Growth of the entire or meromorphic solutions of Differential-difference equations

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Abstract

In this paper, we study the entire or meromorphic solutions for differential-difference equations in $f(z)$, its shifts, its derivatives and derivatives of its shifts, and study some Hayman’s results for differential-difference polynomials.

Mathematics Subject Classification:

Keywords: Entire and Meromorphic functions, Differential-difference polynomial, Shared value, Nevanlinna theory.

INTRODUCTION AND MAIN RESULTS:

It is assumed that the reader is familiar with the basic concepts of Nevanlinna Theory, see e.g. ([1],[2]), such as the characteristic function $T(r, f)$, proximity function $m(r, f)$, counting function $N(r, f)$ and so on. In addition, $S(r, f)$ denotes any quantity that satisfies the condition that $S(r, f) = o(T(r, f))$ as $r$ tends to infinity outside of a possible exceptional set of finite logarithmic measure. In the sequel, a meromorphic function $a(z)$ is called a small function with respect to $f$ if and only if $T(r, a(z)) = o(T(r, f))$ as $r$ tends to infinity outside of a possible exceptional set of finite logarithmic measure. We denote by $S(f)$, the family of all such small meromorphic functions.

We say that two meromorphic functions $f$ and $g$ share the value $a$ (belonging to extended complex plane) CM (IM) provided that

$$f(z) \equiv a$$
if and only if 
\[ g(z) \equiv a, \]
counting multiplicity (ignoring multiplicity).

**DEFINITION 1 :**
Let \( c \) be a non-zero complex constant then for a meromorphic function \( f(z) \), we define its shift by \( f(z+c) \) and its difference operator by
\[
\Delta_c f(z) = f(z + c) - f(z),
\]
\[
\Delta_{mc} f(z) = f(z + mc) - f(z),
\]
where \( m \) is a positive integer
\[
\Delta^n_c f(z) = \Delta^{n-1}_c (\Delta_c f(z)),
\]
\( n \in \mathbb{N}, n \geq 2 \),
\[
= \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} f(z + \frac{n-k}{m}c).
\]
In particular,
\[
\Delta^n_c f(z) = \Delta^n f(z)
\]
for \( c=1 \).
We define **Differential - difference Monomial** as
\[
M[f] = \prod_{i=0}^{k} \prod_{j=0}^{m} [f^{(j)}(z + c_{ij})]^{n_{ij}}
\]
where \( c_{ij} \) are complex constants , and \( n_{ij} \) are natural numbers , \( i = 0, 1, ... , k \) and \( j = 0, 1, ... , m \).

Then the **degree of** \( M[f] \) will be the sum of all the powers in the product on the right hand side. Let us define the **weight of** \( M[f] \) as \( \Gamma_M = \text{sum of powers of } f + 2 \cdot \text{sum of powers of } f' + 3 \cdot \text{sum of powers of } f'' + ... \)

**DEFINITION 2 :**
Let
\[
M_1[f], M_2[f], ...
\]
denote the distinct monomials in \( f \), and
\[
a_1(z), a_2(z), ...
\]
be the small meromorphic functions including complex numbers then
\[
P[f] = P[z, f] = \sum_{j \in \Delta} a_j(z).M_j[f]
\]
where $\Delta$ is a finite set of multi-indices, $a_j(z)$ are small functions of $f$, $M_j[f]$ are differential- difference monomials, will be called a differential- difference polynomial in $f$, which is a finite sum of products of $f$, derivatives of $f$, their shifts, and derivatives of its shifts. Let us define the total degree $d$ and weight $\Gamma$ of $P[z, f]$ in $f$ as

$$d = \max_{j \in \Delta} d_{M_j}, \Gamma = \max_{j \in \Delta} \Gamma_{M_j}.$$ 

If all the terms in the summation of $P[f]$ have same degrees, then $P[f]$ is known as homogeneous differential- difference polynomial. Usually, we take $P[f]$ such that $T(r, P) \neq S(r, f)$.

A finite value $a$ is called the Picard exceptional value of $f$, if $f - a$ has no zeros. The Picard theorem shows that a transcendental entire function has at most one Picard exceptional value, a transcendental meromorphic functions has at most two picard exceptional values.

**SECTION 1:**

W. K. Hayman[1] proved the following theorem:

**THEOREM A[1]:** If $f(z)$ is meromorphic and transcendental in the plane and has only a finite number of zeros and poles, then every non-constant function

$$\varphi(z) = \sum a_j(z).f^j(z), j = 0, \ldots, l$$

assumes every finite value except possibly zero infinitely often.

We will consider the differential- difference polynomial as in definition 2 and prove Theorem A for such polynomials as the following results:

**MAIN RESULTS:**

**THEOREM 1.1:** Let $f$ be a transcendental entire function with finite order and as in definition 2, $P[f]$ be a differential- difference polynomial of degree $d$ defined as

$$P[f] = \sum_{j \in \Delta} a_j(z).M_j[f]; T(r, P[f]) \neq S(r, f),$$

where $\Delta$ is a finite set of multi-indices, $a_j(z)$ are small functions of $f$, $M_j[f]$ are differential- difference monomials, then $P[f] = a(z)$ ( $a(z)$ = small function or complex value, $a(z) \neq 0, \infty$) has infinitely many solutions provided $N(r,0,f) = S(r,f)$.
THEOREM 1.2: Let $f$ be a transcendental meromorphic function with finite order and as in definition 2, $P[f]$ be a differential-difference polynomial of degree $d$ defined as
\[ P[f] = \sum_{j \in \Delta} a_j(z) M_j[f]; \quad T(r, P[f]) \neq S(r, f), \]
where $\Delta$ is a finite set of multi-indices, $a_j(z)$ are small functions of $f$, $M_j[f]$ are differential-difference monomials, then $P[f] = a(z)$ ($a(z) =$ small function or complex value, $a(z) \neq 0, \infty$) has infinitely many solutions provided $N(r,0,f) + N(r,f) = S(r,f)$.

The classical problem of value distributions of differential polynomials is Hayman conjecture [3], i.e. if $f$ is a transcendental meromorphic function and $n \in \mathbb{N}$, then $f^n f' \text{takes every finite non-zero value infinitely often which means that the Picard exceptional value of } f^n f' \text{ may only be zero. This conjecture has been proved by many authors. e.g., Hayman [3] proved that if } f \text{ is a transcendental meromorphic function and } n \geq 3, \text{then } f^n f' \text{takes every finite non-zero complex value infinitely often. The case } n=2 \text{ was proved by Mues[4], and Bergweiler et. al[5] proved the case for } n=1.$

Then many authors started to investigate the uniqueness of meromorphic functions sharing values with their shifts/ q-shifts or difference operators see e.g. ([6]-[9]).

We shall prove the above conjecture for general differential difference polynomials with some condition on power of $f$.

THEOREM 1.3: Let $f$ be a transcendental meromorphic function with finite order and as in definition 2, $P[f]$ be a differential-difference polynomial of degree $d$ and weight $\Gamma$ defined as
\[ P[f] = P[z,f] = \sum_{j \in \Delta} a_j(z) M_j[f] \]
\[ T(r, P[f]) \neq S(r, f), \]
where $\Delta$ is a finite set of multi-indices, $a_j(z)$ are small functions of $f$, $M_j[f]$ are differential-difference monomials, then $f^l (f - 1) P[f] - a(z), a(z) \neq 0, \infty$ has infinitely many zeros provided $l > 2(\Gamma + 1)$.

For the proof of the results we need the following lemmas:

**Lemma 1** [7]: Let $f$ be a non-constant meromorphic function of finite order and $c$ be a non-zero complex constant, then
\[ m(r, \frac{f(z+c)}{f(z)}) = S(r, f), \]
for all $r$ outside a possible exceptional set of finite logarithmic measure.

**Lemma 2** [7]: Let $c$ be a non-zero complex constant, and let $f$ be a meromorphic function of finite order then

$$T(r, f(z+c)) = T(r, f) + S(r, f)$$

$$N(r, f(z+c)) = N(r, f) + S(r, f)$$

$$N(r, 0, f(z+c)) = N(r, 0, f) + S(r, f)$$

**Lemma 3** (Clunie type lemma [6]): Let $f(z)$ be a non-constant meromorphic function of finite order such that

$$f^n P[z, f] = Q[z, f],$$

where $P[z, f], Q[z, f]$ are differential-difference polynomials in $f$. If the degree of $Q[z, f]$ as a polynomial in $f$ and its shifts is at most $n$, then

$$m(r, P[z, f]) = S(r, f).$$

**Lemma 4** ([11]): Let $f$ be a non-constant meromorphic function. If $Q[f]$ is a differential polynomial in $f$ with arbitrary meromorphic coefficients, then

(i) $m(r, Q[f]) \leq \gamma_Q m(r, f) + S(r, f)$

(ii) $N(r, Q[f]) \leq \Gamma_Q N(r, f) + S(r, f)$

Remark: We can obtain Lemma 4 for differential difference polynomials in $f$ of finite order using lemma 2 and definition of weight as in definition 2.

**Proof of Theorem 1.1:**

Let $P[f] = a(z), a(z) \neq 0, \infty$ has finitely many solutions.

Then we get by using Lemma 2, Lemma 3 and given condition:

$$T(r, P[f]) = T(r, \sum_{j \in \Delta} a_j(z) M_j[f])$$

$$= T(r, f^d [a_1 M_1 + a_2 M_2 + \ldots + a_n M_n])$$

$$\geq d T(r, f) - T(r, [a_1 M_1 + a_2 M_2 + \ldots + a_n M_n])$$

But $T(r, [a_1 M_1 + a_2 M_2 + \ldots + a_n M_n])$

$$\leq T(r, M_1) + T(r, M_2) + \ldots + T(r, M_n) + S(r, f)$$
\[
N(r, \frac{M_1}{f^d}) + N(r, \frac{M_2}{f^d}) + \ldots + N(r, \frac{M_n}{f^d}) + S(r, f)
= S(r, f)
\]

Therefore, we have
\[
T(r, P[f]) \geq d T(r, f) + S(r, f)
\]
Since \(f\) is entire, therefore, by using Nevanlinna’s second main theorem, we get
\[
d T(r, f) \leq T(r, P[f]) \leq N(r, \frac{1}{P[f]}) + \tilde{N}(r, P[f]) + S(r, f)
= S(r, f)
\]
which is a contradiction as \(d \geq 1\). Thus our supposition is wrong and hence, \(P[f] = a(z)\) (small function or complex value), \(a(z) \neq 0, \infty\) has infinitely many solutions.

**PROOF OF THEOREM 1.2:**

Let \(P[f] = a(z)\) (small function or complex value), \(a(z) \neq 0, \infty\) has finitely many solutions.

Then we get by using Lemma 2, Lemma 3 and given condition:
\[
T(r, P[f]) = T(r, \sum_{j \in \Delta} a_j(z).M_j[f])
\]
\[
= T(r, P'[a_1 \frac{M_1}{f^d} + a_2 \frac{M_2}{f^d} + \ldots + a_n \frac{M_n}{f^d}])
\]
\[
\geq d T(r, f) - T(r, [a_1 \frac{M_1}{f^d} + a_2 \frac{M_2}{f^d} + \ldots + a_n \frac{M_n}{f^d}])
\]
But \(T(r, [a_1 \frac{M_1}{f^d} + a_2 \frac{M_2}{f^d} + \ldots + a_n \frac{M_n}{f^d}])\)
\[
\leq T(r, \frac{M_1}{f^d}) + T(r, \frac{M_2}{f^d}) + \ldots + T(r, \frac{M_n}{f^d}) + S(r, f)
= N(r, \frac{M_1}{f^d}) + N(r, \frac{M_2}{f^d}) + \ldots + N(r, \frac{M_n}{f^d}) + S(r, f)
= S(r, f)
\]
Therefore, we have
\[
T(r, P[f]) \geq d T(r, f) + S(r, f)
\]
Since $f$ is meromorphic and by using $N(r,0,f) + N(r,f) = S(r,f)$ and Nevanlinna’s second main theorem, we get

$$d \ T(r, f) \leq \ T(r, P[f]) \leq \ \bar{N}(r, \frac{1}{P[f]}) + \bar{N}(r, P[f]) + \bar{N}(r, \frac{1}{P[f] - a(z)}) + S(r, f) = S(r, f)$$

which is a contradiction as $d \geq 1$. Thus our supposition is wrong and hence, $P[f] = a(z)$ (small function or complex value), $a(z) \neq 0, \infty$ has infinitely many solutions.

**PROOF OF THEOREM 1.3:**

Let $G[z] = f^{l}(f - 1)P[z, f]$ where $f$ is a meromorphic function and suppose $G[z]$, $a(z)$, $a(z) \neq 0, \infty$ has finitely many zeros. Then we get by using Lemma 4 for differential difference polynomials

$$T(r, G[z]) = T(r, f^{l}(f - 1) \sum_{j \in \Delta} a_j(z)M_j[f]) \geq (l + 1 - \Gamma)T(r, f) - \Gamma T(r, f)$$

Therefore, we have

$$T(r, G[z]) \geq (1 + 1 - \Gamma)T(r, f) + S(r, f)$$

Since $f$ is meromorphic, therefore, by using Nevanlinna’s second main theorem and lemma 4 for differential difference polynomials, we get

$$(1 + 1 - \Gamma)T(r, f) \leq T(r, G[z]) \leq \bar{N}(r, \frac{1}{G(z)}) + \bar{N}(r, G(z)) + \bar{N}(r, \frac{1}{G(z) - a(z)}) + S(r, G)$$

$$= \bar{N}(r, \frac{1}{G(z)}) + \bar{N}(r, G(z)) + S(r, f) \leq (\Gamma + 2)T(r, f) + S(r, f) = (\Gamma + 3)T(r, f) + S(r, f)$$

So we get

$$l \ T(r, f) \leq 2(\Gamma + 1)T(r, f) + S(r, f)$$

which is a contradiction as $l > 2(\Gamma + 1)$. Thus our supposition is wrong and hence the result.
SECTION 2
Hayman\[3\] proved that a differential polynomial \( f^n + af' - b \) with constant coefficients \( a, b \) admits infinitely many zeros provided that \( f \) is transcendental entire and \( n \geq 3 \). K. Liu and I. Laine in 2010 proved that:

THEOREM B[12]: Let \( f \) be a transcendental entire function of finite order not of period \( c \), then for small non-zero function \( s(z) \), the difference polynomial \( f^n + f(z+c) - f(z) - s(z) \) has infinitely many zeros in the complex plane provided that \( n \geq 3 \).

We prove the above results for general difference polynomials (shifts and difference operators as in definition 1 are part of these) as following:

THEOREM 2.1: Let \( f \) be a transcendental entire function with finite order and as in definition 2, \( P[f] \) be a linear difference polynomial defined as
\[
P[f] = c_0 f(z) + c_1 f(z + c) + c_2 f(z + 2c) + ... + c_n f(z + nc); T(r, P[f]) \neq S(r, f),
\]
where \( c \neq 0 \) and \( c_j, j = 0,1,...,n, \) are complex constants then \( f^l + P[f] - a(z), a(z) \neq 0, \) \( \infty \) has infinitely many zeros provided \( l > 2n + 1 \).

THEOREM 2.2: Let \( f \) be a transcendental meromorphic function with finite order and as in definition 2, \( P[f] \) be a linear difference polynomial defined as
\[
P[f] = c_0 f(z) + c_1 f(z + c) + c_2 f(z + 2c) + ... + c_n f(z + nc); T(r, P[f]) \neq S(r, f),
\]
where \( c \neq 0 \) and \( c_j, j = 0,1,...,n, \) are complex constants then \( f^l + P[f] - a(z), a(z) \neq 0, \) \( \infty \) has infinitely many zeros provided \( l > 3n + 3 \).

PROOF OF THEOREM 2.1:

Let \( G[z] = f^l + P[f] \) where \( f \) is an entire function and suppose \( G[z] - a(z), a(z) \neq 0, \) \( \infty \) has finitely many zeros. Then we get by using Lemma 1 and Lemma 2
\[
T(r, G[z]) \geq (l+1) T(r, f) - T(r, [ c_1 f(z + c) + c_2 f(z + 2c) + ... + c_n f(z + nc)]) \]
\[
\geq (l-n+1) T(r, f) - S(r, f) ...(1)
\]

Therefore, we have
\[
T(r, G[z]) \geq (l-n+1) T(r, f) + S(r, f) ... (1)
\]

Since \( f \) is entire, therefore, by using Nevanlinna’s second main theorem , we get
\[
[1 + 1 - n] T(r, f) \leq T(r, G[z]) \leq \tilde{N}(r, \frac{1}{\Theta(z)}) + \tilde{N}(r, \frac{1}{\Theta'(z)}) + S(r, G)
\]
\[
= \tilde{N}(r, \frac{1}{\Theta(z)}) + S(r, f)
\]
\[(n + 2) N(r, \theta, f) + S(r, f) \leq \]
\[(n + 2) T(r, f) + S(r, f) \]

So we get

\[1 T(r, f) \leq (2n + 1) T(r, f) + S(r, f) \]

which is a contradiction as \( l > 2n + 1 \). Thus our supposition is wrong and hence, \( f^l a(z) - a(z), a(z) \neq 0, \infty \) has infinitely many zeros.

**PROOF OF THEOREM 2.2:**

Let \( G[z] = f^l + P[f] \) where \( f \) is a meromorphic function and suppose \( G[z] - a(z), a(z) \neq 0, \infty \) has finitely many zeros. Then we have

\[ T(r, G[z]) = T(r, f^l + [c_0 f(z) + c_1 f(z + c) + c_2 f(z + 2c) + \ldots + c_n f(z + nc)]) \]

\[ \geq (l + 1) T(r, f) - T(r, [c_1 f(z + c) + c_2 f(z + 2c) + \ldots + c_n f(z + nc)]) \]

Therefore, we have

\[ T(r, G[z]) \geq (l - n + 1) T(r, f) + S(r, f) \]

Since \( f \) is meromorphic, therefore, by using Nevanlinna’s second main theorem and lemma, we get

\[ \left[(l - n + 1) T(r, f) + S(r, f) \right] \leq T(r, G[z]) \leq \tilde{N}(r, \frac{1}{\theta(z)}) + \tilde{N}(r, G(z)) + \tilde{N}(r, \frac{1}{\theta(z) - a(z)}) \]

\[ = \tilde{N}(r, \frac{1}{\theta(z)}) + \tilde{N}(r, G(z)) + S(r, f) \]

\[ \leq (2n + 4) T(r, f) + S(r, f) \]

So we get

\[ 1 T(r, f) \leq (3n + 3) T(r, f) + S(r, f) \]

which is a contradiction as \( l > 3n + 3 \). Thus our supposition is wrong and hence, \( f^l + P[f] - a(z), a(z) \neq 0, \infty \) has infinitely many zeros.

**REFERENCES:**


