then the product $B^{\dagger}A^{\dagger}$ can be written as*

$$B^{\dagger}A^{\dagger} = -[\,B^*,\,0\,]\begin{bmatrix} 0 & A^*AA^* \\ B^*BB^* & B^*A^* \end{bmatrix}^{\dagger}\begin{bmatrix} A^* \\ 0 \end{bmatrix} \triangleq -PJ^{\dagger}Q, \tag{4.107}$$

where the block matrices P, J, and Q satisfy $r(J) = r(A) + r(B)$, $\mathscr{R}(Q) \subseteq \mathscr{R}(J)$, and $\mathscr{R}(P^*) \subseteq \mathscr{R}(J^*)$.

Note that the rank of the product $B^{(i,\dots,j)}A^{(i,\dots,j)}$ may vary with respect to the choice of $A^{(i,\dots,j)}$ and $B^{(i,\dots,j)}$ (the variable matrices in the analytical expressions of $A^{(i,\dots,j)}$ and $B^{(i,\dots,j)}$). Also note from Lemma 4.1(b), (e), and (f) that the ranks of the 64 products $B^{(i,\dots,j)}A^{(i,\dots,j)}$ are involved in the set inclusions for the $\{1,2\}$-, $\{1,2,3\}$-, and $\{1,2,4\}$-generalized inverses of AB. Thus it is imperative to determine the maximum and minimum ranks of $B^{(i,\dots,j)}A^{(i,\dots,j)}$ with respect to the choice of the generalized inverses. In the past several decades, a great achievement in linear algebra is the sufficient development of the matrix rank theory. Thousands of analytical formulas for calculating (maximum and minimum) ranks of matrix expressions have been established, and numerous consequences and applications of these matrix rank formulas have been obtained. In recent two papers [37,38], the present author provided a comprehensive study of the rank problems of matrix expressions composed a pair of matrices and their generalized inverses, including the following analytical formulas for calculating the maximum and minimum ranks of $B^{(i,\dots,j)}A^{(i,\dots,j)}$ with respect to the choice of $A^{(i,\dots,j)}$ and $B^{(i,\dots,j)}$.

Lemma 4.3 ([37]). *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ be given, and denote*

$$M = AB, \quad N = [\,A^*,\,B\,], \quad t = m + p + r(M) - r(A) - r(B). \tag{4.108}$$

Also use $\alpha(B^{(i,\dots,j)}A^{(i,\dots,j)})$ and $\beta(B^{(i,\dots,j)}A^{(i,\dots,j)})$ to denote the maximum and minimum ranks of the product

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$B^{(i,\ldots,j)}A^{(i,\ldots,j)}$ *with respect to the generalized inverses, respectively. Then*

$$
\begin{aligned}
&\alpha(B^{\dagger}A^{(1,3,4)}) = \min\{\, r(B),\ m + r(M) - r(A)\,\}, && \beta(B^{\dagger}A^{(1,3,4)}) = r(M),\\
&\alpha(B^{\dagger}A^{(1,2,4)}) = r(M), && \beta(B^{\dagger}A^{(1,2,4)}) = r(M),\\
&\alpha(B^{\dagger}A^{(1,2,3)}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{\dagger}A^{(1,2,3)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{\dagger}A^{(1,4)}) = \min\{\, r(B),\ m + r(M) - r(A)\,\}, && \beta(B^{\dagger}A^{(1,4)}) = r(M),\\
&\alpha(B^{\dagger}A^{(1,3)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{\dagger}A^{(1,3)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{\dagger}A^{(1,2)}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{\dagger}A^{(1,2)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{\dagger}A^{(1)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{\dagger}A^{(1)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,3,4)}A^{\dagger}) = \min\{\, r(A),\ p + r(M) - r(B)\,\}, && \beta(B^{(1,3,4)}A^{\dagger}) = r(M),\\
&\alpha(B^{(1,3,4)}A^{(1,3,4)}) = \min\{\, m,\ n,\ p,\ t\,\}, && \beta(B^{(1,3,4)}A^{(1,3,4)}) = r(M),\\
&\alpha(B^{(1,3,4)}A^{(1,2,4)}) = \min\{r(A),\ p + r(M) - r(B)\,\}, && \beta(B^{(1,3,4)}A^{(1,2,4)}) = r(M),\\
&\alpha(B^{(1,3,4)}A^{(1,2,3)}) = \min\{\, p,\ r(A)\,\}, && \beta(B^{(1,3,4)}A^{(1,2,3)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,3,4)}A^{(1,4)}) = \min\{\, m,\ n,\ p,\ t\,\}, && \beta(B^{(1,3,4)}A^{(1,4)}) = r(M),\\
&\alpha(B^{(1,3,4)}A^{(1,3)}) = \min\{\, m,\ n,\ p\,\}, && \beta(B^{(1,3,4)}A^{(1,3)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,3,4)}A^{(1,2)}) = \min\{p,\ r(A)\}, && \beta(B^{(1,3,4)}A^{(1,2)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,3,4)}A^{(1)}) = \min\{\, m,\ n,\ p\,\}, && \beta(B^{(1,3,4)}A^{(1)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2,4)}A^{\dagger}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{(1,2,4)}A^{\dagger}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2,4)}A^{(1,3,4)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{(1,2,4)}A^{(1,3,4)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2,4)}A^{(1,2,4)}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{(1,2,4)}A^{(1,2,4)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2,4)}A^{(1,2,3)}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{(1,2,4)}A^{(1,2,3)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1,2,4)}A^{(1,4)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{(1,2,4)}A^{(1,4)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2,4)}A^{(1,3)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{(1,2,4)}A^{(1,3)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1,2,4)}A^{(1,2)}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{(1,2,4)}A^{(1,2)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1,2,4)}A^{(1)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{(1,2,4)}A^{(1)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1,2,3)}A^{\dagger}) = r(M), && \beta(B^{(1,2,3)}A^{\dagger}) = r(M),\\
&\alpha(B^{(1,2,3)}A^{(1,3,4)}) = \min\{r(B),\ m + r(M) - r(A)\,\}, && \beta(B^{(1,2,3)}A^{(1,3,4)}) = r(M),\\
&\alpha(B^{(1,2,3)}A^{(1,2,4)}) = r(M), && \beta(B^{(1,2,3)}A^{(1,2,4)}) = r(M),\\
&\alpha(B^{(1,2,3)}A^{(1,2,3)}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{(1,2,3)}A^{(1,2,3)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2,3)}A^{(1,4)}) = \min\{\, r(B),\ m + r(M) - r(A)\,\}, && \beta(B^{(1,2,3)}A^{(1,4)}) = r(M),\\
&\alpha(B^{(1,2,3)}A^{(1,3)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{(1,2,3)}A^{(1,3)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2,3)}A^{(1,2)}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{(1,2,3)}A^{(1,2)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2,3)}A^{(1)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{(1,2,3)}A^{(1)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,4)}A^{\dagger}) = \min\{\, p,\ r(A)\,\}, && \beta(B^{(1,4)}A^{\dagger}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,4)}A^{(1,3,4)}) = \min\{\, m,\ n,\ p\,\}, && \beta(B^{(1,4)}A^{(1,3,4)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,4)}A^{(1,2,4)}) = \min\{\, p,\ r(A)\,\}, && \beta(B^{(1,4)}A^{(1,2,4)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,4)}A^{(1,2,3)}) = \min\{\, p,\ r(A)\,\}, && \beta(B^{(1,4)}A^{(1,2,3)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1,4)}A^{(1,4)}) = \min\{\, m,\ n,\ p\,\}, && \beta(B^{(1,4)}A^{(1,4)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,4)}A^{(1,3)}) = \min\{\, m,\ n,\ p\,\}, && \beta(B^{(1,4)}A^{(1,3)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1,4)}A^{(1,2)}) = \min\{\, p,\ r(A)\,\}, && \beta(B^{(1,4)}A^{(1,2)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1,4)}A^{(1)}) = \min\{\, m,\ n,\ p\,\}, && \beta(B^{(1,4)}A^{(1)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1,3)}A^{\dagger}) = \min\{\, r(A),\ p + r(M) - r(B)\,\}, && \beta(B^{(1,3)}A^{\dagger}) = r(M),\\
&\alpha(B^{(1,3)}A^{(1,3,4)}) = \min\{\, m,\ n,\ p,\ t\,\}, && \beta(B^{(1,3)}A^{(1,3,4)}) = r(M),\\
&\alpha(B^{(1,3)}A^{(1,2,4)}) = \min\{\, r(A),\ p + r(M) - r(B)\,\}, && \beta(B^{(1,3)}A^{(1,2,4)}) = r(M),\\
&\alpha(B^{(1,3)}A^{(1,2,3)}) = \min\{\, p,\ r(A)\,\}, && \beta(B^{(1,3)}A^{(1,2,3)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,3)}A^{(1,4)}) = \min\{\, m,\ n,\ p,\ t\,\}, && \beta(B^{(1,3)}A^{(1,4)}) = r(M),\\
&\alpha(B^{(1,3)}A^{(1,3)}) = \min\{\, m,\ n,\ p\,\}, && \beta(B^{(1,3)}A^{(1,3)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,3)}A^{(1,2)}) = \min\{\, p,\ r(A)\,\}, && \beta(B^{(1,3)}A^{(1,2)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,3)}A^{(1)}) = \min\{\, m,\ n,\ p\,\}, && \beta(B^{(1,3)}A^{(1)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2)}A^{\dagger}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{(1,2)}A^{\dagger}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2)}A^{(1,3,4)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{(1,2)}A^{(1,3,4)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2)}A^{(1,2,4)}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{(1,2)}A^{(1,2,4)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2)}A^{(1,2,3)}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{(1,2)}A^{(1,2,3)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1,2)}A^{(1,4)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{(1,2)}A^{(1,4)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1,2)}A^{(1,3)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{(1,2)}A^{(1,3)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1,2)}A^{(1,2)}) = \min\{\, r(A),\ r(B)\,\}, && \beta(B^{(1,2)}A^{(1,2)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1,2)}A^{(1)}) = \min\{\, m,\ r(B)\,\}, && \beta(B^{(1,2)}A^{(1)}) = \max\{\, 0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1)}A^{\dagger}) = \min\{\, p,\ r(A)\,\}, && \beta(B^{(1)}A^{\dagger}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1)}A^{(1,3,4)}) = \min\{\, m,\ n,\ p\,\}, && \beta(B^{(1)}A^{(1,3,4)}) = r(A) + r(B) - r(N),
\end{aligned}
$$

$$\begin{aligned}
&\alpha(B^{(1)}A^{(1,2,4)}) = \min\{\,p,\ r(A)\,\}, && \beta(B^{(1)}A^{(1,2,4)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1)}A^{(1,2,3)}) = \min\{\,p,\ r(A)\,\}, && \beta(B^{(1)}A^{(1,2,3)}) = \max\{\,0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1)}A^{(1,4)}) = \min\{\,m,\ n,\ p\,\}, && \beta(B^{(1)}A^{(1,4)}) = r(A) + r(B) - r(N),\\
&\alpha(B^{(1)}A^{(1,3)}) = \min\{\,m,\ n,\ p\,\}, && \beta(B^{(1)}A^{(1,3)}) = \max\{\,0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1)}A^{(1,2)}) = \min\{\,p,\ r(A)\,\}, && \beta(B^{(1)}A^{(1,2)}) = \max\{\,0,\ r(A) + r(B) - n\,\},\\
&\alpha(B^{(1)}A^{(1)}) = \min\{\,m,\ n,\ p\,\}, && \beta(B^{(1)}A^{(1)}) = \max\{0,\ r(A) + r(B) - n\}.
\end{aligned}$$

Notice that the rank formulas in Lemma 4.3 are all given in simple and analytical forms. Thus we can directly use them to describe algebraic performance of the products of generalized inverses in many situations.

For a matrix expression that involves generalized inverses, one of the most fundamental fact people like to know is concerned with the invariance property of the matrix expression with respect to the choice of the generalized inverses; see e.g., [1, 13, 20, 22] for expositions and some precious results. We have seen from Lemma 1.1(h) that the invariance property of the product $B^{(i,\dots,j)}A^{(i,\dots,j)}$ occurs in the characterization of the reverse-order laws $(AB)^{\dagger} = B^{(i,\dots,j)}A^{(i,\dots,j)}$. It is easy to see that the product $B^{(i,\dots,j)}A^{(i,\dots,j)}$ is invariant with respect to the choice of the eight commonly-used types of generalized inverses $A^{(i,\dots,j)}$ and $B^{(i,\dots,j)}$ if and only if

$$B^{(i,\dots,j)}A^{(i,\dots,j)} = B^{\dagger}A^{\dagger} \tag{4.109}$$

holds for all $A^{(i,\dots,j)}$ and $B^{(i,\dots,j)}$. Substituting the 63 matrix-valued functions in (4.8)–(4.70) into (4.109), respectively, will result in 63 linear or multilinear matrix equations. In this situation, applying Lemmas 2.2–2.8 to the 63 matrix equations, we are able to obtain the following results.

Theorem 4.4 ([39]). *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ be given with $AB \neq 0$. Then the following results hold.*

⟨1⟩ $B^{\dagger}A^{(1,3,4)}$ *is invariant* $\Leftrightarrow$ *either* $r(A) = m$ *or* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨2⟩ $B^{\dagger}A^{(1,2,4)}$ *is invariant* $\Leftrightarrow$ $r(A) = m$.

⟨3⟩ $B^{\dagger}A^{(1,2,3)}$ *is invariant* $\Leftrightarrow$ $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨4⟩ $B^{\dagger}A^{(1,4)}$ *is invariant* $\Leftrightarrow$ $r(A) = m$.

⟨5⟩ $B^{\dagger}A^{(1,3)}$ *is invariant* $\Leftrightarrow$ $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨6⟩ $B^{\dagger}A^{(1,2)}$ *is invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨7⟩ $B^{\dagger}A^{(1)}$ *is invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨8⟩ $B^{(1,3,4)}A^{\dagger}$ *is invariant* $\Leftrightarrow$ *either* $r(B) = p$ *or* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨9⟩ $B^{(1,3,4)}A^{(1,3,4)}$ *is invariant* $\Leftrightarrow$ *one of the 4 conditions:* (i) $r(A) = m$ *and* $r(B) = p$; (ii) $r(AB) = n$; (iii) $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$; (iv) $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨10⟩ $B^{(1,3,4)}A^{(1,2,4)}$ *is invariant* $\Leftrightarrow$ *either* $r(B) = p$ *or* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨11⟩ $B^{(1,3,4)}A^{(1,2,3)}$ *is invariant* $\Leftrightarrow$ *either* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$ *or* $r(AB) = n$.

⟨12⟩ $B^{(1,3,4)}A^{(1,4)}$ *is invariant* $\Leftrightarrow$ *either* $r(A) = m$ *and* $r(B) = p$ *or* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨13⟩ $B^{(1,3,4)}A^{(1,3)}$ *is invariant* $\Leftrightarrow$ *either* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$ *or* $r(AB) = n$.

⟨14⟩ $B^{(1,3,4)}A^{(1,2)}$ *is invariant* $\Leftrightarrow$ *either* $r(AB) = m = n$ *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨15⟩ $B^{(1,3,4)}A^{(1)}$ *is invariant* $\Leftrightarrow$ *either* $r(AB) = m = n$ *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨16⟩ $B^{(1,2,4)}A^{\dagger}$ *is invariant* $\Leftrightarrow$ $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨17⟩ $B^{(1,2,4)}A^{(1,3,4)}$ *is invariant* $\Leftrightarrow$ *either* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$ *or* $r(AB) = n$.

⟨18⟩ $B^{(1,2,4)}A^{(1,2,4)}$ *is invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨19⟩ $B^{(1,2,4)}A^{(1,2,3)}$ *is invariant* $\Leftrightarrow$ $r(AB) = n$.

⟨20⟩ $B^{(1,2,4)}A^{(1,4)}$ *is invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨21⟩ $B^{(1,2,4)}A^{(1,3)}$ *is invariant* $\Leftrightarrow$ $r(AB) = n$.

⟨22⟩ $B^{(1,2,4)}A^{(1,2)}$ *is invariant* $\Leftrightarrow$ $r(AB) = m = n$.

⟨23⟩ $B^{(1,2,4)}A^{(1)}$ *is invariant* $\Leftrightarrow$ $r(AB) = m = n$.

⟨24⟩ $B^{(1,2,3)}A^{\dagger}$ *is invariant* $\Leftrightarrow$ $(B) = p$.

⟨25⟩ $B^{(1,2,3)}A^{(1,3,4)}$ *is invariant* $\Leftrightarrow$ *either* $r(A) = m$ *or* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨26⟩ $B^{(1,2,3)}A^{(1,2,4)}$ *is invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $r(B) = p$.

⟨27⟩ $B^{(1,2,3)}A^{(1,2,3)}$ *is invariant* $\Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨28⟩ $B^{(1,2,3)}A^{(1,4)}$ *is invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $r(B) = p$.

⟨29⟩ $B^{(1,2,3)}A^{(1,3)}$ *is invariant* $\Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨30⟩ $B^{(1,2,3)}A^{(1,2)}$ *is invariant* $\Leftrightarrow$ $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨31⟩ $B^{(1,2,3)}A^{(1)}$ *is invariant* $\Leftrightarrow$ $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨32⟩ $B^{(1,4)}A^{\dagger}$ *is invariant* $\Leftrightarrow$ $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨33⟩ $B^{(1,4)}A^{(1,3,4)}$ *is invariant* $\Leftrightarrow$ *either* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$, *or* $r(AB) = n$.

⟨34⟩ $B^{(1,4)}A^{(1,2,4)}$ *is invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨35⟩ $B^{(1,4)}A^{(1,2,3)}$ *is invariant* $\Leftrightarrow$ $r(AB) = n$.

⟨36⟩ $B^{(1,4)}A^{(1,4)}$ *is invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨37⟩ $B^{(1,4)}A^{(1,3)}$ *is invariant* $\Leftrightarrow$ $r(AB) = n$.

⟨38⟩ $B^{(1,4)}A^{(1,2)}$ *is invariant* $\Leftrightarrow$ $r(AB) = m = n$.

⟨39⟩ $B^{(1,4)}A^{(1)}$ *is invariant* $\Leftrightarrow$ $r(AB) = m = n$.

⟨40⟩ $B^{(1,3)}A^{\dagger}$ *is invariant* $\Leftrightarrow$ $r(B) = p$.

⟨41⟩ $B^{(1,3)}A^{(1,3,4)}$ *is invariant* $\Leftrightarrow$ *either* $r(A) = m$ *and* $r(B) = p$, *or* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨42⟩ $B^{(1,3)}A^{(1,2,4)}$ *is invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $r(B) = p$.

⟨43⟩ $B^{(1,3)}A^{(1,2,3)}$ *is invariant* $\Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨44⟩ $B^{(1,3)}A^{(1,4)}$ *is invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $r(B) = p$.

⟨45⟩ $B^{(1,3)}A^{(1,3)}$ *is invariant* $\Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨46⟩ $B^{(1,3)}A^{(1,2)}$ *is invariant* $\Leftrightarrow$ $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨47⟩ $B^{(1,3)}A^{(1)}$ *is invariant* $\Leftrightarrow$ $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨48⟩ $B^{(1,2)}A^{\dagger}$ *is invariant* $\Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨49⟩ $B^{(1,2)}A^{(1,3,4)}$ *is invariant* $\Leftrightarrow$ *either* $r(AB) = n = p$, *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨50⟩ $B^{(1,2)}A^{(1,2,4)}$ *is invariant* $\Leftrightarrow$ $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨51⟩ $B^{(1,2)}A^{(1,2,3)}$ *is invariant* $\Leftrightarrow$ $r(AB) = n = p$.

⟨52⟩ $B^{(1,2)}A^{(1,4)}$ *is invariant* $\Leftrightarrow$ $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨53⟩ $B^{(1,2)}A^{(1,3)}$ *is invariant* $\Leftrightarrow$ $r(AB) = n = p$.

⟨54⟩ $B^{(1,2)}A^{(1,2)}$ *is invariant* $\Leftrightarrow$ $r(AB) = m = n = p$.

⟨55⟩ $B^{(1,2)}A^{(1)}$ *is invariant* $\Leftrightarrow$ $r(AB) = m = n = p$.

⟨56⟩ $B^{(1)}A^{\dagger}$ *is invariant* $\Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨57⟩ $B^{(1)}A^{(1,3,4)}$ *is invariant* $\Leftrightarrow$ *either* $r(AB) = n = p$ *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨58⟩ $B^{(1)}A^{(1,2,4)}$ *is invariant* $\Leftrightarrow$ $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨59⟩ $B^{(1)}A^{(1,2,3)}$ *is invariant* $\Leftrightarrow$ $r(AB) = n = p$.

⟨60⟩ $B^{(1)}A^{(1,4)}$ *is invariant* $\Leftrightarrow$ $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨61⟩ $B^{(1)}A^{(1,3)}$ *is invariant* $\Leftrightarrow$ $r(AB) = n = p$.

⟨62⟩ $B^{(1)}A^{(1,2)}$ *is invariant* $\Leftrightarrow$ $r(AB) = m = n = p$.

⟨63⟩ $B^{(1)}A^{(1)}$ *is invariant* $\Leftrightarrow$ $r(AB) = m = n = p$.

The following results can be established similarly. But the details are rather technical and tedious, and therefore are omitted.

Theorem 4.5. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$ *be given with* $AB \neq 0$, *and denote* $N = [\,A^*,\, B\,]$. *Then the following results hold.*

⟨1⟩ *The products in* (4.72) *are invariant* $\Leftrightarrow$ $r(AB) = r(A) + r(B) - r(N)$.

⟨2⟩ *The products in* (4.73) *are invariant* $\Leftrightarrow$ $r(AB) = r(A) + r(B) - r(N)$.

⟨3⟩ *The products in* (4.74) *are invariant* $\Leftrightarrow$ $r(AB) = r(A) + r(B) - n$.

⟨4⟩ *The products in* (4.76) *are invariant* $\Leftrightarrow$ *either* $r(A) = m$ *or* $r(AB) = r(A) + r(B) - r(N)$.

⟨5⟩ *The products in* (4.77) *are invariant* $\Leftrightarrow r(A) = m$.

⟨6⟩ *The products in* (4.78) *are invariant* $\Leftrightarrow r(AB) = r(A) + r(B) - r(N)$.

⟨7⟩ *The products in* (4.79) *are invariant* $\Leftrightarrow r(A) = m$.

⟨8⟩ *The products in* (4.80) *are invariant* $\Leftrightarrow r(AB) = r(A) + r(B) - r(N)$.

⟨9⟩ *The products in* (4.81) *are invariant* $\Leftrightarrow$ *either* $r(A) = m$ *or* $r(AB) = r(A) + r(B) - r(N)$.

⟨10⟩ *The products in* (4.82) *are invariant* $\Leftrightarrow$ *either* $r(A) = m$ *or* $r(AB) = r(A) + r(B) - r(N)$.

⟨11⟩ *The products in* (4.83) *are invariant* $\Leftrightarrow \mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨12⟩ *The products in* (4.84) *are invariant* $\Leftrightarrow$ *either* $r(AB) = n$ *or* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨13⟩ *The products in* (4.85) *are invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨14⟩ *The products in* (4.86) *are invariant* $\Leftrightarrow r(AB) = n$.

⟨15⟩ *The products in* (4.87) *are invariant* $\Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

⟨16⟩ *The products in* (4.88) *are invariant* $\Leftrightarrow r(A) = r(B) = n$.

⟨17⟩ *The products in* (4.89) *are invariant* $\Leftrightarrow r(AB) = m = n$.

⟨18⟩ *The products in* (4.90) *are invariant* $\Leftrightarrow r(AB) = m = n$.

⟨20⟩ *The products in* (4.92) *are invariant* $\Leftrightarrow$ *either* $r(B) = p$ *or* $r(AB) = r(A) + r(B) - r(N)$.

⟨21⟩ *The products in* (4.93) *are invariant* $\Leftrightarrow r(N) = r(A) + r(B) - r(AB)$.

⟨22⟩ *The products in* (4.94) *are invariant* $\Leftrightarrow r(B) = p$.

⟨23⟩ *The products in* (4.96) *are invariant* $\Leftrightarrow r(AB) = r(A) + r(B) - r(N)$.

⟨24⟩ *The products in* (4.95) *are invariant* $\Leftrightarrow r(B) = p$.

⟨25⟩ *The products in* (4.97) *are invariant* $\Leftrightarrow$ *either* $r(B) = p$ *or* $r(AB) = r(A) + r(B) - r(N)$.

⟨26⟩ *The products in* (4.98) *are invariant* $\Leftrightarrow$ *either* $r(B) = p$ *or* $r(AB) = r(A) + r(B) - r(N)$.

⟨27⟩ *The products in* (4.99) *are invariant* $\Leftrightarrow \mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨28⟩ *The products in* (4.100) *are invariant* $\Leftrightarrow$ *either* $r(AB) = n$ *or* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨29⟩ *The products in* (4.101) *are invariant* $\Leftrightarrow r(AB) = n$.

⟨30⟩ *The products in* (4.102) *are invariant* $\Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨31⟩ *The products in* (4.103) *are invariant* $\Leftrightarrow r(AB) = n$.

⟨32⟩ *The products in* (4.104) *are invariant* $\Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

⟨33⟩ *The products in* (4.105) *are invariant* $\Leftrightarrow r(AB) = n = p$.

⟨34⟩ *The products in* (4.106) *are invariant* $\Leftrightarrow r(AB) = n = p$.

In addition, we need the following rank and range formulas for various products of two matrices.

Lemma 4.6 ([37,38])**.** *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{m\times p}$, *and* $P \in \mathbb{C}^{p\times m}$. *Then the following results hold*

$$r(AA^*ABB^*B) = r(A^*ABB^*) = r(ABB^*A^*) = r(B^*A^*AB) = r(AB), \tag{4.110}$$

$$r[(A^*A)^{1/2}(BB^*)^{1/2}] = r[(BB^*)^{1/2}(A^*A)^{1/2}] = r(AB), \tag{4.111}$$

$$r(B^\dagger A^\dagger) = r(B^*A^\dagger) = r(B^\dagger A^*) = r(AB), \tag{4.112}$$

$$r(A^\dagger ABB^\dagger) = r(A^\dagger ABB^*) = r(A^*ABB^\dagger) = r(AB), \tag{4.113}$$

$$r(BB^\dagger A^\dagger A) = r(BB^\dagger A^*A) = r(BB^*A^\dagger A) = r(AB), \tag{4.114}$$

$$r(ABB^\dagger A^\dagger) = r(ABB^\dagger A^*) = r(ABB^*A^\dagger) = r(AB), \tag{4.115}$$

$$r(B^\dagger A^\dagger AB) = r(B^\dagger A^*AB) = r(B^*A^\dagger AB) = r(AB), \tag{4.116}$$

$$r(ABB^\dagger A^\dagger AB) = r(ABB^\dagger A^*AB) = r(ABB^*A^\dagger AB) = r(AB), \tag{4.117}$$

$$r(B^\dagger A^\dagger ABB^\dagger A^\dagger) = r(B^\dagger A^*ABB^\dagger A^\dagger) = r(B^\dagger A^\dagger ABB^*A^\dagger) = r(AB), \tag{4.118}$$

$$r[(BB^*)^\dagger(A^*A)^\dagger)] = r[(BB^*)^\dagger(A^*A)] = r[(BB^*)(A^*A)^\dagger)] = r(AB), \tag{4.119}$$

$$r[B^\dagger(A^*A)^\dagger)] = r(B^\dagger A^*A) = r[B^*(A^*A)^\dagger)] = r(AB), \tag{4.120}$$

$$r[(BB^*)^\dagger A^\dagger] = r[(BB^*)^\dagger A^*] = r(BB^*A^\dagger) = r(AB), \tag{4.121}$$

$$r[(A^\dagger)^*(B^\dagger)^*] = r[(A^\dagger)^*B] = r[A(B^\dagger)^*] = r(AB), \tag{4.122}$$

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and

$$\text{both } \mathscr{R}(A) \subseteq \mathscr{R}(B) \text{ and } r(A) = r(B) \Rightarrow \mathscr{R}(A) = \mathscr{R}(B), \tag{4.123}$$
$$\mathscr{R}(A) \subseteq \mathscr{R}(B) \Rightarrow \mathscr{R}(PA) \subseteq \mathscr{R}(PB), \tag{4.124}$$
$$\mathscr{R}(A) = \mathscr{R}(B) \Rightarrow \mathscr{R}(PA) = \mathscr{R}(PB), \tag{4.125}$$
$$\mathscr{R}(A) = \mathscr{R}(AA^*) = \mathscr{R}(AA^*A) = \mathscr{R}(AA^\dagger) = \mathscr{R}[(A^\dagger)^*], \tag{4.126}$$
$$\mathscr{R}(A^*) = \mathscr{R}(A^*A) = \mathscr{R}(A^*AA^*) = \mathscr{R}(A^\dagger) = \mathscr{R}(A^\dagger A), \tag{4.127}$$
$$\mathscr{R}(ABB^*A^*) = \mathscr{R}(ABB^*) = \mathscr{R}(AB), \tag{4.128}$$
$$\mathscr{R}(B^*A^*AB) = \mathscr{R}(B^*A^*A) = \mathscr{R}(B^*A^*), \tag{4.129}$$
$$\mathscr{R}(ABB^\dagger A^\dagger AB) = \mathscr{R}(ABB^\dagger A^\dagger) = \mathscr{R}(AB), \tag{4.130}$$
$$\mathscr{R}(B^\dagger A^\dagger ABB^\dagger A^\dagger) = \mathscr{R}(B^\dagger A^\dagger AB) = \mathscr{R}(B^\dagger A^\dagger) = \mathscr{R}(B^\dagger A^*). \tag{4.131}$$

Eqs. (4.110)–(4.131) can be used to establish various formulas for calculating the ranks of matrix expressions or matrix-valued functions composed by products of two matrices with their conjugates and Moore–Penrose inverses, which we can use, as demonstrated below, to describe performance and reveal fundamental properties of the matrix expressions and matrix-valued functions.

Applying the preliminary formulas in Section 2 to these matrix-valued functions, we obtain the following rank formulas.

Theorem 4.7. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and denote $M = AB$, $H = ABB^\dagger A^\dagger AB$, and $N = [\,A^*,\, B\,]$.*

(a) *The following rank formulas*

$$\begin{aligned} & r(M - MB^\dagger A^{(1,3,4)}M) = r(M - MB^\dagger A^{(1,2,4)}M) = r(M - MB^\dagger A^{(1,4)}M) \\ &= r(M - MB^{(1,3,4)}A^\dagger M = r(M - MB^{(1,3,4)}A^{(1,3,4)}M) = r(M - MB^{(1,3,4)}A^{(1,2,4)}M) \\ &= r(M - MB^{(1,3,4)}A^{(1,4)}M) = r(M - MB^{(1,2,3)}A^\dagger M) = r(M - MB^{(1,2,3)}A^{(1,3,4)}M) \\ &= r(M - MB^{(1,2,3)}A^{(1,2,4)}M) = r(M - MB^{(1,2,3)}A^{(1,4)}M) = r(M - MB^{(1,3)}A^\dagger M) \\ &= r(M - MB^{(1,3)}A^{(1,3,4)}M) = r(M - MB^{(1,3)}A^{(1,2,4)}M) = r(M - MB^{(1,3)}A^{(1,4)}M) \\ &= r(\,M - MB^\dagger A^\dagger M) = r(AE_BF_AB) = r(N) + r(AB) - r(A) - r(B) \end{aligned} \tag{4.132}$$

hold for all $A^{(i,\dots,j)}$ and $B^{(i,\dots,j)}$ in them.

(b) *The following rank formulas hold*

$$\begin{aligned} & \max_{A^{(1,2,3)}} r(\,AB - ABB^\dagger A^{(1,2,3)}AB\,) = \max_{A^{(1,3)}} r(\,AB - ABB^\dagger A^{(1,3)}AB\,) \\ &= \max_{A^{(1,2)}} r(\,AB - ABB^\dagger A^{(1,2)}AB\,) = \max_{B^{(1)}} r(\,AB - ABB^\dagger A^{(1)}AB\,) \\ &= \max_{B^{(1,3,4)},A^{(1,2,3)}} r(\,AB - ABB^{(1,3,4)}A^{(1,2,3)}AB\,) = \max_{B^{(1,3,4)},A^{(1,3)}} r(\,AB - ABB^{(1,3,4)}A^{(1,3)}AB\,) \\ &= \max_{B^{(1,3,4)},A^{(1,2)}} r(\,AB - ABB^{(1,3,4)}A^{(1,2)}AB\,) = \max_{B^{(1,3,4)},A^{(1)}} r(\,AB - ABB^{(1,3,4)}A^{(1)}AB\,) \\ &= \max_{B^{(1,2,3)},A^{(1,2,3)}} r(\,AB - ABB^{(1,2,3)}A^{(1,2,3)}AB\,) = \max_{B^{(1,2,3)},A^{(1,3)}} r(\,AB - ABB^{(1,2,3)}A^{(1,3)}AB\,) \\ &= \max_{B^{(1,2,3)},A^{(1,2)}} r(\,AB - ABB^{(1,2,3)}A^{(1,2)}AB\,) = \max_{B^{(1,2,3)},A^{(1)}} r(\,AB - ABB^{(1,2,3)}A^{(1)}AB \qquad (4.133) \\ &= \max_{B^{(1,3)},A^{(1,2,3)}} r(\,AB - ABB^{(1,3)}A^{(1,2,3)}AB\,) = \max_{B^{(1,3)},A^{(1,3)}} r(\,AB - ABB^{(1,3)}A^{(1,3)}AB\,) \\ &= \max_{B^{(1,3)},A^{(1,2)}} r(\,AB - ABB^{(1,3)}A^{(1,2)}AB\,) = \max_{B^{(1,3)},A^{(1)}} r(\,AB - ABB^{(1,3)}A^{(1)}AB\,) \\ &= r(N) + r(AB) - r(A) - r(B). \end{aligned} \tag{4.134}$$

(c) *The following rank formulas hold*

$$\begin{aligned} & \max_{B^{(1,2,4)}} r(\,AB - ABB^{(1,2,4)}A^\dagger AB\,) = \max_{B^{(1,2,4)},A^{(1,3,4)}} r(\,AB - ABB^{(1,2,4)}A^{(1,3,4)}AB\,) \\ &= \max_{B^{(1,2,4)},A^{(1,2,4)}} r(\,AB - ABB^{(1,2,4)}A^{(1,2,4)}AB = \max_{B^{(1,2,3)},A^{(1,4)}} r(\,AB - ABB^{(1,2,4)}A^{(1,4)}AB\,) \\ &= \max_{B^{(1,4)}} r(\,AB - ABB^{(1,4)}A^\dagger AB\,) = \max_{B^{(1,4)},A^{(1,3,4)}} r(\,AB - ABB^{(1,4)}A^{(1,3,4)}AB\,) \\ & \max_{B^{(1,4)},A^{(1,2,4)}} r(\,AB - ABB^{(1,4)}A^{(1,2,4)}AB\,) = \max_{B^{(1,4)},A^{(1,4)}} r(\,AB - ABB^{(1,4)}A^{(1,4)}AB\,) \end{aligned}$$

$$
\begin{aligned}
&= \max_{B^{(1,2)}} r(\, AB - ABB^{(1,2)}A^{\dagger}AB\,) = \max_{B^{(1,4)},A^{(1,2,4)}} r(\, AB - ABB^{(1,2)}A^{(1,3,4)}AB\,)\\
&= \max_{B^{(1,2)},A^{(1,2,4)}} r(\, AB - ABB^{(1,2)}A^{(1,2,4)}AB\,) = \max_{B^{(1,2)},A^{(1,4)}} r(\, AB - ABB^{(1,2)}A^{(1,4)}AB\,)\\
&= \max_{B^{(1)}} r(\, AB - ABB^{(1)}A^{\dagger}AB\,) = \max_{B^{(1)},A^{(1,3,4)}} r(\, AB - ABB^{(1)}A^{(1,3,4)}AB\,)\\
&= \max_{B^{(1)},A^{(1,2,4)}} r(\, AB - ABB^{(1)}A^{(1,2,4)}AB = \max_{B^{(1)},A^{(1,4)}} r(\, AB - ABB^{(1)}A^{(1,4)}AB\,)\\
&= r(N) + r(AB) - r(A) - r(B).
\end{aligned}
\tag{4.135}
$$

(d) *The following rank formulas hold*

$$
\begin{aligned}
&\max_{A^{(1,2,3)}} r(\, H - ABB^{\dagger}A^{(1,2,3)}AB\,) = \max_{A^{(1,3)}} r(\, H - ABB^{\dagger}A^{(1,3)}AB\,)\\
&= \max_{A^{(1,2)}} r(\, H - ABB^{\dagger}A^{(1,2)}AB\,) = \max_{B^{(1)}} r(\, H - ABB^{\dagger}A^{(1)}AB\,)\\
&= \max_{B^{(1,3,4)},A^{(1,2,3)}} r(\, H - ABB^{(1,3,4)}A^{(1,2,3)}AB\,) = \max_{B^{(1,3,4)},A^{(1,3)}} r(\, H - ABB^{(1,3,4)}A^{(1,3)}AB\,)\\
&= \max_{B^{(1,3,4)},A^{(1,2)}} r(\, H - ABB^{(1,3,4)}A^{(1,2)}AB\,) = \max_{B^{(1,3,4)},A^{(1)}} r(\, ABB^{\dagger}A^{\dagger}ABy - ABB^{(1,3,4)}A^{(1)}AB\,)\\
&= \max_{B^{(1,2,3)},A^{(1,2,3)}} r(\, H - ABB^{(1,2,3)}A^{(1,2,3)}AB\,) = \max_{B^{(1,2,3)},A^{(1,3)}} r(\, H - ABB^{(1,2,3)}A^{(1,3)}AB\,)\\
&= \max_{B^{(1,2,3)},A^{(1,2)}} r(\, H - ABB^{(1,2,3)}A^{(1,2)}AB\,) = \max_{B^{(1,2,3)},A^{(1)}} r(\, H - ABB^{(1,2,3)}A^{(1)}AB\\
&= \max_{B^{(1,3)},A^{(1,2,3)}} r(\, H - ABB^{(1,3)}A^{(1,2,3)}AB\,) = \max_{B^{(1,3)},A^{(1,3)}} r(\, ABB^{\dagger}A^{\dagger}ABy - ABB^{(1,3)}A^{(1,3)}AB\,)\\
&= \max_{B^{(1,3)},A^{(1,2)}} r(\, H - ABB^{(1,3)}A^{(1,2)}AB\,) = \max_{B^{(1,3)},A^{(1)}} r(\, H - ABB^{(1,3)}A^{(1)}AB\,)\\
&= r(N) + r(AB) - r(A) - r(B).
\end{aligned}
\tag{4.136}
$$

(e) *The following rank formulas hold*

$$
\begin{aligned}
&\max_{B^{(1,2,4)}} r(\, H - ABB^{(1,2,4)}A^{\dagger}AB\,) = \max_{B^{(1,2,4)},A^{(1,3,4)}} r(\, H - ABB^{(1,2,4)}A^{(1,3,4)}AB\,)\\
&= \max_{B^{(1,2,4)},A^{(1,2,4)}} r(\, H - ABB^{(1,2,4)}A^{(1,2,4)}AB = \max_{B^{(1,2,3)},A^{(1,4)}} r(\, H - ABB^{(1,2,4)}A^{(1,4)}AB\,)\\
&= \max_{B^{(1,4)}} r(\, H - ABB^{(1,4)}A^{\dagger}AB\,) = \max_{B^{(1,4)},A^{(1,3,4)}} r(\, H - ABB^{(1,4)}A^{(1,3,4)}AB\,)\\
&= \max_{B^{(1,4)},A^{(1,2,4)}} r(\, H - ABB^{(1,4)}A^{(1,2,4)}AB\,) = \max_{B^{(1,4)},A^{(1,4)}} r(\, H - ABB^{(1,4)}A^{(1,4)}AB\,)\\
&= \max_{B^{(1,2)}} r(\, H - ABB^{(1,2)}A^{\dagger}AB\,) = \max_{B^{(1,4)},A^{(1,2,4)}} r(\, H - ABB^{(1,2)}A^{(1,3,4)}AB\,)\\
&= \max_{B^{(1,2)},A^{(1,2,4)}} r(\, H - ABB^{(1,2)}A^{(1,2,4)}AB\,) = \max_{B^{(1,2)},A^{(1,4)}} r(\, H - ABB^{(1,2)}A^{(1,4)}AB\,)\\
&= \max_{B^{(1)}} r(\, H - ABB^{(1)}A^{\dagger}AB\,) = \max_{B^{(1)},A^{(1,3,4)}} r(\, H - ABB^{(1)}A^{(1,3,4)}AB\,)\\
&= \max_{B^{(1)},A^{(1,2,4)}} r(\, H - ABB^{(1)}A^{(1,2,4)}AB = \max_{B^{(1)},A^{(1,4)}} r(\, H - ABB^{(1)}A^{(1,4)}AB\,)\\
&= r(N) + r(AB) - r(A) - r(B).
\end{aligned}
\tag{4.137}
$$

Proof. It is easy to verify that $AE_BF_AB = A(I_n - BB^{\dagger} - A^{\dagger}A + BB^{\dagger}A^{\dagger}A)B = MB^{\dagger}A^{\dagger}M - M$. The last rank formula in (4.132) was established in [2]. We next give a direct proof of the last rank formula in (4.132). Applying (3.39) and (4.107) to $M - MB^{\dagger}A^{\dagger}M$ gives

$$
\begin{aligned}
&r(\, M - MB^{\dagger}A^{\dagger}M\,) = r(\, AB + ABPJ^{\dagger}QAB\,)\\
&= r\begin{bmatrix} B^*A^* & B^*BB^* & 0\\ A^*AA^* & 0 & A^*AB\\ 0 & ABB^* & -AB \end{bmatrix} - r(A) - r(B) = r\begin{bmatrix} B^*A^* & B^*B & 0\\ AA^* & 0 & AB\\ 0 & AB & -AB \end{bmatrix} - r(A) - r(B)\\
&= r\begin{bmatrix} B^*A^* & B^*B & 0\\ AA^* & AB & 0\\ 0 & 0 & -AB \end{bmatrix} - r(A) - r(B) = r(\,[\,A^*,\,B\,]^*[\,A^*,\,B\,]\,) + r(AB) - r(A) - r(B)\\
&= r(N) + r(AB) - r(A) - r(B),
\end{aligned}
\tag{4.138}
$$

thus establishing the last two rank formulas in (4.132). The first 16 rank equalities in (4.132) follow directly from (4.71).

Applying (3.52) to the difference of AB and (4.72) gives

$$\max_{U} r(\, AB - H - ABB^{\dagger}F_AUAB\,) = \min\left\{r[\, AB - H,\ ABB^{\dagger}F_A\,],\ \ r(AB)\right\}, \tag{4.139}$$

where by (3.36) and (3.37),

$$\begin{aligned} r[\, AB - H,\ ABB^{\dagger}F_A\,] &= r\begin{bmatrix} AB - H & ABB^{\dagger} \\ 0 & A \end{bmatrix} - r(A) = r\begin{bmatrix} AB & ABB^{\dagger} \\ AB & A \end{bmatrix} - r(A) \\ &= r\begin{bmatrix} AB & 0 \\ 0 & AE_B \end{bmatrix} - r(A) = r(AB) + r(E_BA^*) - r(A) \\ &= r(N) + r(AB) - r(A) - r(B). \end{aligned} \tag{4.140}$$

Substituting (4.140) into (4.139) and noticing that $r(N) + r(AB) - r(A) - r(B) \leqslant r(AB)$, we obtain (4.134). Applying (3.52) to the difference of AB and (4.73), we are also able to obtain

$$\max_{V} r(\, AB - H - ABVE_BA^{\dagger}AB\,) = r(N) + r(AB) - r(A) - r(B),$$

as required for (4.135). Applying (3.52) to to the differences of H with (4.66) and (4.67), respectively, and simplifying by (3.36) and (3.37) we obtain the following two rank formulas

$$\max_{U} r(\, ABB^{\dagger}F_AUAB\,) = r(ABB^{\dagger}F_A) = r(N) + r(AB) - r(A) - r(B),$$
$$\max_{V} r(\, ABVE_BA^{\dagger}AB\,) = r(E_BA^{\dagger}AB) = r(N) + r(AB) - r(A) - r(B),$$

thus establishing (4.136) and (4.137). □

A common feature of (4.132)–(4.137) is in that all the terms on the right-hand sides of these formulas are identical. Thus, setting all sides of these formulas equal to zero will produce many equivalent facts on matrix operations, which will be presented in the next section in the classification study of the 512 reverse-order laws.

Lemma 4.8 ([36]). *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$. *Then*

$$\begin{aligned} \max_{A^{(1)},\, B^{(1)}} r(\, AB - ABB^{(1)}A^{(1)}AB\,) &= \max_{V,\, U} r[\, AB - (\, ABB^{\dagger} + ABVE_B\,)(\, A^{\dagger}AB + F_AUAB\,)\,] \\ &= \min\{\, r(AB),\ \ r(AB) - r(A) - r(B) + n\,\} \\ &= \min\{\, r(AB),\ \ r(F_AE_B)\,\}. \end{aligned} \tag{4.141}$$

5 Set inclusions for $\{1\}$-, $\{1,2\}$-, $\{1,3\}$-, $\{1,4\}$-generalized inverses of AB

From Lemma 4.1(a), the following equivalent facts

$$\{(AB)^{(1)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\} \Leftrightarrow ABB^{(i,\dots,j)}A^{(i,\dots,j)}AB = AB \tag{5.1}$$

hold for all the eight commonly-used types of generalized inverses $A^{(i,\dots,j)}$ and $B^{(i,\dots,j)}$, respectively. In this situation, substituting (4.71)–(4.74) into (5.1), we see that the equalities in (5.1) are converted to 63 linear or multilinear matrix equations with one or two unknown matrices for the eight commonly-used types of generalized inverses of A and B except $ABB^{\dagger}A^{\dagger}AB = AB$.

In this section, we first derive various equivalent statements for the 64 reverse-order laws in (5.1) to hold, which correspond in turn to the statistical fact in (1.26).

Theorem 5.1. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$. *Then the following statements are equivalent:*

⟨1⟩	$\{(AB)^{(1)}\} \supseteq \{B^{\dagger}A^{(1)}\}$.	⟨2⟩	$\{(AB)^{(1)}\} \supseteq \{B^{\dagger}A^{(1,2)}\}$.
⟨3⟩	$\{(AB)^{(1)}\} \supseteq \{B^{\dagger}A^{(1,3)}\}$.	⟨4⟩	$\{(AB)^{(1)}\} \supseteq \{B^{\dagger}A^{(1,4)}\}$.
⟨5⟩	$\{(AB)^{(1)}\} \supseteq \{B^{\dagger}A^{(1,2,3)}\}$.	⟨6⟩	$\{(AB)^{(1)}\} \supseteq \{B^{\dagger}A^{(1,2,4)}\}$.
⟨7⟩	$\{(AB)^{(1)}\} \supseteq \{B^{\dagger}A^{(1,3,4)}\}$.	⟨8⟩	$\{(AB)^{(1)}\} \supseteq \{B^{(1,3,4)}A^{(1)}\}$.
⟨9⟩	$\{(AB)^{(1)}\} \supseteq \{B^{(1,3,4)}A^{(1,2)}\}$.	⟨10⟩	$\{(AB)^{(1)}\} \supseteq \{B^{(1,3,4)}A^{(1,3)}\}$.
⟨11⟩	$\{(AB)^{(1)}\} \supseteq \{B^{(1,3,4)}A^{(1,4)}\}$.	⟨12⟩	$\{(AB)^{(1)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,3)}\}$.

$\langle 13\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,4)}\}.$

$\langle 14\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,3,4)}A^{(1,3,4)}\}.$

$\langle 15\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,3,4)}A^{\dagger}\}.$

$\langle 16\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,4)}A^{(1,4)}\}$

$\langle 17\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,4)}\}.$

$\langle 18\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,4)}A^{(1,3,4)}\}.$

$\langle 19\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,4)}A^{\dagger}\}.$

$\langle 20\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,3)}A^{(1)}\}.$

$\langle 21\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,3)}A^{(1,2)}\}.$

$\langle 22\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,3)}A^{(1,3)}\}.$

$\langle 23\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,3)}A^{(1,4)}\}.$

$\langle 24\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,3)}\}.$

$\langle 25\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,4)}\}.$

$\langle 26\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,3)}A^{(1,3,4)}\}.$

$\langle 27\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2,3)}A^{\dagger}\}.$

$\langle 28\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,4)}A^{(1,4)}\}.$

$\langle 29\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,4)}A^{(1,2,4)}\}.$

$\langle 30\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,4)}A^{(1,3,4)}\}.$

$\langle 31\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,4)}A^{\dagger}\}.$

$\langle 32\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,3)}A^{(1)}\}.$

$\langle 33\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,3)}A^{(1,2)}\}.$

$\langle 34\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,3)}A^{(1,3)}\}.$

$\langle 35\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,3)}A^{(1,4)}\}.$

$\langle 36\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,3)}A^{(1,2,3)}\}.$

$\langle 37\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,3)}A^{(1,2,4)}\}.$

$\langle 38\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,3)}A^{(1,3,4)}\}.$

$\langle 39\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,3)}A^{\dagger}\}.$

$\langle 40\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2)}A^{(1,4)}\}.$

$\langle 41\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2)}A^{(1,2,4)}\}.$

$\langle 42\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2)}A^{(1,3,4)}\}.$

$\langle 43\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1,2)}A^{\dagger}\}.$

$\langle 44\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1)}A^{(1,4)}\}.$

$\langle 45\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1)}A^{(1,2,4)}\}.$

$\langle 46\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1)}A^{(1,3,4)}\}.$

$\langle 47\rangle$ $\{(AB)^{(1)}\} \supseteq \{B^{(1)}A^{\dagger}\}.$

$\langle 48\rangle$ $\{(AB)^{(1)}\} \ni B^{\dagger}A^{\dagger}.$

$\langle 49\rangle$ $\{AB(AB)^{(1)}\} \supseteq \{ABB^{\dagger}A^{(1)}\}.$

$\langle 50\rangle$ $\{AB(AB)^{(1)}\} \supseteq \{ABB^{\dagger}A^{(1,2)}\}.$

$\langle 51\rangle$ $\{AB(AB)^{(1)}\} \supseteq \{ABB^{\dagger}A^{(1,3)}\}.$

$\langle 52\rangle$ $\{AB(AB)^{(1)}\} \supseteq \{ABB^{\dagger}A^{(1,4)}\}.$

$\langle 53\rangle$ $\{AB(AB)^{(1)}\} \supseteq \{ABB^{\dagger}A^{(1,2,3)}\}.$

$\langle 54\rangle$ $\{AB(AB)^{(1)}\} \supseteq \{ABB^{\dagger}A^{(1,2,4)}\}.$

$\langle 55\rangle$ $\{AB(AB)^{(1)}\} \supseteq \{ABB^{\dagger}A^{(1,3,4)}\}.$

$\langle 56\rangle$ $\{AB(AB)^{(1)}\} \ni ABB^{\dagger}A^{\dagger}.$

$\langle 57\rangle$ $\{AB(AB)^{(1)}\} \supseteq \{ABB^{(1)}A^{(1,4)}\}.$

$\langle 58\rangle$ $\{AB(AB)^{(1)}\} \supseteq \{ABB^{(1)}A^{(1,2,4)}\}.$

$\langle 59\rangle$ $\{AB(AB)^{(1)}\} \supseteq \{ABB^{(1)}A^{(1,3,4)}\}.$

$\langle 60\rangle$ $\{AB(AB)^{(1)}\} \supseteq \{ABB^{(1)}A^{\dagger}\}.$

$\langle 61\rangle$ $\{(AB)^{(1)}AB\} \supseteq \{B^{(1)}A^{\dagger}AB\}.$

$\langle 62\rangle$ $\{(AB)^{(1)}AB\} \supseteq \{B^{(1,2)}A^{\dagger}AB\}.$

$\langle 63\rangle$ $\{(AB)^{(1)}AB\} \supseteq \{B^{(1,3)}A^{\dagger}AB\}.$

$\langle 64\rangle$ $\{(AB)^{(1)}AB\} \supseteq \{B^{(1,4)}A^{\dagger}AB\}.$

$\langle 65\rangle$ $\{(AB)^{(1)}AB\} \supseteq \{B^{(1,2,3)}A^{\dagger}AB\}.$

$\langle 66\rangle$ $\{(AB)^{(1)}AB\} \supseteq \{B^{(1,2,4)}A^{\dagger}AB\}.$

$\langle 67\rangle$ $\{(AB)^{(1)}AB\} \supseteq \{B^{(1,3,4)}A^{\dagger}AB\}.$

$\langle 68\rangle$ $\{(AB)^{(1)}AB\} \ni B^{\dagger}A^{\dagger}AB.$

$\langle 69\rangle$ $\{(AB)^{(1)}AB\} \supseteq \{B^{(1,3)}A^{(1)}AB\}.$

$\langle 70\rangle$ $\{(AB)^{(1)}AB\} \supseteq \{B^{(1,4)}A^{(1)}AB\}.$

$\langle 71\rangle$ $\{(AB)^{(1)}AB\} \supseteq \{B^{(1,2,3)}A^{(1)}AB\}.$

$\langle 72\rangle$ $\{(AB)^{(1)}AB\} \supseteq \{B^{(1,3,4)}A^{(1)}AB\}.$

$\langle 73\rangle$ $\{B(AB)^{(1)}A\} \supseteq \{BB^{(1)}A^{(1)}A\}.$

$\langle 74\rangle$ $\{B(AB)^{(1)}A\} \supseteq \{BB^{\dagger}A^{(1)}A\}.$

$\langle 75\rangle$ $\{B(AB)^{(1)}A\} \supseteq \{BB^{(1)}A^{\dagger}A\}.$

$\langle 76\rangle$ $\{B(AB)^{(1)}A\} \ni BB^{\dagger}A^{\dagger}A.$

$\langle 77\rangle$ $\big\{[(A^{\dagger})^*B]^{(1)}\big\} \ni B^{\dagger}A^*$ *and/or* $\big\{[A(B^{\dagger})^*]^{(1)}\big\} \ni B^*A^{\dagger}.$

$\langle 78\rangle$ $\big\{(A^*AB)^{(1)}\big\} \ni B^{\dagger}(A^*A)^{\dagger}$ *and/or* $\big\{(ABB^*)^{(1)}\big\} \ni (BB^*)^{\dagger}A^{\dagger}.$

$\langle 79\rangle$ $\big\{(A^*ABB^*)^{(1)}\big\} \ni (BB^*)^{\dagger}(A^*A)^{\dagger}$ *and/or* $\big\{(BB^*A^*A)^{(1)}\big\} \ni (A^*A)^{\dagger}(BB^*)^{\dagger}.$

$\langle 80\rangle$ $\big\{[(A^*A)^{1/2}(BB^*)^{1/2}]^{(1)}\big\} \ni [(BB^*)^{1/2}]^{\dagger}[(A^*A)^{1/2}]^{\dagger}$ *and/or* $\big\{[(BB^*)^{1/2}(A^*A)^{1/2}]^{(1)}\big\} \ni [(A^*A)^{1/2}]^{\dagger}(BB^*)^{1/2}]^{\dagger}.$

$\langle 81\rangle$ $\big\{(AA^*ABB^*B)^{(1)}\big\} \ni (BB^*B)^{\dagger}(AA^*A)^{\dagger}.$

$\langle 82\rangle$ $\big\{(A^{\dagger}AB)^{(1)}\big\} \ni B^{\dagger}A^{\dagger}A$ *and/or* $\big\{(ABB^{\dagger})^{(1)}\big\} \ni BB^{\dagger}A^{\dagger}.$

$\langle 83\rangle$ $\big\{(A^{\dagger}ABB^{\dagger})^{(1)}\big\} \ni BB^{\dagger}A^{\dagger}A$ *and/or* $\big\{(BB^{\dagger}A^{\dagger}A)^{(1)}\big\} \ni A^{\dagger}ABB^{\dagger}.$

$\langle 84\rangle$ $\big\{(F_ABB^{\dagger})^{(1)}\big\} \ni BB^{\dagger}F_A$ *and/or* $\big\{(BB^{\dagger}F_A)^{(1)}\big\} \ni F_ABB^{\dagger}.$

$\langle 85\rangle$ $\big\{(A^{\dagger}AE_B)^{(1)}\big\} \ni E_BA^{\dagger}A$ *and/or* $\big\{(E_BA^{\dagger}A)^{(1)}\big\} \ni A^{\dagger}AE_B.$

$\langle 86\rangle$ $\big\{(F_AE_B)^{(1)}\big\} \in E_BF_A$ *and/or* $\big\{(E_BF_A)^{(1)}\big\} \ni F_AE_B.$

$\langle 87\rangle$ $ABB^{\dagger}A^{\dagger}AB = AB$ *and/or* $AE_BF_AB = 0.$

$\langle 88\rangle$ $B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger} = B^{\dagger}A^{\dagger}$ *and/or* $B^{\dagger}F_AE_BA^{\dagger} = 0.$

$\langle 89\rangle$ $ABB^{\dagger}F_A = 0$ *and/or* $E_BA^{\dagger}AB = 0.$

$\langle 90\rangle$ $AE_BF_A = 0$ *and/or* $E_BF_AB = 0.$

$\langle 91\rangle$ $A^{\dagger}ABB^{\dagger} = BB^{\dagger}A^{\dagger}A$ *and/or* $F_ABB^{\dagger} = BB^{\dagger}F_A,\ A^{\dagger}AE_B = E_BA^{\dagger}A,\ F_AE_B = E_BF_A.$

$\langle 92\rangle$ $(A^{\dagger}ABB^{\dagger})^2 = A^{\dagger}ABB^{\dagger}$ *and/or* $(BB^{\dagger}A^{\dagger}A)^2 = BB^{\dagger}A^{\dagger}A.$

$\langle 93\rangle$ $(ABB^{\dagger}A^{\dagger})^2 = ABB^{\dagger}A^{\dagger}$ *and/or* $(B^{\dagger}A^{\dagger}AB)^2 = B^{\dagger}A^{\dagger}AB.$

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⟨94⟩ $(F_ABB^\dagger)^2 = F_ABB^\dagger$ *and/or* $(BB^\dagger F_A)^2 = BB^\dagger F_A$.

⟨95⟩ $(A^\dagger AE_B)^2 = A^\dagger AE_B$ *and/or* $(E_BA^\dagger A)^2 = E_BA^\dagger A$.

⟨96⟩ $(F_AE_B)^2 = F_AE_B$ *and/or* $(E_BF_A)^2 = E_BF_A$.

⟨97⟩ $(AE_BA^\dagger)^2 = AE_BA^\dagger$ *and/or* $(B^\dagger F_AB)^2 = B^\dagger F_AB$.

⟨98⟩ $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$ *and/or* $(BB^\dagger A^\dagger A)^\dagger = A^\dagger ABB^\dagger$.

⟨99⟩ $(BB^\dagger F_A)^\dagger = F_ABB^\dagger$ *and/or* $(F_ABB^\dagger)^\dagger = BB^\dagger F_A$.

⟨100⟩ $(A^\dagger AE_B)^\dagger = E_BA^\dagger A$ *and/or* $(E_BA^\dagger A)^\dagger = A^\dagger AE_B$.

⟨101⟩ $(F_AE_B)^\dagger = E_BF_A$ *and/or* $(E_BF_A)^\dagger = F_AE_B$.

⟨102⟩ $(A^\dagger A - BB^\dagger)^\dagger = A^\dagger A - BB^\dagger$ *and/or* $(BB^\dagger - A^\dagger A)^\dagger = BB^\dagger - A^\dagger A$.

⟨103⟩ $(I_n - A^\dagger A - BB^\dagger)^\dagger = I_n - A^\dagger A - BB^\dagger$.

⟨104⟩ $(\, I_m - ABB^\dagger A^\dagger \,)^\dagger = I_m - ABB^\dagger A^\dagger$ *and/or* $(\, I_p - B^\dagger A^\dagger AB \,)^\dagger = I_p - B^\dagger A^\dagger AB$.

⟨105⟩ $(I_n - A^\dagger ABB^\dagger)^\dagger = I_n - A^\dagger ABB^\dagger$ *and/or* $(I_n - BB^\dagger A^\dagger A)^\dagger = I_n - BB^\dagger A^\dagger A$.

⟨106⟩ $(\, I_m - AE_BA^\dagger \,)^\dagger = I_m - AE_BA^\dagger$ *and/or* $(\, I_p - B^\dagger F_AB \,)^\dagger = I_p - B^\dagger F_AB$.

⟨107⟩ $(I_n - F_ABB^\dagger)^\dagger = I_n - F_ABB^\dagger$ *and/or* $(I_n - BB^\dagger F_A)^\dagger = I_n - BB^\dagger F_A$.

⟨108⟩ $(I_n - A^\dagger AE_B)^\dagger = I_n - A^\dagger AE_B$ *and/or* $(I_n - E_BA^\dagger A)^\dagger = I_n - E_BA^\dagger A$.

⟨109⟩ $(I_n - F_AE_B)^\dagger = I_n - F_AE_B$ *and/or* $(I_n - E_BF_A)^\dagger = I_n - E_BF_A$.

⟨110⟩ $[\, A^*,\, B\,][\, A^*,\, B\,]^\dagger = A^\dagger A + BB^\dagger - A^\dagger ABB^\dagger$ *and/or* $[\, A^*,\, B\,][\, A^*,\, B\,]^\dagger = A^\dagger A + BB^\dagger - BB^\dagger A^\dagger A$.

⟨111⟩ $[\, F_A,\, B\,][\, F_A,\, B\,]^\dagger = F_A + BB^\dagger - F_ABB^\dagger$ *and/or* $[\, F_A,\, B\,][\, F_A,\, B\,]^\dagger = F_A + BB^\dagger - BB^\dagger F_A$.

⟨112⟩ $[\, A^*,\, E_B\,][\, A^*,\, E_B\,]^\dagger = A^\dagger A + E_B - A^\dagger AE_B$ *and/or* $[\, A^*,\, E_B\,][\, A^*,\, E_B\,]^\dagger = A^\dagger A + E_B - E_BA^\dagger A$.

⟨113⟩ $[\, F_A,\, E_B\,][\, F_A,\, E_B\,]^\dagger = F_A + E_B - F_AE_B$ *and/or* $[\, F_A,\, E_B\,][\, F_A,\, E_B\,]^\dagger = F_A + E_B - E_BF_A$.

⟨114⟩ $ABB^{(1,3)}A^{(1)}AB$ *is invariant with respect to the choice of* $A^{(1)}$ *and* $B^{(1,3)}$, *i.e.,* $ABB^{(1,3)}A^{(1)}AB = ABB^\dagger A^\dagger AB$ *holds for all* $A^{(1)}$ *and* $B^{(1,3)}$.

⟨115⟩ $ABB^{(1)}A^{(1,4)}AB$ *is invariant with respect to the choice of* $A^{(1,4)}$ *and* $B^{(1)}$, *i.e.,* $ABB^{(1)}A^{(1,4)}AB = ABB^\dagger A^\dagger AB$ *holds for all* $A^{(1,4)}$ *and* $B^{(1)}$.

⟨116⟩ $r[\, A^*,\, B\,] = r(A) + r(B) - r(AB)$.

⟨117⟩ $r[\, F_A,\, B\,] = r(F_A) + r(B) - r(F_AB)$.

⟨118⟩ $r[\, A,\, E_B\,] = r(A) + r(E_B) - r(AE_B)$.

⟨119⟩ $r[\, F_A,\, E_B\,] = r(F_A) + r(E_B) - r(F_AE_B)$.

⟨120⟩ $r(\, I_n - A^\dagger ABB^\dagger \,) = n - r(A^\dagger ABB^\dagger \,)$ *and/or* $r(\, I_n - BB^\dagger A^\dagger A\,) = n - r(BB^\dagger A^\dagger A\,)$.

⟨121⟩ $r(\, I_m - ABB^\dagger A^\dagger \,) = m - r(ABB^\dagger A^\dagger \,)$ *and/or* $r(\, I_p - B^\dagger A^\dagger AB\,) = p - r(B^\dagger A^\dagger AB\,)$.

⟨122⟩ $r(\, I_n - F_ABB^\dagger \,) = n - r(F_ABB^\dagger \,)$ *and/or* $r(\, I_n - BB^\dagger F_A\,) = n - r(BB^\dagger F_A\,)$.

⟨123⟩ $r(\, I_n - A^\dagger AE_B\,) = n - r(A^\dagger AE_B\,)$ *and/or* $r(\, I_n - E_BA^\dagger A\,) = n - r(E_BA^\dagger A\,)$.

⟨124⟩ $r(\, I_n - F_AE_B\,) = n - r(F_AE_B)$ *and/or* $r(\, I_n - E_BF_A\,) = n - r(E_BF_A)$.

⟨125⟩ $r(\, I_m - AE_BA^\dagger \,) = m - r(AE_BA^\dagger \,)$ *and/or* $r(\, I_p - B^\dagger F_AB\,) = p - r(B^\dagger F_AB\,)$.

⟨126⟩ $r(\, A^\dagger A - A^\dagger ABB^\dagger \,) = r(A^\dagger A) - r(A^\dagger ABB^\dagger)$ *and/or* $r(\, BB^\dagger - A^\dagger ABB^\dagger \,) = r(BB^\dagger) - r(A^\dagger ABB^\dagger)$.

⟨127⟩ $r(\, A^\dagger A - A^\dagger AE_B\,) = r(A^\dagger A) - r(A^\dagger AE_B)$ *and/or* $r(\, BB^\dagger - F_ABB^\dagger \,) = r(BB^\dagger) - r(F_ABB^\dagger)$.

⟨128⟩ $r(\, F_A - F_ABB^\dagger \,) = r(F_A) - r(F_ABB^\dagger)$ *and/or* $r(\, E_B - A^\dagger AE_B\,) = r(E_B) - r(A^\dagger AE_B)$.

⟨129⟩ $r(\, F_A - F_AE_B\,) = r(F_A) - r(F_AE_B)$ *and* $r(\, E_B - F_AE_B\,) = r(E_B) - r(F_AE_B)$.

⟨130⟩ $\mathscr{R}(\, I_n - A^\dagger ABB^\dagger \,) \cap \mathscr{R}(A^\dagger ABB^\dagger \,) = \{0\}$ *and/or* $\mathscr{R}(\, I_n - BB^\dagger A^\dagger A\,) \cap \mathscr{R}(BB^\dagger A^\dagger A\,) = \{0\}$.

⟨131⟩ $\mathscr{R}(I_n - F_ABB^{\dagger}) \cap \mathscr{R}(F_ABB^{\dagger}) = \{0\}$ *and/or* $\mathscr{R}(I_n - BB^{\dagger}F_A) \cap \mathscr{R}(BB^{\dagger}F_A) = \{0\}$.

⟨132⟩ $\mathscr{R}(I_n - A^{\dagger}AE_B) \cap \mathscr{R}(A^{\dagger}AE_B) = \{0\}$ *and/or* $\mathscr{R}(I_n - E_BA^{\dagger}A) \cap \mathscr{R}(E_BA^{\dagger}A) = \{0\}$.

⟨133⟩ $\mathscr{R}(I_n - F_AE_B) \cap \mathscr{R}(F_AE_B) = \{0\}$ *and/or* $\mathscr{R}(I_n - E_BF_A) \cap \mathscr{R}(E_BF_A) = \{0\}$.

⟨134⟩ $\mathscr{R}(I_m - ABB^{\dagger}A^{\dagger}) \cap \mathscr{R}(ABB^{\dagger}A^{\dagger}) = \{0\}$ *and/or* $\mathscr{R}[(I_m - ABB^{\dagger}A^{\dagger})^*] \cap \mathscr{R}[(ABB^{\dagger}A^{\dagger})^*] = \{0\}$.

⟨135⟩ $\mathscr{R}(I_p - B^{\dagger}A^{\dagger}AB) \cap \mathscr{R}(B^{\dagger}A^{\dagger}AB) = \{0\}$ *and/or* $\mathscr{R}[(I_p - B^{\dagger}A^{\dagger}AB)^*] \cap \mathscr{R}[(B^{\dagger}A^{\dagger}AB)^*] = \{0\}$.

⟨136⟩ $\mathscr{R}(I_m - AE_BA^{\dagger}) \cap \mathscr{R}(AE_BA^{\dagger}) = \{0\}$ *and/or* $\mathscr{R}[(I_m - AE_BA^{\dagger})^*] \cap \mathscr{R}[(AE_BA^{\dagger})^*] = \{0\}$.

⟨137⟩ $\mathscr{R}(I_p - B^{\dagger}F_AB) \cap \mathscr{R}(B^{\dagger}F_AB) = \{0\}$ *and/or* $\mathscr{R}[(I_p - B^{\dagger}F_AB)^*] \cap \mathscr{R}[(B^{\dagger}F_AB)^*] = \{0\}$.

⟨138⟩ $\mathbb{C}^n = \mathscr{R}(I_n - A^{\dagger}ABB^{\dagger}) \oplus \mathscr{R}(A^{\dagger}ABB^{\dagger})$ *and/or* $\mathbb{C}^n = \mathscr{R}(I_n - BB^{\dagger}A^{\dagger}A) \oplus \mathscr{R}(BB^{\dagger}A^{\dagger}A)$.

⟨139⟩ $\mathbb{C}^n = \mathscr{R}(I_n - F_ABB^{\dagger}) \oplus \mathscr{R}(F_ABB^{\dagger})$ *and/or* $\mathbb{C}^n = \mathscr{R}(I_n - BB^{\dagger}F_A) \oplus \mathscr{R}(BB^{\dagger}F_A)$.

⟨140⟩ $\mathbb{C}^n = \mathscr{R}(I_n - A^{\dagger}AE_B) \oplus \mathscr{R}(A^{\dagger}AE_B)$ *and/or* $\mathbb{C}^n = \mathscr{R}(I_n - E_BA^{\dagger}A) \oplus \mathscr{R}(E_BA^{\dagger}A)$.

⟨141⟩ $\mathbb{C}^n = \mathscr{R}(I_n - F_AE_B) \oplus \mathscr{R}(F_AE_B)$ *and/or* $\mathbb{C}^n = \mathscr{R}(I_n - E_BF_A) \oplus \mathscr{R}(E_BF_A)$.

⟨142⟩ $\mathbb{C}^m = \mathscr{R}(I_m - ABB^{\dagger}A^{\dagger}) \oplus \mathscr{R}(ABB^{\dagger}A^{\dagger})$ *and/or* $\mathbb{C}^m = \mathscr{R}[(I_m - ABB^{\dagger}A^{\dagger})^*] \oplus \mathscr{R}[(ABB^{\dagger}A^{\dagger})^*]$.

⟨143⟩ $\mathbb{C}^p = \mathscr{R}(I_p - B^{\dagger}A^{\dagger}AB) \oplus \mathscr{R}(B^{\dagger}A^{\dagger}AB)$ *and/or* $\mathbb{C}^p = \mathscr{R}[(I_p - B^{\dagger}A^{\dagger}AB)^*] \oplus \mathscr{R}[(B^{\dagger}A^{\dagger}AB)^*] = \{0\}$.

⟨144⟩ $\mathbb{C}^m = \mathscr{R}(I_m - AE_BA^{\dagger}) \oplus \mathscr{R}(AE_BA^{\dagger})$ *and/or* $\mathbb{C}^m = \mathscr{R}[(I_m - AE_BA^{\dagger})^*] \oplus \mathscr{R}[(AE_BA^{\dagger})^*]$.

⟨145⟩ $\mathbb{C}^p = \mathscr{R}(I_p - B^{\dagger}F_AB) \oplus \mathscr{R}(B^{\dagger}F_AB)$ *and/or* $\mathbb{C}^p = \mathscr{R}[(I_p - B^{\dagger}F_AB)^*] \oplus \mathscr{R}[(B^{\dagger}F_AB)^*]$.

⟨146⟩ $\dim[\mathscr{R}(A^*) \cap \mathscr{R}(B)] = r(AB)$ *and/or* $\dim[\mathscr{R}(F_A) \cap \mathscr{R}(B)] = r(F_AB)$, $\dim[\mathscr{R}(A^*) \cap \mathscr{R}(E_B)] = r(AE_B)$, $\dim[\mathscr{R}(F_A) \cap \mathscr{R}(E_B)] = r(F_AE_B)$.

⟨147⟩ $\mathscr{R}(BB^{\dagger}A^{\dagger}A) \subseteq \mathscr{R}(A^{\dagger}A)$ *and/or* $\mathscr{R}(A^{\dagger}ABB^{\dagger}) \subseteq \mathscr{R}(BB^{\dagger})$.

⟨148⟩ $\mathscr{R}(BB^{\dagger}F_A) \subseteq \mathscr{R}(F_A)$ *and/or* $\mathscr{R}(F_ABB^{\dagger}) \subseteq \mathscr{R}(BB^{\dagger})$.

⟨149⟩ $\mathscr{R}(E_BA^{\dagger}A) \subseteq \mathscr{R}(A^{\dagger}A)$ *and/or* $\mathscr{R}(A^{\dagger}AE_B) \subseteq \mathscr{R}(E_B)$.

⟨150⟩ $\mathscr{R}(E_BF_A) \subseteq \mathscr{R}(F_A)$ *and/or* $\mathscr{R}(F_AE_B) \subseteq \mathscr{R}(E_B)$.

⟨151⟩ $\mathscr{R}(A^{\dagger}ABB^{\dagger}) = \mathscr{R}(A^{\dagger}A) \cap \mathscr{R}(BB^{\dagger})$ *and/or* $\mathscr{R}(BB^{\dagger}A^{\dagger}A) = \mathscr{R}(A^{\dagger}A) \cap \mathscr{R}(BB^{\dagger})$.

⟨152⟩ $\mathscr{R}(F_ABB^{\dagger}) = \mathscr{R}(F_A) \cap \mathscr{R}(BB^{\dagger})$ *and/or* $\mathscr{R}(BB^{\dagger}F_A) = \mathscr{R}(F_A) \cap \mathscr{R}(BB^{\dagger})$.

⟨153⟩ $\mathscr{R}(A^{\dagger}AE_B) = \mathscr{R}(A^{\dagger}A) \cap \mathscr{R}(E_B)$ *and/or* $\mathscr{R}(E_BA^{\dagger}A) = \mathscr{R}(A^{\dagger}A) \cap \mathscr{R}(E_B)$.

⟨154⟩ $\mathscr{R}(F_AE_B) = \mathscr{R}(F_A) \cap \mathscr{R}(E_B)$ *and/or* $\mathscr{R}(E_BF_A) = \mathscr{R}(F_A) \cap \mathscr{R}(E_B)$.

⟨155⟩ $\mathscr{R}(A^{\dagger}ABB^{\dagger}) = \mathscr{R}(BB^{\dagger}A^{\dagger}A)$ *and/or* $\mathscr{R}(A^{\dagger}AE_B) = \mathscr{R}(E_BA^{\dagger}A)$, $\mathscr{R}(F_ABB^{\dagger}) = \mathscr{R}(BB^{\dagger}F_A)$, $\mathscr{R}(F_AE_B) = \mathscr{R}(E_BF_A)$.

⟨156⟩ $\mathscr{R}(AB) \cap \mathscr{R}(AE_B) = \{0\}$ *and/or* $\mathscr{R}(F_AB) \cap \mathscr{R}(F_AE_B) = \{0\}$, $\mathscr{R}(B^*A^*) \cap \mathscr{R}(B^*F_A) = \{0\}$, $\mathscr{R}(E_BA^*) \cap \mathscr{R}(E_BF_A) = \{0\}$.

⟨157⟩ $\mathscr{R}(A) = \mathscr{R}(AB) \oplus \mathscr{R}(AE_B)$ *and/or* $\mathscr{N}(A) = \mathscr{R}(F_AB) \oplus \mathscr{R}(F_AE_B)$, $\mathscr{R}(B^*) = \mathscr{R}(B^*A^*) \oplus \mathscr{R}(B^*F_A)$, $\mathscr{N}(B^*) = \mathscr{R}(E_BA^*) \oplus \mathscr{R}(E_BF_A)$.

⟨158⟩ *The matrix equations* $A^*X = BB^{\dagger}A^*$ *and/or* $BY = A^{\dagger}AB$ *are solvable.*

⟨159⟩ *The matrix equations* $F_AX = BB^{\dagger}F_A$ *and/or* $BY = F_AB$ *are solvable.*

⟨160⟩ *The matrix equations* $A^*X = E_BA^*$ *and/or* $E_BY = A^{\dagger}AE_B$ *are solvable.*

⟨161⟩ *The matrix equations* $F_AX = E_BF_A$ *and/or* $E_BY = F_AE_B$ *are solvable.*

Proof. Setting all sides of (4.132)–(4.135) equal to zero The equivalence of ⟨1⟩–⟨48⟩, ⟨87⟩, and ⟨116⟩.

Setting all sides of (4.136) and (4.137) equal to zero leads to the equivalence of ⟨114⟩, ⟨115⟩, and ⟨116⟩.

By (3.36),

$$r(F_A) = n - r(A), \quad r(F_AB) = r[\, A^*,\, B\,] - r(A^*), \tag{5.2}$$
$$r(E_B) = n - r(B), \quad r(AE_B) = r[\, A^*,\, B\,] - r(B), \tag{5.3}$$
$$r[\, F_A,\, B\,] = r(F_A) + r(A^{\dagger}AB) = n - r(A) + r(AB), \tag{5.4}$$
$$r[\, A^*,\, E_B\,] = r(ABB^{\dagger}) + r(E_B) = n - r(B) + r(AB), \tag{5.5}$$
$$r[\, F_A,\, E_B\,] = r(F_A) + r(A^{\dagger}AE_B) = n - r(A) - r(B) - r[\, A^*,\, B\,], \tag{5.6}$$
$$r(F_AE_B) = r[\, A^*,\, E_B\,] - r(A^*) = n - r(A) - r(B) + r(AB). \tag{5.7}$$

Substituting (5.2)–(5.7) into the three rank equalities in ⟨117⟩–⟨119⟩ and simplifying, we obtain the equivalence of ⟨116⟩–⟨119⟩.

Applying (3.40) to $I_n - A^{\dagger}ABB^{\dagger}$, $I_m - ABB^{\dagger}A^{\dagger}$, and $I_p - B^{\dagger}A^{\dagger}AB$, and simplifying by (3.36) and (3.37), we obtain

$$\begin{aligned} r(\, I_n - A^{\dagger}ABB^{\dagger}\,) &= r(\, I_n - BB^{\dagger}A^{\dagger}A\,) = r\begin{bmatrix} AA^* & ABB^{\dagger} \\ A^* & I_n \end{bmatrix} - r(A) = r\begin{bmatrix} AA^* - ABB^{\dagger}A^* & 0 \\ 0 & I_n \end{bmatrix} - r(A) \\ &= n + r(\, AA^* - ABB^{\dagger}A^*\,) - r(A) = n + r\begin{bmatrix} B^*B & B^*A^* \\ AB & AA^* \end{bmatrix} - r(A) - r(B) \\ &= n + r([\, A,\, B^*\,]^*[\, A,\, B^*\,]) - r(A) - r(B) = n + r(N) - r(A) - r(B), \end{aligned} \tag{5.8}$$

$$\begin{aligned} r(\, I_m - ABB^{\dagger}A^{\dagger}\,) &= r\begin{bmatrix} B^*B & B^*A^{\dagger} \\ AB & I_m \end{bmatrix} - r(B) = r\begin{bmatrix} B^*B - B^*A^{\dagger}AB & 0 \\ 0 & I_m \end{bmatrix} - r(B) \\ &= r(B^*F_AB) - r(B) + m = r(F_AB) - r(B) + m = r(N) - r(A) - r(B) + m, \end{aligned} \tag{5.9}$$

$$\begin{aligned} r(\, I_p - B^{\dagger}A^{\dagger}AB\,) &= r\begin{bmatrix} AA^* & AB \\ B^{\dagger}A^* & I_p \end{bmatrix} - r(A) = r\begin{bmatrix} AA^* - ABB^{\dagger}A^* & 0 \\ 0 & I_p \end{bmatrix} - r(A) \\ &= r(\, AE_BA^*\,) - r(A) + p = r(E_BA^*) - r(A) + p = r(N) - r(A) - r(B) + p. \end{aligned} \tag{5.10}$$

Combining (5.8)–(5.10) with (4.114)–(4.116) yields

$$\begin{aligned} r(\, I_n - A^{\dagger}ABB^{\dagger}\,) - n + r(A^{\dagger}ABB^{\dagger}) &= r(\, I_n - BB^{\dagger}A^{\dagger}A\,) - n + r(BB^{\dagger}A^{\dagger}A) \\ &= r(N) - r(A) - r(B) + r(AB), \end{aligned} \tag{5.11}$$
$$r(\, I_m - ABB^{\dagger}A^{\dagger}\,) - m + r(ABB^{\dagger}A^{\dagger}) = r(N) - r(A) - r(B) + r(AB), \tag{5.12}$$
$$r(\, I_p - B^{\dagger}A^{\dagger}AB\,) - p + r(B^{\dagger}A^{\dagger}AB) = r(N) - r(A) - r(B) + r(AB). \tag{5.13}$$

Setting all sides of (5.11)–(5.13) equal to zero leads to the equivalence of ⟨120⟩, ⟨121⟩, and ⟨116⟩. Replacing $A^{\dagger}A$ with F_A and $BB^{\dagger}$ with E_B respectively in ⟨120⟩ and ⟨121⟩ leads to the equivalence of ⟨122⟩–⟨125⟩ with ⟨116⟩–⟨118⟩.

Applying (4.132) to $B^{\dagger}F_AE_BA^{\dagger} = B^{\dagger}A^{\dagger} - B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger}$ and simplifying by Lemma 4.6, we obtain

$$\begin{aligned} r(B^{\dagger}F_AE_BA^{\dagger}) = r(\, B^{\dagger}A^{\dagger} - B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger}) &= r[\,(B^{\dagger})^*,\, A^{\dagger}] + r(B^{\dagger}A^{\dagger}) - r(B^{\dagger}) - r(A^{\dagger}) \\ &= r(N) + r(AB) - r(B) - r(A). \end{aligned} \tag{5.14}$$

Setting both sides of (5.14) equal to zero leads to the equivalence of ⟨88⟩ and ⟨116⟩.

Replacing A and B in (5.14) with $(A^{\dagger})^*$, A^*A, $(A^*A)^{1/2}$, AA^*A, $A^{\dagger}A$, $(B^{\dagger})^*$, BB^*, $(BB^*)^{1/2}$, BB^*B, and $BB^{\dagger}$, respectively, and simplifying by Lemma 4.6, we also obtain the following rank formulas

$$r[\,(A^{\dagger})^*B - (A^{\dagger})^*BB^{\dagger}A^*(A^{\dagger})^*B\,] = r(N) + r(AB) - r(A) - r(B), \tag{5.15}$$
$$r[\,A(B^{\dagger})^* - A(B^{\dagger})^*B^*A^{\dagger}A(B^{\dagger})^*\,] = r(N) + r(AB) - r(A) - r(B), \tag{5.16}$$
$$r[\,(A^*A)B - (A^*A)BB^{\dagger}(A^*A)^{\dagger}(A^*A)B\,] = r(N) + r(AB) - r(A) - r(B), \tag{5.17}$$
$$r[\,A(BB^*) - A(BB^*)(BB^*)^{\dagger}A^{\dagger}A(BB^*)\,] = r(N) + r(AB) - r(A) - r(B), \tag{5.18}$$
$$r[\,(A^*A)(BB^*) - (A^*A)(BB^*)(BB^*)^{\dagger}(A^*A)^{\dagger}(A^*A)(BB^*)\,] = r(N) + r(AB) - r(A) - r(B), \tag{5.19}$$
$$\begin{aligned} &r\{\,(A^*A)^{1/2}(BB^*)^{1/2} - (A^*A)^{1/2}(BB^*)^{1/2}[(BB^*)^{1/2}]^{\dagger}[(A^*A)^{1/2}]^{\dagger}(A^*A)^{1/2}(BB^*)^{1/2}\} \\ &= r(N) + r(AB) - r(A) - r(B), \end{aligned} \tag{5.20}$$
$$\begin{aligned} &r[\,(AA^*A)(BB^*B) - (AA^*A)(BB^*B)(BB^*B)^{\dagger}(AA^*A)^{\dagger}(AA^*A)(BB^*B)\,] \\ &= r(N) + r(AB) - r(A) - r(B), \end{aligned} \tag{5.21}$$

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$$r[\,A^{\dagger}AB - (A^{\dagger}AB)B^{\dagger}A^{\dagger}A(A^{\dagger}AB)\,] = r(N) + r(AB) - r(A) - r(B), \tag{5.22}$$
$$r[\,ABB^{\dagger} - (ABB^{\dagger})BB^{\dagger}A^{\dagger}(ABB^{\dagger})\,] = r(N) + r(AB) - r(A) - r(B), \tag{5.23}$$
$$r[\,A^{\dagger}ABB^{\dagger} - (A^{\dagger}ABB^{\dagger})BB^{\dagger}A^{\dagger}A(A^{\dagger}ABB^{\dagger})\,] = r(N) + r(AB) - r(A) - r(B). \tag{5.24}$$

Setting all sides of (5.15)–(5.24) equal to zero leads to the equivalence of ⟨77⟩–⟨83⟩ and ⟨116⟩.

Replacing A and B in (5.14) with $AA^{\dagger}$, F_A, $BB^{\dagger}$, and E_B, we further obtain

$$r[\,F_ABB^{\dagger} - (F_ABB^{\dagger})BB^{\dagger}F_A(F_ABB^{\dagger})\,] = r[\,F_A,\, B\,] - r(F_A) - r(B) + r(F_AB), \tag{5.25}$$
$$r[\,A^{\dagger}AE_B - (A^{\dagger}AE_B)E_BA^{\dagger}A(A^{\dagger}AE_B)\,] = r[\,A,\, E_B\,] - r(A) - r(E_B) + r(AE_B), \tag{5.26}$$
$$r[\,F_AE_B - (F_AE_B)E_BF_A(F_AE_B)\,] = r[\,F_A,\, E_B\,] - r(F_A) - r(E_B) + r(F_AAE_B). \tag{5.27}$$

Setting all sides of (5.25)–(5.27) equal to zero leads to the equivalence of ⟨84⟩–⟨86⟩ and ⟨117⟩–⟨119⟩.

Applying (3.36) to $ABB^{\dagger}F_A$ and $E_BA^{\dagger}AB$ and simplifying by Lemma 3.6(c), we obtain

$$\begin{aligned} r(ABB^{\dagger}F_A) &= r\begin{bmatrix} ABB^{\dagger} \\ A \end{bmatrix} - r(A) = r\begin{bmatrix} ABB^{\dagger} \\ A - ABB^{\dagger} \end{bmatrix} - r(A) \\ &= r(ABB^{\dagger}) + r(A - ABB^{\dagger}) - r(A) = r(N) - r(A) - r(B) + r(AB), \end{aligned} \tag{5.28}$$
$$\begin{aligned} r(E_BA^{\dagger}AB) &= r[\,B,\ A^{\dagger}AB\,] - r(B) = r[\,B - A^{\dagger}AB,\ A^{\dagger}AB\,] - r(B) \\ &= r(B - A^{\dagger}AB) + r(A^{\dagger}AB) - r(B) = r(N) - r(A) - r(B) + r(AB). \end{aligned} \tag{5.29}$$

Setting all sides of (5.28) and (5.29) equal to zero leads to the equivalence of ⟨89⟩ and ⟨116⟩. Replacing both $A^{\dagger}A$ and $BB^{\dagger}$ with F_A and E_B in ⟨89⟩ respectively leads to the equivalence of ⟨90⟩ and ⟨117⟩ and ⟨118⟩.

Applying (3.49) to $A^{\dagger}ABB^{\dagger} - BB^{\dagger}A^{\dagger}A$ and simplifying, we obtain

$$\begin{aligned} & r(\,A^{\dagger}ABB^{\dagger} - BB^{\dagger}A^{\dagger}A\,) = r(\,F_ABB^{\dagger} - BB^{\dagger}F_A\,) \\ &= r(\,A^{\dagger}AE_B - E_BA^{\dagger}A\,) = r(\,F_AE_B - E_BF_A\,) \\ &= 2r[\,A^{\dagger}A,\, BB^{\dagger}\,] - 2r(A^{\dagger}A) - 2r(BB^{\dagger}) + 2r(A^{\dagger}ABB^{\dagger}) \\ &= 2r(N) - 2r(A) - 2r(B) + 2r(AB). \end{aligned} \tag{5.30}$$

Setting both sides of (5.30) equal to zero leads to the equivalence of ⟨91⟩ and ⟨116⟩.

Applying (3.44) to $(A^{\dagger}ABB^{\dagger})^2 - A^{\dagger}ABB^{\dagger}$, $(BB^{\dagger}A^{\dagger}A)^2 - BB^{\dagger}A^{\dagger}A$, $(ABB^{\dagger}A^{\dagger})^2 - ABB^{\dagger}A^{\dagger}$, and $(B^{\dagger}A^{\dagger}AB)^2 - B^{\dagger}A^{\dagger}AB$, and simplifying by (5.8)–(5.10) yield

$$r[\,(A^{\dagger}ABB^{\dagger})^2 - A^{\dagger}ABB^{\dagger}\,] = r(\,I_n - A^{\dagger}ABB^{\dagger}\,) + r(A^{\dagger}ABB^{\dagger}) - n = r(N) - r(A) - r(B) + r(AB), \tag{5.31}$$
$$r[\,(BB^{\dagger}A^{\dagger}A)^2 - BB^{\dagger}A^{\dagger}A\,] = r(\,I_n - BB^{\dagger}A^{\dagger}A\,) + r(BB^{\dagger}A^{\dagger}A) - n = r(N) - r(A) - r(B) + r(AB), \tag{5.32}$$
$$r[\,(ABB^{\dagger}A^{\dagger})^2 - ABB^{\dagger}A^{\dagger}\,] = r(\,I_m - ABB^{\dagger}A^{\dagger}\,) + r(ABB^{\dagger}A^{\dagger}) - m = r(N) - r(A) - r(B) + r(AB), \tag{5.33}$$
$$r[\,(B^{\dagger}A^{\dagger}AB)^2 - B^{\dagger}A^{\dagger}AB\,] = r(\,I_p - B^{\dagger}A^{\dagger}AB\,) + r(B^{\dagger}A^{\dagger}AB) - p = r(N) - r(A) - r(B) + r(AB). \tag{5.34}$$

Setting all sides of (5.31)–(5.34) equal to zero leads to the equivalence of ⟨92⟩, ⟨93⟩, and ⟨116⟩. Replacing $A^{\dagger}A$ with F_A in ⟨92⟩ leads to the equivalence of ⟨94⟩ and ⟨117⟩. Replacing $BB^{\dagger}$ with E_B in ⟨92⟩ leads to the equivalence of ⟨95⟩ and ⟨118⟩. Replacing both $A^{\dagger}A$ and and $BB^{\dagger}$ with F_A and E_B in ⟨92⟩ leads to the equivalence of ⟨96⟩ and ⟨119⟩. Replacing $A^{\dagger}A$ and and $BB^{\dagger}$ with F_A and E_B in ⟨93⟩ respectively leads to the equivalence of ⟨97⟩ with ⟨117⟩ and ⟨118⟩.

Applying (3.43) to $(A^{\dagger}ABB^{\dagger})^{\dagger} - BB^{\dagger}A^{\dagger}A$ and simplifying by (3.44) and (5.8), we obtain

$$\begin{aligned} & r[\,(A^{\dagger}ABB^{\dagger})^{\dagger} - BB^{\dagger}A^{\dagger}A\,] = r[\,(A^{\dagger}ABB^{\dagger})^{\dagger} - (A^{\dagger}ABB^{\dagger})^{*}\,] \\ &= r[\,(A^{\dagger}ABB^{\dagger}) - (A^{\dagger}ABB^{\dagger})(A^{\dagger}ABB^{\dagger})^{*}(A^{\dagger}ABB^{\dagger})\,] = r[\,(A^{\dagger}ABB^{\dagger}) - (A^{\dagger}ABB^{\dagger})^2\,] \\ &= r(A^{\dagger}ABB^{\dagger}) + r(\,I_n - A^{\dagger}ABB^{\dagger}\,) - n = r(N) + r(AB) - r(A) - r(B). \end{aligned} \tag{5.35}$$

Setting all sides of (5.35) equal to zero leads to the equivalence of ⟨98⟩ and ⟨116⟩. Replacing $A^{\dagger}A$ with F_A in ⟨98⟩ leads to the equivalence of ⟨99⟩ and ⟨117⟩. Replacing $BB^{\dagger}$ with E_B in ⟨98⟩ leads to the equivalence of ⟨100⟩ and ⟨118⟩. Replacing both $A^{\dagger}A$ and $BB^{\dagger}$ with F_A and E_B in ⟨98⟩ leads to the equivalence of ⟨101⟩ and ⟨119⟩.

Applying (3.43) to the difference $(A^{\dagger}A - BB^{\dagger})^{\dagger} - (A^{\dagger}A - BB^{\dagger})$ and simplifying by (3.45)–(3.48) gives

$$\begin{aligned} & r\left[(A^{\dagger}A - BB^{\dagger})^{\dagger} - (A^{\dagger}A - BB^{\dagger})\right] \\ &= r[\,(A^{\dagger}A - BB^{\dagger}) - (A^{\dagger}A - BB^{\dagger})^3\,] \\ &= r(A^{\dagger}A - BB^{\dagger}) + r(\,I_n - A^{\dagger}A + BB^{\dagger}) + r(\,I_n + A^{\dagger}A - BB^{\dagger}) - 2n \\ &= 2r[\,A^{\dagger}A,\, BB^{\dagger}\,] - r(A^{\dagger}A) - r(BB^{\dagger}) + r[\,I_n - A^{\dagger}A,\, BB^{\dagger}\,] + r[\,I_n - BB^{\dagger},\, A^{\dagger}A\,] - 2n \\ &= 2r(N) - r(A) - r(B) + n - r(A) + r(AB) + n - r(B) + r(AB) - 2n \\ &= 2r(N) - 2r(A) - 2r(B) + 2r(AB). \end{aligned} \tag{5.36}$$

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Setting both sides of (5.36) equal to zero leads to the equivalence of ⟨102⟩ and ⟨116⟩. Replacing $A^{\dagger}A$ with F_A in (5.36) yields

$$r\left[(I_n - A^{\dagger}A - BB^{\dagger})^{\dagger} - (I_n - A^{\dagger}A - BB^{\dagger})\right] = r[\,F_A,\, B\,] - r(F_A) - r(B) + r(F_AB). \tag{5.37}$$

Setting both sides of (5.37) equal to zero leads to the equivalence of ⟨103⟩ and ⟨117⟩.

It can be derived from (3.40) that

$$\begin{aligned} r(\,2I_m - ABB^{\dagger}A^{\dagger}\,) &= r\begin{bmatrix} B^*B & B^*A^{\dagger} \\ AB & 2I_m \end{bmatrix} - r(B) = r\begin{bmatrix} B^*B - B^*A^{\dagger}AB/2 & 0 \\ 0 & 2I_m \end{bmatrix} - r(B) \\ &= m + r[\,B^*(I_n - A^{\dagger}A/2)B\,] - r(B) = m + r(B) - r(B) = m, \end{aligned} \tag{5.38}$$

$$\begin{aligned} r(\,2I_p - B^{\dagger}A^{\dagger}AB\,) &= r\begin{bmatrix} AA^* & AB \\ B^{\dagger}A^* & 2I_p \end{bmatrix} - r(A) = r\begin{bmatrix} AA^* - ABB^{\dagger}A^*/2 & 0 \\ 0 & 2I_p \end{bmatrix} - r(A) \\ &= p + r[A(\,I_p - BB^{\dagger}/2\,)A^*] - r(A) = p + r(A) - r(A) = p, \end{aligned} \tag{5.39}$$

$$\begin{aligned} r(\,2I_n - A^{\dagger}ABB^{\dagger}\,) &= r(\,2I_n - BB^{\dagger}A^{\dagger}A\,) = r\begin{bmatrix} B^*B & B^*A^{\dagger}A \\ B & 2I_n \end{bmatrix} - r(B) \\ &= r\begin{bmatrix} B^*B - B^*A^{\dagger}AB/2 & 0 \\ 0 & 2I_n \end{bmatrix} - r(B) = r[B^*(\,I_n - A^{\dagger}A/2\,)B] + n - r(B) \\ &= r(B) + n - r(B) = n. \end{aligned} \tag{5.40}$$

We then derive from (3.43), (5.8)–(5.10), and (5.38)–(5.40) the following formulas

$$\begin{aligned} & r[\,(\,I_m - ABB^{\dagger}A^{\dagger}\,)^{\dagger} - (\,I_m - ABB^{\dagger}A^{\dagger}\,)\,] \\ &= r[\,(\,I_m - ABB^{\dagger}A^{\dagger}\,) - (\,I_m - ABB^{\dagger}A^{\dagger}\,)^3\,] \\ &= r(\,I_m - ABB^{\dagger}A^{\dagger}\,) + r(\,2I_m - ABB^{\dagger}A^{\dagger}\,) + r(ABB^{\dagger}A^{\dagger}) - 2m \\ &= r(N) - r(A) - r(B) + 2m + r(AB) - 2m \\ &= r(N) - r(A) - r(B) + r(AB), \end{aligned} \tag{5.41}$$

$$\begin{aligned} & r[\,(\,I_p - B^{\dagger}A^{\dagger}AB\,)^{\dagger} - (\,I_p - B^{\dagger}A^{\dagger}AB\,)\,] \\ &= r[\,(\,I_p - B^{\dagger}A^{\dagger}AB\,) - (\,I_p - B^{\dagger}A^{\dagger}AB\,)^3\,] \\ &= r(\,I_p - B^{\dagger}A^{\dagger}AB\,) + r(\,2I_p - B^{\dagger}A^{\dagger}AB\,) + r(B^{\dagger}A^{\dagger}AB) - 2p \\ &= r(N) - r(A) - r(B) + 2p + r(AB) - 2p \\ &= r(N) - r(A) - r(B) + r(AB), \end{aligned} \tag{5.42}$$

$$\begin{aligned} & r[\,(\,I_n - A^{\dagger}ABB^{\dagger}\,)^{\dagger} - (\,I_n - A^{\dagger}ABB^{\dagger}\,)\,] \\ &= r[\,(\,I_n - A^{\dagger}ABB^{\dagger}\,) - (\,I_n - A^{\dagger}ABB^{\dagger}\,)^3\,] \\ &= r(\,I_n - A^{\dagger}ABB^{\dagger}\,) + r(\,2I_n - A^{\dagger}ABB^{\dagger}\,) + r(A^{\dagger}ABB^{\dagger}) - 2n \\ &= r(N) - r(A) - r(B) + 2n + r(AB) - 2n \\ &= r(N) - r(A) - r(B) + r(AB). \end{aligned} \tag{5.43}$$

Setting all sides of (5.41)–(5.43) equal to zero leads to the equivalence of ⟨104⟩, ⟨105⟩, and ⟨116⟩. Replacing $A^{\dagger}A$ with F_A and $BB^{\dagger}$ with E_B in ⟨104⟩ and ⟨105⟩ respectively leads to the equivalence of ⟨106⟩–⟨109⟩ and ⟨117⟩–⟨119⟩.

The following rank identities

$$\begin{aligned} r\left(NN^{\dagger} - A^{\dagger}A - BB^{\dagger} + A^{\dagger}ABB^{\dagger}\right) &= r\left(NN^{\dagger} - A^{\dagger}A - BB^{\dagger} + BB^{\dagger}A^{\dagger}A\right) \\ &= r(N) - r(A) - r(B) + r(AB) \end{aligned}$$

were proved in [44]. Setting the three sides of these two identities equal to zero leads to the equivalence of ⟨110⟩ and ⟨116⟩. Replacing A^* and B with F_A and E_B in ⟨110⟩ respectively leads to the equivalence of ⟨111⟩–⟨113⟩ and ⟨117⟩–⟨119⟩.

It follows from (3.36) and (3.37) that

$$r(\,A^{\dagger}A - A^{\dagger}ABB^{\dagger}\,) = r(E_BA^*) = r(N) - r(B), \quad r(\,BB^{\dagger} - A^{\dagger}ABB^{\dagger}\,) = r(F_AB) = r(N) - r(A).$$

Thus the two rank equalities in ⟨126⟩ are equivalent to ⟨116⟩. Replacing $A^{\dagger}A$ and $BB^{\dagger}$ with F_A and E_B in ⟨126⟩ respectively leads to the equivalence of ⟨127⟩–⟨129⟩ and ⟨117⟩–⟨119⟩.

The equivalence of ⟨120⟩–⟨125⟩ and ⟨130⟩–⟨137⟩ respectively follows from Lemma 3.6(f). The equivalence of ⟨130⟩–⟨137⟩ and ⟨138⟩–⟨145⟩ are obvious.

The equivalence of ⟨146⟩ and ⟨116⟩–⟨119⟩ follows from the following well-known dimension formulas

$$\dim[\mathscr{R}(A^*)\cap\mathscr{R}(B)] = r(A)+r(B)-r[\,A^*,\,B\,],$$
$$\dim[\mathscr{R}(F_A)\cap\mathscr{R}(B)] = r(F_A)+r(B)-r[\,F_A,\,B\,],$$
$$\dim[\mathscr{R}(A^*)\cap\mathscr{R}(E_B)] = r(A)+r(E_B)-r[\,A^*,\,E_B\,],$$
$$\dim[\mathscr{R}(F_A)\cap\mathscr{R}(E_B)] = r(F_A)+r(E_B)-r[\,F_A,\,E_B\,].$$

Applying Lemma 3.6(a) and (b) to ⟨89⟩ leads to the equivalence of ⟨89⟩ and ⟨149⟩. Replacing $A^\dagger A$ and $BB^\dagger$ with F_A and E_B in ⟨149⟩ respectively leads to the equivalence of ⟨148⟩–⟨150⟩ and ⟨117⟩–⟨119⟩.

It follows first from ⟨147⟩ that

$$\mathscr{R}(A^\dagger ABB^\dagger)\subseteq\mathscr{R}(A^\dagger A)\cap\mathscr{R}(BB^\dagger)\ \text{ and/or }\ \mathscr{R}(BB^\dagger A^\dagger A)\subseteq\mathscr{R}(A^\dagger A)\cap\mathscr{R}(BB^\dagger), \tag{5.44}$$

and from ⟨116⟩ that

$$\dim[\mathscr{R}(A^\dagger A)\cap\mathscr{R}(BB^\dagger)] = r(A^\dagger A)+r(BB^\dagger)-r[\,A^\dagger A,\,BB^\dagger\,] = r(A^\dagger ABB^\dagger) = r(BB^\dagger A^\dagger A). \tag{5.45}$$

Applying (4.123) to (5.44) and (5.45) leads to the range identities in ⟨151⟩. Conversely, ⟨151⟩ obviously implies ⟨147⟩. Replacing $A^\dagger A$ and $BB^\dagger$ with F_A and E_B in ⟨151⟩ respectively leads to the equivalence of ⟨152⟩–⟨154⟩ and ⟨117⟩–⟨119⟩.

By Lemma 3.5(c) and (3.36),

$$\begin{aligned} r[\,A^\dagger ABB^\dagger,\,BB^\dagger A^\dagger A\,] &= r[\,(I_n-BB^\dagger)A^\dagger ABB^\dagger,\,BB^\dagger A^\dagger A\,] = r[\,(I_n-BB^\dagger)A^\dagger ABB^\dagger\,]+r(BB^\dagger A^\dagger A)\\ &= r[\,B,\,A^\dagger AB\,]-r(B)+r(AB) = r[\,(I_n-A^\dagger A)B,\,A^\dagger AB\,]-r(B)+r(AB)\\ &= r[\,(I_n-A^\dagger A)B]+r(A^\dagger AB)-r(B)+r(AB)\\ &= r(N)-r(A)-r(B)+2r(AB). \end{aligned} \tag{5.46}$$

Combining (5.46) with (4.113) and (4.114), we obtain

$$\begin{aligned} r[\,A^\dagger ABB^\dagger,\,BB^\dagger A^\dagger A\,]-r(A^\dagger ABB^\dagger) &= r[\,A^\dagger ABB^\dagger,\,BB^\dagger A^\dagger A\,]-r(BB^\dagger A^\dagger A)\\ &= r(N)-r(A)-r(B)+r(AB). \end{aligned} \tag{5.47}$$

Setting both sides of (5.47) equal to zero leads to the equivalence of the first range equality in ⟨155⟩ and ⟨116⟩. Replacing $A^\dagger A$ and and $BB^\dagger$ with F_A and E_B in the first range equality of ⟨155⟩ respectively leads to the equivalences of the second, third, and fourth range equalities in ⟨155⟩ and those in ⟨117⟩–⟨119⟩.

By (3.36), the dimension of the intersection $\mathscr{R}(AB)\cap\mathscr{R}(AE_B)$ is reduced to

$$\begin{aligned} \dim[\mathscr{R}(AB)\cap\mathscr{R}(AE_B)] &= r(AB)+r(AE_B)-r[\,AB,\,AE_B\,]\\ &= r(AB)+r(N)-r(B)-r[\,AB,\,A\,]\\ &= r(N)-r(A)-r(B)+r(AB). \end{aligned} \tag{5.48}$$

Setting both sides of (5.48) equal to zero, we obtain the equivalence of the first range equality in ⟨156⟩ and ⟨116⟩. The equivalence of the other three range equalities in ⟨128⟩ with ⟨116⟩–⟨119⟩ can be shown similarly.

The equivalence of ⟨156⟩ and ⟨157⟩ follows from the definition of direct sum of linear subspaces.

Applying (2.2) to ⟨147⟩–⟨150⟩ leads to the equivalence of ⟨147⟩–⟨150⟩ and ⟨158⟩–⟨161⟩. □

Theorem 5.2. *Let $A\in\mathbb{C}^{m\times n}$ and $B\in\mathbb{C}^{n\times p}$ be given with $AB\neq 0$, and denote $N=[\,A^*,\,B\,]$. Then the following 28 statements are equivalent:*

⟨1⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,2,4)}A^{(1,2,3)}\}$.
⟨2⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,2,4)}A^{(1,3)}\}$.
⟨3⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,2,4)}A^{(1,2)}\}$.
⟨4⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,2,4)}A^{(1)}\}$.
⟨5⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,4)}A^{(1,2,3)}\}$.
⟨6⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,4)}A^{(1,3)}\}$.
⟨7⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,4)}A^{(1,2)}\}$.
⟨8⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,4)}A^{(1)}\}$.
⟨9⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,2)}A^{(1,2,3)}\}$.
⟨10⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,2)}A^{(1,3)}\}$.
⟨11⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,2)}A^{(1,2)}\}$.
⟨12⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1,2)}A^{(1)}\}$.
⟨13⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1)}A^{(1,2,3)}\}$.
⟨14⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1)}A^{(1,3)}\}$.
⟨15⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1)}A^{(1,2)}\}$.
⟨16⟩ $\{(AB)^{(1)}\}\supseteq\{B^{(1)}A^{(1)}\}$.
⟨17⟩ $\{AB(AB)^{(1)}\}\supseteq\{ABB^{(1)}A^{(1,2,3)}\}$.
⟨18⟩ $\{AB(AB)^{(1)}\}\supseteq\{ABB^{(1)}A^{(1,3)}\}$.
⟨19⟩ $\{AB(AB)^{(1)}\}\supseteq\{ABB^{(1)}A^{(1,2)}\}$.
⟨20⟩ $\{AB(AB)^{(1)}\}\supseteq\{ABB^{(1)}A^{(1)}\}$.
⟨21⟩ $\{(AB)^{(1)}AB\}\supseteq\{B^{(1,2,4)}A^{(1)}AB\}$.
⟨22⟩ $\{(AB)^{(1)}AB\}\supseteq\{B^{(1,4)}A^{(1)}AB\}$.
⟨23⟩ $\{(AB)^{(1)}AB\}\supseteq\{B^{(1,2)}A^{(1)}AB\}$.
⟨24⟩ $\{(AB)^{(1)}AB\}\supseteq\{B^{(1)}A^{(1)}AB\}$.
⟨25⟩ $\{B(AB)^{(1)}A\}\supseteq\{BB^{(1)}A^{(1)}A\}$.
⟨26⟩ *Either* $AB=0$ *or* $r(AB)=r(A)+r(B)-n$.
⟨27⟩ *Either* $AB=0$ *or* $F_AE_B=0$.
⟨28⟩ *Either* $\mathscr{N}(A)\supseteq\mathscr{R}(B)$ *or* $\mathscr{R}(A^*)\subseteq\mathscr{N}(B^*)$.

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Proof. The equivalence of ⟨1⟩–⟨16⟩, ⟨26⟩, ⟨27⟩, and ⟨28⟩ follows from (4.141) and Lemma 4.8. The equivalence of ⟨1⟩ and ⟨17⟩–⟨25⟩ follows from Lemma 4.1(a). □

We next characterize the set inclusions in (1.32) for $\{1,2\}$-, $\{1,3\}$-, and $\{1,4\}$-generalized inverses of AB.

Theorem 5.3. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ be given with $M = AB \neq 0$, and denote $N = [\,A^*,\, B\,]$ and $t = m+p+r(M)-r(A)-r(B)$. Then the following results hold.*

(a) *The following 5 statements are equivalent:*

⟨1⟩ $\{(AB)^{(1,2)}\} \ni B^{\dagger}A^{\dagger}$. ⟨2⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{\dagger}A^{(1,2,4)}\}$.
⟨3⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,3)}A^{\dagger}\}$. ⟨4⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,4)}\}$.
⟨5⟩ $ABB^{\dagger}A^{\dagger}AB = AB$ *and/or* $B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger} = B^{\dagger}A^{\dagger}$. ⟨6⟩ $r(M) = r(A)+r(B)-r(N)$.

(b) *The following 9 statements are equivalent:*

⟨1⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{\dagger}A^{(1,2,3)}\}$. ⟨2⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{\dagger}A^{(1,2)}\}$.
⟨3⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,4)}A^{\dagger}\}$. ⟨4⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,4)}\}$.
⟨5⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,3)}\}$. ⟨6⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,3)}A^{(1,2)}\}$.
⟨7⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2)}A^{\dagger}\}$. ⟨8⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2)}A^{(1,2,4)}\}$.
⟨9⟩ *Either* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$ *or* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

(c) *The following 9 statements are equivalent:*

⟨1⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3,4)}A^{(1,2)}\}$
⟨3⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,4)}A^{\dagger}\}$. ⟨4⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,4)}A^{(1,2,4)}\}$.
⟨5⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3)}A^{(1,2,3)}\}$. ⟨6⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3)}A^{(1,2)}\}$.
⟨7⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1)}A^{\dagger}\}$. ⟨8⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1)}A^{(1,2,4)}\}$.
⟨9⟩ *Either* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$, *or* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$ *and* $r(B) = p$.

(d) *The following 9 statements are equivalent:*

⟨1⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{\dagger}A^{(1,3)}\}$. ⟨2⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{\dagger}A^{(1)}\}$.
⟨3⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,4)}A^{(1,3,4)}\}$. ⟨4⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,4)}A^{(1,4)}\}$.
⟨5⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,3)}A^{(1,3)}\}$. ⟨6⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,3)}A^{(1)}\}$.
⟨7⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2)}A^{(1,3,4)}\}$. ⟨8⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2)}A^{(1,4)}\}$.
⟨9⟩ *Either* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$, *or* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

(e) *The following 5 statements are equivalent:*

⟨1⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3,4)}A^{\dagger}\}$. ⟨2⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,4)}\}$.
⟨3⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3)}A^{\dagger}\}$. ⟨4⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3)}A^{(1,2,4)}\}$.
⟨5⟩ *Either* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$, *or* $r(B) = p$ *and* $r(M) = r(A)+r(B)-r(N)$.

(f) *The following 5 statements are equivalent:*

⟨1⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{\dagger}A^{(1,3,4)}\}$. ⟨2⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{\dagger}A^{(1,4)}\}$.
⟨3⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,3)}A^{(1,3,4)}\}$. ⟨4⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,3)}A^{(1,4)}\}$.
⟨5⟩ *Either* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$, *or* $r(A) = m$ *and* $r(M) = r(A)+r(B)-r(N)$.

(g) *The following 9 statements are equivalent:*

⟨1⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3,4)}A^{(1,3)}\}$. ⟨2⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3,4)}A^{(1)}\}$.
⟨3⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,4)}A^{(1,3,4)}\}$. ⟨4⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,4)}A^{(1,4)}\}$.
⟨5⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3)}A^{(1,3)}\}$. ⟨6⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3)}A^{(1)}\}$.
⟨7⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1)}A^{(1,3,4)}\}$. ⟨8⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1)}A^{(1,4)}\}$.
⟨9⟩ $r(M) = r(A)+r(B)-r(N) = \min\{\,m,\ n,\ p\,\}$.

(h) *The following 5 statements are equivalent:*

⟨1⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3,4)}A^{(1,3,4)}\}$. ⟨2⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3,4)}A^{(1,4)}\}$.
⟨3⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3)}A^{(1,3,4)}\}$. ⟨4⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,3)}A^{(1,4)}\}$.
⟨5⟩ $r(M) = r(A)+r(B)-r(N) = \min\{\,m,\ n,\ p,\ t\,\}$.

(i) *The following 17 statements are equivalent:*

⟨1⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,4)}A^{(1,2)}\}$.
⟨3⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2)}A^{(1,2,3)}\}$. ⟨4⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2)}A^{(1,2)}\}$.
⟨5⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,4)}A^{(1,3)}\}$. ⟨6⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2,4)}A^{(1)}\}$.
⟨7⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2)}A^{(1,3)}\}$. ⟨8⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,2)}A^{(1)}\}$.
⟨9⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,4)}A^{(1,2,3)}\}$. ⟨10⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,4)}A^{(1,2)}\}$.
⟨11⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1)}A^{(1,2,3)}\}$. ⟨12⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1)}A^{(1,2)}\}$.
⟨13⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,4)}A^{(1,3)}\}$. ⟨14⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1,4)}A^{(1)}\}$.
⟨15⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1)}A^{(1,3)}\}$. ⟨16⟩ $\{(AB)^{(1,2)}\} \supseteq \{B^{(1)}A^{(1)}\}$.
⟨17⟩ $r(A) = r(B) = n$ *or equivalently,* $r(M) = n$.

Proof. Applying (4.132) to $B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger} - B^{\dagger}A^{\dagger}$ and simplifying by Lemma 4.6, we obtain

$$\begin{aligned} r(\, B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger} - B^{\dagger}A^{\dagger}) &= r[\,(B^{\dagger})^{*},\, A^{\dagger}\,] - r(B^{\dagger}A^{\dagger}) - r(A^{\dagger}) - r(B^{\dagger}) \\ &= r(N) + r(M) - r(A) - r(B). \end{aligned} \tag{5.49}$$

Setting both sides of (5.49) equal to zero leads to the equivalence of the second rank equality in ⟨5⟩ and ⟨6⟩. All the remaining equivalences in (a)–(i) follow from Lemma 4.1(b), Lemma 4.3, Theorem 5.1, as well as the simplification of the combined conditions. □

Theorem 5.4. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$ *be given with* $M = AB \neq 0$, *and denote* $N = [\,A^{*},\, B\,]$. *Then the following results hold.*

(a) *The following 26 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \ni B^{\dagger}A^{\dagger}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{\dagger}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{\dagger}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{\dagger}\}$.
⟨5⟩ $MM^{\dagger} = MB^{\dagger}A^{\dagger}$. ⟨6⟩ $M^{*}MB^{\dagger}A^{\dagger} = M^{*}$.
⟨7⟩ $ABB^{\dagger}A^{*}M = AA^{*}M$. ⟨8⟩ $BB^{\dagger}A^{*}M = A^{*}M$.
⟨9⟩ $(ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger}$. ⟨10⟩ $B^{*}(ABB^{\dagger})^{\dagger}AA^{*} = M^{*}$.
⟨11⟩ $(AE_B)^{\dagger} = E_BA^{\dagger}$. ⟨12⟩ $A(AE_B)^{\dagger} = AE_BA^{\dagger}$.
⟨13⟩ $MM^{\dagger}A = MB^{\dagger}$. ⟨14⟩ $A^{*}ABB^{\dagger} = BB^{\dagger}A^{*}A$.
⟨15⟩ $r[\,A^{*}M,\, B\,] = r(B)$. ⟨16⟩ $r[\,A^{*}AE_B,\, E_B\,] = r(E_B)$.
⟨17⟩ $\mathscr{R}(A^{*}M) \subseteq \mathscr{R}(B)$. ⟨18⟩ $\mathscr{R}(A^{*}AE_B) \subseteq \mathscr{N}(B^{*})$.
⟨19⟩ $\mathscr{R}(A^{*}M) = \mathscr{R}(A^{*}) \cap \mathscr{R}(B)$. ⟨20⟩ $\mathscr{R}(A^{*}AE_B) = \mathscr{R}(A^{*}) \cap \mathscr{N}(B^{*})$.
⟨21⟩ $\mathscr{R}(A^{*}ABB^{\dagger}) = \mathscr{R}(BB^{\dagger}A^{*}A)$. ⟨22⟩ $\mathscr{R}(A^{*}AE_B) = \mathscr{R}(E_BA^{*}A)$.
⟨23⟩ $MB^{\dagger}A^{\dagger}$ *is an orthogonal projector, i.e.,* $(MB^{\dagger}A^{\dagger})^2 = MB^{\dagger}A^{\dagger} = (MB^{\dagger}A^{\dagger})^{*}$.
⟨24⟩ $AE_BA^{\dagger}$ *is an orthogonal projector, i.e.,* $(AE_BA^{\dagger})^2 = AE_BA^{\dagger} = (AE_BA^{\dagger})^{*}$.
⟨25⟩ *Both* $MB^{\dagger}A^{\dagger}M = M$ *and* $(MB^{\dagger}A^{\dagger})^{*} = MB^{\dagger}A^{\dagger}$.
⟨26⟩ *Both* $r(N) = r(A) + r(B) - r(M)$ *and* $r[\,AA^{*}M,\, M\,] = r(M)$.

(b) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{\dagger}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,3,4)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{(1,3,4)}\}$.
⟨5⟩ $M^{*}MB^{\dagger}F_AUE_A \equiv M^{*} - M^{*}MB^{\dagger}A^{\dagger}$ *for all* U.
⟨6⟩ $\mathscr{R}(A^{*}M) \subseteq \mathscr{R}(B)$.

(c) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{\dagger}A^{(1,2,4)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,4)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,4)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{(1,2,4)}\}$.
⟨5⟩ $M^{*}MB^{\dagger}A^{\dagger}AUE_A \equiv M^{*} - M^{*}MB^{\dagger}A^{\dagger}$ *for all* U.
⟨6⟩ *Both* $r(A) = m$ *and* $\mathscr{R}(A^{*}M) \subseteq \mathscr{R}(B)$.

(d) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{\dagger}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,3)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,3)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{(1,2,3)}\}$.
⟨5⟩ $M^{*}MB^{\dagger}F_AUAA^{\dagger} \equiv M^{*} - M^{*}MB^{\dagger}A^{\dagger}$ *for all* U.
⟨6⟩ $\mathscr{R}(A^{*}M) \subseteq \mathscr{R}(B)$.

(e) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{\dagger}A^{(1,4)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,4)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,4)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{(1,4)}\}$.
⟨5⟩ $M^{*}MB^{\dagger}UE_A \equiv M^{*} - M^{*}MB^{\dagger}A^{\dagger}$ *for all* U.
⟨6⟩ *Both* $r(A) = m$ *and* $\mathscr{R}(A^{*}M) \subseteq \mathscr{R}(B)$.

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(f) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{\dagger}A^{(1,3)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,3)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{(1,3)}\}$.
⟨5⟩ $M^*MB^{\dagger}F_AU \equiv M^* - M^*MB^{\dagger}A^{\dagger}$ *for all* U.
⟨6⟩ $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$.

(g) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{\dagger}A^{(1,2)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,2)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{(1,2)}\}$.
⟨5⟩ $(M^*MB^{\dagger}A^{\dagger} + M^*MB^{\dagger}F_AU_1)(AA^{\dagger} + AU_2E_A) \equiv M^*$ *for all* U_1 *and* U_2.
⟨6⟩ *Both* $r(A) = m$ *and* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$.

(h) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{\dagger}A^{(1)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{(1)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{(1)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{(1)}\}$.
⟨5⟩ $M^*MB^{\dagger}F_AU_1 + M^*MB^{\dagger}U_2E_A \equiv M^* - M^*MB^{\dagger}A^{\dagger}$ *for all* U_1 *and* U_2.
⟨6⟩ *Both* $r(A) = m$ *and* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$.

(i) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,4)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,4)}A^{\dagger}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2)}A^{\dagger}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1)}A^{\dagger}\}$.
⟨5⟩ $M^*MVE_BA^{\dagger} \equiv M^* - M^*MB^{\dagger}A^{\dagger}$ *for all* V.
⟨6⟩ $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

(j) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,4)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2)}A^{(1,3,4)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1)}A^{(1,3,4)}\}$.
⟨5⟩ $(M^*MB^{\dagger} + M^*MVE_B)(A^{\dagger} + F_AUE_A) \equiv M^*$ *for all* U *and* V.
⟨6⟩ *Either* $r(B) = n$, *or* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

(k) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,4)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,4)}A^{(1,2,4)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2)}A^{(1,2,4)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1)}A^{(1,2,4)}\}$.
⟨5⟩ $(M^*MB^{\dagger} + M^*MVE_B)(A^{\dagger} + A^{\dagger}AUE_A) \equiv M^*$ *for all* U *and* V.
⟨6⟩ *Both* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

(l) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,4)}A^{(1,2,3)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2)}A^{(1,2,3)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1)}A^{(1,2,3)}\}$.
⟨5⟩ $(M^*MB^{\dagger} + M^*MVE_B)(A^{\dagger} + F_AUAA^{\dagger}) \equiv M^*$ *for all* U *and* V.
⟨6⟩ $r(B) = n$.

(m) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,4)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,4)}A^{(1,4)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2)}A^{(1,4)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1)}A^{(1,4)}\}$.
⟨5⟩ $(M^*MB^{\dagger} + M^*MVE_B)(A^{\dagger} + UE_A) \equiv M^*$ *for all* U *and* V.
⟨6⟩ *Both* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

(n) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,3)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,4)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2)}A^{(1,3)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1)}A^{(1,3)}\}$.
⟨5⟩ $(M^*MB^{\dagger} + M^*MVE_B)(A^{\dagger} + F_AU) \equiv M^*$ *for all* U *and* V.
⟨6⟩ $r(B) = n$.

(o) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,2)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,4)}A^{(1,2)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1)}A^{(1,2)}\}$.
⟨5⟩ $(M^*MB^{\dagger} + M^*MVE_B)(A^{\dagger} + F_AU_1)(AA^{\dagger} + AU_2E_A) \equiv M^*$ *for all* U_1, *and* U_2, *and* V.
⟨6⟩ *Both* $r(A) = m$ *and* $r(B) = n$.

(p) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2,4)}A^{(1)}\}$. ⟨2⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,4)}A^{(1)}\}$.
⟨3⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1,2)}A^{(1)}\}$. ⟨4⟩ $\{M^{(1,3)}\} \supseteq \{B^{(1)}A^{(1)}\}$.
⟨5⟩ $(M^*MB^\dagger + M^*MVE_B)(A^\dagger + F_AU_1 + U_2E_A) \equiv M^*$ *for all* U_1, *and* U_2, *and* V.
⟨6⟩ *Both* $r(A) = m$ *and* $r(B) = n$.

Proof. The equivalence of ⟨1⟩ and ⟨25⟩ in (a) follows from the definition of {1, 3}-generalized inverses of a matrix.

The equivalence of ⟨1⟩–⟨6⟩ in (a) follows from (4.5) and (4.75).

Applying (3.42) and simplifying gives

$$\begin{aligned} r(\,MM^\dagger - MB^\dagger A^\dagger\,) &= r(\,M^* - M^*MB^\dagger A^\dagger\,) = r[\,M^*AA^* - M^*MB^\dagger A^*\,) \\ &= r(\,AA^*AB - ABB^\dagger A^*AB\,) = r\begin{bmatrix} B^*B & B^*A^*AB \\ AB & AA^*AB \end{bmatrix} - r(B) \\ &= r\left(\begin{bmatrix} B^* \\ A \end{bmatrix}[\,B,\ A^*AB\,]\right) - r(B) = r\left(\begin{bmatrix} B^* \\ B^*A^*A \end{bmatrix}[\,B,\ A^*AB\,]\right) - r(B) \\ &= r[\,A^*AB,\ B\,] - r(B). \end{aligned} \tag{5.50}$$

Setting all sides of (5.50) equal to zero leads to the equivalence of ⟨5⟩–⟨7⟩, ⟨15⟩, and ⟨17⟩.

By (3.50),

$$r(\,BB^\dagger A^*M - A^*M\,) = r(E_BA^*M) = r[\,A^*AB,\ B\,] - r(B). \tag{5.51}$$

Setting all sides of (5.51) equal to zero leads to the equivalence of ⟨8⟩ and ⟨15⟩.

By (3.36),

$$r[\,A^*AE_B,\ E_B\,] - r(E_B) = r(BB^\dagger A^*AE_B) = r(B^*A^*AE_B) = r[\,A^*AB,\ B^*\,] - r(B). \tag{5.52}$$

Setting all sides of (5.52) equal to zero leads to the equivalence of ⟨15⟩, ⟨16⟩, and ⟨18⟩.

It follows from $t_5 \geq t_6$ in (3.54) that

$$r[\,A^*AB,\ B\,] - r(B) \geqslant r(M) + r(M) - r(A) - r(B) \geqslant 0. \tag{5.53}$$

Also by (4.124),

$$\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B) \Rightarrow \mathscr{R}(AA^*AB) \subseteq \mathscr{R}(AB). \tag{5.54}$$

Combining (5.53) and (5.54) yields

$$r[\,A^*AB,\ B\,] = r(B) \Rightarrow \text{both } r(M) = r(A) + r(B) - r(M) \text{ and } r[\,AA^*AB,\ AB\,] = r(M). \tag{5.55}$$

So that ⟨28⟩ implies ⟨15⟩. Conversely, substituting the two rank equalities in ⟨28⟩ into both sides of $t_4 \geq t_5$ in (3.54) yields $r[\,A^*AB,\ B\,] \leqslant r(B)$, which in fact implies $r[\,A^*AB,\ B\,] = r(B)$. So that ⟨15⟩ is equivalent to ⟨26⟩.

Applying (3.40) to $(ABB^\dagger)^\dagger - BB^\dagger A^\dagger$ and simplifying yields

$$\begin{aligned} r[(ABB^\dagger)^\dagger - BB^\dagger A^\dagger] &= r[B^*(ABB^\dagger)^\dagger AA^* - B^*A^*] \\ &= r\begin{bmatrix} (ABB^\dagger)^*(ABB^\dagger)(ABB^\dagger)^* & (ABB^\dagger)^*AA^* \\ B^*(ABB^\dagger)^* & B^*A^* \end{bmatrix} - r(ABB^\dagger) \\ &= r\begin{bmatrix} B^*A^*ABB^\dagger A^* & B^*A^*AA^* \\ B^*A^* & B^*A^* \end{bmatrix} - r(M) \\ &= r\begin{bmatrix} 0 & B^*A^*AA^* - B^*A^*ABB^\dagger A^* \\ B^*A^* & 0 \end{bmatrix} - r(M) \\ &= r(\,AA^*AB - ABB^\dagger A^*AB\,) \\ &= r[\,A^*AB,\ B\,] - r(B) \ \ (\text{by } (5.50)). \end{aligned} \tag{5.56}$$

Setting all sides of (5.56) equal to zero leads to the equivalence of ⟨9⟩, ⟨10⟩, and ⟨15⟩.

Replace $BB^\dagger$ with E_B in (5.56) and applying (5.52) yields

$$r[(AE_B)^\dagger - E_BA^\dagger] = r[\,A^*AE_B,\ E_B\,] - r(E_B) = r[\,A^*AB,\ B^*\,] - r(B). \tag{5.57}$$

Setting all sides of (5.57) equal to zero leads to the equivalence of ⟨11⟩ and ⟨15⟩.

By (5.50) and (5.52),

$$\begin{aligned} r[A(AE_B)^\dagger - AE_BA^\dagger] &= r[(AE_B)(AE_B)^\dagger - (AE_B)(E_B)^\dagger A^\dagger] \\ &= r[\,A^*AE_B,\, E_B\,] - r(E_B) \\ &= r[\,A^*AB,\, B^*\,] - r(B). \end{aligned} \tag{5.58}$$

Setting all sides of (5.58) equal to zero leads to the equivalence of ⟨12⟩ and ⟨15⟩.

By (3.36),

$$r(\,MM^\dagger A - MB^\dagger\,) = r[\,M^*A - M^*ABB^\dagger\,] = r(E_BA^*M) = r[\,A^*M,\, B\,] - r(B). \tag{5.59}$$

Setting both sides of (5.59) equal to zero leads to the equivalence of ⟨13⟩ and ⟨15⟩.

By (3.50),

$$r(\,A^*ABB^\dagger - BB^\dagger A^*A\,) = 2r[\,A^*ABB^\dagger,\, BB^\dagger\,] - 2r(BB^\dagger) = 2r[\,A^*AB,\, B\,] - 2r(B). \tag{5.60}$$

Setting all sides of (5.60) equal to zero leads to the equivalence of ⟨14⟩ and ⟨15⟩.

It follows from ⟨17⟩ that

$$\mathscr{R}(A^*AB) \subseteq \mathscr{R}(A^*) \cap \mathscr{R}(B), \tag{5.61}$$

and from (3.34) and ⟨28⟩ that

$$\dim[\mathscr{R}(A^*) \cap \mathscr{R}(B)] = r(A) + r(B) - r[\,A^*,\, B\,] = r(M). \tag{5.62}$$

Applying (4.123) to (5.61) and (5.62) leads to the range identities in ⟨19⟩. Conversely, ⟨19⟩ obviously implies ⟨17⟩. The equivalence of ⟨18⟩ and ⟨20⟩ can be shown similarly.

Replace $BB^\dagger$ with E_B in (5.63) and applying (5.52) yields

$$\mathscr{R}(A^*AE_B) = \mathscr{R}(E_BA^*A) \Leftrightarrow r[\,A^*AE_B,\, E_B\,] = r(E_B),$$

establishing the equivalence of ⟨16⟩ and ⟨22⟩.

Since $MM^\dagger$ is both idempotent and Hermitian, ⟨5⟩ implies ⟨23⟩. Conversely, the first equality in ⟨23⟩ is equivalent to $MB^\dagger A^\dagger M = M$ by Theorem 5.1⟨48⟩ and ⟨93⟩. This equality together with the second equality in ⟨23⟩ implies ⟨1⟩.

Replace $BB^\dagger$ with E_B in ⟨23⟩ leads to the equivalence of ⟨16⟩ and ⟨24⟩.

Applying Lemma 4.6 and (3.36) to $[\,A^*ABB^\dagger,\, BB^\dagger A^*A]$ yields

$$\begin{aligned} r[\,A^*ABB^\dagger,\, BB^\dagger A^*A] &= r[\,A^*AB,\, BB^\dagger A^*\,] \\ &= r[\,A^*AB - BB^\dagger A^*AB,\, BB^\dagger A^*\,] \\ &= r[\,(\,I_n - BB^\dagger\,)A^*AB,\, BB^\dagger A^*\,] \\ &= r[\,(\,I_n - BB^\dagger\,)A^*AB\,] + r(BB^\dagger A^*) \\ &= r[\,A^*AB,\, B\,] + r(M) - r(B). \end{aligned}$$

Hence,

$$\begin{aligned} \mathscr{R}(A^*ABB^\dagger) = \mathscr{R}(BB^\dagger A^*A) &\Leftrightarrow r[\,A^*ABB^\dagger,\, BB^\dagger A^*A] = r(A^*ABB^\dagger) = r(BB^\dagger A^*A) = r(M) \\ &\Leftrightarrow r[\,A^*AB,\, B\,] = r(B), \end{aligned} \tag{5.63}$$

establishing the equivalence of ⟨15⟩ and ⟨21⟩. Replace $BB^\dagger$ with E_B in ⟨21⟩ leads to the equivalence of ⟨22⟩ and ⟨16⟩.

The equivalence of ⟨1⟩–⟨5⟩ in (b) follows from (4.76). By Lemma 2.2, the matrix equation in ⟨5⟩ holds for all U if and only if

$$[\,M^*MB^\dagger A^\dagger - M^*,\, M^*MB^\dagger F_A\,] = 0 \;\text{ or }\; \begin{bmatrix} M^*MB^\dagger A^\dagger - M^* \\ E_A \end{bmatrix} = 0, \tag{5.64}$$

where by (3.37)

$$\begin{aligned} & r[\,M^*MB^\dagger A^\dagger - M^*,\, M^*MB^\dagger F_A\,] \\ &= r\begin{bmatrix} M^*MB^\dagger A^\dagger - M^* & M^*MB^\dagger \\ 0 & A \end{bmatrix} - r(A) \\ &= r\begin{bmatrix} -B^*A^* & M^*MB^\dagger \\ -AA^\dagger & A \end{bmatrix} - r(A) = r\begin{bmatrix} B^*A^*A & M^*MB^\dagger \\ A & A \end{bmatrix} - r(A) \\ &= r\begin{bmatrix} 0 & M^*MB^\dagger - B^*A^*A \\ A & 0 \end{bmatrix} - r(A) = r(BB^\dagger A^*M - A^*M) \\ &= r[\,B,\, A^*M\,] - r(B) \quad (\text{by } (3.36)), \end{aligned} \tag{5.65}$$

and

$$r\begin{bmatrix} M^*MB^\dagger A^\dagger - M^* \\ E_A \end{bmatrix} = r(M^*MB^\dagger A^\dagger - M^*) + r(E_A) \quad \text{(by Lemma 3.5(d))}$$
$$= m - r(A) + r[\,B,\, A^*M\,] - r(B). \tag{5.66}$$

Combining (5.64) with (5.65) and (5.66) leads to

$$[\,M^*MB^\dagger F_A,\, M^*MB^\dagger A^\dagger - M^*\,] = 0 \Leftrightarrow M^*MB^\dagger A^\dagger - M^* = 0 \Leftrightarrow \mathscr{R}(A^*M) \subseteq \mathscr{R}(B), \tag{5.67}$$
$$\begin{bmatrix} M^*MB^\dagger A^\dagger - M^* \\ E_A \end{bmatrix} = 0 \Leftrightarrow r(A) = m \ \text{ and } \ \mathscr{R}(A^*M) \subseteq \mathscr{R}(B). \tag{5.68}$$

Combining (5.67) with (5.68) and (5.64) leads to the equivalence of $\langle 5\rangle$ and $\langle 6\rangle$ in (b).

The equivalence of $\langle 1\rangle$–$\langle 5\rangle$ in (c) follows from (4.78). By Lemma 2.2, the matrix equation in $\langle 5\rangle$ holds for all U if and only if

$$[\,M^*MB^\dagger A^\dagger - M^*,\, M^*MB^\dagger A^\dagger A\,] = 0 \text{ or } \begin{bmatrix} M^*MB^\dagger A^\dagger - M^* \\ E_A \end{bmatrix} = 0, \tag{5.69}$$

where the rank of the first block matrix in (5.69) is

$$r[\,M^*MB^\dagger A^\dagger - M^*,\, M^*MB^\dagger A^\dagger A\,] = r[\,-M^*,\, M^*MB^\dagger A^\dagger A\,] = r[\,M^*,\, 0\,] = r(M) \neq 0, \tag{5.70}$$

a contradiction to the first equality in (5.69). In this case, combining (5.68) with (5.69) leads to the equivalence of $\langle 5\rangle$ and $\langle 6\rangle$ in (c).

The equivalence of $\langle 1\rangle$–$\langle 5\rangle$ in (d) follows from (4.79). By Lemma 2.2, the matrix equation in $\langle 5\rangle$ holds for all U if and only if

$$[\,M^*MB^\dagger A^\dagger - M^*,\, M^*MB^\dagger F_A\,] = 0 \text{ or } \begin{bmatrix} M^*MB^\dagger A^\dagger - M^* \\ AA^\dagger \end{bmatrix} = 0, \tag{5.71}$$

where the first equality in (5.71) is equivalent to (5.67), and the second equality is a contradiction to $A \neq 0$ Thus $\langle 5\rangle$ and $\langle 6\rangle$ in (d) are equivalent.

The equivalence of $\langle 1\rangle$–$\langle 5\rangle$ in (e) follows from (4.80). By Lemma 2.2, the matrix equation in $\langle 5\rangle$ holds for all U if and only if

$$[\,M^*MB^\dagger A^\dagger - M^*,\, M^*MB^\dagger\,] = 0 \text{ or } \begin{bmatrix} M^*MB^\dagger A^\dagger - M^* \\ E_A \end{bmatrix} = 0, \tag{5.72}$$

where the rank of the first block matrix in (5.72) is

$$r[\,M^*MB^\dagger A^\dagger - M^*,\, M^*MB^\dagger\,] = r[\,-M^*,\, M^*MB^\dagger\,] = r[\,M^*,\, 0\,] = r(M) \neq 0, \tag{5.73}$$

a contradiction to the first equality in (5.72), while the second equality in (5.72) is equivalent to (5.68). Combining (5.72) with (5.68) and (5.73) leads to the equivalence of $\langle 5\rangle$ and $\langle 6\rangle$ in (e).

The equivalence of $\langle 1\rangle$–$\langle 5\rangle$ in (f) follows from (4.81). By Lemma 2.2, the matrix equation in $\langle 5\rangle$ holds for all U if and only if $[\,M^*MB^\dagger A^\dagger - M^*,\, M^*MB^\dagger F_A\,] = 0$, which is equivalent to (5.67). Thus $\langle 5\rangle$ and $\langle 6\rangle$ in (f) are equivalent.

The equivalence of $\langle 1\rangle$–$\langle 5\rangle$ in (g) follows from (4.82). By Lemma 2.4(e), the matrix equation in $\langle 5\rangle$ holds for all U_1 and U_2 if and only if

$$\begin{bmatrix} M^*MB^\dagger A^\dagger - M^* & M^*MB^\dagger F_A \\ E_A & 0 \end{bmatrix} = 0, \tag{5.74}$$

where by Lemma 3.5(d)

$$r\begin{bmatrix} M^*MB^\dagger A^\dagger - M^* & M^*MB^\dagger F_A \\ E_A & 0 \end{bmatrix} = r[\,M^*MB^\dagger A^\dagger - M^*,\, M^*MB^\dagger F_A\,] + r(E_A)$$
$$= m - r(A) + r[\,B,\, A^*M\,] - r(B) \quad \text{(by (5.65))}. \tag{5.75}$$

Thus

$$\begin{bmatrix} M^*MB^\dagger A^\dagger - M^* & M^*MB^\dagger F_A \\ E_A & 0 \end{bmatrix} = 0 \Leftrightarrow r(A) = m \text{ and } \mathscr{R}(A^*M) \subseteq \mathscr{R}(B), \tag{5.76}$$

establishing the equivalence of $\langle 5\rangle$ and $\langle 6\rangle$ in (g).

The equivalence of ⟨1⟩–⟨5⟩ in (h) follows from (4.83). By Lemma 2.3(b), the matrix equation in ⟨5⟩ holds for all U_1 and U_2 if and only if $\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* & M^*MB^{\dagger}F_A \\ E_A & 0 \end{bmatrix} = 0$, which is equivalent to ⟨6⟩ in (h) by (5.76).

The equivalence of ⟨1⟩–⟨5⟩ in (i) follows from (4.84). By Lemma 2.2, the matrix equation in ⟨5⟩ holds for all V if and only if

$$[\, M^*MB^{\dagger}A^{\dagger} - M^*,\ M^*M\,] = 0 \text{ or } \begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* \\ E_BA^{\dagger} \end{bmatrix} = 0, \tag{5.77}$$

where the rank of the first block matrix in (5.77) is

$$r[\, M^*MB^{\dagger}A^{\dagger} - M^*,\ M^*M\,] = r[\, 0,\ M^*M\,] = r(M) \neq 0, \tag{5.78}$$

a contradiction to the first equality in (5.77), and the rank of the second block matrix in (5.77) is

$$\begin{aligned} r\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* \\ E_BA^{\dagger} \end{bmatrix} &= r\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* & 0 \\ A^{\dagger} & B \end{bmatrix} - r(B) = r\begin{bmatrix} -M^* & -M^*M \\ A^{\dagger} & B \end{bmatrix} - r(B) \\ &= r\begin{bmatrix} (AB)^*AA^* & (AB)^*AB \\ A^* & B \end{bmatrix} - r(B) = r\begin{bmatrix} 0 & 0 \\ A^* & B \end{bmatrix} - r(B) \\ &= r[A^*,\ B\,] - r(B). \end{aligned} \tag{5.79}$$

Thus

$$\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* \\ E_BA^{\dagger} \end{bmatrix} = 0 \Leftrightarrow r[A^*,\ B\,] = r(B) \Leftrightarrow \mathscr{R}(A^*M) \subseteq \mathscr{R}(B). \tag{5.80}$$

Combining (5.77) and (5.78) with (5.80) leads to the equivalence of ⟨5⟩ and ⟨6⟩ in (i).

The equivalence of ⟨1⟩–⟨5⟩ in (j) follows from (4.85). By Lemma 2.4(a), the matrix equation in ⟨5⟩ holds for all U and V if and only if

$$\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* & M^*MB^{\dagger}F_A \\ E_BA^{\dagger} & F_BF_A \end{bmatrix} = 0 \text{ or } \begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* \\ E_BA^{\dagger} \\ E_A \end{bmatrix} = 0, \tag{5.81}$$

where the ranks of the two block matrices in (5.81) are

$$\begin{aligned} r\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* & M^*MB^{\dagger}F_A \\ E_BA^{\dagger} & E_BF_A \end{bmatrix} &= r\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* & M^*ABB^{\dagger}(I_n - A^{\dagger}A) \\ E_BA^{\dagger} & E_B(I_n - A^{\dagger}A) \end{bmatrix} \\ &= r\begin{bmatrix} M^*ABB^{\dagger}A^{\dagger} - M^* & -M^*AE_B \\ E_BA^{\dagger} & E_B \end{bmatrix} \\ &= r\begin{bmatrix} 0 & 0 \\ 0 & E_B \end{bmatrix} = n - r(B), \end{aligned} \tag{5.82}$$

$$\begin{aligned} r\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* \\ E_BA^{\dagger} \\ E_A \end{bmatrix} &= r\begin{bmatrix} M^*ABB^{\dagger}A^{\dagger} - M^* \\ E_BA^{\dagger} \end{bmatrix} + r(E_A) = r\begin{bmatrix} 0 \\ E_BA^{\dagger} \end{bmatrix} + r(E_A) \\ &= r[\, A^*,\ B\,] - r(B) + m - r(A). \end{aligned} \tag{5.83}$$

Combining (5.81) with (5.82) and (5.83) yields

$$\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* & M^*MB^{\dagger}F_A \\ E_BA^{\dagger} & E_BF_A \end{bmatrix} = 0 \Leftrightarrow r(B) = n, \tag{5.84}$$

$$\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* \\ E_BA^{\dagger} \\ E_A \end{bmatrix} = 0 \Leftrightarrow r(A) = m \text{ and } \mathscr{R}(A^*) \subseteq \mathscr{R}(B), \tag{5.85}$$

establishing the equivalence of ⟨5⟩ and ⟨6⟩ in (j).

The two groups of equivalence in ⟨1⟩–⟨5⟩ of (k) and (m) follow from (4.86) and (4.88), respectively. By Lemma 2.4(b), the two matrix equations in ⟨5⟩ of (k) and (m) hold for all U and V if and only if $\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* \\ E_BA^{\dagger} \\ E_A \end{bmatrix} = 0$, which, by (5.85), is equivalent to ⟨6⟩ in (k) and (m), respectively.

The two groups of equivalence in ⟨1⟩–⟨5⟩ of (l) and (n) follow from (4.87) and (4.89), respectively. By Lemma 2.4(b), the two matrix equations in ⟨5⟩ holds for all U and V if and only if $\begin{bmatrix} M^*MB^{\dagger}A^{\dagger} - M^* & M^*MB^{\dagger}F_A \\ E_BA^{\dagger} & F_BF_A \end{bmatrix} = 0$, which, by (5.84), is equivalent to ⟨6⟩ in (l) and (n), respectively.

The equivalence of ⟨1⟩–⟨5⟩ in (o) follows from (4.90). By Lemma 2.5(b), the matrix equation in ⟨5⟩ holds for all U_1, U_2, and V if and only if

$$\begin{bmatrix} M^*MB^\dagger A^\dagger - M^* & M^*MB^\dagger F_A \\ E_BA^\dagger & E_BF_A \\ E_A & 0 \end{bmatrix} = 0, \tag{5.86}$$

where by Lemma 3.5(d) and (5.82)

$$r\begin{bmatrix} M^*MB^\dagger A^\dagger - M^* & M^*MB^\dagger F_A \\ E_BA^\dagger & E_BF_A \\ E_A & 0 \end{bmatrix} = r\begin{bmatrix} M^*MB^\dagger A^\dagger - M^* & M^*MB^\dagger F_A \\ E_BA^\dagger & E_BF_A \end{bmatrix} + r(E_A)$$
$$= m - r(A) + n - r(B). \tag{5.87}$$

Thus

$$\begin{bmatrix} M^*MB^\dagger A^\dagger - M^* & M^*MB^\dagger F_A \\ E_BA^\dagger & E_BF_A \\ E_A & 0 \end{bmatrix} = 0 \Leftrightarrow r(A) = m \text{ and } r(B) = n, \tag{5.88}$$

establishing the equivalence of ⟨5⟩ and ⟨6⟩ in (o).

The equivalence of ⟨1⟩–⟨5⟩ in (p) follows from (4.90). By Lemma 2.5(b), the matrix equation in ⟨5⟩ holds for all U_1, U_2, and V if and only if (5.87) holds, which is equivalent to ⟨6⟩ in (p) by (5.88). □

The following theorem can be established by a similar approach, and the details are therefore omitted.

Theorem 5.5. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ be given with $M = AB \neq 0$, and denote $N = [\,A^*,\, B\,]$. Then the following results hold.*

(a) *The following 26 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^\dagger\}$.
⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,4)}\}$.
⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,4)}\}$.
⟨5⟩ $M^\dagger M = B^\dagger A^\dagger M$.
⟨6⟩ $B^\dagger A^\dagger MM^* = M^*$.
⟨7⟩ $B^*A^\dagger ABM^* = B^*BM^*$.
⟨8⟩ $A^\dagger ABM^* = BM^*$.
⟨9⟩ $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$.
⟨10⟩ $B^*B(A^\dagger AB)^\dagger A^* = M^*$.
⟨11⟩ $(F_AB)^\dagger = B^\dagger F_A$.
⟨12⟩ $(F_AB)^\dagger B = B^\dagger F_AB$.
⟨13⟩ $BM^\dagger M = A^\dagger M$.
⟨14⟩ $A^\dagger ABB^* = BB^*A^\dagger A$.
⟨15⟩ $r[\,BM^*,\, A^*\,] = r(A)$.
⟨16⟩ $r[\,BB^*F_A,\, F_A\,] = r(F_A)$.
⟨17⟩ $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.
⟨18⟩ $\mathscr{R}(BB^*F_A) \subseteq \mathscr{N}(A)$.
⟨19⟩ $\mathscr{R}(BM^*) = \mathscr{R}(B) \cap \mathscr{R}(A^*)$.
⟨20⟩ $\mathscr{R}(BB^*F_A) = \mathscr{R}(B) \cap \mathscr{N}(A)$.
⟨21⟩ $\mathscr{R}(A^\dagger ABB^*) = \mathscr{R}(BB^*A^\dagger A)$.
⟨22⟩ $\mathscr{R}(F_ABB^*) = \mathscr{R}(BB^*F_A)$.
⟨23⟩ *$B^\dagger A^\dagger M$ is an orthogonal projector, i.e.,* $(B^\dagger A^\dagger M)^2 = B^\dagger A^\dagger M = (B^\dagger A^\dagger M)^*$.
⟨24⟩ *$B^\dagger F_AB$ is an orthogonal projector, i.e.,* $(B^\dagger F_AB)^2 = B^\dagger F_AB = (B^\dagger F_AB)^*$.
⟨25⟩ *Both* $MB^\dagger A^\dagger M = M$ *and* $(B^\dagger A^\dagger M)^* = B^\dagger A^\dagger M$.
⟨26⟩ *Both* $r(N) = r(A) + r(B) - r(M)$ *and* $r[\,B^*BM^*,\, M^*\,] = r(M)$.

(b) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^\dagger\}$.
⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,4)}\}$.
⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,4)}\}$.
⟨5⟩ $F_BVE_BA^\dagger MM^* \equiv M^* - B^\dagger A^\dagger MM^*$ *for all* V.
⟨6⟩ $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(c) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^\dagger\}$.
⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,4)}\}$.
⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,4)}\}$.
⟨5⟩ $B^\dagger BVE_BA^\dagger MM^* \equiv M^* - B^\dagger A^\dagger MM^*$ *for all* V.
⟨6⟩ $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(d) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,3)}A^\dagger\}$.
⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,4)}\}$.
⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,4)}\}$.
⟨5⟩ $F_BVBB^\dagger A^\dagger MM^* \equiv M^* - B^\dagger A^\dagger MM^*$ *for all* V.
⟨6⟩ *Both* $r(B) = p$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(e) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{(1,2,4)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{(1,4)}\}$.
⟨5⟩ $VE_BA^{\dagger}MM^* \equiv M^* - B^{\dagger}A^{\dagger}MM^*$ *for all* V.
⟨6⟩ $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(f) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3)}A^{(1,2,4)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3)}A^{(1,4)}\}$.
⟨5⟩ $F_BVA^{\dagger}MM^* \equiv M^* - B^{\dagger}A^{\dagger}MM^*$ *for all* V.
⟨6⟩ *Both* $r(B) = p$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(g) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2)}A^{(1,2,4)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2)}A^{(1,4)}\}$.
⟨5⟩ $(B^{\dagger} + F_BV_1)(BB^{\dagger}A^{\dagger}MM^* + BV_2E_BA^{\dagger}MM^*) \equiv M^*$ *for all* V_1 *and* V_2.
⟨6⟩ *Both* $r(B) = p$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(h) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1)}A^{(1,2,4)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1)}A^{(1,4)}\}$.
⟨5⟩ $F_BV_1A^{\dagger}MM^* + V_2E_BA^{\dagger}MM^* \equiv M^* - B^{\dagger}A^{\dagger}MM^*$ *for all* V_1 *and* V_2.
⟨6⟩ *Both* $r(B) = p$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(i) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{\dagger}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{\dagger}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{\dagger}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{\dagger}A^{(1)}\}$.
⟨5⟩ $B^{\dagger}F_AUMM^* \equiv M^* - B^{\dagger}A^{\dagger}MM^*$ *for all* U.
⟨6⟩ $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

(j) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1)}\}$.
⟨5⟩ $(B^{\dagger} + F_BVE_B)(A^{\dagger}MM^* + F_AUMM^*) \equiv M^*$ *for all* U *and* V.
⟨6⟩ *Either* $r(A) = n$, *or* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

(k) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,3)}A^{(1)}\}$.
⟨5⟩ $(B^{\dagger} + F_BVBB^{\dagger})(A^{\dagger}MM^* + F_AUMM^*) \equiv M^*$ *for all* U *and* V.
⟨6⟩ *Both* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

(l) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^{(1)}\}$.
⟨5⟩ $(B^{\dagger} + B^{\dagger}BVE_B)(A^{\dagger}MM^* + F_AUMM^*) \equiv M^*$ *for all* U *and* V.
⟨6⟩ $r(A) = n$.

(m) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,3)}A^{(1)}\}$.
⟨5⟩ $(B^{\dagger} + F_BV)(A^{\dagger}MM^* + F_AUMM^*) \equiv M^*$ *for all* U *and* V.
⟨6⟩ *Both* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

(n) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{(1)}\}$.
⟨5⟩ $(B^{\dagger} + VE_B)(A^{\dagger}MM^* + F_AUMM^*) \equiv M^*$ *for all* U *and* V.
⟨6⟩ $r(A) = n$.

(o) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1,2)}A^{(1)}\}$.
⟨5⟩ $(B^{\dagger} + F_BV_1)B(B^{\dagger} + V_2E_B)(A^{\dagger}MM^* + F_AUMM^*) \equiv M^*$ *for all* U, V_1, *and* V_2.
⟨6⟩ *Both* $r(A) = n$ *and* $r(B) = p$.

(p) *The following 6 statements are equivalent:*

⟨1⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,4)}\} \supseteq \{B^{(1)}A^{(1)}\}$.
⟨5⟩ $(B^{\dagger} + F_BV_1 + V_2E_B)(A^{\dagger}MM^* + F_AUMM^*) \equiv M^*$ *for all* U, V_1, *and* V_2.
⟨6⟩ *Both* $r(A) = n$ *and* $r(B) = p$.

6 Set inclusions for $\{1,2,3\}$-, $\{1,2,4\}$-, $\{1,3,4\}$-generalized inverses of AB

Applying Lemma 4.1(e), (f), and (g) to Theorems 5.3–5.5, we obtain the following theorems. The details of the proofs are omitted.

Theorem 6.1. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$ *be given with* $AB \neq 0$. *Also denote* $M = AB$ *and* $t = m + p + r(M) - r(A) - r(B)$. *Then the following results hold.*

(a) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \ni B^{\dagger}A^{\dagger}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,3)}A^{\dagger}\}$. ⟨3⟩ $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$.

(b) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{\dagger}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,3)}\}$.
⟨3⟩ *Either* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$ *or* $\mathscr{R}(A^*M) = \mathscr{R}(B)$.

(c) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{\dagger}A^{(1,2,4)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,4)}\}$.
⟨3⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{\dagger}A^{(1,4)}\}$. ⟨4⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,4)}\}$.
⟨5⟩ *Both* $r(A) = m$ *and* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$.

(d) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{\dagger}A^{(1,2)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,2)}\}$.
⟨3⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3)}A^{(1,2)}\}$.
⟨5⟩ *Either* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$, *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*M) = \mathscr{R}(B)$.

(e) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,4)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,4)}A^{\dagger}\}$.
⟨3⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2)}A^{\dagger}\}$. ⟨4⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1)}A^{\dagger}\}$.
⟨5⟩ $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

(f) *The following 9 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,4)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2)}A^{(1,2,4)}\}$.
⟨3⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,4)}A^{(1,2,4)}\}$. ⟨4⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1)}A^{(1,2,4)}\}$.
⟨5⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,4)}\}$. ⟨6⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2)}A^{(1,4)}\}$.
⟨7⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,4)}A^{(1,4)}\}$. ⟨8⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1)}A^{(1,4)}\}$.
⟨9⟩ *Both* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

(g) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3)}A^{(1,2,3)}\}$.
⟨3⟩ *Either* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$, *or* $r(B) = p$ *and* $\mathscr{R}(A^*M) = \mathscr{R}(B)$.

(h) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{\dagger}A^{(1,3)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,3)}\}$.
⟨3⟩ *Either* $\mathscr{R}(A^*M) = \mathscr{R}(B)$, *or* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

(i) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{\dagger}A^{(1)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,3)}A^{(1)}\}$.
⟨3⟩ *Either* $r(A) = m$ *and* $\mathscr{R}(A^*M) = \mathscr{R}(B)$, *or* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

(j) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,4)}A^{(1,3,4)}\}$. ⟨4⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1)}A^{(1,3,4)}\}$.
⟨5⟩ *Either* $r(M) = n$, *or* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$.

(k) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3,4)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3)}A^{\dagger}\}$.
⟨3⟩ *Either* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$, *or* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$ *and* $r(B) = p$.

(l) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,4)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3)}A^{(1,2,4)}\}$.
⟨3⟩ *Either* $r(A) = m$ *and* $\mathscr{R}(A^*) \subseteq \mathscr{R}(B)$, *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$.

(m) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{\dagger}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,3,4)}\}$.
⟨3⟩ *Either* $\mathscr{R}(A^*M) = \mathscr{R}(B)$, *or* $r(A) = m$ *and* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$.

(n) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,3)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3)}A^{(1,3)}\}$.
⟨3⟩ *Both* $r(M) = \min\{m, n, p\}$ *and* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$.

(o) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3,4)}A^{(1)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3)}A^{(1)}\}$.
⟨3⟩ $r(A) = m$, $r(M) = \min\{m, n, p\}$, *and* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$.

(p) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3)}A^{(1,3,4)}\}$.
⟨3⟩ *Both* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$ *and* $r(M) = \min\{m, n, p, t\}$.

(q) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,4)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,3)}A^{(1,4)}\}$.
⟨3⟩ $r(A) = m$, $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$, *and* $r(M) = \min\{m, p, t\}$.

(r) *The following 9 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2)}A^{(1,2,3)}\}$.
⟨3⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,3)}\}$. ⟨4⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2)}A^{(1,3)}\}$.
⟨5⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,4)}A^{(1,2,3)}\}$. ⟨6⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1)}A^{(1,2,3)}\}$.
⟨7⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,4)}A^{(1,3)}\}$. ⟨8⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1)}A^{(1,3)}\}$.
⟨9⟩ $r(M) = n$.

(s) *The following 9 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,4)}A^{(1,2)}\}$. ⟨2⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2)}A^{(1,2)}\}$.
⟨3⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2,4)}A^{(1)}\}$. ⟨4⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,2)}A^{(1)}\}$.
⟨5⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,4)}A^{(1,2)}\}$. ⟨6⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1)}A^{(1,2)}\}$.
⟨7⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1,4)}A^{(1)}\}$. ⟨8⟩ $\{M^{(1,2,3)}\} \supseteq \{B^{(1)}A^{(1)}\}$.
⟨9⟩ $r(M) = m = n$.

Theorem 6.2. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$ *be given with* $AB \neq 0$. *Also denote* $M = AB$ *and* $t = m + p + r(M) - r(A) - r(B)$. *Then the following results hold.*

(a) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \ni B^{\dagger}A^{\dagger}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{\dagger}A^{(1,2,4)}\}$. ⟨3⟩ $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(b) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,4)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,4)}\}$.
⟨3⟩ *Either* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$ *or* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$.

(c) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,3)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,4)}\}$.
⟨3⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,4)}A^{\dagger}\}$. ⟨4⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3)}A^{(1,2,4)}\}$.
⟨5⟩ *Both* $r(B) = p$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(d) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2)}A^{(1,2,4)}\}$.
⟨3⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2)}A^{(1,3,4)}\}$. ⟨4⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2)}A^{(1,4)}\}$.
⟨5⟩ *Either* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$, *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$.

(e) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{\dagger}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{\dagger}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{\dagger}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{\dagger}A^{(1)}\}$.
⟨5⟩ $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

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(f) *The following 9 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,2)}\}$.
⟨3⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,3)}\}$. ⟨4⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,3)}A^{(1)}\}$.
⟨5⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3)}A^{(1,2,3)}\}$. ⟨6⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3)}A^{(1,2)}\}$.
⟨7⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3)}A^{(1,3)}\}$. ⟨8⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3)}A^{(1)}\}$.
⟨9⟩ *Both* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

(g) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,4)}\}$.
⟨3⟩ *Either* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$, *or* $r(A) = m$ *and* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$.

(h) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,4)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,4)}A^{(1,2,4)}\}$.
⟨3⟩ *Either* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$, *or* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

(i) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1)}A^{(1,2,4)}\}$
⟨3⟩ *Either* $r(B) = p$ *and* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$, *or* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

(j) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,2)}\}$.
⟨3⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,3)}\}$. ⟨4⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3,4)}A^{(1)}\}$.
⟨5⟩ *Either* $r(M) = n$, *or* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$.

(k) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{\dagger}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{\dagger}A^{(1,4)}\}$.
⟨3⟩ *Either* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$, *or* $r(A) = m$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(l) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,4)}\}$.
⟨3⟩ *Either* $r(B) = p$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(B)$, *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(m) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3,4)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,4)}\}$.
⟨3⟩ *Either* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$, *or* $r(B) = p$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(n) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,4)}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,4)}A^{(1,4)}\}$.
⟨3⟩ *Both* $r(M) = \min\{m,\ n,\ p\}$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(o) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1)}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1)}A^{(1,4)}\}$.
⟨3⟩ $r(B) = p$, $r(M) = \min\{m,\ n,\ p\}$, *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(p) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,4)}\}$.
⟨3⟩ *Both* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$ *and* $r(M) = \min\{m,\ n,\ p,\ t\}$.

(q) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3)}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3)}A^{(1,4)}\}$.
⟨3⟩ $r(B) = p$, $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$, *and* $r(M) = \min\{m,\ p,\ t\}$.

(r) *The following 9 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,2)}\}$.
⟨3⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,4)}A^{(1,2,3)}\}$. ⟨4⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,4)}A^{(1,2)}\}$.
⟨5⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,3)}\}$. ⟨6⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2,4)}A^{(1)}\}$.
⟨7⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,4)}A^{(1,3)}\}$. ⟨8⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,3)}A^{(1)}\}$.
⟨9⟩ $r(M) = n$.

(s) *The following 9 statements are equivalent:*

⟨1⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2)}A^{(1,2)}\}$.
⟨3⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1)}A^{(1,2,3)}\}$. ⟨4⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1)}A^{(1,2)}\}$.
⟨5⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2)}A^{(1,3)}\}$. ⟨6⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1,2)}A^{(1)}\}$.
⟨7⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1)}A^{(1,3)}\}$. ⟨8⟩ $\{M^{(1,2,4)}\} \supseteq \{B^{(1)}A^{(1)}\}$.
⟨9⟩ $r(M) = n = p$.

Theorem 6.3. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$ *be given with* $AB \neq 0$, *and denote* $M = AB$. *Then the following results hold.*

(a) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \ni B^{\dagger}A^{\dagger}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{\dagger}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3,4)}A^{\dagger}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,3,4)}\}$.
⟨5⟩ *Both* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(b) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{\dagger}A^{(1,2,4)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{\dagger}A^{(1,4)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,4)}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,4)}\}$.
⟨5⟩ $r(A) = m$, $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(c) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,4)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,4)}A^{\dagger}\}$. ⟨3⟩ $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$.

(d) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{\dagger}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{\dagger}A^{(1,3)}\}$. ⟨3⟩ $\mathscr{R}(A^*M) = \mathscr{R}(B)$.

(e) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,4)}A^{(1,3,4)}\}$.
⟨3⟩ *Either* $r(B) = n$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$, *or* $r(A) = m$ *and* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$.

(f) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,4)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,4)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,4)}A^{(1,2,4)}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,4)}A^{(1,4)}\}$.
⟨5⟩ *Both* $r(A) = m$ *and* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$.

(g) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,3)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,3,4)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3)}A^{\dagger}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3)}A^{(1,3,4)}\}$.
⟨5⟩ $r(B) = p$, $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(h) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,4)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,4)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3)}A^{(1,2,4)}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3)}A^{(1,4)}\}$.
⟨5⟩ $r(A) = m$, $r(B) = p$, $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

(i) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2)}A^{\dagger}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1)}A^{\dagger}\}$. ⟨3⟩ $r(B) = p$ *and* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$.

(j) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2)}A^{(1,3,4)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1)}A^{(1,3,4)}\}$.
⟨3⟩ *Either* $r(B) = n = p$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$, *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$.

(k) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2)}A^{(1,2,4)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2)}A^{(1,4)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1)}A^{(1,2,4)}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1)}A^{(1,4)}\}$.
⟨5⟩ $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$.

(l) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3)}A^{(1,2)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3)}A^{(1)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,3)}A^{(1)}\}$.
⟨5⟩ $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*M) = \mathscr{R}(B)$.

(m) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{\dagger}A^{(1,2)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{\dagger}A^{(1)}\}$.
⟨3⟩ *Both* $r(A) = m$ *and* $\mathscr{R}(BM^*) = \mathscr{R}(A^*)$.

(n) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,3)}\}$.
⟨3⟩ *Either* $r(A) = n$ *and* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B^*)$, *or* $r(B) = p$ *and* $\mathscr{R}(A^*M) = \mathscr{R}(B)$.

(o) *The following 3 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,2)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3,4)}A^{(1)}\}$.
⟨3⟩ *Either* $r(A) = m = n$ *and* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$, *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*M) = \mathscr{R}(B)$.

(p) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,3)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3)}A^{(1,2,3)}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3)}A^{(1,3)}\}$.
⟨5⟩ *Both* $r(B) = p$ *and* $\mathscr{R}(A^*M) = \mathscr{R}(B)$.

(q) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,4)}A^{(1,2,3)}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,4)}A^{(1,3)}\}$.
⟨5⟩ $r(M) = n$.

(r) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,2)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2,4)}A^{(1)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,4)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,4)}A^{(1)}\}$.
⟨5⟩ $r(M) = m = n$.

(s) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2)}A^{(1,2,3)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2)}A^{(1,3)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1)}A^{(1,2,3)}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1)}A^{(1,3)}\}$.
⟨5⟩ $r(M) = n = p$.

(t) *The following 5 statements are equivalent:*

⟨1⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2)}A^{(1,2)}\}$. ⟨2⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,2)}A^{(1)}\}$.
⟨3⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1)}A^{(1,2)}\}$. ⟨4⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1)}A^{(1)}\}$.
⟨5⟩ $r(M) = m = n = p$.

Since the product $B^{\dagger}A^{\dagger}$ is unique, the last 64 cases in (1.32) can be written as

$$(AB)^{\dagger} = \{(AB)^{(1,2,3,4)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\} \tag{6.1}$$

for the eight commonly-used types of generalized inverses of matrices. From Lemma 4.1(h),

$$(AB)^{\dagger} = \{(AB)^{(1,2,3,4)}\} \supseteq \{B^{(i,\dots,j)}A^{(i,\dots,j)}\} \Leftrightarrow B^{(i,\dots,j)}A^{(i,\dots,j)} \text{ is invariant and } (AB)^{\dagger} = B^{\dagger}A^{\dagger}. \tag{6.2}$$

The invariance property of $B^{(i,\dots,j)}A^{(i,\dots,j)}$ is characterized in Theorem 4.4. Also by the definition of the Moore–Penrose inverse,

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow \{(AB)^{(1,2,3)}\} \ni B^{\dagger}A^{\dagger} \text{ and } \{(AB)^{(1,2,4)}\} \ni B^{\dagger}A^{\dagger}. \tag{6.3}$$

Thus we obtain from Theorems 6.1(a) and 6.2(a) that

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow \mathscr{R}(A^*AB) \subseteq \mathscr{R}(B) \text{ and } \mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*). \tag{6.4}$$

This fact was well known in the theory of generalized inverses and was first established in [7]. Finally, combining Theorem 4.4 with (6.4), we obtain the following results.

Theorem 6.4. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ be given with $AB \neq 0$. Then the following results hold.*

⟨1⟩ $(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow$ *both* $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$ *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$.

⟨2⟩ $(AB)^{\dagger} = B^{\dagger}A^{(1,3,4)}$ *holds for all* $A^{(1,3,4)} \Leftrightarrow$ *either* $r(A) = m$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$, *or* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨3⟩ $(AB)^{\dagger} = B^{\dagger}A^{(1,2,4)}$ *holds for all* $A^{(1,2,4)} \Leftrightarrow r(A) = m$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$.

⟨4⟩ $(AB)^{\dagger} = B^{\dagger}A^{(1,2,3)}$ *holds for all* $A^{(1,2,3)} \Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨5⟩ $(AB)^{\dagger} = B^{\dagger}A^{(1,4)}$ *holds for all* $A^{(1,4)} \Leftrightarrow r(A) = m$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$.

⟨6⟩ $(AB)^{\dagger} = B^{\dagger}A^{(1,3)}$ *holds for* $A^{(1,3)}$ *all* $\Leftrightarrow \mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨7⟩ $(AB)^{\dagger} = B^{\dagger}A^{(1,2)}$ *holds for all* $A^{(1,2)} \Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨8⟩ $(AB)^{\dagger} = B^{\dagger}A^{(1)}$ *holds for all* $A^{(1)} \Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨9⟩ $(AB)^{\dagger} = B^{(1,3,4)}A^{\dagger}$ *holds for all* $B^{(1,3,4)} \Leftrightarrow$ *either* $r(B) = p$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$, *or* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨10⟩ $(AB)^{\dagger} = B^{(1,3,4)}A^{(1,3,4)}$ *holds for all* $B^{(1,3,4)}$ *and* $A^{(1,3,4)} \Leftrightarrow$ *one of the 4 conditions:* (i) $r(M) = n$; (ii) $r(A) = m$, $r(B) = p$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$; (iii) $r(A) = m$ *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$; (iv) $r(B) = p$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨11⟩ $(AB)^{\dagger} = B^{(1,3,4)}A^{(1,2,4)}$ *holds for all* $B^{(1,3,4)}$ *and* $A^{(1,2,4)} \Leftrightarrow$ *either* $r(B) = p$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$, *or* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨12⟩ $(AB)^{\dagger} = B^{(1,3,4)}A^{(1,2,3)}$ *holds for all* $B^{(1,3,4)}$ *and* $A^{(1,2,3)} \Leftrightarrow$ *either* $r(M) = n$, *or* $r(B) = p$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨13⟩ $(AB)^{\dagger} = B^{(1,3,4)}A^{(1,4)}$ *holds for all* $B^{(1,3,4)}$ *and* $A^{(1,4)} \Leftrightarrow$ *either* $r(A) = m$, $r(B) = p$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$, *or* $r(A) = m$ *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨14⟩ $(AB)^{\dagger} = B^{(1,3,4)}A^{(1,3)}$ *holds for all* $B^{(1,3,4)}$ *and* $A^{(1,3)} \Leftrightarrow$ *either* $r(M) = n$, *or* $r(B) = p$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨15⟩ $(AB)^{\dagger} = B^{(1,3,4)}A^{(1,2)}$ *holds for all* $B^{(1,3,4)}$ *and* $A^{(1,2)} \Leftrightarrow$ *either* $r(M) = m = n$, *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨16⟩ $(AB)^{\dagger} = B^{(1,3,4)}A^{(1)}$ *holds for all* $B^{(1,3,4)}$ *and* $A^{(1)} \Leftrightarrow$ *either* $r(M) = m = n$, *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨17⟩ $(AB)^{\dagger} = B^{(1,2,4)}A^{\dagger}$ *holds for all* $B^{(1,2,4)} \Leftrightarrow \mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨18⟩ $(AB)^{\dagger} = B^{(1,2,4)}A^{(1,3,4)}$ *holds for all* $B^{(1,2,4)}$ *and* $A^{(1,3,4)} \Leftrightarrow$ *either* $r(M) = n$, *or* $r(A) = m$ *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨19⟩ $(AB)^{\dagger} = B^{(1,2,4)}A^{(1,2,4)}$ *holds for all* $B^{(1,2,4)}$ *and* $A^{(1,2,4)} \Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨20⟩ $(AB)^{\dagger} = B^{(1,2,4)}A^{(1,2,3)}$ *holds for all* $B^{(1,2,4)}$ *and* $A^{(1,2,3)} \Leftrightarrow r(M) = n$.

⟨21⟩ $(AB)^{\dagger} = B^{(1,2,4)}A^{(1,4)}$ *holds for all* $B^{(1,2,4)}$ *and* $A^{(1,4)} \Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨22⟩ $(AB)^{\dagger} = B^{(1,2,4)}A^{(1,3)}$ *holds for all* $B^{(1,2,4)}$ *and* $A^{(1,3)} \Leftrightarrow r(M) = n$.

⟨23⟩ $(AB)^{\dagger} = B^{(1,2,4)}A^{(1,2)}$ *holds for all* $B^{(1,2,4)}$ *and* $A^{(1,2)} \Leftrightarrow r(M) = m = n$.

⟨24⟩ $(AB)^{\dagger} = B^{(1,2,4)}A^{(1)}$ *holds for all* $B^{(1,2,4)}$ *and* $A^{(1)} \Leftrightarrow r(M) = m = n$.

⟨25⟩ $(AB)^{\dagger} = B^{(1,2,3)}A^{\dagger}$ *holds for all* $B^{(1,2,3)} \Leftrightarrow (B) = p$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$.

⟨26⟩ $(AB)^{\dagger} = B^{(1,2,3)}A^{(1,3,4)}$ *holds for all* $B^{(1,2,3)}$ *and* $A^{(1,3,4)} \Leftrightarrow$ *either* $r(A) = m$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$, *or* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨27⟩ $(AB)^{\dagger} = B^{(1,2,3)}A^{(1,2,4)}$ *holds for all* $B^{(1,2,3)}$ *and* $A^{(1,2,4)} \Leftrightarrow r(A) = m$, $r(B) = p$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$.

⟨28⟩ $(AB)^{\dagger} = B^{(1,2,3)}A^{(1,2,3)}$ *holds for all* $B^{(1,2,3)}$ *and* $A^{(1,2,3)} \Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨29⟩ $(AB)^{\dagger} = B^{(1,2,3)}A^{(1,4)}$ *holds for all* $B^{(1,2,3)}$ *and* $A^{(1,4)} \Leftrightarrow r(A) = m$, $r(B) = p$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$.

⟨30⟩ $(AB)^{\dagger} = B^{(1,2,3)}A^{(1,3)}$ *holds for all* $B^{(1,2,3)}$ *and* $A^{(1,3)} \Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨31⟩ $(AB)^{\dagger} = B^{(1,2,3)}A^{(1,2)}$ *holds for all* $B^{(1,2,3)}$ *and* $A^{(1,2)} \Leftrightarrow r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨32⟩ $(AB)^{\dagger} = B^{(1,2,3)}A^{(1)}$ *holds for all* $B^{(1,2,3)}$ *and* $A^{(1)} \Leftrightarrow r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨33⟩ $(AB)^{\dagger} = B^{(1,4)}A^{\dagger}$ *holds for all* $B^{(1,4)} \Leftrightarrow \mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨34⟩ $(AB)^{\dagger} = B^{(1,4)}A^{(1,3,4)}$ *holds for all* $B^{(1,4)}$ *and* $A^{(1,3,4)} \Leftrightarrow$ *either* $r(M) = n$, *or* $r(A) = m$ *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨35⟩ $(AB)^{\dagger} = B^{(1,4)}A^{(1,2,4)}$ *holds for all* $B^{(1,4)}$ *and* $A^{(1,2,4)} \Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨36⟩ $(AB)^{\dagger} = B^{(1,4)}A^{(1,2,3)}$ *holds for all* $B^{(1,4)}$ *and* $A^{(1,2,3)} \Leftrightarrow r(M) = n$.

⟨37⟩ $(AB)^{\dagger} = B^{(1,4)}A^{(1,4)}$ *holds for all* $B^{(1,4)}$ *and* $A^{(1,4)} \Leftrightarrow$ *both* $r(A) = m$ *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨38⟩ $(AB)^{\dagger} = B^{(1,4)}A^{(1,3)}$ *holds for all* $B^{(1,4)}$ *and* $A^{(1,3)} \Leftrightarrow r(M) = n$.

⟨39⟩ $(AB)^{\dagger} = B^{(1,4)}A^{(1,2)}$ *holds for all* $B^{(1,4)}$ *and* $A^{(1,2)} \Leftrightarrow r(M) = m = n$.

⟨40⟩ $(AB)^{\dagger} = B^{(1,4)}A^{(1)}$ *holds for all* $B^{(1,4)}$ *and* $A^{(1)} \Leftrightarrow r(M) = m = n$.

⟨41⟩ $(AB)^{\dagger} = B^{(1,3)}A^{\dagger}$ *holds for all* $B^{(1,3)} \Leftrightarrow r(B) = p$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$.

⟨42⟩ $(AB)^{\dagger} = B^{(1,3)}A^{(1,3,4)}$ *holds for all* $B^{(1,3)}$ *and* $A^{(1,3,4)} \Leftrightarrow$ *either* $r(A) = m$, $r(B) = p$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$, *or* $r(B) = p$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨43⟩ $(AB)^{\dagger} = B^{(1,3)}A^{(1,2,4)}$ *holds for all* $B^{(1,3)}$ *and* $A^{(1,2,4)} \Leftrightarrow r(A) = m$, $r(B) = p$, $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$.

⟨44⟩ $(AB)^{\dagger} = B^{(1,3)}A^{(1,2,3)}$ *holds for all* $B^{(1,3)}$ *and* $A^{(1,2,3)} \Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨45⟩ $(AB)^{\dagger} = B^{(1,3)}A^{(1,4)}$ *holds for all* $B^{(1,3)}$ *and* $A^{(1,4)} \Leftrightarrow r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*)$.

⟨46⟩ $(AB)^{\dagger} = B^{(1,3)}A^{(1,3)}$ *holds for all* $B^{(1,3)}$ *and* $A^{(1,3)} \Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨47⟩ $(AB)^{\dagger} = B^{(1,3)}A^{(1,2)}$ *holds for all* $B^{(1,3)}$ *and* $A^{(1,2)} \Leftrightarrow r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨48⟩ $(AB)^{\dagger} = B^{(1,3)}A^{(1)}$ *holds for all* $B^{(1,3)}$ *and* $A^{(1)} \Leftrightarrow r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(A^*AB) = \mathscr{R}(B)$.

⟨49⟩ $(AB)^\dagger = B^{(1,2)}A^\dagger$ *holds for all* $B^{(1,2)} \Leftrightarrow$ *both* $r(B) = p$ *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨50⟩ $(AB)^\dagger = B^{(1,2)}A^{(1,3,4)}$ *holds for all* $B^{(1,2)}$ *and* $A^{(1,3,4)} \Leftrightarrow$ *either* $r(M) = n = p$ *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨51⟩ $(AB)^\dagger = B^{(1,2)}A^{(1,2,4)}$ *holds for all* $B^{(1,2)}$ *and* $A^{(1,2,4)} \Leftrightarrow r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨52⟩ $(AB)^\dagger = B^{(1,2)}A^{(1,2,3)}$ *holds for all* $B^{(1,2)}$ *and* $A^{(1,2,3)} \Leftrightarrow r(M) = n = p$.

⟨53⟩ $(AB)^\dagger = B^{(1,2)}A^{(1,4)}$ *holds for all* $B^{(1,2)}$ *and* $A^{(1,4)} \Leftrightarrow r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨54⟩ $(AB)^\dagger = B^{(1,2)}A^{(1,3)}$ *holds for all* $B^{(1,2)}$ *and* $A^{(1,3)} \Leftrightarrow r(M) = n = p$.

⟨55⟩ $(AB)^\dagger = B^{(1,2)}A^{(1,2)}$ *holds for all* $B^{(1,2)}$ *and* $A^{(1,2)} \Leftrightarrow r(M) = m = n = p$.

⟨56⟩ $(AB)^\dagger = B^{(1,2)}A^{(1)}$ *holds for all* $B^{(1,2)}$ *and* $A^{(1)} \Leftrightarrow r(M) = m = n = p$.

⟨57⟩ $(AB)^\dagger = B^{(1)}A^\dagger$ *holds for all* $B^{(1)} \Leftrightarrow r(B) = p$ *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨58⟩ $(AB)^\dagger = B^{(1)}A^{(1,3,4)}$ *holds for all* $B^{(1)}$ *and* $A^{(1,3,4)} \Leftrightarrow$ *either* $r(M) = n = p$ *or* $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨59⟩ $(AB)^\dagger = B^{(1)}A^{(1,2,4)}$ *holds for all* $B^{(1)}$ *and* $A^{(1,2,4)} \Leftrightarrow r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨60⟩ $(AB)^\dagger = B^{(1)}A^{(1,2,3)}$ *holds for all* $B^{(1)}$ *and* $A^{(1,2,3)} \Leftrightarrow r(M) = n = p$.

⟨61⟩ $(AB)^\dagger = B^{(1)}A^{(1,4)}$ *holds for all* $\Leftrightarrow B^{(1)}$ *and* $A^{(1,4)}$ $r(A) = m$, $r(B) = p$, *and* $\mathscr{R}(BB^*A^*) = \mathscr{R}(A^*)$.

⟨62⟩ $(AB)^\dagger = B^{(1)}A^{(1,3)}$ *holds for all* $B^{(1)}$ *and* $A^{(1,3)} \Leftrightarrow r(M) = n = p$.

⟨63⟩ $(AB)^\dagger = B^{(1)}A^{(1,2)}$ *holds for all* $B^{(1)}$ *and* $A^{(1,2)} \Leftrightarrow r(M) = m = n = p$.

⟨64⟩ $(AB)^\dagger = B^{(1)}A^{(1)}$ *holds for all* $B^{(1)}$ *and* $A^{(1)} \Leftrightarrow r(M) = m = n = p$.

7 Miscellaneous results on reverse-order laws

As demonstrated in the previous sections, the reverse-order law $(AB)^\dagger = B^\dagger A^\dagger$ for the Moore–Penrose inverses is one of the most important forms in (1.27), while both (6.3) and (6.4) show that $(AB)^\dagger = B^\dagger A^\dagger$ has essential links with other types of the reverse-order laws in (1.27), and can be characterized by many equivalent statements. In this section, we reconsider this reverse-order law and present a bunch of necessary and sufficient conditions for it to hold.

Lemma 7.1. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$, *and denote* $M = AB$. *Then the following 11 statements are equivalent*:

⟨1⟩ $\{(A^*M)^{(1,3)}\} \ni M^\dagger (A^*)^\dagger$.

⟨2⟩ $M^\dagger = (A^\dagger M)^\dagger A^\dagger$.

⟨3⟩ $M^\dagger = (A^*M)^\dagger A^*$.

⟨4⟩ $(A^\dagger M)^\dagger = M^\dagger A$.

⟨5⟩ $A^\dagger MM^\dagger A$ *is Hermitian, i.e.,* $(A^\dagger MM^\dagger A)^* = A^\dagger MM^\dagger A$.

⟨6⟩ $MM^\dagger$ *and* AA^* *commute, i.e.,* $MM^\dagger AA^* = AA^*MM^\dagger$.

⟨7⟩ $MB^\dagger A^\dagger$ *is EP, i.e.,* $\mathscr{R}(MB^\dagger A^\dagger) = \mathscr{R}[(MB^\dagger A^\dagger)^*]$.

⟨8⟩ *The range reverse-order law* $\mathscr{R}[(M^*)^\dagger] = \mathscr{R}[(A^*)^\dagger (B^*)^\dagger]$ *for the Moore–Penrose inverse of matrix product holds.*

⟨9⟩ $r[\, AA^*M, \, M\,] = r(M)$.

⟨10⟩ $\mathscr{R}(AA^*M) = \mathscr{R}(M)$.

⟨11⟩ $r[\, AA^*M, \, M\,] = r(A) + r[\, A^*M, \, B\,] - r[\, A^*, \, B\,]$ *and* $r[\, A^*M, \, B\,] = r[\, A^*, \, B\,] + r(M) - r(A)$.

Proof. By Theorem 5.4⟨1⟩ and ⟨15⟩, Result ⟨1⟩ holds if and only if $r[\, AA^*M,\, M\,] = r(M)$, establishing the equivalence of ⟨1⟩, ⟨9⟩, and ⟨10⟩.

The equivalence of ⟨9⟩ and ⟨11⟩ follows from $t_4 \geq t_5 \geq t_6$ in (3.54).

It is easy to verify that $(AB)^{\dagger}$, $(A^{\dagger}AB)^{\dagger}A^{\dagger}$, and $(A^*AB)^{\dagger}A^*$ are {2}-inverses of AB. Then we obtain by (3.46) that

$$\begin{aligned} r[\, M^{\dagger} - (A^{\dagger}AB)^{\dagger}A^{\dagger}\,] &= r\begin{bmatrix} M^{\dagger} \\ (A^{\dagger}M)^{\dagger}A^{\dagger} \end{bmatrix} + r[\, M^{\dagger},\, (A^{\dagger}M)^{\dagger}A^{\dagger}\,] - r(M^{\dagger}) - r[\,(A^{\dagger}M)^{\dagger}A^{\dagger}\,] \\ &= r\begin{bmatrix} M^* \\ (A^{\dagger}M)^*A^{\dagger} \end{bmatrix} + r[\, M^*,\, (A^{\dagger}M)^*\,] - 2r(M) \\ &= r\begin{bmatrix} M^*AA^* \\ M^* \end{bmatrix} - r(M) \\ &= r[\, AA^*M,\, M\,] - r(M), \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} r[\, M^{\dagger} - (A^*M)^{\dagger}A^*\,] &= r\begin{bmatrix} M^{\dagger} \\ (A^*M)^{\dagger}A^* \end{bmatrix} + r[\, M^{\dagger},\, (A^*M)^{\dagger}A^*\,] - r(M^{\dagger}) - r[\,(A^*M)^{\dagger}A^*\,] \\ &= r\begin{bmatrix} M^* \\ (A^*M)^*A^* \end{bmatrix} + r[\, M^*,\, (A^*M)^*\,] - 2r(M) \\ &= r\begin{bmatrix} M^* \\ (AA^*M)^* \end{bmatrix} - r(M) \\ &= r[\, AA^*M,\, M\,] - r(M). \end{aligned} \tag{7.2}$$

It is also easy to verify that $(A^{\dagger}M)^{\dagger}$ and $M^{\dagger}A$ are {2}-inverses of $A^{\dagger}M$. Then we obtain by (3.46) that

$$r[\,(A^{\dagger}M)^{\dagger} - M^{\dagger}A\,] = r[\, AA^*M,\, M\,] - r(M). \tag{7.3}$$

Setting all sides (7.1)–(7.3) of equal to zero leads to the equivalence of ⟨2⟩–⟨4⟩, and ⟨9⟩.

The rank of $(A^{\dagger}MM^{\dagger}A)^* - A^{\dagger}MM^{\dagger}A$ is

$$\begin{aligned} r[\,(A^{\dagger}MM^{\dagger}A)^* - A^{\dagger}MM^{\dagger}A\,] &= r[\, A^*MM^{\dagger}(A^{\dagger})^* - A^{\dagger}MM^{\dagger}A\,] \\ &= r(\, AA^*MM^{\dagger} - MM^{\dagger}AA^*\,) \\ &= 2r[\, AA^*MM^{\dagger},\, MM^{\dagger}\,] - 2r(MM^{\dagger}) \quad \text{(by (3.50)} \\ &= 2r[\, AA^*M,\, M\,] - 2r(M). \end{aligned} \tag{7.4}$$

Setting all sides of (7.4) equal to zero leads to the equivalence of ⟨5⟩, ⟨6⟩, and ⟨9⟩.

By (3.35) and Lemma 4.7,

$$\begin{aligned} r[\, MB^{\dagger}A^{\dagger},\, (MB^{\dagger}A^{\dagger})^*] &= r[\, M,\, (A^{\dagger})^*B\,] = r[\, AA^*M,\, M\,], \\ r[\,(M^{\dagger})^*,\, (A^{\dagger})^*(B^{\dagger})^*\,] &= r[\, M,\, (A^{\dagger})^*B\,] = r[\, AA^*M,\, M\,]. \end{aligned}$$

Hence,

$$\begin{aligned} \mathscr{R}(MB^{\dagger}A^{\dagger}) = \mathscr{R}[(MB^{\dagger}A^{\dagger})^*] &\Leftrightarrow r[\,(M^{\dagger})^*,\, (A^{\dagger})^*(B^{\dagger})^*\,] = r(M) \Leftrightarrow r[\, AA^*M,\, M\,] = r(M), \\ \mathscr{R}[(M^{\dagger})^*] = \mathscr{R}[(A^{\dagger})^*(B^{\dagger})^*] &\Leftrightarrow r[\,(M^{\dagger})^*,\, (A^{\dagger})^*(B^{\dagger})^*\,] = r(M) \Leftrightarrow r[\, AA^*M,\, M\,] = r(M), \end{aligned}$$

establishing the equivalence of ⟨7⟩, ⟨8⟩, and ⟨9⟩. □

Lemma 7.2. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and denote $M = AB$. Then the following 11 statements are equivalent:*

⟨1⟩ $\{(MB^*)^{(1,4)}\} \ni (B^*)^{\dagger}M^{\dagger}$.

⟨2⟩ $M^{\dagger} = B^{\dagger}(MB^{\dagger})^{\dagger}$.

⟨3⟩ $M^{\dagger} = B^*(MB^*)^{\dagger}$.

⟨4⟩ $(MB^{\dagger})^{\dagger} = BM^{\dagger}$.

⟨5⟩ $BM^{\dagger}MB^{\dagger}$ *is Hermitian, i.e.,* $(BM^{\dagger}MB^{\dagger})^* = BM^{\dagger}MB^{\dagger}$.

⟨6⟩ $M^{\dagger}M$ *and* B^*B *commute, i.e.,* $M^{\dagger}MB^*B = B^*BM^{\dagger}M$.

⟨7⟩ $B^{\dagger}A^{\dagger}M$ *is EP, i.e.,* $\mathscr{R}(B^{\dagger}A^{\dagger}M) = \mathscr{R}[(B^{\dagger}A^{\dagger}M)^*]$.

⟨8⟩ *The range reverse-order law* $\mathscr{R}(M^{\dagger}) = \mathscr{R}(B^{\dagger}A^{\dagger})$ *for the Moore–Penrose inverse of matrix product holds.*

⟨9⟩ $r[\, B^*BM^*,\, M^* \,] = r(M)$.

⟨10⟩ $\mathscr{R}(B^*BM^*) = \mathscr{R}(M^*)$.

⟨11⟩ $r[\, B^*BM^*,\, M^* \,] = r(B) + r[\, BM^*,\, A^* \,] - r[\, A^*,\, B \,]$ *and* $r[\, BM^*,\, A^* \,] = r[\, A^*,\, B \,] + r(M) - r(B)$.

Theorem 7.3. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$*, and denote* $M = AB$*. Then the following 70 statements are equivalent:*

⟨1⟩ $M^{\dagger} = B^{\dagger}A^{\dagger}$, *i.e.*, $MB^{\dagger}A^{\dagger}M = M$, $(MB^{\dagger}A^{\dagger})^* = MB^{\dagger}A^{\dagger}$, *and* $(B^{\dagger}A^{\dagger}M)^* = B^{\dagger}A^{\dagger}M$.

⟨2⟩ $M^{\dagger} \in \{B^{\dagger}A^{(1,3,4)}\}$.

⟨3⟩ $M^{\dagger} \in \{B^{\dagger}A^{(1,2,4)}\}$.

⟨4⟩ $M^{\dagger} \in \{B^{\dagger}A^{(1,4)}\}$.

⟨5⟩ $M^{\dagger} \in \{B^{(1,3,4)}A^{\dagger}\}$.

⟨6⟩ $M^{\dagger} \in \{B^{(1,3,4)}A^{(1,3,4)}\}$.

⟨7⟩ $M^{\dagger} \in \{B^{(1,3,4)}A^{(1,2,4)}\}$.

⟨8⟩ $M^{\dagger} \in \{B^{(1,3,4)}A^{(1,4)}\}$.

⟨9⟩ $M^{\dagger} \in \{B^{(1,2,3)}A^{\dagger}\}$.

⟨10⟩ $M^{\dagger} \in \{B^{(1,2,3)}A^{(1,3,4)}\}$.

⟨11⟩ $M^{\dagger} \in \{B^{(1,2,3)}A^{(1,2,4)}\}$.

⟨12⟩ $M^{\dagger} \in \{B^{(1,2,3)}A^{(1,4)}\}$.

⟨13⟩ $M^{\dagger} \in \{B^{(1,3)}A^{\dagger}\}$.

⟨14⟩ $M^{\dagger} \in \{B^{(1,3)}A^{(1,3,4)}\}$.

⟨15⟩ $M^{\dagger} \in \{B^{(1,3)}A^{(1,2,4)}\}$.

⟨16⟩ $M^{\dagger} \in \{B^{(1,3)}A^{(1,4)}\}$.

⟨17⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{\dagger}A^{(1,3,4)}\}$.

⟨18⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3,4)}A^{\dagger}\}$.

⟨19⟩ $\{M^{(1,3,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,3,4)}\}$.

⟨20⟩ *Both* $\{M^{(1,3)}\} \ni B^{\dagger}A^{\dagger}$ *and* $\{M^{(1,4)}\} \ni B^{\dagger}A^{\dagger}$.

⟨21⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{\dagger}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{\dagger}A^{(1,3,4)}\}$.

⟨22⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{\dagger}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{\dagger}A^{(1,2,4)}\}$.

⟨23⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{\dagger}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{\dagger}A^{(1,4)}\}$.

⟨24⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{\dagger}A^{(1,3,4)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{\dagger}\}$.

⟨25⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,3,4)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,3,4)}\}$.

⟨26⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,3,4)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,4)}\}$.

⟨27⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{(1,3,4)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,3,4)}A^{(1,3)}\}$.

⟨28⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{\dagger}A^{(1,2,3)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^{\dagger}\}$.

⟨29⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,2,3)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,3,34)}\}$.

⟨30⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,2,3)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,2,4)}\}$.

⟨31⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{(1,2,3)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,2,4)}A^{(1,4)}\}$.

⟨32⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{\dagger}A^{(1,3)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{\dagger}\}$.

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⟨33⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,3,4)}A^{(1,3)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{(1,3,4)}\}$.

⟨34⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,2,3)}A^{(1,3)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{(1,2,4)}\}$.

⟨35⟩ *Both* $\{M^{(1,3)}\} \supseteq \{B^{(1,3)}A^{(1,3)}\}$ *and* $\{M^{(1,4)}\} \supseteq \{B^{(1,4)}A^{(1,4)}\}$.

⟨36⟩ $\{(A^*M)^{(1)}\} \ni B^{\dagger}(A^*A)^{\dagger}$, $\{(A^*M)^{(1,3)}\} \ni M^{\dagger}(A^*)^{\dagger}$, $\{(MB^*)^{(1)}\} \ni (BB^*)^{\dagger}A^{\dagger}$, *and* $\{(MB^*)^{(1,4)}\} \ni (B^*)^{\dagger}M^{\dagger}$.

⟨37⟩ $M^{\dagger} = B^{\dagger}A^{\dagger}MB^{\dagger}A^{\dagger}$.

⟨38⟩ $BM^{\dagger}A = BB^{\dagger}A^{\dagger}A$.

⟨39⟩ $B^*BM^{\dagger}AA^* = M^*$.

⟨40⟩ *Both* $MM^{\dagger} = MB^{\dagger}A^{\dagger}$ *and* $M^{\dagger}M = B^{\dagger}A^{\dagger}M$.

⟨41⟩ *Both* $MM^{\dagger}A = MB^{\dagger}$ *and* $A^{\dagger}M = BM^{\dagger}M$.

⟨42⟩ *Both* $BB^{\dagger}A^*AB = A^*AB$ *and* $ABB^*A^{\dagger}A = ABB^*$.

⟨43⟩ $BB^{\dagger}A^*ABB^*A^{\dagger}A = A^*ABB^*$.

⟨44⟩ *Both* $M^{\dagger} = (A^{\dagger}AB)^{\dagger}A^{\dagger}$ *and* $(A^{\dagger}AB)^{\dagger} = B^{\dagger}A^{\dagger}A$.

⟨45⟩ *Both* $M^{\dagger} = B^{\dagger}(ABB^{\dagger})^{\dagger}$ *and* $(ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger}$.

⟨46⟩ *Both* $M^{\dagger} = (A^*AB)^{\dagger}A^*$ *and* $(A^*AB)^{\dagger} = B^{\dagger}(A^*A)^{\dagger}$.

⟨47⟩ *Both* $M^{\dagger} = B^*(ABB^*)^{\dagger}$ *and* $(ABB^*)^{\dagger} = (BB^*)^{\dagger}A^{\dagger}$.

⟨48⟩ *Both* $M^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$ *and* $(A^{\dagger}ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger}A$.

⟨49⟩ *Both* $M^{\dagger} = B^*(BB^*)^{k-1}[(A^*A)^k(BB^*)^k]^{\dagger}(A^*A)^{k-1}A^*$ *and* $[(A^*A)^k(BB^*)^k]^{\dagger} = [(BB^*)^k]^{\dagger}[(A^*A)^k]^{\dagger}$ *for any integer* $k \geq 1$.

⟨50⟩ *Both* $M^{\dagger} = B^*B(AA^*ABB^*B)^{\dagger}AA^*$ *and* $(AA^*ABB^*B)^{\dagger} = (BB^*B)^{\dagger}(AA^*A)^{\dagger}$.

⟨51⟩ *Both* $(ABB^{\dagger})^{\dagger} = BB^{\dagger}A^{\dagger}$ *and* $(A^{\dagger}AB)^{\dagger} = B^{\dagger}A^{\dagger}A$.

⟨52⟩ *Both* $MB^{\dagger}A^{\dagger}$ *and* $B^{\dagger}A^{\dagger}M$ *are orthogonal projectors.*

⟨53⟩ *Both* $AE_BA^{\dagger}$ *and* $B^{\dagger}F_AB$ *are orthogonal projectors.*

⟨54⟩ *Both* $MM^{\dagger}A = MB^{\dagger}$ *and* $BM^{\dagger}M = A^{\dagger}M$.

⟨55⟩ *Both* $A^*ABB^{\dagger} = BB^{\dagger}A^*A$ *and* $A^{\dagger}ABB^* = BB^*A^{\dagger}A$.

⟨56⟩ *Both* $MB^{\dagger}A^{\dagger}M = M$ *and* $M^{\dagger} = (A^{\dagger}M)^{\dagger}A^{\dagger} = B^{\dagger}(MB^{\dagger})^{\dagger}$.

⟨57⟩ *Both* $MB^{\dagger}A^{\dagger}M = M$ *and* $M^{\dagger} = (A^*M)^{\dagger}A^* = B^*(MB^*)^{\dagger}$.

⟨58⟩ $MB^{\dagger}A^{\dagger}M = M$, $(A^{\dagger}M)^{\dagger} = M^{\dagger}A$, *and* $(MB^{\dagger})^{\dagger} = BM^{\dagger}$.

⟨59⟩ $MB^{\dagger}A^{\dagger}M = M$, $(A^{\dagger}MM^{\dagger}A)^* = A^{\dagger}MM^{\dagger}A$, *and* $(BM^{\dagger}MB^{\dagger})^* = BM^{\dagger}MB^{\dagger}$.

⟨60⟩ $MB^{\dagger}A^{\dagger}M = M$, $MM^{\dagger}AA^* = AA^*MM^{\dagger}$, *and* $M^{\dagger}MB^*B = B^*BM^{\dagger}M$.

⟨61⟩ *Both* $\mathscr{R}(A^*ABB^{\dagger}) = \mathscr{R}(BB^{\dagger}A^*A)$ *and* $\mathscr{R}(A^{\dagger}ABB^*) = \mathscr{R}(BB^*A^{\dagger}A)$.

⟨62⟩ $\mathscr{R}(A^{\dagger}ABB^{\dagger}) = \mathscr{R}(BB^{\dagger}A^{\dagger}A)$, $\mathscr{R}(M^{\dagger}) = \mathscr{R}(B^{\dagger}A^{\dagger})$, *and* $\mathscr{R}[(M^*)^{\dagger}] = \mathscr{R}[(A^*)^{\dagger}(B^*)^{\dagger}]$.

⟨63⟩ *Both* $\mathscr{R}(A^{\dagger}ABB^{\dagger}) = \mathscr{R}(BB^*A^*A)$ *and* $\mathscr{R}(BB^{\dagger}A^{\dagger}A) = \mathscr{R}(A^*ABB^*)$.

⟨64⟩ $\mathscr{R}(A^*ABB^*) = \mathscr{R}(BB^*A^*A)$.

⟨65⟩ *Both* $\mathscr{R}(A^*M) \subseteq \mathscr{R}(B)$ *and* $\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*)$.

⟨66⟩ *Both* $\mathscr{R}(A^*M) = \mathscr{R}(A^*) \cap \mathscr{R}(B)$ *and* $\mathscr{R}(BM^*) = \mathscr{R}(B) \cap \mathscr{R}(A^*)$.

⟨67⟩ *Both* $r[\,A^*M,\ B\,] = r(B)$ *and* $r[\,BM^*,\ A^*\,] = r(A)$.

⟨68⟩ $r[\,A^*,\ B\,] = r(A) + r(B) - r(M)$, $r[\,AA^*M,\ M\,] = r(M)$, *and* $r[\,B^*BM^*,\ M^*\,] = r(M)$.

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⟨69⟩ $r\begin{bmatrix} MM^*M & MB^*B \\ AA^*M & M \end{bmatrix} = r\left(\begin{bmatrix} MB^* \\ A \end{bmatrix} [\, A^*M,\, B\,]\right) = r(M).$

⟨70⟩ *The matrix equation* $BXA = A^*ABB^*$ *is consistent.*

Proof. Result ⟨1⟩ obviously implies ⟨2⟩–⟨16⟩ by Lemma 3.1(c). Conversely, if one of the following holds

$$M^{\dagger} = B^{\dagger}A^{(1,3,4)}, \qquad M^{\dagger} = B^{\dagger}A^{(1,2,4)}, \tag{7.5}$$
$$M^{\dagger} = B^{\dagger}A^{(1,4)}, \qquad M^{\dagger} = B^{(1,3,4)}A^{\dagger}, \tag{7.6}$$
$$M^{\dagger} = B^{(1,3,4)}A^{(1,3,4)}, \qquad M^{\dagger} = B^{(1,3,4)}A^{(1,2,4)}, \tag{7.7}$$
$$M^{\dagger} = B^{(1,3,4)}A^{(1,4)}, \qquad M^{\dagger} = B^{(1,2,3)}A^{\dagger}, \tag{7.8}$$
$$M^{\dagger} = B^{(1,2,3)}A^{(1,3,4)}, \qquad M^{\dagger} = B^{(1,2,3)}A^{(1,2,4)}, \tag{7.9}$$
$$M^{\dagger} = B^{(1,2,3)}A^{(1,4)}, \qquad M^{\dagger} = B^{(1,3)}A^{\dagger}, \tag{7.10}$$
$$M^{\dagger} = B^{(1,3)}A^{(1,3,4)}, \qquad M^{\dagger} = B^{(1,3)}A^{(1,2,4)}, \tag{7.11}$$
$$M^{\dagger} = B^{(1,3)}A^{(1,4)}, \tag{7.12}$$

then pre- and post-multiplying $B^{\dagger}B$ and $AA^{\dagger}$ to both sides of the equalities and applying (3.15), (3.17), and $B^{\dagger}BM^{\dagger}AA^{\dagger} = M^{\dagger}$ lead to

$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{\dagger}BB^{\dagger}A^{(1,3,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.13}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{\dagger}BB^{\dagger}A^{(1,2,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.14}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{\dagger}BB^{\dagger}A^{(1,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.15}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{\dagger}BB^{\dagger}BB^{(1,3,4)}A^{\dagger}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.16}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{(1,3,4)}A^{(1,3,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.17}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{(1,3,4)}A^{(1,2,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.18}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{(1,3,4)}A^{(1,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.19}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{(1,2,3)}A^{\dagger}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.20}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{(1,2,3)}A^{(1,3,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.21}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{(1,2,3)}A^{(1,2,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.22}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{(1,2,3)}A^{(1,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.23}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{(1,3)}A^{\dagger}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.24}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{(1,3)}A^{(1,3,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.25}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{(1,3)}A^{(1,2,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}, \tag{7.26}$$
$$M^{\dagger} = B^{\dagger}BM^{\dagger}AA^{\dagger} = B^{(1,3)}A^{(1,4)}AA^{\dagger} = B^{\dagger}A^{\dagger}. \tag{7.27}$$

Eqs. (7.13)–(7.27) show that each of ⟨2⟩–⟨16⟩ implies ⟨1⟩.

The equivalence of ⟨1⟩ and ⟨17⟩–⟨19⟩ follows from Theorem 6.3(a).

The equivalence of ⟨20⟩–⟨35⟩ and ⟨67⟩ follows from Theorem 5.4(a), (b), (d), and (f), and Theorem 5.5(a), (b), (d), and (f).

The equivalence of ⟨36⟩ and ⟨68⟩ follows from Theorem 5.1⟨78⟩ and ⟨116⟩, Lemma 7.1⟨1⟩ and ⟨9⟩, and Lemma 7.2⟨1⟩ and ⟨9⟩.

Result ⟨1⟩ obviously implies ⟨37⟩. Conversely, pre-multiplying the equality in ⟨37⟩ with AB yields $ABM^{\dagger} = ABB^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger} = (ABB^{\dagger}A^{\dagger})^2$, where $ABM^{\dagger}$ is idempotent. So that $(ABB^{\dagger}A^{\dagger})^2 = ABB^{\dagger}A^{\dagger}$, which is equivalent to $B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger} = B^{\dagger}A^{\dagger}$ by Theorem 5.1⟨88⟩ and ⟨93⟩. Thus, ⟨37⟩ implies ⟨1⟩.

Pre- and post-multiplying A and B to both sides of ⟨1⟩ yields ⟨38⟩. Conversely, pre- and post-multiplying $A^{\dagger}$ and $B^{\dagger}$ to both sides of ⟨38⟩ yields ⟨1⟩.

Pre- and post-multiplying A^* and B^* to both sides of ⟨1⟩ yields ⟨39⟩. Conversely, pre- and post-multiplying $(A^{\dagger})^*$ and $(B^{\dagger})^*$ to both sides of ⟨39⟩ yields ⟨1⟩.

The equivalence of ⟨1⟩ and ⟨40⟩ follows from Theorem 5.4(a)⟨1⟩ and ⟨5⟩ and Theorem 5.5(a)⟨1⟩ and ⟨5⟩.

The equivalence of ⟨1⟩ and ⟨41⟩ follows from Theorem 5.4(a)⟨1⟩ and ⟨5⟩ and Theorem 5.5(a)⟨1⟩ and ⟨5⟩.

The equivalence of ⟨42⟩ and ⟨67⟩ follows from Theorem 5.4(a)⟨8⟩ and ⟨17⟩ and Theorem 5.5(a)⟨8⟩ and ⟨17⟩.

The equivalence of ⟨43⟩ and ⟨67⟩ is derived from Lemma 2.2(d) and (e).

The equivalence of ⟨44⟩ and ⟨68⟩ follows from Lemma 7.1⟨2⟩ and ⟨9⟩ and Theorem 5.5(a)⟨9⟩ and ⟨26⟩.

The equivalence of ⟨45⟩ and ⟨68⟩ follows from Lemma 7.2⟨2⟩ and ⟨9⟩ and Theorem 5.4(a)⟨9⟩ and ⟨26⟩.

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Applying the equivalence of ⟨1⟩ and ⟨65⟩ to the two equalities in ⟨46⟩, we first see that

$$M^{\dagger} = (A^*AB)^{\dagger}A^* \Leftrightarrow \mathscr{R}(AA^*M) = \mathscr{R}(M), \tag{7.28}$$

$$(A^*AB)^{\dagger} = B^{\dagger}(A^*A)^{\dagger} \Leftrightarrow \text{both } \mathscr{R}[(AA^*)^2AB] \subseteq \mathscr{R}(B) \text{ and } \mathscr{R}(BB^*A^*) \subseteq \mathscr{R}(A^*). \tag{7.29}$$

Also by (4.124),

$$\mathscr{R}(AA^*AB) \subseteq \mathscr{R}(B) \Rightarrow \mathscr{R}[(AA^*)^2AB] \subseteq \mathscr{R}(AA^*B) \Rightarrow \mathscr{R}[(AA^*)^2AB] \subseteq \mathscr{R}(B). \tag{7.30}$$

Thus if ⟨1⟩ holds, combining ⟨65⟩ and ⟨68⟩ with (7.28), (7.29), and (7.30), we see that the two equalities in ⟨46⟩ hold. Conversely, merging the two equalities in ⟨46⟩ and simplifying yields the equality in ⟨1⟩.

Merging the two equalities in ⟨48⟩ and simplifying yields the equality in ⟨1⟩. Conversely, notice that $B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$ is a {2}-inverses of AB. Then we obtain by (3.46) that

$$\begin{aligned} & r[\, M^{\dagger} - B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}\,] \\ &= r\begin{bmatrix} M^{\dagger} \\ B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger} \end{bmatrix} + r[\, M^{\dagger},\, B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}\,] - r(M^{\dagger}) - r[B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}\,] \\ &= r\begin{bmatrix} M^* \\ (A^{\dagger}ABB^{\dagger})^*A^{\dagger} \end{bmatrix} + r[\, M^*,\, (A^{\dagger}ABB^{\dagger})^*\,] - 2r(M) \\ &= r\begin{bmatrix} M^* \\ B^*A^{\dagger} \end{bmatrix} + r[\, M^*,\, B^{\dagger}(A^{\dagger}ABB^{\dagger})^*\,] - 2r(M) \\ &= r\begin{bmatrix} M \\ MB^*B \end{bmatrix} + r[\, M,\, AA^*M\,] - 2r(M). \end{aligned} \tag{7.31}$$

So that

$$M^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger} \Leftrightarrow r[\, AA^*M,\, M\,] = r(M) \text{ and } r[\, B^*BM^*,\, M^*\,] = r(M). \tag{7.32}$$

Thus if ⟨1⟩ holds, combining ⟨68⟩ with (7.32) and Theorem 5.1⟨98⟩ and ⟨116⟩, we see that the two equalities in ⟨48⟩ hold.

Merging the two equalities in ⟨49⟩ and simplifying yields the equality in ⟨1⟩. Conversely, notice that $B^*(BB^*)^{k-1}[(A^*A)^k(BB^*)^k]^{\dagger}(A^*A)^{k-1}A^*$ is a {2}-inverses of AB. Then we obtain by (3.46) that

$$\begin{aligned} & r\{\, M^{\dagger} - B^*(BB^*)^{k-1}[(A^*A)^k(BB^*)^k]^{\dagger}(A^*A)^{k-1}A^*\,\} \\ &= r\begin{bmatrix} M^{\dagger} \\ B^*(BB^*)^{k-1}[(A^*A)^k(BB^*)^k]^{\dagger}(A^*A)^{k-1}A^* \end{bmatrix} \\ &\quad + \ r[\, M^{\dagger},\, B^*(BB^*)^{k-1}[(A^*A)^k(BB^*)^k]^{\dagger}(A^*A)^{k-1}A^*\,] \\ &\quad - \ r(M^{\dagger}) - r\{\, B^*(BB^*)^{k-1}[(A^*A)^k(BB^*)^k]^{\dagger}(A^*A)^{k-1}A^*\,\} \\ &= r\begin{bmatrix} M^* \\ B^*(A^*A)^{2k-1}A^* \end{bmatrix} + r[\, M^*,\, B^*(BB^*)^{2k-1}A^*\,] - 2r(M) \\ &= r\begin{bmatrix} M \\ A(BB^*)^{2k-1}B \end{bmatrix} + r[\, M,\, A(A^*A)^{2k-1}B\,] - 2r(M). \end{aligned} \tag{7.33}$$

So that

$$\begin{aligned} & M^{\dagger} = B^*(BB^*)^{k-1}[(A^*A)^k(BB^*)^k]^{\dagger}(A^*A)^{k-1}A^* \\ & \Leftrightarrow \mathscr{R}[A(A^*A)^{2k-1}B] = \mathscr{R}(M) \text{ and } \mathscr{R}[B^*(BB^*)^{2k-1}A^*] = \mathscr{R}(M^*). \end{aligned} \tag{7.34}$$

Applying the equivalence of ⟨1⟩ and ⟨65⟩ to the product $(A^*A)^k(BB^*)^k$ yields

$$[(A^*A)^k(BB^*)^k]^{\dagger} = [(BB^*)^k]^{\dagger}[(A^*A)^k]^{\dagger} \Leftrightarrow \mathscr{R}[(A^*A)^{2k}B] \subseteq \mathscr{R}(B) \text{ and } \mathscr{R}[(BB^*)^{2k}A^*] \subseteq \mathscr{R}(A^*). \tag{7.35}$$

Also note from (7.30) that

$$\mathscr{R}(A^*M) \subseteq \mathscr{R}(B) \Rightarrow \mathscr{R}[\,(A^*A)^{2k}B\,] \subseteq \mathscr{R}(B) \text{ and } \mathscr{R}[A(A^*A)^{2k-1}B] = \mathscr{R}(M), \tag{7.36}$$

$$\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*) \Rightarrow \mathscr{R}[(BB^*)^{2k}A^*] \subseteq \mathscr{R}(A^*) \text{ and } \mathscr{R}[B^*(BB^*)^{2k-1}A^*] = \mathscr{R}(M^*). \tag{7.37}$$

Thus if ⟨1⟩ holds, combining ⟨65⟩ with (7.34)–(7.37), we see that the two equalities in ⟨49⟩ hold.

Merging the two equalities in ⟨50⟩ and simplifying yields the equality in ⟨1⟩. Conversely, notice that $B^*B(AA^*ABB^*B)^\dagger AA^*$ is a {2}-inverses of AB. Then we obtain by (3.46) that

$$\begin{aligned} & r[\, M^\dagger - B^*B(AA^*ABB^*B)^\dagger AA^* \,] \\ & = r\begin{bmatrix} M^\dagger \\ B^*B(AA^*ABB^*B)^\dagger AA^* \end{bmatrix} + r[\, M^\dagger,\ B^*B(AA^*ABB^*B)^\dagger AA^* \,] \\ & \quad - \ r(M^\dagger) - r[\, B^*B(AA^*ABB^*B)^\dagger AA^* \,] \\ & = r\begin{bmatrix} M^* \\ B^*A^*(AA^*)^2 \end{bmatrix} + r[\, M^*,\ (B^*B)^2B^*A^* \,] - 2r(M) \\ & = r\begin{bmatrix} M \\ M(B^*B)^2 \end{bmatrix} + r[\, M,\ (AA^*)^2M \,] - 2r(M). \end{aligned} \tag{7.38}$$

So that

$$M^\dagger = B^*B(AA^*ABB^*B)^\dagger AA^* \Leftrightarrow \mathscr{R}[(AA^*)^2M] = \mathscr{R}(M) \ and \ \mathscr{R}[(B^*B)^2M^*] = \mathscr{R}(M^*). \tag{7.39}$$

Applying the equivalence of ⟨1⟩ and ⟨65⟩ to the product AA^*ABB^*B yields

$$(AA^*ABB^*B)^\dagger = (BB^*B)^\dagger(AA^*A)^\dagger \Leftrightarrow \mathscr{R}[(A^*A)^3B] \subseteq \mathscr{R}(B) \text{ and } \mathscr{R}[(BB^*)^3A^*] \subseteq \mathscr{R}(A^*). \tag{7.40}$$

Also by (4.124),

$$\mathscr{R}(A^*M) \subseteq \mathscr{R}(B) \Rightarrow \mathscr{R}[\,(A^*A)^3B\,] \subseteq \mathscr{R}(B) \ and \ \mathscr{R}[(AA^*)^2M] = \mathscr{R}(M), \tag{7.41}$$

$$\mathscr{R}(BM^*) \subseteq \mathscr{R}(A^*) \Rightarrow \mathscr{R}[(BB^*)^3A^*] \subseteq \mathscr{R}(A^*) \ and \ \mathscr{R}[(B^*B)^2M^*)] = \mathscr{R}(M^*). \tag{7.42}$$

Thus if ⟨1⟩ holds, combining ⟨65⟩ with (7.39)–(7.42), we see that the two equalities in ⟨50⟩ hold.

The equivalence of ⟨51⟩ and ⟨65⟩ follows from Theorem 5.4(a)⟨9⟩ and ⟨26⟩ and Theorem 5.5(a)⟨9⟩ and ⟨26⟩.

The equivalence of ⟨52⟩ and ⟨65⟩ follows from Theorem 5.4(a)⟨23⟩ and ⟨26⟩ and Theorem 5.5(a)⟨23⟩ and ⟨26⟩.

The equivalence of ⟨53⟩ and ⟨65⟩ follows from Theorem 5.4(a)⟨23⟩ and ⟨26⟩ and Theorem 5.5(a)⟨23⟩ and ⟨26⟩.

The equivalence of ⟨54⟩ and ⟨65⟩ follows from Theorem 5.4(a)⟨13⟩ and ⟨26⟩ and Theorem 5.5(a)⟨13⟩ and ⟨26⟩.

The equivalence of ⟨55⟩ and ⟨65⟩ follows from Theorem 5.4(a)⟨14⟩ and ⟨26⟩ and Theorem 5.5(a)⟨14⟩ and ⟨26⟩.

The equivalence of ⟨56⟩ and ⟨68⟩ follows from Theorem 5.1(a)⟨87⟩ and ⟨116⟩, Theorem 7.1⟨2⟩ and ⟨9⟩, and Theorem 7.2⟨2⟩ and ⟨9⟩.

The equivalence of ⟨57⟩ and ⟨68⟩ follows from Theorem 5.1(a)⟨87⟩ and ⟨116⟩, Theorem 7.1⟨3⟩ and ⟨9⟩, and Theorem 7.2⟨3⟩ and ⟨9⟩.

The equivalence of ⟨58⟩ and ⟨68⟩ follows from Theorem 5.1(a)⟨87⟩ and ⟨116⟩, Theorem 7.1⟨4⟩ and ⟨9⟩, and Theorem 7.2⟨4⟩ and ⟨9⟩.

The equivalence of ⟨59⟩ and ⟨68⟩ follows from Theorem 5.1(a)⟨87⟩ and ⟨116⟩, Theorem 7.1⟨5⟩ and ⟨9⟩, and Theorem 7.2⟨5⟩ and ⟨9⟩.

The equivalence of ⟨60⟩ and ⟨68⟩ follows from Theorem 5.1(a)⟨87⟩ and ⟨116⟩, Theorem 7.1⟨6⟩ and ⟨9⟩, and Theorem 7.2⟨6⟩ and ⟨9⟩.

The equivalence of ⟨61⟩ and ⟨68⟩ follows from Theorem 5.4(a)⟨21⟩ and ⟨26⟩, and Theorem 5.5⟨21⟩ and ⟨26⟩.

The equivalence of ⟨62⟩ and ⟨68⟩ follows from Theorem 5.1(a)⟨116⟩ and ⟨155⟩, Theorem 7.1⟨8⟩ and ⟨9⟩, and Theorem 7.2⟨8⟩ and ⟨9⟩.

It can be derived from (3.40) that

$$r[\, A^\dagger ABB^\dagger,\ BB^*A^*A \,] = r[\, BM^*,\ A^*] + r(M) - r(A), \tag{7.43}$$

$$r[\, BB^\dagger A^\dagger A,\ A^*ABB^* \,] = r[\, A^*M,\ B \,] + r(M) - r(B). \tag{7.44}$$

Thus

$$\mathscr{R}(A^\dagger ABB^\dagger) = \mathscr{R}(BB^*A^*A) \Leftrightarrow r[\, BM^*,\ A^*] = r(A), \tag{7.45}$$

$$\mathscr{R}(BB^\dagger A^\dagger A) = \mathscr{R}(A^*ABB^*) \Leftrightarrow r[\, A^*M,\ B \,] = r(B). \tag{7.46}$$

If ⟨1⟩ holds, then ⟨63⟩ implies ⟨64⟩ by (7.45), (7.46), and Theorem 5.1(a)⟨1⟩ and ⟨155⟩. Conversely, ⟨64⟩ obviously implies ⟨65⟩, and thus it implies ⟨1⟩ as well.

Applying (3.40) to $M^* - B^*BM^\dagger AA^*$ gives

$$r(\, M^* - B^*BM^\dagger AA^* \,) = r\begin{bmatrix} M^*MM^* & M^*A^* \\ B^*M^* & B^*A^* \end{bmatrix} - r(M) = r\begin{bmatrix} MM^*M & MB^*B \\ AA^*M & M \end{bmatrix} - r(M). \tag{7.47}$$

Setting both sides of (7.47) equal to zero leads to the equivalence of ⟨39⟩ yields ⟨69⟩.

The equivalence of ⟨65⟩ yields ⟨70⟩ are derived from Lemma 2.2(b) and (e). □

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Some equivalent statements in Theorem 7.3 were formulated by different authors and were scattered in the literature. But we prefer to giving complete proofs for the equivalences of all these statements in order to sufficiently recognize and use this collection of results in different situations.

8 Concluding remarks

We have seen in the previous sections that reverse-order law problems link many results and facts in matrix theory together. So that people can distinguish and utilize these results and facts in many different situations. The whole work is based on establishing various matrix equations associated with the reverse-order laws and calculating the ranks of various block matrices associated with the matrix equations. Especially, all the results obtained in the paper are presented in simple and explicit forms, so that they are easy to understand and to accept in comparison with many other tedious and ambiguous results in the literature on the reverse-order laws of generalized inverses. It has already been recognized that the three fundamental and valued methods—BMRM, MEM, and MRM have deep and solid roots in matrix theory, which are easy to understand within elementary linear algebra, and have been taken as reliable and efficient tools to deal with various equality problems on generalized inverses of matrix products.

Furthermore, the present author remarks that the results and methods demonstrated in this paper will have certain influential impact on the development of matrix equality theory, and many more deep and fruitful investigations can be conducted on equality problems of matrix-valued functions and generalized inverses of matrix products. For instance,

(I) For a given matrix A of order m and two scalars λ and μ, the matrix product

$$(\lambda I_m - A)(\mu I_m - A), \qquad \lambda \neq \mu \tag{8.1}$$

is a simplest form of matrix polynomial matrices. Applying the results in Sections 5–7 to the matrix polynomial will lead to necessary and sufficient conditions for the following matrix set inclusions

$$\{[(\lambda I_m - A)(\mu I_m - A)]^{(i,\ldots,j)}\} \supseteq \{(\mu I_m - A)^{(i,\ldots,j)}(\lambda I_m - A)^{(i,\ldots,j)}\} \quad (8^3 = 512 \text{ situations}) \tag{8.2}$$

to hold for the eight commonly-used types of generalized inverses, including the special cases for $A - A^2$ and $I_m - A^2$ in (8.2).

(II) Let $P,\, Q \in \mathbb{C}^{m\times m}$ be a pair of idempotent matrices (or orthogonal projectors). Then the anti-commutator $PQ + QP$ and the commutator $PQ - QP$ satisfy the following decomposition identities

$$PQ \pm QP = (\, P \pm Q\,)(\, P + Q - I_m\,). \tag{8.3}$$

In this situation, applying the results in Sections 5–7 to (8.3) will lead to necessary and sufficient conditions for the set inclusions

$$\{(PQ \pm QP)^{(i,\ldots,j)}\} \supseteq \{(P + Q - I_m)^{(i,\ldots,j)}(P \pm Q)^{(i,\ldots,j)}\} \quad (2 \times 8^3 = 1,024 \text{ situations}) \tag{8.4}$$

to hold for the eight commonly-used types of generalized inverses, respectively.

(III) Rewriting the matrix product AB as the two alternative forms

$$AB = A(A^{\dagger}AB) = (ABB^{\dagger})B, \tag{8.5}$$

and applying the results in Sections 5–7 to (8.5) will lead to necessary and sufficient conditions for the following set inclusions

$$\{(AB)^{(i,\ldots,j)}\} \supseteq \{(A^{\dagger}AB)^{(i,\ldots,j)}A^{(i,\ldots,j)}\} \qquad (8^3 = 512 \text{ situations}), \tag{8.6}$$
$$\{(AB)^{(i,\ldots,j)}\} \supseteq \{B^{(i,\ldots,j)}(ABB^{\dagger})^{(i,\ldots,j)}\} \qquad (8^3 = 512 \text{ situations}), \tag{8.7}$$
$$\{(A^{\dagger}AB)^{(i,\ldots,j)}\} \supseteq \{B^{(i,\ldots,j)}(A^{\dagger}A)^{(i,\ldots,j)}\} \qquad (8^3 = 512 \text{ situations}), \tag{8.8}$$
$$\{(ABB^{\dagger})^{(i,\ldots,j)}\} \supseteq \{(BB^{\dagger})^{(i,\ldots,j)}A^{(i,\ldots,j)}\} \qquad (8^3 = 512 \text{ situations}), \tag{8.9}$$
$$\{(A^{\dagger}ABB^{\dagger})^{(i,\ldots,j)}\} \supseteq \{(BB^{\dagger})^{(i,\ldots,j)}(A^{\dagger}A)^{(i,\ldots,j)}\} \qquad (8^3 = 512 \text{ situations}) \tag{8.10}$$

to hold respectively for the eight commonly-used types of generalized inverses. It should be pointed out that many situations in (8.6)–(8.10) are in fact equivalent to those in (1.32).

(IV) Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{m\times p}$ be given. Then a group of set inclusions for the generalized inverses of for the eight commonly-used types of generalized inverses of A, B, and C are given by

$$\{C^{(i,\ldots,j)}\} \supseteq \{B^{(i,\ldots,j)}A^{(i,\ldots,j)}\} \qquad (8^3 = 512 \text{ situations}), \tag{8.11}$$

which are direct extensions of the matrix product AB in (1.32) to a third matrix C.

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(V) For the triple matrix product ABC, the sum of two matrices $A+B$, and the two-by-two block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the three groups of reasonable set inclusion problems for the eight commonly-used types of generalized inverses of the matrices are given by

$$\{(ABC)^{(i,\dots,j)}\} \supseteq \{C^{(i,\dots,j)}B^{(i,\dots,j)}A^{(i,\dots,j)}\} \qquad (8^4 = 4,096 \text{ situations}), \tag{8.12}$$

$$\{(A+B)^{(i,\dots,j)}\} \supseteq \{A^{(i,\dots,j)} + B^{(i,\dots,j)}\} \qquad (8^3 = 512 \text{ situations}), \tag{8.13}$$

$$\left\{\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{(i,\dots,j)}\right\} \supseteq \left\{\begin{bmatrix} A^{(i,\dots,j)} & C^{(i,\dots,j)} \\ B^{(i,\dots,j)} & D^{(i,\dots,j)} \end{bmatrix}\right\} \qquad (8^5 = 32,768 \text{ situations}). \tag{8.14}$$

(VI) In addition, rewriting ABC in the alternative tripe matrix products

$$ABC = (AB)B^{\dagger}(BC) = (ABB^{\dagger})B(B^{\dagger}BC),$$

and applying (7.7) leads to the following matrix set inclusions

$$\{(ABC)^{(i,\dots,j)}\} \supseteq \{(BC)^{(i,\dots,j)}B(AB)^{(i,\dots,j)}\} \qquad (8^3 = 512 \text{ situations}), \tag{8.15}$$

$$\{(ABC)^{(i,\dots,j)}\} \supseteq \{(B^{\dagger}BC)^{(i,\dots,j)}B^{(i,\dots,j)}(ABB^{\dagger})^{(i,\dots,j)}\} \qquad (8^4 = 4,096 \text{ situations}), \tag{8.16}$$

as well as

$$\{(A+B)^{(i,\dots,j)}\} \supseteq \left\{\begin{bmatrix} A \\ B \end{bmatrix}^{(i,\dots,j)} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [\,A,\, B\,]^{(i,\dots,j)}\right\} \qquad (8^3 = 512 \text{ situations}), \tag{8.17}$$

$$\{(A+B)^{(i,\dots,j)}\} \supseteq \left\{\begin{bmatrix} A^{\dagger}A \\ B^{\dagger}B \end{bmatrix}^{(i,\dots,j)} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^{(i,\dots,j)} [\,AA^{\dagger},\, BB^{\dagger}\,]^{(i,\dots,j)}\right\} \qquad (8^4 = 4,096 \text{ situations}) \tag{8.18}$$

for the eight commonly-used types of generalized inverses of the matrices.

Many special cases of matrix equalities involved in (8.11)–(8.18) were considered in the literature; see e.g., [8,16,28,29,33,42]. It is no doubt that necessary and sufficient conditions for all these designated set inclusions can be derived clearly and systematically by mean of the three valued algebraic methods, but people have to make tremendous preparation on various analytical rank formulas and invariance properties of the matrix expressions appeared in (8.11)–(8.18) before finishing this task; see [37,38,41].

Finally, the author remarks that all the results and facts presented in this paper can symbolically be extended to the same topics on reverse-order laws in other algebraic structures (mainly rings and operator algebras) in which generalized inverses of elements are also defined by the four Penrose equations. It should be pointed out that for the same reverse-order law problem, any result derived from other methods or established for elements in other algebraic structures must be consistent with these derived by the BMRM, MEM, and MRM for real or complex matrices. Otherwise, there do exist some flaws in their derivations, and thus the corresponding results are not acceptable as reasonable extensions of the work on real or complex matrices.

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