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Article

Integral representations and algebraic decompositions of the Fox-Wright Type of special functions

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Abstract: The manuscript surveys the special functions of the Fox-Wright type. These functions are generalizations of the hypergeometric functions. Notable representatives of the type are the Mittag-Leffler functions and the Wright function. The integral representations of such functions are given and the conditions under which these function can be represented by simpler functions are demonstrated. The connection with generalized fractional differential and integral operators is demonstrated and discussed.

Keywords: Wright function; Gamma function, Beta function; fractional calculus

MSC: Primary: 33C20; Secondary: 33C60, 33C99, 26A33, 33B15

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1. Introduction

This paper is concerned with partial integral representations of the Fox-Wright functions. The first characteristic exemplar of this function family has been introduced by E. M. Wright, who generalized the concept in a series of papers in 1930s. The Fox-Wright special functions have very broad applications in mathematical physics, notably in descriptions of wave phenomena, heat and mass transfer. They encompass generalized hypergeometric functions and are related to the family of the Bessel functions. Many authors introduce the Fox-Wright functions from their representation as H-functions, which are in turn defined as Mellin transforms. Such presentation tends to obfuscate the utility of Fox-Wright functions. The objective of the paper is to give a self-contained treatment of the Fox-Wright functions as generalized hypergeoemtric series (GHG) and related them to the theory of the Euler Gamma and Beta functions. In addition the relationship with some generalized fractional calculus Erdélyi-Kober operators is discussed.

2. Notation

The generalized hypergeometric functions are defined by the infinite series

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q,x) \quad := \quad \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(m+1)} \prod_{k=1}^{p} \frac{\Gamma(a_k+m)}{\Gamma(a_k)} \prod_{k=1}^{q} \frac{\Gamma(b_k)}{\Gamma(b_k+m)} $$

The defining property fo HG series is that the coefficients are rational functions of the index variable (i.e. k). onditions for the existence of the generalized Wright function together with its representation in terms of the Mellin-Barnes integral and of the H-function were established in [1].

In the following sections we will use the parametric notation similar to the one adopted by Oldham and Spanier [2].

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q,x) \equiv \left[\begin{array}{c|c} a_1,\ldots,a_p & x \\ b_1,\ldots,b_q & \end{array} \right] $$

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The Fox-Wright functions are further generalizations of the hypergeometric (HG) functions of the form

$$ {}_p\bar{\Psi}_q(z) \equiv \bar{\Psi}\left[\begin{array}{c|c} (A_1,a_1)\ldots,(A_p,a_p) & z \\ (B_1,b_1)\ldots,(B_q,b_q) & \end{array}\right] := \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(m+1)} \prod_{k=1}^{p} \frac{\Gamma(a_k m + A_k)}{\Gamma(A_k)} \prod_{k=1}^{q} \frac{\Gamma(B_k)}{\Gamma(b_k m + B_k)} $$

for this generalization one can not expect that in general the coefficients are rational functions of the index variable. The following simplifying convention will be used further:

$$ \left[\begin{array}{c|c|c} a_1,\ldots & - & z \\ b_1,\ldots & - & \end{array}\right] = \left[\begin{array}{c|c} a_1,\ldots & z \\ b_1,\ldots & \end{array}\right] \tag{1} $$

and

$$ \left[\begin{array}{c|c|c} a_1,\ldots,a_p & (A,1) & z \\ b_1,\ldots,b_q & - & \end{array}\right] = \left[\begin{array}{c|c} a_1,\ldots,a_p,A & z \\ b_1,\ldots,b_q & \end{array}\right] \tag{2} $$

For convergence of the series the condition

$$ \sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k > -1 $$

will be assumed everywhere [3,4]. At this point we introduce some extended notation under the convention

$$ {}_{p+1}\bar{\Psi}_q(z) = \left[\begin{array}{c|c|c} a_1,\ldots,a_p & (A,a) & z \\ b_1,\ldots,b_q & - & \end{array}\right], \quad {}_{p+1}\bar{\Psi}_q(0) = 1, $$

such that the monomial terms are multiplied by factors of the form $\frac{\Gamma(ka+A)}{\Gamma(A)}$ or their reciprocals, respectively.

The order in the parametric convention for the arguments of the Gamma function follows the usual convention, used for example in [5,6]. This is unfortunately converse to the order of the more conventional Wright function.

3. Algebraic Decomposition

The coefficients of the GH series can be identified by means of the following Lemma:

Lemma 1 (HG Recurrence). *Suppose that*

$$ S = {}_1\bar{\Psi}_0(z) = \sum_{k=0}^{\infty} \frac{c_k z^k}{\Gamma(k+1)}, \quad c_k = \Gamma(q_k) $$

or

$$ S = {}_0\bar{\Psi}_1(z), \quad c_k = \frac{1}{\Gamma(q_k)} $$

under the same convention. Then

$$ S = c_0 \left[\begin{array}{c|c|c} - & (A,a) & z \\ - & - & \end{array}\right] $$

or

$$ S = c_0 \left[\begin{array}{c|c|c} - & - & z \\ - & (A,a) & \end{array}\right] $$

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respectively, where

$$a = q_{k+1} \bmod q_k$$

and

$$A = q_k \bmod k$$

Proof. We prove the first case only since the second one follows identical reasoning. By hypothesis $c_k = \Gamma(A + ka)$ for some unknown A and a. Let's form the ratio

$$Q_k = \frac{\Gamma(A + a + ka)}{\Gamma(A + ka)}$$

Then $(A + a + ka) \mod (A + k\,a) = a$ and $A + k\,a \bmod k = A$. □

The generalized hypergeometric series can be decomposed in symmetric (even) and anti-symmetric (odd) series as follows:

Theorem 1 (GH Series Parity Decomposition). *Suppose that the generalized hypergeometric series $S = S_e + S_o$ is absolutely convergent at z. Denote S_e as the even part while S_o as the odd part.*

If S is of the form

$$S = \left[\begin{array}{c|c|c} - & (A, a) & z \\ - & \dots & \end{array} \right]$$

then

$$S_e = \left[\begin{array}{c|c|c} - & (A, 2a) & \frac{z^2}{4} \\ \frac{1}{2} & \dots & \end{array} \right], \qquad S_o = z\frac{\Gamma(a + A)}{\Gamma(A)} \left[\begin{array}{c|c|c} - & (a + A, 2\,a) & \frac{z^2}{4} \\ \frac{3}{2} & \dots & \end{array} \right]$$

If S is of the form

$$S = \left[\begin{array}{c|c|c} - & \dots & z \\ - & (A, a) & \end{array} \right]$$

then

$$S_e = \left[\begin{array}{c|c|c} - & \dots & \frac{z^2}{4} \\ \frac{1}{2} & (A, 2\,a) & \end{array} \right], \qquad S_o = z\frac{\Gamma(A)}{\Gamma(a + A)} \left[\begin{array}{c|c|c} - & \dots & \frac{z^2}{4} \\ \frac{3}{2} & (a + A, 2\,a) & \end{array} \right]$$

Proof. Le the even part and odd series be S_e and S_o, respectively. We prove only the first statement because the second one can be proved in identical way. For simplicity of calculations suppose that S is of the form

$$\left[\begin{array}{c|c|c} - & (A, a) & z \\ - & - & \end{array} \right]$$

For the even part:

$$k = 2m + 2: \quad \frac{\Gamma(ak + A)}{\Gamma(k + 1)} z^k \mapsto \frac{\Gamma(2a\,m + 2a + A)}{\Gamma(2m + 3)} z^{2m+2}$$

$$k = 2\,m: \quad \frac{\Gamma(ak + A)}{\Gamma(k + 1)} z^k \mapsto \frac{\Gamma(2a\,m + A)}{\Gamma(2m + 1)} z^{2m}$$

so that the ratio of the coefficients is

$$\frac{\Gamma(2m + 1)}{\Gamma(2m + 3)} \frac{\Gamma(2a\,m + 2a + A)}{\Gamma(2a\,m + A)} z^2 = \frac{z^2}{4\,(m + 1)\left(m + \frac{1}{2}\right)} \frac{\Gamma(2a\,m + 2a + A)}{\Gamma(2a\,m + A)}$$

Therefore,

$$S_e = \left[\begin{array}{c|c|c} - & (A, 2a) & z \\ \frac{1}{2} & - & \end{array} \right]$$

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For the odd part:

$$k = 2m+1: \quad \frac{\Gamma(ak+A)}{\Gamma(k+1)} z^k \mapsto \frac{\Gamma(2a\,m+a+A)}{\Gamma(2m+2)} z^{2m+2}$$

$$k = 2\,m-1: \quad \frac{\Gamma(ak+A)}{\Gamma(k+1)} z^k \mapsto \frac{\Gamma(2a\,m-a+A)}{\Gamma(2m)} z^{2m}$$

so the the ratio of the coefficients is

$$\frac{\Gamma(2\,m)}{\Gamma(2m+2)} \frac{\Gamma(2a\,m+a+A)}{\Gamma(2a\,m-a+A)} z^{2m+2} = \frac{z^2}{4\,m\left(m+\frac{1}{2}\right)} \frac{\Gamma(2a\,m+a+A)}{\Gamma(2a\,m-a+A)}$$

Therefore,

$$S_o = z\frac{\Gamma(a+A)}{\Gamma(A)} \left[\begin{array}{c|c|c} - & (a+A, 2\,a) & \frac{z^2}{4} \\ \frac{3}{2} & - & \end{array} \right]$$

□

The simplest example is given by the exponential series.

Example 1 (The exponential function decomposition)**.**

$$e^z = \left[\begin{array}{c|c} - & z \\ - & \end{array} \right] = \left[\begin{array}{c|c} - & \frac{z^2}{4} \\ \frac{1}{2} & \end{array} \right] + z \left[\begin{array}{c|c} - & \frac{z^2}{4} \\ \frac{3}{2} & \end{array} \right] = \cosh z + \sinh z$$

and

$$e^{i\,z} = \left[\begin{array}{c|c} - & i\,z \\ - & \end{array} \right] = \left[\begin{array}{c|c} - & -\frac{z^2}{4} \\ \frac{1}{2} & \end{array} \right] + iz \left[\begin{array}{c|c} - & -\frac{z^2}{4} \\ \frac{3}{2} & \end{array} \right] = \cos z + i \sin z$$

as expected.

The negative multiplicative parameters can be raised to the numerator by the application of the following Theorem:

Theorem 2. *Suppose that $z \in \mathcal{R}$, $A > 0$ and $-1 < a < 0$. Then*

$$\left[\begin{array}{c|c|c} \ldots & - & z \\ \ldots & (A, -a) & \end{array} \right] = Im\, \frac{q_A}{\pi} \left[\begin{array}{c|c|c} \ldots & (1-A, a) & q_a z \\ \ldots & - & \end{array} \right]$$

where $q_a = e^{-i\pi a}$, $q_A = e^{i\pi A}$.

Proof. Consider the monomial

$$B_k = \frac{z^k}{\Gamma(k+1)\Gamma(-ka+A)}$$

By the reflection formula

$$B_k = \frac{z^k}{\Gamma(k+1)\Gamma(-ka+A)} \frac{\Gamma(1+ka-A)}{\Gamma(1+ka-A)} = \frac{z^k \sin\left(-\pi ka + \pi A\right)\Gamma(1+ka-A)}{\Gamma(k+1)\pi}$$

This can be embedded in the complex monomial expression

$$C_k = \frac{e^{i\pi A}}{\pi} \frac{\Gamma(1+ka-A)}{\Gamma(k+1)} z^k e^{-ika}$$

Assuming that z is real the original expression B_k is the imaginary part of C_k.

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Further, C_k has modulus

$$|C_k| = \frac{1}{\pi} \frac{\Gamma(1+ka-A)}{\Gamma(k+1)} |z|^k$$

so that the infinite series for C_k converges and so does its imaginary part. □

4. Integral Representations

4.1. Integral Representations by Beta Integrals

The main result of this section is given by the theorem below. The result allows for the representation of a GHG function of order $(p+1, q+1)$ in terms of an integral of a GHG function of order $(p, q+1)$ or in special cases (p, q).

Theorem 3 (Beta integral representation). *For $B > A > 0$ and $b \geq a$ the following representation holds*

$$\left[\begin{array}{c|c|c} a_1, \ldots, a_p & (A, a) & z \\ b_1, \ldots, b_q & (B, b) & \end{array} \right] = \frac{1}{\mathcal{B}(A, B-A)} \int_0^1 \tau^{A-1} (1-\tau)^{B-A-1} \left[\begin{array}{c|c|c} a_1, \ldots, a_p & - & z\, \tau^a (1-\tau)^{(b-a)} \\ b_1, \ldots, b_q & (B-A, b-a) & \end{array} \right] d\tau$$

By change of variables $t = 1/(1+u)$

$$\left[\begin{array}{c|c|c} a_1, \ldots, a_p & (A, a) & z \\ b_1, \ldots, b_q & (B, b) & \end{array} \right] = \frac{1}{\mathcal{B}(A, B-A)} \int_0^\infty \frac{u^{B-A-1}}{(u+1)^B} \left[\begin{array}{c|c|c} a_1, \ldots, a_p & - & \frac{z\, u^{b-a}}{(u+1)^a} \\ b_1, \ldots, b_q & (B-A, b-a) & \end{array} \right] du$$

Proof. The proof follows from the hypothesis of absolute convergence of the series. Therefore, the order of integration and summation can be switched.

Let $w = \Gamma(B)/\Gamma(A)$ and suppose that $a \neq b$. Observe that by eq. A3

$$\frac{\mathcal{B}(A, B)}{\Gamma(B-A)} = \frac{\Gamma(A)}{\Gamma(B)}, \quad B > A > 0$$

Therefore,

$$\frac{\Gamma(ak+A)}{\Gamma(bk+B)} = \frac{\mathcal{B}(ka+A, k(b-a)+B-A)}{\Gamma(k(b-a)+B-A)}$$

Therefore, by absolute convergence of the series

$$w \sum_{k=0}^{\infty} \frac{\Gamma(ak+A)}{\Gamma(bk+B)} \frac{z^k c_k}{\Gamma(k+1)} = w \int_0^1 \sum_{k=0}^{\infty} \tau^{ka+A-1} (1-\tau)^{k(b-a)+B-A-1} \frac{z^k c_k}{\Gamma(k+1)} d\tau$$

Therefore,

$$w \int_0^1 \tau^{A-1} (1-\tau)^{B-A-1} \sum_{k=0}^{\infty} \tau^{ka} (1-\tau)^{k(b-a)} \frac{z^k c_k}{\Gamma(k+1)} d\tau = \frac{1}{\mathcal{B}(A, B-A)} \int_0^1 \tau^{A-1} (1-\tau)^{B-A-1} \left[\begin{array}{c|c|c} \cdots & - & z\, \tau^a (1-\tau)^{(b-a)} \\ \cdots & (B-A, b-a) & \end{array} \right] d\tau$$

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Furthermore, let now $a = b = 1$. It can be further observed that for the monomial term

$$\frac{1}{\mathcal{B}(A,B-A)}\int_0^1 \tau^{A-1}(1-\tau)^{B-A-1}\frac{c_k\tau^k}{\Gamma(k+1)}\,d\tau = \frac{\mathcal{B}(k+A,B-A)\,\Gamma(B)}{\Gamma(A)\,\Gamma(B-A)}\frac{c_k}{\Gamma(k+1)} = \frac{c_k}{\Gamma(k+1)}\frac{\Gamma(k+A)}{\Gamma(A)}\frac{\Gamma(B)}{\Gamma(k+B)}$$

Therefore,

$$\frac{1}{\mathcal{B}(A,B-A)}\int_0^1 \tau^{A-1}(1-\tau)^{B-a-1}\left[\begin{array}{c|c|c} \ldots & \ldots & z\,\tau \\ \ldots & \ldots & \end{array}\right]d\tau = \left[\begin{array}{c|c|c} \ldots,A & \ldots & z \\ \ldots,B & \ldots & \end{array}\right]$$

□

This representation step reduces a $(p+1, q+1)$ GHG series into a $(p, q+1)$ GHG series. It can be seen that the reduction via Beta integral is not complete except if $a = b$. Therefore, it can be instructive to distinguish *homogeneous* GHG series with indices $a_i = b_i$ and different multiplicies. This is the subject of the following results:

Corollary 1 (Homogeneous Euler reduction). *For $B > A$ and $a > 0$*

$$\left[\begin{array}{c|c|c} a_1,\ldots,a_p & (A,a) & z \\ b_1,\ldots,b_q & (B,a) & \end{array}\right] = \frac{1}{\mathcal{B}(A,B-A)}\int_0^1 \tau^{A-1}(1-\tau)^{B-A-1}\left[\begin{array}{c|c} a_1,\ldots,a_p & z\,\tau^a \\ b_1,\ldots,b_q & \end{array}\right]d\tau$$

Furthermore, for $a = 1$ the usual Euler reduction holds

$$\left[\begin{array}{c|c|c} a_1,\ldots,a_p & (A,1) & z \\ b_1,\ldots,b_q & (B,1) & \end{array}\right] = \left[\begin{array}{c|c} a_1,\ldots,a_p,A & z \\ b_1,\ldots,b_q,B & \end{array}\right] = \frac{1}{\mathcal{B}(A,B-A)}\int_0^1 \tau^{A-1}(1-\tau)^{B-A-1}\left[\begin{array}{c|c} a_1,\ldots,a_p & z\,\tau \\ b_1,\ldots,b_q & \end{array}\right]d\tau$$

By change of variables the reduction can be expressed as an improper integral:

Corollary 2. *By change of variables $t = 1/(1+u)$ for $a > 0$*

$$\left[\begin{array}{c|c|c} a_1,\ldots,a_p & (A,a) & z \\ b_1,\ldots,b_q & (B,a) & \end{array}\right] = \frac{1}{\mathcal{B}(A,B-A)}\int_0^\infty \frac{u^{B-A-1}}{(u+1)^B}\left[\begin{array}{c|c} a_1,\ldots,a_p & \frac{z}{(u+1)^a} \\ b_1,\ldots,b_q & \end{array}\right]du$$

and for $a = b = 1$

$$\left[\begin{array}{c|c|c} a_1,\ldots,a_p & (A,1) & z \\ b_1,\ldots,b_q & (B,1) & \end{array}\right] = \left[\begin{array}{c|c} a_1,\ldots,a_p,A & z \\ b_1,\ldots,b_q,B & \end{array}\right] = \frac{1}{\mathcal{B}(A,B-A)}\int_0^\infty \frac{u^{B-A-1}}{(u+1)^B}\left[\begin{array}{c|c} a_1,\ldots,a_p & \frac{z}{(u+1)} \\ b_1,\ldots,b_q & \end{array}\right]du$$

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4.2. Integral Representations by Gamma Integrals

Theorem 4 (Complex GH Series Representation)**.** *Suppose that all indices a_i and b_i are real. Then for real z and $B > -1$*

$$\left[\begin{array}{c|c|c} a_1,\dots,a_p & \dots & z \\ B,b_1,\dots,b_q & \dots & \end{array}\right] = \frac{(-1)^{-B}\Gamma(B)}{2\pi i}\int_{Ha^+}\frac{e^{-\tau}}{\tau^B}\left[\begin{array}{c|c|c} a_1,\dots,a_p & \dots & -\frac{z}{\tau} \\ b_1,\dots,b_q & \dots & \end{array}\right]d\tau = \frac{\Gamma(B)}{2\pi i}\int_{Ha^-}\frac{e^{\tau}}{\tau^B}\left[\begin{array}{c|c|c} a_1,\dots,a_p & \dots & \frac{z}{\tau} \\ b_1,\dots,b_q & \dots & \end{array}\right]d\tau$$

where Hankel path Ha^+ starts at infinity on the real axis, encircling 0 in a positive sense, and returns to infinity along the real axis, respecting the cut along the positive real axis, while Ha^- is its reflection.

Proof. From the Heine's formula for the reciprocal Gamma function representation

$$\frac{1}{\Gamma(z)} = \frac{(-1)^{-z}}{2\pi i}\int_{Ha^+}\frac{e^{-\tau}}{\tau^z}\,d\tau = \frac{1}{2\pi i}\int_{Ha^-}\frac{e^{\tau}}{\tau^z}\,d\tau$$

It follows that

$$\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+b)}c_k(-z)^k = \frac{(-1)^{-b}}{2\pi i}\sum_{k=0}^{\infty}\int_{Ha}\frac{e^{-\tau}}{\tau^{k+b}}c_k z^k d\tau = \frac{(-1)^{-b}}{2\pi i}\int_{Ha^+}\frac{e^{-\tau}}{\tau^b}\sum_{k=0}^{\infty}c_k\left(\frac{z}{\tau}\right)^k$$

Therefore,

$$\left[\begin{array}{c|c} a_1,\dots,a_p & z \\ b,b_1,\dots,b_q & \end{array}\right] = \frac{(-1)^{-b}\Gamma(b)}{2\pi i}\int_{Ha^+}\frac{e^{-\tau}}{\tau^b}\left[\begin{array}{c|c} a_1,\dots,a_p & -\frac{z}{\tau} \\ b_1,\dots,b_q & \end{array}\right]d\tau$$

and

$$\left[\begin{array}{c|c|c} a_1,\dots,a_p & \dots & z \\ b,b_1,\dots,b_q & \dots & \end{array}\right] = \frac{(-1)^{-b}\Gamma(b)}{2\pi i}\int_{Ha^+}\frac{e^{-\tau}}{\tau^b}\left[\begin{array}{c|c|c} a_1,\dots,a_p & \dots & -\frac{z}{\tau} \\ b_1,\dots,b_q & \dots & \end{array}\right]d\tau$$

by extension. □

Corollary 3. *For $B > -1$ the following representation holds*

$$\left[\begin{array}{c|c|c} a_1,\dots,a_p & \dots & z \\ b_1,\dots,b_q & (B,b) & \end{array}\right] = \frac{(-1)^{-B}\Gamma(B)}{2\pi i}\int_{Ha^+}\frac{e^{-\tau}}{\tau^B}\left[\begin{array}{c|c|c} a_1,\dots,a_p & \dots & \frac{z}{(-\tau)^b} \\ b_1,\dots,b_q & \dots & \end{array}\right]d\tau = \frac{\Gamma(B)}{2\pi i}\int_{Ha^-}\frac{e^{\tau}}{\tau^B}\left[\begin{array}{c|c|c} a_1,\dots,a_p & \dots & \frac{z}{\tau^b} \\ b_1,\dots,b_q & \dots & \end{array}\right]d\tau$$

Theorem 5 (Real GH Series Representation)**.** *Suppose that all indices a_i and b_i are real. Then for some real $A > 0$ and $z < 1$*

$$\left[\begin{array}{c|c|c} A,a_1,\dots,a_p & \dots & z \\ b_1,\dots,b_q & \dots & \end{array}\right] = \frac{1}{\Gamma(A)}\int_0^{\infty}e^{-\tau}\tau^{A-1}\left[\begin{array}{c|c|c} a_1,\dots,a_p & \dots & z\tau \\ b_1,\dots,b_q & \dots & \end{array}\right]d\tau$$

Proof. From the Gamma function representation

$$\Gamma(z) = \int_0^{\infty}e^{-\tau}\tau^{z-1}\,d\tau, \quad z > 0$$

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It follows that

$$\sum_{k=0}^{\infty} \Gamma(k+a) c_k z^k = \sum_{k=0}^{\infty} \int_0^{\infty} e^{-\tau} \tau^{k+a-1} c_k z^k d\tau = \int_0^{\infty} e^{-\tau} \tau^{a-1} \sum_{k=0}^{\infty} c_k \, (z\tau)^k$$

Therefore,

$$\left[\begin{array}{c|c|c} a, a_1, \ldots, a_p & \ldots & z \\ b_1, \ldots, b_q & \ldots & \end{array} \right] = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-\tau} \tau^{a-1} \left[\begin{array}{c|c|c} a_1, \ldots, a_p & \ldots & z\tau \\ b_1, \ldots, b_q & \ldots & \end{array} \right] d\tau$$

provided all parameters are real. □

Corollary 4. *For the real* $A > 0$ *and* $a > 0$

$$\left[\begin{array}{c|c|c} a_1, \ldots, a_p & (A, a) & z \\ b_1, \ldots, b_q & \ldots & \end{array} \right] = \frac{1}{\Gamma(A)} \int_0^{\infty} e^{-\tau} \tau^{A-1} \left[\begin{array}{c|c|c} a_1, \ldots, a_p & \ldots & z\,\tau^a \\ b_1, \ldots, b_q & \ldots & \end{array} \right] d\tau$$

In summary, the section shows that a (p, q) GHG series can be reduced to a $p + q$ multiple integrals of the Euler type.

5. Applications

5.1. Mittag-Leffler Functions

The 2 parameter Mittag-Leffler function [7,8] under the present convention will be denoted as

$$E_{a,b}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)} = \frac{1}{\Gamma(b)} \left[\begin{array}{c|c|c} 1 & - & z \\ - & (b, a) & \end{array} \right]$$

This immediately gives the complex integral representation according to Cor. 4

$$E_{a,b}(z) = \frac{1}{2\pi i} \int_{Ha^-} \frac{e^{\tau}}{\tau^b} \left[\begin{array}{c|c} 1 & \frac{z}{\tau^a} \\ - & \end{array} \right] d\tau = \frac{1}{2\pi i} \int_{Ha^-} \frac{e^{\tau}}{\tau^b} \frac{d\tau}{1 - \frac{z}{\tau^a}} = \frac{1}{2\pi i} \int_{Ha^-} \frac{\tau^{a-b} e^{\tau}}{\tau^a - z} d\tau$$

However, in this case the contour encloses the curve $|1 - z/\tau^a| = 1$.

Another example is the 3-parameter Mittag-Leffler function generalization, that is the Prabhakar function [9] defined as

$$E^{\gamma}_{a,b}(z) := \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma)\, z^k}{\Gamma(\gamma)\Gamma(ak+b)\, k!} = \frac{1}{\Gamma(b)} \left[\begin{array}{c|c|c} \gamma & - & z \\ - & (b, a) & \end{array} \right]$$

In this case

$$E^{\gamma}_{a,b}(z) = \frac{1}{\Gamma(\gamma)\Gamma(b)} \int_0^{\infty} e^{-\tau} \tau^{\gamma-1} W\left(a, b | \, z\tau\right) d\tau$$

which leads to an integral involving the Wright function.

An interesting special case is the function $E^{\gamma}_{a,1}(z)$ which is a confluent Kummer $({}_1F_1)$ hypergeometric function. In this case for $a > \gamma$

$$E^{\gamma}_{a,1}(z) = \left[\begin{array}{c|c} \gamma & z \\ a & \end{array} \right] = \frac{1}{\mathcal{B}(a, a-\gamma)} \int_0^1 \tau^{\gamma-1} (1-\tau)^{a-\gamma-1} e^{z\tau} d\tau$$

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5.2. The Kummer-Wright Function

In particular the following proposition can be stated for the basic GH function (the Kummer-Wright function)

$$\left[\begin{array}{c|c|c} - & (A,a) & z \\ - & (B,b) & \end{array}\right] = \frac{1}{\mathcal{B}(A,B-A)}\int_0^1 \tau^{A-1}(1-\tau)^{B-A-1}\left[\begin{array}{c|c|c} - & - & z\,\tau^a(1-\tau)^{(b-a)} \\ - & (B-A,b-a) & \end{array}\right] d\tau = \frac{\Gamma(A)}{\Gamma(B)}\int_0^1 \tau^{A-1}(1-\tau)^{B-A-1} W\left(b-a, B-A|\,z\,\tau^a(1-\tau)^{(b-a)}\right) d\tau$$

And also

$$\left[\begin{array}{c|c|c} - & (A,a) & z \\ - & (B,b) & \end{array}\right] = \frac{1}{\mathcal{B}(A,B-A)}\int_0^\infty \frac{u^{B-A-1}}{(u+1)^B}\left[\begin{array}{c|c|c} - & - & \frac{z\,u^{b-a}}{(u+1)^a} \\ - & (B-A,b-a) & \end{array}\right] du = \frac{\Gamma(A)}{\Gamma(B)}\int_0^\infty \frac{u^{B-A-1}}{(u+1)^B} W\left(b-a, B-A|\,\frac{z\,u^{b-a}}{(u+1)^a}\right) du$$

5.3. Generalized Fractional Operations

The theory of GHG series has an interesting relationship with the generalized fractional calculus. The Erdélyi-Kober (E-K) fractional integrals are defined as [10]:

$$I_\beta^{\gamma,\delta} f(z) := \frac{1}{\Gamma(\delta)}\int_0^1 \tau^\gamma (1-\tau)^{\delta-1} f(\tau^{1/\beta} z) d\tau$$

Therefore, it follows that

$$I_{1/\beta}^{\gamma,\delta}\left[\begin{array}{c|c|c} \ldots & \ldots & z \\ \ldots & \ldots & \end{array}\right] = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\delta+1)}\left[\begin{array}{c|c|c} \ldots & \ldots,(\gamma+1,\beta) & z \\ \ldots & \ldots,(\gamma+\delta+1,\beta) & \end{array}\right] \tag{3}$$

This corresponds to the findings of Kiryakova [11].

The operator reduces to the Riemann-Liouville fractional integral for $\beta = 1$ as

$$I_{R-L}^\delta f(z) = z^\delta I_1^{0,\delta} f(z)$$

Conversely,

$$I_{R-L}^\delta z^\gamma f(z) = z^{\gamma+\delta} I_1^{\gamma,\delta} f(z)$$

The corresponding generalized fractional derivative is defined as

$$D_\beta^{\gamma,\delta} f(z) = \prod_{j=1}^{[\delta]+1}\left(\frac{z}{\beta}\partial_z + \gamma + j\right) I_\beta^{\gamma+\delta,1-<\delta>} f(z)$$

where $<\delta>$ is the fractional part and $[\delta]$ is the integral part of the number.

The operator reduces to the Riemann-Liouville fractional derivative for $\beta = 1$ as

$$D_{R-L}^\delta f(z) = D_1^{0,\delta} z^{-\delta} f(z)$$

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The operator is left-inverse of the E-K integral for suitable classes of functions. That is

$$D^{\gamma,\delta}_{\beta} I^{\gamma,\delta}_{\beta} f(z) = f(z)$$

Therefore,

$$\left[\begin{array}{c} \dots \\ \dots \end{array} \middle| \begin{array}{c} \dots \\ \dots \end{array} \middle| z \right] = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\delta+1)} D^{\gamma,\delta}_{1/\beta} \left[\begin{array}{c} \dots \\ \dots \end{array} \middle| \begin{array}{c} \dots, (\gamma+1,\beta) \\ \dots, (\gamma+\delta+1,\beta) \end{array} \middle| z \right]$$

and

$$\left[\begin{array}{c} \dots \\ \dots \end{array} \middle| \begin{array}{c} \dots \\ \dots \end{array} \middle| z \right] = \frac{\Gamma(\gamma)}{\Gamma(\delta)} D^{\gamma-1,\delta-\gamma}_{1/\beta} \left[\begin{array}{c} \dots \\ \dots \end{array} \middle| \begin{array}{c} \dots, (\gamma,\beta) \\ \dots, (\delta,\beta) \end{array} \middle| z \right] \tag{4}$$

which can be used for index reduction.

These results demonstrate that certain classes of GHG series (homogeneous) are closed with respect to (generalized) fractional calculus operations. Therefore, the main consequence of the stated results is that all GHG functions of the Fox-Wright type can be represented as multiple (complex) integrals of three primitive functions of orders $(1,0)$, $(0,1)$ and $(1,1)$ respectively. This corroborates the fundings of Kiryakova [5]. These multiple integrals can be denoted as generalized fractional differeitegrals [6], however this line of representation is superfluous to the necessities of the numerical (i.e. physical) modeling.

Appendix A. Euler Integrals

The Gamma integral i.e. the Euler integral of second kind is defined as

$$\Gamma(z) = \int_0^\infty e^{-\tau}\tau^{z-1}\, d\tau, \quad \mathrm{Re} z > 0 \tag{A1}$$

while for all $z \notin \mathbb{Z}^-$

$$\Gamma(z) = \frac{1}{2i \sin \pi z} \int_{Ha^-} e^{\tau}\tau^{z-1}\, d\tau, \quad \tau \in \mathbb{C}$$

The complex representation for the reciprocal Gamma function is given by the Heine's integral as

$$\frac{1}{\Gamma(z)} = \frac{(-1)^{-z}}{2\pi i} \int_{Ha^+} \frac{e^{-\tau}}{\tau^z}\, d\tau = \frac{1}{2\pi i} \int_{Ha^-} \frac{e^{\tau}}{\tau^z}\, d\tau \tag{A2}$$

The contour is depicted in Fig. A1. For non-integral arguments the branch cut is selected as the negative real axis.

The Beta integral (i.e. the Euler integral of first kind) is given as

$$\mathcal{B}(a,b) = \int_0^1 \tau^{a-1}(1-\tau)^{b-a-1} d\tau = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a > 0,\ b > 0 \tag{A3}$$

The Beta function can be continued analytically along the self-intersecting Pochhammer contour as

$$\mathcal{B}(a,b) = \frac{1}{\left(1-e^{2\pi a}\right)\left(1-e^{2\pi b}\right)} \int_0^1 \tau^{a-1}(1-\tau)^{b-a-1} d\tau, \quad \tau \in \mathbb{C}$$

Appendix B. The Wright Function

The function $W(\lambda,\mu|\, z)$, named after E. M. Wright, is defined as the infinite series

$$W(\lambda,\mu|\, z) := \Gamma(\mu) \left[\begin{array}{c} - \\ - \end{array} \middle| \begin{array}{c} - \\ (\mu,\lambda) \end{array} \middle| z \right] = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\lambda k+\mu)}, \quad \lambda > -1,\ \mu \in \mathbb{C}, \tag{A4}$$

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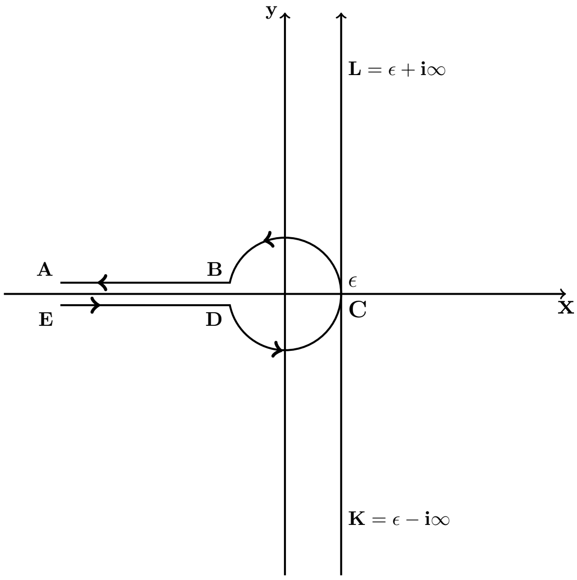

Figure A1. The Hankel contour $Ha^{-}(\epsilon)$ and the Bromwich contour $Br^{+}(\epsilon)$.

$W(\lambda, \mu | z)$ is an entire function of z. The summation is carried out in steps where $\lambda k + \mu \neq 0$. The function is related to the Bessel functions $J_{\nu}(z)$ and $I_{\nu}(z)$ as

$$W\left(1, \nu+1 \middle| -\frac{1}{4}z^2\right) = \left(\frac{z}{2}\right)^{-\nu} J_{\nu}(z), \quad W\left(1, \nu+1 \middle| \frac{1}{4}z^2\right) = \left(\frac{z}{2}\right)^{-\nu} I_{\nu}(z)$$

and is sometimes called generalized Bessel function. A recent survey about the properties of the function can be found in [12].

The integral representation of the Wright function is noteworthy because it can be used for numerical calculations

$$W(\lambda, \mu | z) = \frac{1}{2\pi i} \int_{Ha^{-}} e^{\zeta + z\zeta^{-\lambda}} \zeta^{-\mu} \, d\zeta, \quad \lambda > -1, \ \mu \in \mathbb{C} \tag{A5}$$

where Ha^{-} denotes the Hankel contour in the complex ζ-plane with a cut along the negative real semi-axis $\arg \zeta = \pi$. The contour is depicted in Figure A1.

Furthermore,

$$\frac{d}{dz} W(\lambda, \mu | z) = W(\lambda, \lambda + \mu | z) \tag{A6}$$

The proof follows immediately from the integral representation by Azrelá's theorem:

$$\frac{d}{dz} \frac{1}{2\pi i} \int_{Ha^{-}} e^{\zeta + z\zeta^{-\lambda}} \zeta^{-\mu} \, d\zeta = \frac{1}{2\pi i} \int_{Ha^{-}} e^{\zeta + z\zeta^{-\lambda}} \zeta^{-\mu-\lambda} \, d\zeta = W(\lambda, \lambda + \mu | z)$$

and formally

$$\int W(\lambda, \mu | z) \, dz = W(\lambda, \mu - \lambda | z) + C \tag{A7}$$

by the properties of anti-differentiation.

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Appendix B.1. The M-Wright Function

Mainardi introduces a specialization of the Wright function, which is called here the M-Wright function, which is important in the applications to fractional transport problems [13].

$$M_\nu(z) := W(-\nu, 1-\nu|-z)$$

Special cases of the M-Wright function are

ν	$M_\nu(z)$
$+0$	e^{-z}
$1/2$	$\frac{1}{\sqrt{\pi}} e^{-z^2/4}$
$1/3$	$\sqrt[3]{3^2} Ai\left(z/\sqrt[3]{3}\right)$

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