Peer-reviewed version available at *Mathematics* **2019**, *7*, 126; doi:10.3390/math7020126

Article

A Method of Solving Compressible Navier Stokes Equations in Cylindrical Coordinates Using Geometric Algebra

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Abstract

A method of solution to solve the compressible unsteady 3D Navier-Stokes Equations in cylindrical co-ordinates coupled to the continuity equation in cylindrical coordinates is presented in terms of an additive solution of the three principle directions in the radial, azimuthal and z directions of flow. A dimensionless parameter is introduced whereby in the large limit case a method of solution is sought for in the boundary layer of the tube. A reduction to a single partial differential equation is possible and integral calculus methods are applied for the case of a body force directed to the centre of the tube to obtain an integral form of the Hunter-Saxton equation. Also an extension for a more general body force is shown where in addition there is a rotational force applied.

1. Introduction

Compressible flow has many applications some of which are of physics, mathematics and engineering interest. In general we have two types of flows, internal and external. Internal flows in ducts are important in industry and nozzles and diffusers used in engines are also an applied area for these types of flows. In general, density changes are related to temperature changes. External flows can be important for airplanes and projectiles where compressiblilty effects are important. The Navier Stokes equations have been dealt with extensively in the literature for both analytical [8] and numerical solutions [9], [3]. Some work converting the compressible Navier-Stokes equations to the Schrödinger equation in quantum mechanics by means of transformations has been carried out by Vadasz. [10]. General mathematical and computational methods for compressible flow have been outlined in [6]. Methods in more general fluid mechanics are also addressed in [1]. In the context of functional analysis it has been shown in [5] that generally the motion of a compressible fluid with fixed initial velocity field and constant initial density converges to that of an incompressible fluid as it's sound speed goes to infinity. Some analytical methods such as in [2] have been successfully carried out for one and two dimensional isentropic unsteady compressible flow. Little is known though for analytical methods for three dimensional compressible unsteady flow for both isentropic and non-isentropic flow. In the present work I introduce a new procedure to write the compressible unsteady Navier Stokes equations with a general spatial and temporal varying density term in terms of an additive solution of the three principle directions in the radial, azimuthal and z directions of flow. A dimensionless parameter is introduced whereby in the large limit case a method of solution is sought for in the boundary layer of the tube. It is concluded that the total divergence of the flow can be expressed as the integral with respect to time of the line integral of the dot product of inertial and azimuthal velocity. The line integral is evaluated on a contour that is annular and traces the boundary layer as time increases in the flow. A reduction to a single partial differential equation is possible and integral calculus methods are applied for the case of a body force directed to the centre of the tube to obtain an integral form of the Hunter-Saxton equation. Also an extension for a more general body force is shown where in addition there is a rotational force applied.

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37 ## 2. A new composite velocity formulation

The 3D compressible cylindrical unsteady Navier-Stokes equations are written in expanded form, for each component, u_r,u_θ and u_z:

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}-\frac{\mu}{\rho}\left(-\frac{u_r}{r^2}+\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right)-$$
$$\frac{\mu}{3\rho}\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{\rho}\frac{\partial p}{\partial r}-Fg_r=0 \tag{1}$$

$$\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{1}{r}u_r u_\theta+u_z\frac{\partial u_\theta}{\partial z}-\frac{\mu}{\rho}\left(-\frac{u_\theta}{r^2}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right)-$$
$$\frac{1}{r}\frac{\mu}{3\rho}\frac{\partial}{\partial \theta}\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right)+\frac{1}{\rho r}\frac{\partial p}{\partial \theta}-Fg_\theta=0 \tag{2}$$

$$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}-\frac{\mu}{\rho}\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right)-$$
$$\frac{\mu}{3\rho}\frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial r}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_z}{\partial z}\right)+\frac{1}{\rho}\frac{\partial p}{\partial z}-Fg_z=0 \tag{3}$$

where u_r is the radial component of velocity, u_θ is the azimuthal component and u_z is the velocity component in the direction along tube, ρ is density, μ is dynamic viscosity ,Fg_r, Fg_θ, Fg_z are body forces on fluid. The total gravity force vector is expressed as $\vec{F_T}=(Fg_r, Fg_\theta, Fg_z)$.

The following relationships between starred and non-starred dimensional quantities together with a non-dimensional quantity δ are used:

$$u_r=\frac{1}{\delta^2}u_r^* \tag{4}$$

$$u_\theta=\frac{1}{\delta}u_\theta^* \tag{5}$$

$$u_z=\frac{1}{\delta}u_z^* \tag{6}$$

$$r=\frac{r^*}{\delta} \tag{7}$$

$$\theta=\theta^* \tag{8}$$

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$$z=\frac{1}{\delta}z^*, t=t^* \tag{9}$$

where $\delta=\frac{1}{r^*}\frac{\partial\rho}{\partial\theta^*}\left(\frac{\partial\rho}{\partial r^*}+\frac{\rho}{r^*}\right)^{-1}$ as part of the continuity equation (Eq(12)). The region of interest is confined to within the boundary layer which is created in an assumed compressible viscous flow in the tube. Using Eqs.(4-9),multiplying Eqs(1-3) by cartesian unit vectors $\vec{e_{r^*}}=(1,0,0)$, $\vec{e_{\theta^*}}=(0,1,0)$ and $\vec{k}=(0,0,1)$ respectively and adding Equations (1-3) gives the following equation, for the resulting composite vector $\vec{L_1}=\frac{1}{\delta^2}u^*_{r^*}\vec{e_{r^*}}+\frac{1}{\delta}u^*_{\theta^*}\vec{e_{\theta^*}}+\frac{1}{\delta}u^*_{z^*}\vec{k}$,

$$\delta\left(\frac{1}{\delta}\frac{\partial\vec{L_1}}{\partial t^*}+\frac{u^*_{r^*}}{\delta^2}\frac{\partial\vec{L_1}}{\partial r^*}+\frac{u^*_{\theta^*}}{\delta r^*}\frac{\partial\vec{L_1}}{\partial\theta^*}+\frac{1}{\delta}u^*_{z^*}\frac{\partial\vec{L_1}}{\partial z^*}-\frac{1}{\delta^2 r^*}u^{*2}_{\theta^*}\vec{e_{r^*}}+\frac{1}{\delta^3 r^*}u^*_{r^*}u^*_{\theta^*}\vec{e_{\theta^*}}\right)-$$
$$\frac{\mu}{\rho}\left(-\delta^2\frac{\vec{L_1}}{r^{*2}}+\delta^2\frac{\partial^2\vec{L_1}}{\partial r^{*2}}+\delta^2\frac{1}{r^*}\frac{\partial\vec{L_1}}{\partial r^*}+\delta^2\frac{1}{r^{*2}}\frac{\partial^2\vec{L_1}}{\partial\theta^{*2}}+\frac{2\delta^2}{r^{*2}}\left(\frac{1}{\delta^2}\frac{\partial u^*_{r^*}}{\partial\theta^*}\vec{e_{\theta^*}}-\frac{1}{\delta}\frac{\partial u^*_{\theta^*}}{\partial\theta^*}\vec{e_{r^*}}\right)+\delta^2\frac{\partial^2\vec{L_1}}{\partial z^{*2}}\right)-$$
$$\frac{\mu}{3\rho}\frac{\partial}{\partial r^*}\left(\frac{\partial u^*_{r^*}}{\partial r^*}+\frac{1}{r^*}\frac{\partial u^*_{r^*}}{\partial\theta^*}+\frac{\partial u^*_{r^*}}{\partial z^*}\right)\vec{e_{r^*}}-\frac{\delta}{r^*}\frac{\mu}{3\rho}\frac{\partial}{\partial\theta^*}\left(\frac{\partial u^*_{\theta^*}}{\partial r^*}+\frac{1}{r^*}\frac{\partial u^*_{\theta^*}}{\partial\theta^*}+\frac{\partial u^*_{\theta^*}}{\partial z^*}\right)\vec{e_{\theta^*}}$$
$$-\delta\frac{\mu}{3\rho}\frac{\partial}{\partial z^*}\left(\frac{\partial u^*_{z^*}}{\partial r^*}+\frac{1}{r^*}\frac{\partial u^*_{z^*}}{\partial\theta^*}+\frac{\partial u^*_{z^*}}{\partial z^*}\right)\vec{k}+\delta\frac{1}{\rho}\frac{\partial p}{\partial r^*}\vec{e_{r^*}}+\delta\frac{1}{r^*\rho}\frac{\partial p}{\partial\theta^*}\vec{e_{\theta^*}}+\delta\frac{1}{\rho}\frac{\partial p}{\partial z^*}\vec{k}-\vec{F_T}=0 \tag{10}$$

Furthermore one can take the cross product of Equation (10) with $\vec{e_{r^*}}$ and $\vec{e_{\theta^*}}$ to form two equations respectively where in the second case multiplication by δ to cancel squared nonlinear terms,$-\frac{1}{\delta^2 r^*}u^{*2}_{\theta^*}\vec{e_{r^*}}$, $\frac{1}{\delta^3 r^*}u^*_{r^*}u^*_{\theta^*}\vec{e_{\theta^*}}$, leads to,

$$\delta\left(\frac{1}{\delta}\frac{\partial\vec{L_A}}{\partial t^*}+\frac{u^*_{r^*}}{\delta^2}\frac{\partial\vec{L_A}}{\partial r^*}+\frac{u^*_{\theta^*}}{\delta r^*}\frac{\partial\vec{L_A}}{\partial\theta^*}+\frac{1}{\delta}u^*_{z^*}\frac{\partial\vec{L_A}}{\partial z^*}\right)+\delta^{-2}\frac{u^*_{\theta^*}}{r^*}\frac{-\rho\vec{L}-\frac{\partial\rho}{\partial t}-u^*_{z^*}\frac{\partial\rho}{\partial z^*}}{\frac{\partial\rho}{\partial r^*}}\vec{k}-$$
$$\frac{\mu}{\rho}\left(-\delta^2\frac{\vec{L_A}}{r^{*2}}+\delta^2\frac{\partial^2\vec{L_A}}{\partial r^{*2}}+\delta^2\frac{1}{r^*}\frac{\partial\vec{L_A}}{\partial r^*}+\delta^2\frac{1}{r^{*2}}\frac{\partial^2\vec{L_A}}{\partial\theta^{*2}}+\frac{2\delta^2}{r^{*2}}\left(\frac{1}{\delta^2}\frac{\partial u^*_{r^*}}{\partial\theta^*}\vec{k}-\frac{1}{\delta}\frac{\partial u^*_{\theta^*}}{\partial\theta^*}\vec{k}\right)+\delta^2\frac{\partial^2\vec{L_A}}{\partial z^{*2}}\right)+$$
$$+\frac{\delta}{r^*}\frac{\mu}{3\rho}\frac{\partial}{\partial\theta^*}\left(\frac{\partial u^*_{\theta^*}}{\partial r^*}+\frac{1}{r^*}\frac{\partial u^*_{\theta^*}}{\partial\theta^*}+\frac{\partial u^*_{\theta^*}}{\partial z^*}\right)\vec{k}+\delta\frac{\mu}{3\rho}\frac{\partial}{\partial z^*}\left(\frac{\partial u^*_{z^*}}{\partial r^*}+\frac{1}{r^*}\frac{\partial u^*_{z^*}}{\partial\theta^*}+\frac{\partial u^*_{z^*}}{\partial z^*}\right)\vec{e_{\theta^*}}-$$
$$\delta\frac{\mu}{3\rho}\frac{\partial}{\partial r^*}\left(\frac{\partial u^*_{r^*}}{\partial r^*}+\frac{1}{r^*}\frac{\partial u^*_{r^*}}{\partial\theta^*}+\frac{\partial u^*_{r^*}}{\partial z^*}\right)\vec{k}+\delta^2\frac{\mu}{3\rho}\frac{\partial}{\partial z^*}\left(\frac{\partial u^*_{z^*}}{\partial r^*}+\frac{1}{r^*}\frac{\partial u^*_{z^*}}{\partial\theta^*}+\frac{\partial u^*_{z^*}}{\partial z^*}\right)\vec{e_{r^*}}-\delta\frac{1}{r^*\rho}\frac{\partial p}{\partial\theta^*}\vec{k}-$$
$$\delta\frac{1}{\rho}\frac{\partial p}{\partial z^*}\vec{e_{\theta^*}}+\delta^2\frac{1}{\rho}\frac{\partial p}{\partial r^*}\vec{k}-\delta^2\frac{1}{\rho}\frac{\partial p}{\partial z^*}\vec{e_{r^*}}+\vec{F_T}=0 \tag{11}$$

where $\vec{L_A}=-u^*_{z^*}\vec{e_{r^*}}-\frac{1}{\delta}u^*_{z^*}\vec{e^*_{\theta}}+\frac{1}{\delta}(u^*_{r^*}-u^*_{\theta^*})\vec{k}$, $\vec{L_A}=\vec{L_1}\times\vec{e_{r^*}}+\delta\vec{L_1}\times\vec{e^*_{\theta}}$ and continuity equation, Eq(12) below has been used. Equation (10) is sufficient to carry out the subsequent analysis.

3. A Solution Procedure for δ arbitrarily large in quantity

Multiplication of Eq.(10) by $\frac{\rho}{\delta}$ and Eq.(12) below by $\frac{\vec{L_1}}{\delta}$, addition of the resulting equations [7], and using the ordinary product rule of differential multivariable calculus a form as in Eq.(13) is obtained whereby I set $\vec{a}=\rho\vec{L_1}$.

 doi:10.20944/preprints201812.0081.v2

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The continuity equation in cylindrical co-ordinates is

$$\frac{\partial \rho}{\partial t} + \frac{1}{\delta} u^*_{r^*} \frac{\partial \rho}{\partial r^*} + \frac{u^*_{\theta^*}}{r^*}\frac{\partial \rho}{\partial \theta^*} + u^*_{z^*}\frac{\partial \rho}{\partial z^*} = -\rho \left(\frac{1}{\delta}\frac{u^*_{r^*}}{r^*} + \frac{1}{\delta}\frac{\partial u^*_{r^*}}{\partial r^*} + \frac{1}{r^*}\frac{\partial u^*_{\theta^*}}{\partial \theta^*} + \frac{\partial u^*_{z^*}}{\partial z^*} \right) = -\rho \overline{L} \tag{12}$$

$$\rho \frac{\partial \vec{a}}{\partial t} + \vec{a} \cdot \nabla \vec{a} + \rho^2 \vec{b} \nabla \cdot \vec{b} = \mu(\delta) \left(\nabla^2 \vec{b} + \frac{1}{3} \nabla (\nabla \cdot \vec{b}) \right) + \nabla P + \vec{F_T} \tag{13}$$

where $\mu(\delta) = \mu\delta$. I denote it by μ below.

Taking the geometric product in the previous equation with the inertial vector term,

$$\vec{f} = \vec{a} \cdot \nabla \vec{a} \tag{14}$$

43 where $\vec{b} = \frac{\vec{a}}{\rho}$ is defined, where in the context of Geometric Algebra, the following scalar and vector
44 grade equations arise,
45

$$\vec{f} \cdot \left(\rho^2 \frac{\partial \vec{b}}{\partial t} + \rho \vec{b} \frac{\partial \rho}{\partial t} \right) + \left\| \vec{f} \right\|^2 + \vec{b} \cdot \vec{f} \rho^2 \nabla \cdot \vec{b} = \mu \vec{f} \cdot \nabla^2 \vec{b} + \frac{\mu}{3} \vec{f} \cdot \nabla (\nabla \cdot \vec{b}) + \vec{f} \cdot \nabla P \tag{15}$$

$$\frac{\partial \vec{b}}{\partial t} + \frac{1}{\rho} \vec{b} \frac{\partial \rho}{\partial t} + \vec{b} \rho^2 \nabla \cdot \vec{b} = \mu \left(\nabla^2 \vec{b} + \frac{1}{3} \nabla (\nabla \cdot \vec{b}) \right) + \nabla P + \vec{F_T} \tag{16}$$

The geometric product of two vectors [4], is defined by $\vec{A}\vec{B} = \vec{A} \cdot \vec{B} + \vec{A} \times \vec{B}$.

Taking the divergence of Eq.(16) results in

$$\begin{aligned} \frac{\partial}{\partial t} \left(\nabla \cdot \vec{b} \right) + \frac{1}{\rho}\frac{\partial \rho}{\partial t} \nabla \cdot \vec{b} + \vec{b} \cdot \nabla \left(\frac{1}{\rho}\frac{\partial \rho}{\partial t}\right) + \nabla \cdot (\vec{b} \rho^2 \nabla \cdot \vec{b}) = \nabla \cdot \left(\mu \nabla^2 (\vec{b} \right) + \\ \frac{\mu}{3} \nabla \cdot \left(\nabla (\nabla \cdot \vec{b}) \right) + \nabla \cdot (\nabla P) + \nabla \cdot \vec{F_T} \end{aligned} \tag{17}$$

Upon multiplication of Eq(17) by,

$$H = \frac{\rho^3 \vec{b} \cdot \vec{f}}{\frac{\partial \rho}{\partial t}} \tag{18}$$

the resulting equation is

$$\begin{aligned} H \frac{\partial}{\partial t} (\nabla \cdot \vec{b}) + \rho^2 \vec{b} \cdot \vec{f} \nabla \cdot \vec{b} + H \vec{b} \cdot \nabla \left(\frac{1}{\rho}\frac{\partial \rho}{\partial t}\right) + H \nabla \cdot (\vec{b} \rho^2 \nabla \cdot \vec{b}) = \\ \mu H \nabla \cdot \nabla^2 \vec{b} + \frac{\mu}{3} H \nabla \cdot (\nabla (\nabla \cdot \vec{b})) + H \nabla \cdot (\nabla P) + H \nabla \cdot \vec{F_T} \end{aligned} \tag{19}$$

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which results upon using Eq(15) in,

$$H\frac{\partial}{\partial t}(\nabla\cdot\vec{b}) - \vec{f}\cdot\left(\rho^2\frac{\partial\vec{b}}{\partial t} + \rho\vec{b}\frac{\partial\rho}{\partial t}\right) - \left\|\vec{f}\right\|^2 - \mu\vec{f}\cdot\nabla^2\vec{b} - \frac{\mu}{3}\vec{f}\cdot\nabla(\nabla\cdot\vec{b}) - \vec{f}\cdot\nabla P + H\vec{b}\cdot\nabla(\frac{1}{\rho}\frac{\partial\rho}{\partial t}) +$$
$$H\nabla\cdot(\vec{b}\rho^2\nabla\cdot\vec{b}) = \mu H(\nabla\cdot\nabla^2\vec{b}) + \frac{\mu}{3}H\nabla\cdot(\nabla(\nabla\cdot\vec{b})) + H\nabla\cdot(\nabla P) + H\nabla\cdot\vec{F_T} \tag{20}$$

The continuity equation is written in terms of $\vec{b}$ as,

$$\frac{\partial\rho}{\partial t} + \rho^2\nabla\cdot\vec{b} + \rho\vec{b}\cdot\nabla\rho + \rho\nabla\rho\cdot\vec{b} = 0 \tag{21}$$

and

$$\nabla\cdot\vec{b} = -\frac{1}{\rho^2}\frac{\partial\rho}{\partial t} - \frac{2}{\rho}\nabla\rho\cdot\vec{b} \tag{22}$$

where the following compact expression is given,

$$Y^* = \nabla\cdot\vec{b} \tag{23}$$

Multiplying by $(\vec{f}\cdot\vec{f})^{-1}$ in Eq.(20) and, using properties of third derivatives involving the gradient and in particular the fact that the Laplacian of the divergence of a vector field is equivalent to the divergence of the Laplacian of a vector field, leads to the following form,

$$W^*\frac{\partial Y^*}{\partial t} - G(\rho,\frac{\partial\rho}{\partial t})W^* - F(\rho,\frac{\partial\rho}{\partial t})\vec{b}\vec{f}(1+\vec{f}\cdot\nabla P) - V(\mu)W^*\nabla^2(Y^*) - \vec{b}\vec{f}\frac{\partial\rho}{\partial t}\frac{\rho^{-3}}{\left\|\vec{f}\right\|^2}\vec{f}\cdot\left(\frac{\partial\vec{b}}{\partial t}\right) + \Omega -$$
$$\vec{b}\vec{f}\frac{\partial\rho}{\partial t}\frac{\rho^{-3}}{\left\|\vec{f}\right\|^2}\mu\vec{f}\cdot\nabla^2\vec{b} - \vec{b}\vec{f}\frac{\partial\rho}{\partial t}\frac{\rho^{-3}}{\left\|\vec{f}\right\|^2}\frac{\mu}{3}\vec{f}\cdot\nabla(\nabla\cdot\vec{b}) - \vec{b}\vec{f}\frac{1}{\left\|\vec{f}\right\|^2}\vec{b}\cdot\vec{f}\,\nabla\cdot\left(\vec{b}\rho^2\nabla\cdot\vec{b} + \nabla P + \vec{F_T}\right) = 0 \tag{24}$$

where $\Omega = H\vec{b}\cdot\nabla(\frac{1}{\rho}\frac{\partial\rho}{\partial t})$ in Eq.(24) and,

$$W^* = \left(\frac{\vec{f}\cdot\vec{b}}{\left\|\vec{f}\right\|^2}\vec{f}\right)\vec{b} = \left(\frac{\vec{f}\cdot\vec{b}}{\left\|\vec{f}\right\|^2}\vec{f}\right)\cdot\vec{b} + \left(\frac{\vec{f}\cdot\vec{b}}{\left\|\vec{f}\right\|^2}\vec{f}\right)\times\vec{b} = \xi + \vec{f}\times\left(\frac{\vec{f}\cdot\vec{b}}{\left\|\vec{f}\right\|^2}\vec{b}\right) \tag{25}$$

This involves the vector projection of $\vec{b}$ onto $\vec{f}$ which is written in the conventional form,

$$\mathrm{proj}_{\mathbf{f}}\,\mathbf{b} = \frac{\mathbf{f}\cdot\mathbf{b}}{\|\mathbf{f}\|^2}\mathbf{f} \tag{26}$$

Equation (24) can be written compactly as

$$\frac{\partial Y^*}{\partial t} - G(\rho,\frac{\partial\rho}{\partial t}) - V(\mu)\nabla^2 Y^* - \nabla^2 P = \frac{U_{\vec{f}}\left[\mathbf{Q}(\rho,\frac{\partial\rho}{\partial t},\vec{b},\frac{\partial\vec{b}}{\partial t},\nabla P,\vec{F_T}) + \vec{f}\right]}{U_{\vec{f}}\vec{b}} \tag{27}$$

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where $U_{\vec{f}}\vec{\xi}$ is the scalar projection for $\vec{b}$, $\mathbf{Q}$ and hence for a constant α,

$$\frac{\partial Y^*}{\partial t} - G(\rho,\frac{\partial \rho}{\partial t}) - V(\mu)\nabla^2 Y^* - \nabla^2 P = \frac{\left\|\mathbf{Q}(\rho,\frac{\partial \rho}{\partial t},\vec{b},\frac{\partial \vec{b}}{\partial t},\nabla P,\vec{F_T}) + \vec{f}\right\|}{\left\|\vec{b}\right\|} = \alpha \tag{28}$$

with solution in terms of a function $\mathcal{B}$,

$$Y^* = \nabla\cdot\vec{b} = \mathcal{B}(\alpha,r,\theta,z,t) \tag{29}$$

If the rate of change of ρ with respect to t is changing exponentially in time as $\frac{1}{r}B(\theta,z)e^{-ct}$ for some real constant c such that $O(c^2)\approx 0$, B a general function which vanishes in $G(\rho,\frac{\partial \rho}{\partial t}) = \frac{(\frac{\partial \rho}{\partial t})^2}{\rho^2}$, and since ρ is $O(1/r)$ in boundary layer, it may be proven that $\mathcal{B}$ is written linearly in terms of a separable function, that is

$$\mathcal{B}(\alpha,r,\theta,z,t) = F_1(r)\,F_2(\theta)\,F_3(z)\,F_4(t) - 1/4\,\frac{\left(C_1 + C_2\,\ln(r) + C_3\,r^2\right)\left(c^2+\alpha\right)}{\mu\,C_3} \tag{30}$$

where F_i are the separable parts of the function and C_j are arbitrary constants. Also it is assumed that higher order derivatives are negligible in the boundary layer, so that the Laplacian of pressure is dropped and the gradient of pressure is preserved.

For c^2 approaching zero and using the properties of the norm,

$$\left\|\left[Y^*(r,\theta,z,t) - \left(F1(r)\,F2(\theta)\,F3(z)\,F4(t)\right)\right]\vec{b} - M(r)\left[\mathbf{Q}\left(\rho,\frac{\partial \rho}{\partial t},\vec{b},\frac{\partial \vec{b}}{\partial t},\nabla P,\vec{F_T}\right) + \vec{f}\right]\right\| = 0 \tag{31}$$

Furthermore using the property for a norm on a vector space of continuous functions(since away from $r=0$) ,$\|A\| = 0$ iff $A = 0$, and Eq.(16) and the second line of Eq.(24) gives,

$$\begin{aligned}&\left[Y^*(r,\theta,z,t) - \left(F1(r)\,F2(\theta)\,F3(z)\,F4(t)\right)\right]\vec{b} - \\ &K(\rho,\frac{\partial \rho}{\partial t})M(r)\left[-2\frac{\partial \vec{b}}{\partial t} - \frac{1}{\rho}\vec{b}\frac{\partial \rho}{\partial t} - F(\rho,\frac{\partial \rho}{\partial t})(\vec{f}+\nabla P) - \vec{b}\nabla\cdot\left(-\vec{b}\rho^2\nabla\cdot\vec{b} + \vec{F_T}\right)\right] = 0\end{aligned} \tag{32}$$

where $M(r)$ is the function on the very right side of Eq.(30).

$$\begin{aligned}&\left[\nabla\cdot\vec{b} - \left(F1(r)\,F2(\theta)\,F3(z)\,F4(t)\right)\right]\vec{b} - \\ &K(\rho,\frac{\partial \rho}{\partial t})M(r)\left[-2\frac{\partial \vec{b}}{\partial t} - \frac{1}{\rho}\vec{b}\frac{\partial \rho}{\partial t} - F(\rho,\frac{\partial \rho}{\partial t})(\vec{f}+\nabla P) - \vec{b}\nabla\cdot\vec{F_T}\right] = 0\end{aligned} \tag{33}$$

where $F(\rho,\frac{\partial \rho}{\partial t}) = \rho^{-3}\frac{\partial \rho}{\partial t}$. Ω in Eq.(24) vanishes due to assumption on rate of change of density with respect to t and $c^2\approx 0$, and finally Eq.(22) has been used with a calculation done to show that the expression reduces in going from Eq (32) to Eq(33). This occurs in the boundary layer due to density being $O(1/r)$.

In the boundary layer it can be proven that $F1(r)$ is decaying as a J_0 Bessel function and K is

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an increasing function radially there with the following resulting upon dividing by K and $M(r)$,

$$\left[-2\frac{\partial\vec{b}}{\partial t}-\frac{1}{\rho}\vec{b}\frac{\partial\rho}{\partial t}-F(\rho,\frac{\partial\rho}{\partial t})(\vec{f}+\nabla P)-\vec{b}\nabla\cdot\vec{F_T}\right]=0 \tag{34}$$

Dividing by $F(\rho,\frac{\partial\rho}{\partial t})$ and taking the curl of Eq.(34) simplifies to the following in the boundary layer,

$$-2F^{-1}\frac{\partial}{\partial t}\nabla\times\vec{b}-\nabla\times\vec{f}-F^{-1}\nabla\times\left(\vec{b}\nabla\cdot\vec{F_T}\right)=0 \tag{35}$$

The inertial vector $\vec{f}=\nabla(f^2/2)-\vec{a}\times\nabla\times\vec{a}$.

Multiply Eq.(35) by the normal vector $\cos(\theta)\vec{a}$ which is the normal component of $\vec{a}$ at wall of moving control volume (CV) in Figure 1,

$$\cos(\theta)\vec{a}\cdot\left[-2F^{-1}\frac{\partial}{\partial t}\nabla\times\vec{b}-\nabla\times\vec{f}-F^{-1}\nabla\times\left(\vec{b}\nabla\cdot\vec{F_T}\right)\right]=0 \tag{36}$$

Choosing ρ so that $F=2$ the following is used,

Recalling Divergence theorem and Stoke's theorem, for general **F**,

$$\iiint_V(\nabla\cdot\mathbf{F})dV=\oint\!\!\!\oint_{S(V)}\mathbf{F}\cdot\hat{\mathbf{n}}dS$$

$$\iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{S}=\oint_C\mathbf{F}(\mathbf{r})\cdot d\mathbf{r} \tag{37}$$

where $C=C_1\cup C_2$ defines the union of two contours where the first is at the boundary layer edge and the other is along the wall (Fig. 2) and S consists of all surfaces of control volume.

Defining the following vector field,

$$\vec{W}=-\frac{\partial}{\partial t}\nabla\times\vec{b}-\frac{1}{2}\nabla\times\left(\vec{b}\nabla\cdot\vec{F_T}\right), \tag{38}$$

$$\oint\!\!\!\oint_{S(V)}\mathbf{W}\cdot\hat{\mathbf{n}}dS=\iiint_V(\nabla\cdot\mathbf{W})dV \tag{39}$$

$$=\oint_C\mathbf{f}(\mathbf{r})\cdot d\mathbf{r} \tag{40}$$

where $\hat{\mathbf{n}}=\cos(\theta)\vec{a}$, and Stoke's theorem has been used.

$$\oint_C\vec{f}(\mathbf{r})\cdot\vec{T}ds=-\oint_C(\frac{\partial\vec{b}}{\partial t}+\frac{1}{2}\vec{b}\nabla\cdot\vec{F_T})\cdot d\mathbf{r} \tag{41}$$

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where $\vec{T}$ is unit tangent vector to closed curve C and ds is arc length,

$$\oint_C (\vec{f}(\mathbf{r}) + \frac{\partial \vec{b}}{\partial t} + \frac{1}{2}\vec{b}\nabla \cdot \vec{F_T}) \cdot d\mathbf{r} = 0 \tag{42}$$

The third term in the parenthesis in Eq(42) is integrated by parts for line integral and I obtain the following,

$$\oint_C (\vec{f}(\mathbf{r}) + \frac{\partial \vec{b}}{\partial t} - \frac{1}{2}\vec{F_T}\nabla \cdot \vec{b}) \cdot d\mathbf{r} = 0 \tag{43}$$

Parametrizing the circle as $r = g(\theta)$ in polar coordinates it can be proven that the line integral in Eq(43) is,

$$\oint_C (f_1 + \frac{\partial b_1}{\partial t} - \frac{1}{2}F_{T_1}\nabla \cdot \vec{b}) \cdot d\theta - \oint_C (f_2 + \frac{\partial b_2}{\partial t} - \frac{1}{2}F_{T_2}\nabla \cdot \vec{b}) \cdot dr = 0 \tag{44}$$

The normal form of Green's theorem can be used for the line integral in Eq(44),setting first,

$$M = f_1 + \frac{\partial b_1}{\partial t} - \frac{1}{2}F_{T_1}\nabla \cdot \vec{b} \tag{45}$$

$$N = f_2 + \frac{\partial b_2}{\partial t} - \frac{1}{2}F_{T_2}\nabla \cdot \vec{b} \tag{46}$$

The line integral in Eq(44) is equal to the following,

$$\iint_R \left(\frac{\partial M}{\partial r} + \frac{\partial N}{\partial \theta} \right) dr d\theta \tag{47}$$

where M and N are given by Eqs(45) and (46) respectively and R is the annulus with boundary C. The external force F_{T_1} is expressed as follows.(see Appendix),

$$F_{T_1} = \mu \sin(\theta) \int_{r_{B_L}}^{r} \frac{\partial b_2}{\partial \theta} dr \tag{48}$$

46 It follows that Eq.(47) becomes,

47

$$\iint_R \left[\mu \sin(\theta) \int_{r_{B_L}}^{r} \frac{\partial b_2}{\partial \theta} dr \left(\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial b_2}{\partial \theta} \right) + \mu \sin(\theta) \frac{1}{r} \left(\frac{\partial b_2}{\partial \theta} \right)^2 + (f_2 + \frac{\partial b_2}{\partial t})_\theta \right] dr d\theta \tag{49}$$

and since the boundary layer is thin the first integral vanishes and the Eq.(47) becomes,

$$\iint_R \left[\mu \sin(\theta) \frac{1}{r} \left(\frac{\partial b_2}{\partial \theta} \right)^2 + (f_2 + \frac{\partial b_2}{\partial t})_\theta \right] dr d\theta \tag{50}$$

In the boundary layer the function $b_2 = b_2(\theta, t)$ and $b_1 = b_1(r, t)$ and thus,

$$\int_0^{2\pi} \left((r - r_{B_L})(b_2 b_{2\theta} + b_{2_t})_\theta - \mu \sin(\theta) \ln(\frac{r}{r_{B_L}}) b_{2_\theta}^2 \right) d\theta \tag{51}$$

 doi:10.20944/preprints201812.0081.v2

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Dividing by $r - r_{B_L}$,for $\theta \approx \pi/2$, and using L'Hopital's rule, I obtain the Hunter Saxton equation with $\mu r_{B_L} = 1$.

4. The Hunter-Saxton Equation

Substituting f_2 and F_{T_1} into Eq(45) and (46) since $F_{T_2} = 0$ and f_1, b_1 terms are negligible in the boundary layer(Recall $u_r = u_r^*/\delta^2, t = t^*/\delta$ then one can obtain an integral form of the Hunter-Saxton equation using Eq.(47),

$$\left(b_{2_t}^* + b_2^* b_{2_\theta}^*\right)_\theta = \frac{1}{2} b_{2_\theta}^{*\ 2} \tag{52}$$

It necessarily follows that for small θ approximation that $\frac{\partial b_2(0,t)}{\partial \theta} = 0$, $\frac{\partial^2 b_2(0,t)}{\partial^2 \theta} = 0$, and higher derivatives at zero are zero.

It is of interest that a more complicated partial differential equation is manifest upon taking F_{T_2} to be non zero and thus defining a rotational force related to the vorticity of the fluid elements in the boundary layer.

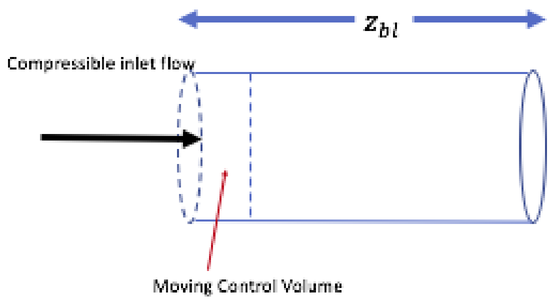

Figure 1. Compressible Viscous Flow in a Tube, z_{bl} is the distance to achieve maximum boundary layer height.

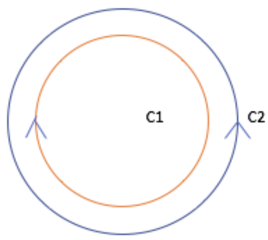

Figure 2. A Typical set of contour lines between edge of boundary layer and tube wall.

In the interval $[0, t]$ the boundary layer starts to form at $t = 0$ and reaches maximum height at time t. See Fig. 1 for control volume over the growth of the boundary layer. The time dependence is shown in the inertial term $\vec{f}$. The right side of Eq.(40) consists of nonlinear inertial term $\vec{f}$ and Eq.(39) shows the dependence on these gradients and rate of change of curl of $\vec{b}$ with respect to t.

5. Conclusion

An attempt has been made to reduce the compressible Navier Stokes equations coupled to the continuity equation in cylindrical co-ordinates to a simpler problem in terms of an additive solution of the three principle directions in the radial, azimuthal and z directions of flow. A dimensionless

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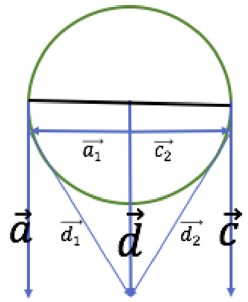

Figure 3. Vectors $\vec{a}, \vec{c}, \vec{d}$ as well as $\vec{d}_1$ and $\vec{d}_2$ which are pointing in the direction of increasing gravitational force.(Appendix).

parameter is introduced whereby in the large limit case a method of solution is sought for in the boundary layer of the tube. By seeking an additive solution using the continuity equation and the simplified vector equation obtained by a similar procedure and using the product rule of differentiable calculus, and using Geometric Algebra as a starting point, it follows through analysis that the integral of total divergence of a specific vector field over time can be expressed as the integral with respect to time of the line integral of the dot product of inertial and azimuthal velocity. The line integral is evaluated on a contour that is annular and traces the boundary layer as time increases in the flow. It has been shown that a reduction of the 3D compressible unsteady Navier-Stokes equations to a single partial differential equation is possible and integral calculus methods are applied for the case of a body force directed to the centre of the tube to obtain an integral form of the Hunter-Saxton equation. Also an extension for a more general body force has been shown where in addition there is a rotational force applied.

6. Appendix

Referring to Fig.3, let,

$$\vec{d} = k\vec{a} \tag{53}$$

$$\vec{d} = m\vec{c} \tag{54}$$

It follows that,

$$\frac{\vec{d}+\vec{d}}{2} = \vec{d} = k\vec{a} + m\vec{c} \tag{55}$$

If $\vec{a}$ and $\vec{c}$ are in the same direction then $km > 0$ and k, m are either both positive or both negative. Also according to Fig.3 it follows that,

$$\left\|\vec{d}\right\|^2 = \|\vec{c_2}\|^2 + \left\|\vec{d_2}\right\|^2 \tag{56}$$

$$\left\|\vec{d}\right\|^2 = \|\vec{a_1}\|^2 + \left\|\vec{d_1}\right\|^2 \tag{57}$$

If the circle shrinks to an arbitrary small circle in diameter then

$$\left\|\vec{d}\right\|^2 \approx \left\|\vec{d_2}\right\|^2 \tag{58}$$

$$\left\|\vec{d}\right\|^2 \approx \left\|\vec{d}_1\right\|^2 \tag{59}$$

Summing up all the d_i components and letting $b_{2_i} e_\theta$ represent the tangential velocity for each point on the circumference of disk R, it follows for N distinct points that,

$$\sum_{i=1}^{N} d_i = \sum_{i=1}^{N} b_{2_i} \tag{60}$$

Thus in the limit as the circle diameter gets small, introduce a function g defined as,

$$\|g\| = \frac{\partial b_2}{\partial \theta} \tag{61}$$

where $\vec{b} = b_2 \vec{e}_\theta$ and $\frac{\partial \vec{b}}{\partial \theta}$ is the vector pointing towards the center of the disk in R.

Acknowledgment

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