On a duality between Time and Space Cones

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Abstract

We give an exact mathematical construction of a spacelike order $<$, which is dual to the standard chronological order $\ll$ in the n-dimensional Minkowski space M^n and we discuss its order-theoretic, geometrical as well as its topological implications, conjecturing a possible extension to curved spacetimes.

1 Introduction.

Many important theorems, within the frame of general relativity, refer to spacelike, such as singularity theorems which require the existence of a smooth spacelike Cauchy surface Σ (see [Hawking, Ellis]), schematic conformal diagrams depicting causal independence (see for example [Penrose, 2007]), etc. In all cases, spacelike is synonymous to locally acausal [1], where there is no timelike relation or information traveling to the speed of light. In this article we show that the structure of the null-cone is induced, in a topological sense, by a spacelike order which creates a spacelike orientation in an analogous way to the timelike orientation.

[1] It was highlighted to us by a Reviewer of this article, and we consider it useful to mention this here as well, that spacelike is a purely local property, but acausal is a global property. For instance, the Lorentzian cylinder $M = \mathbb{R}^1 \times \mathbb{S}^1$, with metric $-dt^2 + d\theta^2$, and the submanifold which is the image of the map $f : \mathbb{R} \to M$ given by $f(s) = (\frac{1}{2}s, s)$: this is manifestly spacelike at all points, but $f(s) \ll f(s + 2\pi)$ for all s.

2 Definitions and notation.

Let M^n be the n-dimensional Minkowski space. Let Q be the characteristic quadratic form on M^n, defined by $Q(x) = \{x_0^2+x_1^2+x_2^2+\cdots+x_{n-2}^2-x_{n-1}^2 : x = (x_0, x_1, x_2, \cdots, x_{n-1}) \in M^n\}$.

For an event $x \in M^n$, we consider the following sets:

1. $C^T(x) = \{y : y = x \text{ or } Q(y-x) < 0\}$, the *time-cone* of x,
2. $C^L(x) = \{y : Q(y-x) = 0\}$, the *light-cone* of x,
3. $C^S(x) = \{y : y = x \text{ or } Q(y-x) > 0\}$, the *space-cone* [2] of x,
4. $C^{LT}(x) = C^T(x) \cup C^L(x)$ the union of the time- and light-cones of x, also known as the *causal cone* of x, and
5. $C^{LS}(x) = C^S(x) \cup C^L(x)$ the union of the space- and light-cones of x.

We also consider the half-planes:

1. $P_m^+(x) = \{y : g(m, y-x) \geq 0 \text{ and } y \neq x\}$ and
2. $P_m^-(x) = \{y : g(m, y-x) \leq 0 \text{ and } y \neq x\}$

in M^n, for any plane $P_m(x)$, where $m \in M^n$, $m \neq 0$ is the normal to $P_m(x)$, $P_m(x) = \{y : g(m, y-x) = 0\}$ and where g denotes the spacetime metric.

For abbreviation, we will write $P_m^+(x) := P_+(x)$, $P_m^-(x) := P_-(x)$ and $P_m(x) := P(x)$.

We observe that $P^+(x) \cup P^-(x) = \{x\}^c$, where the superscript c -here and throughout the text- denotes the complement of a set. So, for an event $y \in M$, $y \in P_+(x) \cup P_-(x) = \{x\}^c$ if and only if $y \neq x$.

Last, but not least, we define the following subspaces:

1. $C_+^L(x) = P_+(x) \cap C^L(x)$;
2. $C_-^L(x) = P_-(x) \cap C^L(x)$;
3. $C_+^S(x) = P_+(x) \cap C^S(x)$;
4. $C_-^S(x) = P_-(x) \cap C^S(x)$;
5. $C_+^{LS}(x) = P_+(x) \cap C^{LS}(x)$;
6. $C_-^{LS}(x) = P_-(x) \cap C^{LS}(x)$.

It is standard (see [Penrose,1972]) to consider two partial orders, the *chronological* order $\ll$, which is irreflexive, and the *causal order* $\prec$, which is reflexive, defined not only in M^n but in general in any spacetime, as follows:

1. $x \ll y$ iff $y \in C_+^T(x)$ and

[2] Here the word"cone" is used in a generalised sense, i.e. it is a cone on $I \times \mathbb{S}^{n-2}$.

2. $x \prec y$ iff $y \in C^T_+(x) \cup C^L_+(x)$

In addition, the reflexive relation *horismos* $\rightarrow$ is defined as $x \rightarrow y$ iff $x \prec y$ but not $x \ll y$.

3 The weak interval topology.

We will consider a topology that we call the *weak interval topology*, which is constructed in an analogous way to the interval topology in [Gierz et al.], but which does not apply only to lattices. In fact, when restricted to the 2-dimensional Minkowski space M^2, under the causal order $\prec$, the weak interval topology will coincide with the interval topology, but in general it will not be restricted to lattices. For its construction, we need a relation R defined on a set X. We then consider the sets $I^+(x) = \{y \in X : xRy\}$ and $I^-(x) = \{y \in X : yRx\}$, as well as the collections $\mathcal{S}^+ = \{X \setminus I^-(x) : x \in X\}$ and $\mathcal{S}^- = \{X \setminus I^+(x) : x \in X\}$. A basic-open set U in the weak interval topology T^{in} is defined as $U = A \cap B$, where $A \in \mathcal{S}^+$ and $B \in \mathcal{S}^-$; in other words, $\mathcal{S}^+ \cup \mathcal{S}^-$ forms a subbase for T^{in}.

The topology T^{in} with respect to the relation $\rightarrow$ was studied in [Papadopoulos et al.] and also in [Antoniadis et al.] and with respect to the order $\ll$ in [Papadopoulos, 2018], where there was conjectured a possibility to create a duality between timelike and spacelike, based on a spacelike order $<$ dual to chronology $\ll$. In this paper we give the exact mathematical construction of $<$ as well as of the induced topology $T^{in}_<$, for n-dimensional Minkowski space M^n.

4 The order on the space-cone and its induced topology.

We define a partial spacelike order $<$ dual to the chronological order $\ll$. This order is obviously not causal, but it brings an interesting duality between the time cone $C^T(x)$ and the space cone $C^S(x)$, of an event x. Through $<$, the "cone" $C^S(x)$ exhibits similar properties to C^T under $\ll$. Since "chronological" comes from the word "chronos", which means time, we name $<$ "chorological", as it refers to "choros", space.

Definition 4.1. *For non causally-related events $x, y \in M^n$, $x < y$ iff $y \in C^S_+(x)$, where we have defined $C^S_+(x)$ for some fixed choice of $m \in M^n$.*

It follows that $x < y$ iff $x \in C^S_-(y)$. In addition, $\leq$ denotes $<$ including the boundary, in a dual way as $\prec$ is to $\ll$, that is, $x \leq y$ iff $y \in C_+^{LS}(x)$.

We remark that $<$ is a partial order; the transitivity is obvious, as soon as it is highlighted that $<$ refers to events which are not causally related; thus, if x, y, z are mutually not causally related (x is not causally related to y, y is not causally related to z and x is not causally related to z), then $x < y$ and $y < z$ implies that $x < z$.

Lemma 4.1. $C_+^{LS}(x) \cup C_-^{LS}(x) = C^{LS}(x) - \{x\}$.

Proof. $C_+^{LS}(x) \cup C_-^{LS}(x) = C^{LS}(x) \cap (P_+(x) \cup P_-(x)) = C^{LS}(x) \cap \{x\}^c = C^{LS}(x) - \{x\}$. □

Proposition 4.1. $[C_+^{LS}(x)]^c \cap [C_-^{LS}(x)]^c = C^T(x)$

Proof. $[C_+^{LS}(x)]^c \cap [C_-^{LS}(x)]^c$
$= [C_+^{LS}(x) \cup C_-^{LS}(x)]^c$
$= [C^{LS}(x) - \{x\}]^c$
$= [C^{LS}(x)]^c \cup \{x\}$
$= [C^T(x) - \{x\}] \cup \{x\}$
$= C^T(x)$ □

Remark 4.1. *The vector m (normal to plane P) is perpendicular to the vector $< x_0, x_1, ..., x_{n-1} >=< 0, .., 0, 1 >$ where x_{n-1} corresponds to time.*

Also, both Lemma 4.1 and Proposition 4.1 if expressed for C^S instead of for C^{LS}, i.e. if the boundary was omitted, would give $[C_+^S(x)]^c \cap [C_-^S(x)]^c$ which would equal $C^L(x) \cup C^T(x)$, i.e the causal cone $C^{LT}(x)$.

Proposition 4.2. *The weak interval topology $T_{\leq}^{in}$, generated by the space-like order $\leq$, has as basic-open sets the time-cones $C^T(x)$.*

Proof. The proof follows from Proposition 4.1, by observing that $[C_+^{LS}(x)]^c \in \mathcal{S}^-$ and $[C_-^{LS}(x)]^c \in \mathcal{S}^+$, where $I^+(x)$ and $I^-(x)$ are defined with respect to $R = \leq$ (see Section 3). □

For the last result of our discussion, we will need to use Reed's definition of *intersection topology* (see [Reed, 1986]):

Definition 4.2. *If T_1 and T_2 are two topologies on a set X, then the* intersection topology *T^{int} with respect to T_1 and T_2, is the topology on X such that the set $\{U_1 \cap U_2 : U_1 \in T_1, U_2 \in T_2\}$ forms a base for (X, T).*

The topology Z^T is defined to be the intersection topology, according to Reed's definition, of the topologies $T_{\leq}^{in}$ and the natural topology of $\mathbb{R}^n$, in M^n. This topology, in M, coincides with one of the three topologies that were suggested by Zeeman in [Zeeman, 1967], as alternatives to his Fine topology. Its characteristic is that its open sets are time-cones bounded by Euclidean-open balls, in M^n, and its general relativistic analogue is actually the Path topology of Hawking-King-McCarthy (see [Hawking et al.]). Agrawal and Shrivastava have reviewed several topological properties of Z^T in [Agrawal, Shrivastava], showing that (due to its equality to the Path topology) it is Hausdorff, separable, first countable, and path-connected, not regular, not metrisable, non-Lindelöf and not simply connected, and have studied in depth Zeno sequences in Z^T. In particular, Zeno sequences were introduced by Zeeman in [Zeeman, 1967] with respect to his "Fine" topology F; a sequence $\{z_n\}_{n\in\mathbb{N}}$ which converges to some z in M^n under the natural topology of $\mathbb{R}^n$ and not under the topology F (or any other topology in the class of Zeeman topologies, [Göbel, 1976]) is called a Zeno sequence. Agrawal-Shrivastava have shown that, within the n-dimensional Minkowski space M^n, for a Zeno sequence under topology Z^T converging to $z \in M^n$ there exists a subsequence of this sequence whose image is closed under Z^T but not under the natural topology of $\mathbb{R}^n$. They have shown, in addition, that again within M^n and for a nonempty open-set G in the natural topology of $\mathbb{R}^n$, if $z \in G$, then G contains a completed image of a Zeno sequence under Z^T converging to z. With respect to convergence of causal-curves, Low has shown that under the Path topology, that is under the general-relativistic analogue of Z^T, the Limit Curve Theorem fails to hold (see [Low]), and so basic arguments for building contradiction in singularity theorems fail as well under the Path topology.

The topology Z^{LT} is defined to be the intersection topology, according to Reed's definition, of the topologies $T^{in}_{<}$ and the natural topology of $\mathbb{R}^n$, in M^n. This topology fully incorporates the causal structure of M^n. This is so, because it admits a countable base of open sets of the form $C^{LT}(x) \cap B_\epsilon(x)$, where $B_\epsilon(x)$ is a ball in $\mathbb{R}^n$ centered at x and of radius $\epsilon > 0$.

5 Questions.

It would be desirable if the results of section 4 generalised to any curved spacetime, in the frame of general relativity; this hope comes for the following intuition. In a relativistic spacetime manifold, wherever there is spacetime there are events and for every event there is a light-cone. Since our construction of $T^{in}_{\leq}$ is topological, depending exclusively on the interior (time-cone), boundary (light-cone) and exterior (space-cone) of an event x, independently of the geometry of the space-time, one could consider the general-relativistic analogue of $T^{in}_{\leq}$ as the topology with basic-open sets the time-cones $C^T(x)$ and the general-relativistic analogue of Z^T as the topology which has basic-open sets the bounded time-cones $C^T(x) \cap B^h_\epsilon(x)$, for a Riemannian metric h on the spacetime manifold, and by considering Riemann-open balls $B^h_\epsilon(x)$. This is trivial, from a topological perspective, when we already know the open sets of the special-relativistic topology Z^T, without needing any information about the order $\leq$.

The question in this case is how could one define the general-relativistic analogue of the order $\leq$. How could one describe the general-relativistic analogues of the half-planes $P^+_m(x)$ and $P^-_m(x)$, that we examined in Section 2, since they will not be "flat" planes in the Euclidean sense anymore, but will follow the geometry of the particular spacetime manifold, so that their union will give $\{x\}^c$. So, it would be vital to also express the general relativistic analogue of the normal m to the plane $P_m(x)$ (Section 2), in a rigorous algebraic way; such an algebraic development should open further directions to our discussion about the duality between causal and locally acausal orders in a spacetime, a duality which might play a role for the passage from locality to non-locality (see, for example, [Vagenas, 2018]). An answer to such a question will also give a solution to the orderability problem (see [Papadopoulos, 2014]) in the particular case of the Path topology of Hawking-King-McCarthy.

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