Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 December 2018 doi:10.20944/preprints201812.0207.v1

Classical and Quantum Kepler's Third Law of N-Body System

Bohua Sun[1]
[1]*Newlands, Cape Town, South Africa*
sunbohua@yahoo.com

Inspired by amazing result obtained by Semay [7], this study revisits generalised Kepler's third law of an n-body system from the perspective of dimension analysis. To be compatible with Semay's quantum n-body result, this letter reports a conjecture which not be included in author's early publication [1] but formulated in the author's research memo. The new conjecture is proposed as follows: $T_N|E_N|^{3/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{(\sum_{i=1}^N\sum_{j=i+1}^N m_im_j)^3}{\sum_{k=1}^N m_k}\right]^{1/2}$. This formulae is, of course, consistent with the Kepler's third law of 2-body system, and exact same as Semay's quantum result.

Keywords: Kepler's third law; n-body system; periodic orbits; dimensional analysis, classical and quantum mechanics

The study of the motion between the two bodies was solved by Kepler (1609) and Newton (1687) early in the 17th century. For the elliptic periodic orbit of 2-body system, Kepler's third law of the two-body system is given by $T|E|^{3/2} = \frac{\pi}{\sqrt{2}}Gm_1m_2\sqrt{\frac{m_1m_2}{m_1+m_2}}$, where the gravitation constant, $G = 6.673\times10^{-11}m^3kg^{-1}s^{-2}$, the orbit period, T, the total energy of the 2-body system, $|E|$, and point masses m_1 and m_2.

Bohua Sun proposed a conjecture on generalization of Kepler's third law from dimensional analysis [1]. For 3-body system (Fig.1), the Sun's kepler third law is given by

$$T_3|E_3|^{3/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{(m_1m_2)^3+(m_2m_3)^3+(m_3m_1)^3}{m_1+m_2+m_3}\right]^{1/2}, \quad (1)$$

and for N-body system (Fig.2) , the Sun's kepler third law is given by

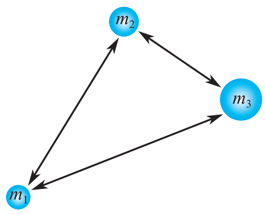

FIG. 1: 3-body system

$$T_N|E_N|^{3/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{\sum_{j=1}^N\sum_{i=j+1}^N(m_jm_i)^3}{\sum_{k=1}^N m_k}\right]^{1/2}, \quad (2)$$

where T_N is the period of the orbit and E_N os the total energy (kinetic and potential) of the orbit of N-bodies system. Eq.(2) gives results in good agreement with computations based a great number of periodic planer collisionless orbits for N=3 of [2, 3]. Since the publication of [1], it has received a great attentions [4–7].

Very recently C. Semay [7] reported an amazing result on the quantum supporting of my conjecture [1]. Semay [7] formulated quantum Kepler's third law for identical bodies as follows

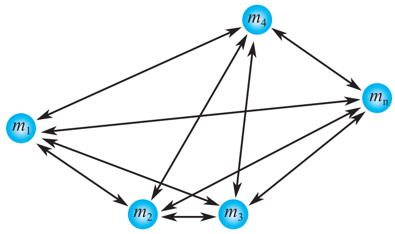

FIG. 2: N-body system

$$T_q|E_q|^{3/2} = \frac{\pi}{4}Gm^{5/2}N(N-1)^{3/2}, \quad (3)$$

where the quantum period T_q and quantum energy E_q of N self-gravitating particles with a mass m in D-dimensional space [7–9].

Semay [7] pointed out the above result is similar to the following

$$T_N|E_N|^{3/2} = \frac{\pi}{2}Gm^{5/2}(N-1)^{1/2}, \quad (4)$$

which is the result for identical bodies reduced from Eq.(2) when $m_1 = m_2 = \cdots = m$.

However, the Eqs.(3) and (4) have different factors, one has $N(N-1)^{3/2}$ and another has $(N-1)^{1/2}$. The factor difference may give us a cue that another form of Eq.(2) may be existed.

I went through my research memorandum and seen that, from dimensional analysis and symmetry of mass product, initially I proposed two expressions for N-body system, the first one is

$$T_N|E_N|^{3/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{\left(\sum_{j=1}^N\sum_{i=j+1}^N m_jm_i\right)^3}{\sum_{k=1}^N m_k}\right]^{1/2}. \quad (5)$$

Unfortunately, Eq.(5) does not give results in good agreement with computations of [2, 3], while Eq.(2) is good agree with [3]. To be compatible with numerical results of [2, 3], only Eq.(2) was reported in the publication [1], while Eq.(5) was not included in [1].

However, from dimensional perspectives, both Eq.(2) and Eq.(5) are valid invariants. They might be valid for different situations of the N-body system. From Semay's formula on identical bodies in Eq.(3), it would be an natural attempts to apply Eq.(5) to the case of identical bodies and see what going to be happen.

For identical bodies system, Eq.(5) is reduced to

$$T_N|E_N|^{3/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{\left(\sum_{j=1}^{N}\sum_{i=j+1}^{N}mm\right)^3}{\sum_{k=1}^{N}m}\right]^{1/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{\left(\sum_{j=1}^{N}\sum_{i=j+1}^{N}mm\right)^3}{\sum_{k=1}^{N}m}\right]^{1/2}, \quad (6)$$

noting $\sum_{j=1}^{N}\sum_{i=j+1}^{N}mm = \frac{N(N-1)}{2}m^2$ and $\sum_{k=1}^{N}m = Nm$, hence

$$T_N|E_N|^{3/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{\left(\frac{N(N-1)}{2}m^2\right)^3}{Nm}\right]^{1/2} = \frac{\pi}{4}Gm^{5/2}N(N-1)^{3/2}. \quad (7)$$

Ii is surprise to see that the Eq.(5) predicts exact same result as Semay in Eq.(3) [7] for quantum case. The coincidence of classical and quantum results for a generalization of the Keplers third law for N-body systems is more than a simple happy coincidence, which might implies that Eq.(2) is a second valid conjecture of generalized Kepler's third law for self gravitating particles with same mass m.

Therefore, analogue to classical generalized Kepler's third law in Eq.(2), it might be a natural thinking to propose quantum Kepler's third law with unequal mass as follows

$$T_q|E_q|^{3/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{\left(\sum_{j=1}^{N}\sum_{i=j+1}^{N}m_jm_i\right)^3}{\sum_{k=1}^{N}m_k}\right]^{1/2}. \quad (8)$$

Comparing the above relation with Eq.(5) of [7], the quantum energy E_q of N self gravitating particles with unequal masses can be predicted as follows

$$E_q = -\frac{\pi}{2}\frac{G^2}{Q^2\hbar^2}\frac{\left(\sum_{j=1}^{N}\sum_{i=j+1}^{N}m_jm_i\right)^3}{\sum_{k=1}^{N}m_k}, \quad (9)$$

where Q is a global quantum obtained by Semay [8].

For the Lagrange restricted 3-body system with equal mass ($m_1 = m_2 = m_3 = m$), in which the three masses form an equilateral triangle at each instant, it might be interesting to show that Eq.(5) gives a correct predication as follows

$$T_3|E_3|^{3/2} = \frac{\pi}{\sqrt{2}}G\left[\frac{(m_1m_2+m_2m_3+m_3m_1)^3}{m_1+m_2+m_3}\right]^{1/2} = \frac{3\pi}{\sqrt{2}}m^{5/2}. \quad (10)$$

In summary, this letter reported the Eq.(5) which was formulated by the author in his research memo and not be included in [1]. The reduced expression for identical bodies from Eq.(5) fully supports the quantum result obtained by Semay in Eq.(3) [7]. From this study, we must extend conjecture to include both Eqs.(2) and Eq.(5) as a candidate of generalization of Kepler's third law for both classical and quantum mechanics.

[1] B. H. Sun, Keplers third law of n-body periodic orbits in a Newtonian gravitation field, Sci. China-Phys. Mech. Astron. 61, 054721 (2018), https://doi.org/10.1007/s11433-017-9154-0.

[2] X.M. Li and S.J. Liao, More than six hundred new families of Newtonian periodic planar collisionless 3-body orbits, Sci. China-Phys. Mech. Astron. **60**, 129511 (2017).

[3] X.M. Li, Y.P Jing and S.J. Liao, The 1223 new periodic orbits of planar 3-body problem with unequal mass and zero angular momentum, arxiv.org/abs/1709.04775 (2017).

[4] Z.-S. She, Keplers third law of n-body system, Sci. China-Phys. Mech. Astron. 61, 094531 (2018), https://doi.org/10.1007/s11433-018-9221-y

[5] C.Y. Zhao and M.J. Zhang, A conjecture on Keplers third law of n-body periodic orbits, arXiv:1811.00735v1 (2018).

[6] X.L. Xu, Research Highlits-Editorss selection of papers from China's academic journals, National Science Review 5: 447-449, 2018 doi: 10.1093/nsr/nwy061

[7] C. Semay, Quantum support to BoHua Sun's conjecture, arXiv:1812.00807v1 (2018).

[8] C. Semay and G. Sicorello, Many-body forces with the envelope theory, Few-Body Syst. 59, 119 (2018)

[9] C. Semay, Numerical Tests of the Envelope Theory for Few-Boson Systems, Few-Body Syst. 56, 149 (2015)