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DGJ Method for Linear and Nonlinear Third order Fractional Differential Equation

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Absract: DGJ (Daftardar-Gejii-Jafaris) method is used to obtain numerical solution of the third order fractional differential equation. Providing the DGJ method converges, the approximate solution is a good and effective numerical result which is close to the exact solution or the exact solution. For this,the examples of the explaning the method are presented. The proposed method is implemented for the approximation solution of the third order nonlinear fractional partial differential equations. The method was shown to be unsuitable and inconsistent for an example of a nonlinear fractional partial differential equation depend on initial-boundary value conditions. The fact that these numerical results are not consistent can be explained by the fact that the method is not convergent.

Keywords: DGJ method, third order fractional differential equation, nonlinear differential equation, convergence of the method.

1 Introduction

Fractional partial differential equations have gained considerable importance recently in the literature. Fractional differential equations have various applications in the fields of finance, applied sciences, seismology engineering, physics and biology.[1, 2, 3]. This fractional differential equations can be solved separately depending on the time and space variables. There are some methods for approximate solutions of fractional differential equations due to space and time variables [4, 6, 7, 5]. These methods are the radial basis function, Chebyshew Tau method, thin plate splines method, variational iteration method, finite difference schemes method and DGJ method [9, 10, 11, 12, 23]. DGJ method was used for evolution and nonlinear functional equation and fractional order nonlinear systems [13, 14, 15]. Then, telegraph equation was solved by DGJ method [8]. Finally, Daftardar-Gejji and Jafari method was applied to solve fractional heat-like and wave-like models with variable coefficients [20]. Fractional derivative was studied by many authors for different methods [17, 18, 19].

In this study, we will study the initial-boundary value problems of the third-order fractional differential equations defined by Caputo derivative

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} + \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} + u(t,x) = \lambda \frac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), \\ 0 < x < L,\ 0 < t < T,\ 0 < \alpha \leq 1, \\ u(0,x) = g_1(x),\ u_t(0,x) = g_2(x),\ u_{tt}(0,x) = g_3(x),\ 0 \leq t \leq T, \\ u(t,X_L) = r_1(t), u(t,X_R) = r_2(t),\ X_L < x < X_R. \end{cases} \tag{1}$$

Where λ is known constant coefficient, g_1, g_2, g_3, r_1 and r_2 are known functions and u is the unknown function.

Unlike in the study [20], the DGJ method was applied to a third-order and non-linear fractional partial differential equation. In this study, the advantages and disadvantages of the DGJ method were clearly demonstrated by the examples given.

Now, we give some basic definitions and properties of fractional calculus theory for DGJ method.

Definition 1. The definition of gamma function is given the following form as:

$$\Gamma(z) =_0^t e^{-t} t^{z-1} dt,\ \text{for all } z \in C.$$

Definition 2. The Caputo fractional derivative $D_t^\alpha u(t,x)$ of order α with respect to time is defined as:

$$\begin{aligned} \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} &= D_t^\alpha u(t,x) \\ &= \frac{1}{\Gamma(n-\alpha)}{}_0^t \frac{1}{(t-p)^{\alpha-n+1}} \frac{\partial^\alpha u(p,x)}{\partial p^\alpha} dp,\ \ (n-1 < \alpha < n) \end{aligned} \tag{2}$$

and for $\alpha = n \in N$ defined as:

$$D_t^\alpha u(t,x) = \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^n u(t,x)}{\partial t^n}.$$

Definition 3. By using gamma function and the formula (2), the following formula can be written

$$D_t^\alpha (t^\beta) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}.$$

2 DGJ Iteration Method

Using the method [15], we can write the general form as:

$$u(x) = N(u(x)) + f(x) \tag{3}$$

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where N is an operator and f is a known function $x = (x_1, x_{2,} ..., x_n)$. A solution of u the equation (3) is given the following series form

$$u(x) =_{j=0}^{\infty} u_j(x). \tag{4}$$

The operator N can be written as:

$$N(_{j=0}^{\infty} u_j(x)) = N(u_0) +_{j=1}^{\infty} \{_{i=0}^{j} u_j(x)) - N(_{i=0}^{j-1} u_j(x))\}. \tag{5}$$

Written the formulas (4) and (5) into the formula (3), we have

$$_{j=0}^{\infty} u_j(x) = f(x) + N(u_0) +_{j=1}^{\infty} \{_{i=0}^{j} u_j(x)) - N(_{i=0}^{j-1} u_j(x))\}. \tag{6}$$

From the above formulas, we can obtain the following formula

$$\begin{cases} u_0 = f, \\ u_1 = N(u_0), \\ u_n = N(u_0 + u_1 + ... + u_n) - N(u_0 + u_1 + ... + u_{n-1}), \;\; n = 1, 2, ... \end{cases} \tag{7}$$

Thus,

$$(u_0 + u_1 + ... + u_{n+1}) = N(u_0 + u_1 + ... + u_n), n = 1, 2, ... \tag{8}$$

and

$$_{j=0}^{\infty} u_j = f + N(_{j=0}^{\infty} u_j). \tag{9}$$

Now, we shall apply the DGJ method to third order fractional differential equation. For this, the initial conditions consider operator $L_{t.t} = \frac{\partial^3}{\partial t^3}$. The inverse operator of $L_{t.t}$ $L_{t.t}^{-1} =_0^t \,_{0\,0}^{t\,t}(.)dtdtdt$.Applying the inverse operator to both sides of the equation (1), the following integral equation is obtained

$$\begin{aligned} u(t,x) \;\; = \;\; & u(t,0) + \frac{\partial u(t,0)}{\partial t} + \frac{\partial^2 u(t,0)}{\partial t^2} +_0^t \,_{0\,0}^{t\,t} f(t,x) dtdtdt \\ & +_{0\,0\,0}^{t\,t\,t} (\lambda \frac{\partial^2 u(t,x)}{\partial x^2} - \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} - u(t,x)) dtdtdt. \end{aligned} \tag{10}$$

DGJ algorithnm is applied the folowing as:

$$\begin{aligned} u_0 \;\; &= \;\; g_1 + g_2 t + g_3 \frac{t^2}{2} +_0^t \,_{0\,0}^{t\,t} f(t,x) dtdtdt, \\ u_1 \;\; &= \;\; N(u_0) =_0^t \,_{0\,0}^{t\,t} (\lambda \frac{\partial^2 u_0(t,x)}{\partial x^2} - \frac{\partial^\alpha u_0(t,x)}{\partial t^\alpha} - u_0(t,x)) dtdtdt, \\ u_{m+1} \;\; &= \;\; N(u_0 + u_1 + ... + u_m) - N(u_0 + u_1 + ... + u_{m-1}), \;\; m = 1, 2, ... \end{aligned} \tag{11}$$

Convergences of the method and convergeness conditions can be seen in the reference [15].

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3 Numerical implementation

Now, the proposed method is implemented to solve some examples of variable form of the third order fractional partial differential equation. The first example have obtained the exact solutions. For the Example 2, the method has been observed for some values are moving away from the exact solution while for some values approaching the exact solution.

Example 1. Investigate the following third order fractional partial differential equation for initial boundary value problems

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} + \frac{\partial^{\frac{1}{2}} u(t,x)}{\partial t^{\frac{1}{2}}} + u(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2} + e^x(6 + 6\frac{t^{\frac{5}{2}}}{\Gamma(\frac{7}{2})}), \\ 0 < x, \ 0 < t < 1, \ 0 < \alpha \leq 1, \\ u(0,x) = u_t(0,x) = u_{tt}(0,x) = 0, \ 0 \leq t. \end{cases} \tag{12}$$

Using the formula (11) and initial condition of the formula (12) for DGJ method, we obtain

$$\begin{aligned} u_0 &= g_1 + g_2 t + g_3 \frac{t^2}{2} +_0^t {}_0^t {}_0^t e^x(6 + 6\frac{s^{\frac{5}{2}}}{\Gamma(\frac{7}{2})})dsdsds \\ &= 6e^x{}_0^t {}_0^t (s + \frac{s^{\frac{7}{2}}}{\Gamma(\frac{9}{2})})dsds = 6e^x_0{}^t(\frac{s^2}{2} + \frac{s^{\frac{9}{2}}}{\Gamma(\frac{11}{2})})ds \\ &= 6e^x(\frac{t^3}{6} + \frac{t^{\frac{11}{2}}}{\Gamma(\frac{13}{2})}), \\ u_1 &= N(u_0) =_0^t {}_0^t {}_0^t (\frac{\partial^2 u_0(s,x)}{\partial x^2} - \frac{\partial^\alpha u_0(s,x)}{\partial t^\alpha} - u_0(t,x))dsdsds \\ &= -6e^x{}_0^t {}_0^t {}_0^t (\frac{s^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} + \frac{s^5}{\Gamma(6)})dsdsds \\ &= -6e^x(\frac{t^{\frac{11}{2}}}{\Gamma(\frac{13}{2})} + \frac{t^8}{\Gamma(9)}, \\ u_2 &= 6e^x(\frac{t^8}{\Gamma(9)} + \frac{t^{\frac{21}{2}}}{\Gamma(\frac{23}{2})}), \\ & . \\ & . \\ & . \\ u_n &= (-1)^n 6e^x(\frac{t^{3+5n}}{\Gamma(4+5n)} + \frac{t^{\frac{9}{2}+5n}}{\Gamma(\frac{11}{2}+5n)}), \end{aligned}$$

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When $n \to \infty$, the result converges to exact solution as follow:

$$\begin{aligned} u(t,x) &= \lim_{n\to\infty} \sum_{j=0}^{n} u_j = 6e^x \lim_{n\to\infty} \sum_{j=0}^{n} (-1)^n e^x \Big(\frac{t^{3+5n}}{\Gamma(4+5n)} + \frac{t^{\frac{9}{2}+5n}}{\Gamma(\frac{11}{2}+5n)} \\ &= 6e^x \Big(\frac{t^3}{6} - \lim_{n\to\infty} \frac{t^{\frac{9}{2}+5n}}{\Gamma(\frac{11}{2}+5n)} = e^x t^3 \text{ for every } 0 \le t \le 1. \end{aligned}$$

Finding $u(t,x) = e^x t^3$ are the exact solution for the problem (12).

Example 2. Investigate the following nonlinear third order fractional partial differential equation for initial boundary value problems

$$\begin{cases} \frac{\partial^3 u(t,x)}{\partial t^3} + \frac{\partial^{\frac{1}{2}} u(t,x)}{\partial t^{\frac{1}{2}}} = 3\frac{\partial^2 u(t,x)}{\partial x^2} u(t,x) + 6(x-x^2)(t^6 + \frac{t^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} + 1), \\ 0 < x,\ 0 < t < 1,\ 0 < \alpha \le 1, \\ u(0,x) = u_t(0,x) = u_{tt}(0,x) = 0,\ 0 \le t \le 1, \\ u(t,0) = u(t,1) = 0,\ 0 \le x \le 1. \end{cases} \tag{13}$$

Using the formula (11) and initial condition of the formula (12) for DGJ method, we have

$$\begin{aligned} u_0 &= g_1 + g_2 t + g_3 \frac{t^2}{2} +{}_0^t{}_0^t{}_0^t 6(x-x^2)(s^6 + \frac{s^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} + 1) dsdsds \\ &= 6(x-x^2){}_0^t{}_0^t \Big(\frac{s^7}{7} + \frac{s^{\frac{7}{2}}}{\Gamma(\frac{9}{2})} + s\Big) dsds = 6(x-x^2){}_0^t \Big(\frac{s^8}{7.8} + \frac{s^{\frac{9}{2}}}{\Gamma(\frac{11}{2})} + \frac{s^2}{2}\Big) ds \\ u_0 &= 6(x-x^2)\Big(\frac{t^9}{7.8.9} + \frac{t^{\frac{11}{2}}}{\Gamma(\frac{13}{2})} + \frac{t^3}{6}\Big), \end{aligned} \tag{14}$$

and

$$\begin{aligned} u_1 &= N(u_0) ={}_0^t{}_0^t{}_0^t \Big(3\frac{\partial^2 u_0(t,x)}{\partial x^2} u(t,x) - \frac{\partial^\alpha u_0(t,x)}{\partial t^\alpha}\Big) dsdsds \\ &= -(x-x^2){}_0^t{}_0^t{}_0^t \Big(6\frac{s^{\frac{29}{2}}}{7\Gamma(\frac{13}{2})} + \frac{s^{18}}{24.49} + \frac{s^{12}}{7} \\ &\quad +216\frac{s^{11}}{(\Gamma(\frac{13}{2}))^2} + \Big(\frac{6.\Gamma(7)}{\Gamma(19/2)} + \frac{72}{\Gamma(\frac{13}{2})}\Big) s^{\frac{17}{2}} \\ &\quad + \frac{s^{\frac{11}{2}}}{\Gamma(\frac{13}{2})} + \frac{s^3}{6} + 6t^6\Big) dsdsds. \end{aligned}$$

From that, we get

$$
\begin{aligned}
u_1 &= -(x-x^2)(\frac{48}{7.31.33.35.\Gamma(\frac{13}{2})}t^{\frac{35}{2}} + \frac{t^{21}}{19.20.21.24.49} + \frac{t^{15}}{7.13.14.15} \qquad (15)\\
&+\frac{216}{12.13.14.(\Gamma(\frac{13}{2}))^2}t^{14} + (\frac{6.\Gamma(7)}{\Gamma(25/2)} + \frac{576}{19.21.23.\Gamma(\frac{13}{2})})t^{\frac{23}{2}}\\
&+\frac{t^{\frac{17}{2}}}{\Gamma(\frac{19}{2})} + \frac{t^6}{480} + \frac{t^9}{84}).
\end{aligned}
$$

Using (14) and (15) formulas for the numerical results by DGJ method, we obtain the following error analysis Table 1 with $\epsilon = \max\left|u_{exact} -_{i=0}^{k} u_i\right|$ for $i = 0, 1$. Here $u_{exact}(t, x) = (x - x^2)t^3$ is exact solution for the nonlinear problem (13) that can obtain by using Laplace transform method and u_i are the approximation solutions by using DGJ method obtained the above procedure.

Table 1.

$t\ ,x$	$u_i(k=1)$	u_{exact}	$\epsilon(k=1)$
$t=0.5, x=0.5$	-4.142853×10^{3}	0.0312500	4.142884×10^{3}
$t=0.25, x=0.25$	0.186714	0.002929	0.189644
$t=0.1, x=0.1$	8.976138×10^{-5}	9.000000×10^{-5}	2.386131×10^{-7}
$t=0.05, x=0.05$	5.937559×10^{-6}	5.937500×10^{-6}	5.975908×10^{-11}
$t=0.02, x=0.02$	1.568001×10^{-7}	1.568000×10^{-7}	1.822421×10^{-13}
$t=0.01, x=0.01$	9.900002×10^{-9}	9.900000×10^{-9}	2.042696×10^{-15}
$t=0.001, x=0.001$	9.990000×10^{-13}	9.990000×10^{-13}	6.563302×10^{-22}

By helping the Matlab program, to obtain numerical solution of this problem can be applied for further steps. Moving from the steps found, it is seen that the approximate solution is very close to each other with exact solution in larger steps. On the other hand, for $t = 0.25$ and $x = 1.9$ obtained approximations results are far from the exact solution.

Now, we explain the convergence or not convergence of the DGJ method. Using the methods in [14],[15], we obtain

$$
\begin{aligned}
\|N(u_0(t,x)\| &= \Bigg\| -(x-x^2)_{000}^{ttt}(6\frac{s^{\frac{29}{2}}}{7\Gamma(\frac{13}{2})} + \frac{s^{18}}{24.49} + \frac{s^{12}}{7}\\
&\quad +216\frac{s^{11}}{(\Gamma(\frac{13}{2}))^2} + (\frac{6.\Gamma(7)}{\Gamma(19/2)} + \frac{72}{\Gamma(\frac{13}{2})})s^{\frac{17}{2}} + +\frac{s^{\frac{11}{2}}}{\Gamma(\frac{13}{2})} + \frac{s^3}{6} + 6t^6)dsdsds\Bigg\|\\
&= \Bigg\|(x-x^2)\frac{48}{7.31.33.35.\Gamma(\frac{13}{2})}t^{\frac{35}{2}} + \frac{t^{21}}{19.20.21.24.49} + \frac{t^{15}}{7.13.14.15}\\
&\quad +\frac{216}{12.13.14.(\Gamma(\frac{13}{2}))^2}t^{14} + (\frac{6.\Gamma(7)}{\Gamma(25/2)} + \frac{576}{19.21.23.\Gamma(\frac{13}{2})})t^{\frac{23}{2}}\\
&\quad +\frac{t^{\frac{17}{2}}}{\Gamma(\frac{19}{2})} + \frac{t^6}{480} + \frac{t^9}{84})\Bigg\|\\
&= 0.0286 < \frac{1}{e},
\end{aligned}
$$

$$
\begin{aligned}
\|N_t(u_0(t,x)\| &= \Big\| -(x-x^2)\big(\frac{24}{7.31.33.\Gamma(\frac{13}{2})}t^{\frac{33}{2}} + \frac{t^{20}}{19.20.24.49} + \frac{t^{14}}{7.13.14} \\
&\quad + \frac{216}{12.13.(\Gamma(\frac{13}{2}))^2}t^{13} + \big(\frac{6.\Gamma(7)}{\Gamma(23/2)} + \frac{288}{19.21.\Gamma(\frac{13}{2})}\big)t^{\frac{21}{2}} \\
&\quad + \frac{t^{\frac{15}{2}}}{\Gamma(\frac{17}{2})} + \frac{t^5}{80} + 9\frac{t^8}{84}\big)\Big\| \\
&= 0.2410 < \frac{1}{e}, \\
&\quad \ldots \\
N_{tt} &= 2.386494384604877 \times 10^{10} > \frac{1}{e}.
\end{aligned}
$$

This last formula shows that Example 2. is not convergence for $n > 1$ derivatives as t. Because DJM is equivalent to Taylor series expansion around u_0 and Taylor series condititions aren't satisfied as in reference [16].

The vast majority of articles so far have just written the advantages of numerical methods. In this study, both advantages and disadvantages of the method were examined.

4 Conclusion

In this paper, The DGJ method is constructed. for the third order linear and nonlinear fractional differential equation with Caputo fractional definition. This method gives the exact solution that is obtained by Laplace transform method depend on initial-boundary value problems for the first example. The second example is a nonlinear third order fractional partial differential equation. for solving telegraph partial differential equations. Approximate solutions for numerical experiments are found by these-method. These results are compared with the exact solutions. MATLAB is used for numerical calculations for the Example 2.

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