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Growth of the entire or meromorphic solutions of Differential- difference equations

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Abstract

In this paper, we study the entire or meromorphic solutions for differential-difference equations in f(z) , its shifts, its derivatives and derivatives of its shifts. and study some Hayman's results for differential- difference polynomials .

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INTRODUCTION AND MAIN RESULTS:

It is assumed that the reader is familiar with the basic concepts of Nevanlinna Theory, see e.g. ([1],[2]), such as the characteristic function T(r, f), proximity function m(r, f), counting function N(r, f) and so on. In addition, S(r, f) denotes any quantity that satisfies the condition that S(r, f)= o(T(r, f)) as r tends to infinity outside of a possible exceptional set of finite logarithimic measure. In the sequel, a meromorphic function a(z)is called a small function with respect to f if and only if T[r, a (z)] = o(T(r, f)) as r tends to infinity outside of a possible exceptional set of finite logarithimic measure. We denote by S(f), the family of all such small meromorphic functions.

We say that two meromorphic functions f and g share the value a (belonging to extended complex plane) CM (IM)

provided that

$$f(z) \equiv a$$

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if and only if

$$g(z) \equiv a,$$

counting multiplicity (ignoring multiplicity).

DEFINITION 1 :

Let c be a non-zero complex costant then for a meromorphic function f(z) , we define its shift by f(z+c) and its difference operator by

$$\Delta_c f(z) = f(z+c) - f(z),$$

$$\Delta_{mc} f(z) = f(z+mc) - f(z),$$

where m is a positive integer

$$\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)),$$

n$\in \mathbf{N}, n \geq 2,$

$$= \sum_{k=0}^{n} \frac{(-1)^k.n!}{k!.(n-k)!} f(z + \overline{n-k}.c).$$

In particular,

$$\Delta_c^n f(z) = \Delta^n f(z)$$

for c=1.

We define **Differential - difference Monomial** as

$$M[f] = \prod_{i=0}^{k} \prod_{j=0}^{m} [f^{(j)}(z + c_{ij})]^n ij$$

where c_{ij} are complex constants , and n_{ij} are natural numbers , i= 0, 1, ... ,k and j=0, 1, ... ,m.

Then the **degree of M[f]** will be the sum of all the powers in the product on the right hand side. Let us define the **weight of M[f]** as Γ_M = sum of powers of f + 2.sum of powers of f'+ 3. sum of powers of f" + ...

DEFINITION 2 :

Let

$$M_1[f], M_2[f], ...$$

denote the distinct monomials in f, and

$$a_1(z), a_2(z), ...$$

be the small meromorphic functions including complex numbers then

$$P[f] = P[z, f] = \sum_{j \in \Delta} a_j(z).M_j[f]$$

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where Δ is a finite set of multi- indices, $a_j(z)$ are small functions of f, $M_j[f]$ are differential- difference monomials,
will be called a differential- difference polynomial in f, which is a finite sum of products of f , derivatives of f, their shifts, and derivatives of its shifts.
Let us define the total degree d and weight Γ of P[z, f] in f as

$$d = \underbrace{Max.}_{j\in\Delta} d_{M_j}; \Gamma = \underbrace{Max.}_{j\in\Delta} \Gamma_{M_j}.$$

If all the terms in the summation of P[f] have same degrees, then P[f] is known as homogeneous differential- difference polynomial. Usually, we take P[f] such that T(r, P) $\neq S(r, f)$.

A finite value a is called the Picard exceptional value of f, if f - a has no zeros. The Picard theorem shows that a transcendental entire function has at most one Picard exceptional value, a transcendental meromorphic functions has at most two picard exceptional values.

SECTION 1:

W. K.Hayman[1] proved the following theorem:

THEOREM A[1]: If f(z) is meromorphic and transcendental in the plane and has only a finite number of zeros and poles, then every non-constant function

$$\varphi(z) = \Sigma a_j(z).f^j(z), j = 0, ..., l$$

assumes every finite value except possibly zero infinitely often.

We will consider the differential- difference polynomial as in definition 2 and prove Theorem A for such polynomials as the following results:

MAIN RESULTS:

THEOREM 1.1: Let f be a transcendental entire function with finite order and as in definition 2, P[f] be a differential- difference polynomial of degree d defined as

$$P[f] = \sum_{j\in\Delta} a_j(z).M_j[f]; T(r, P[f]) \neq S(r, f),$$

where Δ is a finite set of multi- indices, $a_j(z)$ are small functions of f, $M_j[f]$ are differential- difference monomials, then P[f] = a(z) (a(z) = small function or complex value, a(z)$\neq$ 0, ∞) has infinitely many solutions provided N(r,0,f) = S(r,f).

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THEOREM 1.2: Let f be a transcendental meromorphic function with finite order and as in definition 2, P[f] be a differential- difference polynomial of degree d defined as

$$P[f] = \sum_{j\in\Delta} a_j(z).M_j[f]; T(r, P[f]) \neq S(r, f),$$

where Δ is a finite set of multi- indices, $a_j(z)$ are small functions of f, $M_j[f]$ are differential- difference monomials, then P[f] = a(z) (a(z) = small function or complex value, a(z)$\neq$ 0, ∞) has infinitely many solutions provided N(r,0,f) + N(r, f) = S(r,f).

The classical problem of value distributions of differential polynomials is Hayman conjecture [3],i.e. if f is a transcendental meromorphic function and n $\in$ N, *then* $f^n f'$ takes every finite non-zero value infinitely often which means that the Picard exceptional value of $f^n f'$ may only be zero. This conjecture has been proved by many authors. e.g., Hayman [3] proved that if f is a transcendental meromorphic function and n$\geq$ 3,then $f^n f'$takes every finite non-zero complex value infinitely often. The case n=2 was proved by Mues[4], and Bergweiler et. al[5] proved the case for n=1.

Then many authors started to investigate the uniqueness of meromorphic functions sharing values with their shifts/ q-shifts or difference operators see e.g. ([6]-[9]).

We shall prove the above conjecture for general differential difference polynomials with some condition on power of f.

THEOREM 1.3: Let f be a transcendental meromorphic function with finite order and as in definition 2, P[f] be a differential- difference polynomial of degree d and weight Γ defined as

$$P[f] = P[z, f] = \sum_{j\in\Delta} a_j(z).M_j[f]$$

$T(r, P[f])\neq$ S(r, f),
where Δ is a finite set of multi- indices, $a_j(z)$ are small functions of f, $M_j[f]$ are differential- difference monomials,
then $f^l(f-1)P[f] - a(z), a(z)\neq 0, \infty$ has infinitely many zeros provided l $>$ $2(\Gamma + 1)$.

For the proof of the results we need the following lemmas:

LEMMA 1[7]: Let f be a non- constant meromorphic function of finite order and c be a non- zero complex constant, then

$m(r, \frac{f(z+c)}{f(z)})$ = S(r, f),

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for all r outside a possible exceptional set of finite logarithmic measure.

LEMMA 2[7]: Let c be a non-zero complex constant, and let f be a meromorphic function of finite order then

T(r, f(z+c)) = T(r, f) + S(r, f)

N(r, f(z+c)) = N(r, f) + S(r, f)

N(r, 0, f(z+c)) = N(r, 0, f) + S(r, f)

LEMMA 3(Clunie type lemma[6]): Let f(z) be a non- constant meromorphic function of finite order such that

$$f^n P[z, f] = Q[z, f],$$

where P[z, f], Q[z, f] are differential-difference polynomials in f. If the degree of Q[z, f] as a polynomial in f and its shifts is at most n, then

$$m(r, P[z, f]) = S(r, f).$$

LEMMA 4([11]): Let f be a nonconstant meromorphic function. If Q[f] is a differential polynomial in f with arbitrary meromorphic coefficients, then

(i) m(r, Q[f]) $\leq \gamma_Q m(r, f)$ + S(r, f)

(ii) N(r, Q[f]) $\leq \Gamma_Q N(r, f) + S(r, f)$

Remark: We can obtain Lemma 4 for differential difference polynomials in f of finite order using lemma 2 and definition of weight as in definition 2.

PROOF OF THEOREM 1.1:

Let P[f] = a(z), a(z)$\neq$ 0, ∞ has finitely many solutions.
. Then we get by using Lemma 2, Lemma 3 and given condition:

$$T(r, P[f]) = T(r, \sum_{j\in\Delta} a_j(z).M_j[f])$$

=T(r,f^d[$a_1\frac{M_1}{f^d} + a_2\frac{M_2}{f^d} + ... + a_n\frac{M_n}{f^d}$])

$\geq$ d T(r, f) - T(r, [a$_1\frac{M_1}{f^d} + a_2\frac{M_2}{f^d} + ... + a_n\frac{M_n}{f^d}$])

But T(r, [a$_1\frac{M_1}{f^d} + a_2\frac{M_2}{f^d} + ... + a_n\frac{M_n}{f^d}$])

$\leq$ T(r, $\frac{M_1}{f^d}$) + T(r, $\frac{M_2}{f^d}$) + ... + T(r, $\frac{M_n}{f^d}$) + S(r, f)

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= N(r, $\frac{M_1}{f^d}$) + N(r, $\frac{M_2}{f^d}$) + ... + N(r, $\frac{M_n}{f^d}$) + S(r, f)

= S(r, f)

Therefore, we have

T(r, P[f])
$\geq$ d T(r, f) + S(r, f)

Since f is entire, therefore, by using Nevanlinna's second main theorem, we get

d T(r, f) $\leq$ T(r, P[f]) $\leq$ $\bar{N}(r, \frac{1}{P[f]})$ + $\bar{N}(r, P[f])$ + $\bar{N}(r, \frac{1}{P[f]-a(z)})$ + S(r, f)

= S(r, f)

which is a contradiction as d $\geq$ 1. Thus our supposition is wrong and hence, P[f] = a(z) (small function or complex value), a(z)$\neq$ 0, ∞ has infinitely many solutions.

PROOF OF THEOREM 1.2:

Let P[f] = a(z) (small function or complex value), a(z)$\neq$ 0, ∞ has finitely many solutions.
Then we get by using Lemma 2, Lemma 3 and given condition:

$$T(r, P[f]) = T(r, \sum_{j\in\Delta} a_j(z).M_j[f])$$

=T(r,f^d[$a_1\frac{M_1}{f^d} + a_2\frac{M_2}{f^d} + ... + a_n\frac{M_n}{f^d}$])

$\geq$ d T(r, f) - T(r, [a$_1\frac{M_1}{f^d} + a_2\frac{M_2}{f^d} + ... + a_n\frac{M_n}{f^d}$])

But T(r, [a$_1\frac{M_1}{f^d} + a_2\frac{M_2}{f^d} + ... + a_n\frac{M_n}{f^d}$])

$\leq$ T(r, $\frac{M_1}{f^d}$) + T(r, $\frac{M_2}{f^d}$) + ... + T(r, $\frac{M_n}{f^d}$) + S(r, f)

= N(r, $\frac{M_1}{f^d}$) + N(r, $\frac{M_2}{f^d}$) + ... + N(r, $\frac{M_n}{f^d}$) + S(r, f)

= S(r, f)

Therefore, we have

T(r, P[f])
$\geq$ d T(r, f) + S(r, f)

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Since f is meromorphic and by using N(r,0,f) + N(r, f) = S(r,f) and Nevanlinna's second main theorem, we get

d T(r, f) $\leq$ T(r, P[f]) $\leq \bar{N}(r, \frac{1}{P[f]}) + \bar{N}(r, P[f]) + \bar{N}(r, \frac{1}{P[f]-a(z)})$ + S(r, f)

= S(r, f)

which is a contradiction as d $\geq$ 1. Thus our supposition is wrong and hence, P[f] = a(z) (small function or complex value), a(z)$\neq$ 0, ∞ has infinitely many solutions.

PROOF OF THEOREM 1.3:

Let G[z] = f$^l(f-1)P[z,f]$ where f is a meromorphic function and suppose G[z]- a(z), a(z)$\neq$ 0, ∞ has finitely many zeros. Then we get by using Lemma 4 for differential difference polynomials

T(r, G[z]) = T(r,f$^l(f-1)\sum_{j\in\Delta}\ a_j(z).M_j[f]$)

$\geq$ (l + 1) T(r, f) - $\Gamma T(r,f)$

Therefore, we have

T(r, G[z])
$\geq$ (l + 1 -Γ)$T(r,f) + S(r,f)$

Since f is meromorphic, therefore, by using Nevanlinna's second main theorem and lemma 4 for differential difference polynomials, we get

(l + 1 -Γ)$T(r,f) \leq$ T(r, G[z]) $\leq \bar{N}(r, \frac{1}{G(z)}) + \bar{N}(r, G(z)) + \bar{N}(r, \frac{1}{G(z)-a(z)})$ + S(r, G)

$= \bar{N}(r, \frac{1}{G(z)}) + \bar{N}(r, G(z)) + S(r,f)$

$\leq (\Gamma+2)T(r,f) + \bar{N}(r,f) + S(r,f)$

$= (\Gamma+3)T(r,f) + S(r,f)$

So we get

l T(r, f) $\leq 2(\Gamma+1)T(r,f)$+ S(r, f)

which is a contradiction as l $> 2(\Gamma+1)$. Thus our supposition is wrong and hence the result.

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SECTION 2

Hayman[3] proved that a differential polynomial f$^n + af' - b$ with constant coefficients a, b admits infinitely many zeros provided that f is transcendental entire and n $\geq$ 3. K. Liu and I. Laine in 2010 proved that:

THEOREM B[12]: Let f be a transcendental entire function of finite order-not of period c, then for small non-zero function s(z), the difference polynomial f$^n + f(z+c) - f(z) - s(z)$ has infinitely many zeros in the complex plane provided that n $\geq$ 3.
We prove the above results for general difference polynomials (shifts and difference operators as in definition 1 are part of these) as following:

THEOREM 2.1: Let f be a transcendental entire function with finite order and as in definition 2, P[f] be a linear difference polynomial defined as $P[f] = c_0f(z) + c_1f(z+c) + c_2f(z+2c) + ... + c_nf(z+nc); T(r, P[f]) \neq$ S(r, f), where c$\neq$ 0 and c$_j, j = 0, 1, ..., n,$ are complex constants then f$^l + P[f] - a(z), a(z) \neq$ 0, ∞ has infinitely many zeros provided l $>$ 2n + 1.

THEOREM 2.2: Let f be a transcendental meromorphic function with finite order and as in definition 2, P[f] be a linear difference polynomial defined as $P[f] = c_0f(z) + c_1f(z+c) + c_2f(z+2c) + ... + c_nf(z+nc); T(r, P[f]) \neq$ S(r, f), where c$\neq$ 0 and c$_j, j = 0, 1, ..., n,$ are complex constants then f$^l + P[f] - a(z), a(z) \neq$ 0, ∞ has infinitely many zeros provided l $>$ 3n + 3.

PROOF OF THEOREM 2.1:

Let G[z] = f$^l + P[f]$ where f is an entire function and suppose G[z]- a(z), a(z)$\neq$ 0, ∞ has finitely many zeros. Then we get by using Lemma 1 and Lemma 2

T(r, G[z]) = T(r,f$^l + [c_0f(z) + c_1f(z+c) + c_2f(z+2c) + ... + c_nf(z+nc)]$)

$\geq$ (l+1) T(r, f) - T(r, [$c_1f(z+c) + c_2f(z+2c) + ... + c_nf(z+nc)$])

Therefore, we have

T(r, G[z])
$\geq$ (l-n+1)T(r, f)+ S(r, f) ...(1)

Since f is entire, therefore, by using Nevanlinna's second main theorem , we get

[l + 1 - n] T(r, f) $\leq$ T(r, G[z]) $\leq \bar{N}(r, \frac{1}{G(z)}) + \bar{N}(r, \frac{1}{G(z)-a(z)})$ + S(r, G)

$= \bar{N}(r, \frac{1}{G(z)})$ + S(r, f)

$\leq$ (n + 2) N(r, 0, f)+$S(r, f)$

$\leq$ (n + 2) T(r, f) +$S(r, f)$

So we get

l T(r, f) $\leq$ (2n + 1)T(r, f) +$S(r, f)$

which is a contradiction as l > 2n + 1. Thus our supposition is wrong and hence, $f^l P[f] - a(z), a(z) \neq 0$, ∞ has infinitely many zeros.

PROOF OF THEOREM 2.2:

Let G[z] = $f^l + P[f]$ where f is a meromorphic function and suppose G[z]- a(z), a(z)$\neq$ 0, ∞ has finitely many zeros. Then we have

T(r, G[z]) = T(r,$f^l + [c_0 f(z) + c_1 f(z+c) + c_2 f(z+2c) + ... + c_n f(z+nc)]$)

$\geq$ (l+1) T(r, f) - T(r, [$c_1 f(z+c) + c_2 f(z+2c) + ... + c_n f(z+nc)$])

Therefore, we have

T(r, G[z])
$\geq$ (l -n+1) T(r, f)+ S(r, f)

Since f is meromorphic, therefore, by using Nevanlinna's second main theorem and lemma , we get

[l -n+1] T(r, f) $\leq$ T(r, G[z]) $\leq \bar{N}(r, \frac{1}{G(z)}) + \bar{N}(r, G(z)) + \bar{N}(r, \frac{1}{G(z)-a(z)})$ + S(r, G)

$= \bar{N}(r, \frac{1}{G(z)}) + \bar{N}(r, G(z)) + S(r, f)$

$\leq$ (2n + 4) T(r, f)+$S(r, f)$

So we get

l T(r, f) $\leq$ (3n + 3)T(r, f) +$S(r, f)$

which is a contradiction as l > 3n + 3. Thus our supposition is wrong and hence, $f^l + P[f] - a(z), a(z) \neq 0$, ∞ has infinitely many zeros.

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