REPRESENTING SUMS OF FINITE PRODUCTS OF CHEBYSHEV POLYNOMIALS OF THE FIRST KIND AND LUCAS POLYNOMIALS BY CHEBYSHEV POLYNOMIALS

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Abstract. In this paper, we study sums of finite products of Chebyshev polynomials of the first kind and Lucas polynomials and represent each of them in terms of Chebyshev polynomials of all kinds. Here the coefficients involve terminating hypergeometric functions ${}_2F_1$ and these representations are obtained by explicit computations.

1. Introduction and preliminaries

In this section, after fixing some notations, we will recall some basic facts that are needed in this paper and state our results.

For any nonnegative integer n, the falling factorial polynomials $(x)_n$ and the rising factorial polynomials $<x>_n$ are respectively given by

$$(x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1),\ (x)_0 = 1, \tag{1.1}$$

$$<x>_n = x(x+1)\cdots(x+n-1),\ (n \geq 1),\ <x>_0 = 1. \tag{1.2}$$

The two factorial polynomials are related by

$$(-1)^n (x)_n = <-x>_n,\ (-1)^n <x>_n = (-x)_n. \tag{1.3}$$

The hypergeometric function ${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x)$ is defined by

$$\begin{aligned} &{}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x) \\ &= \sum_{n=0}^{\infty} \frac{<a_1>_n \cdots <a_p>_n}{<b_1>_n \cdots <b_q>_n} \frac{x^n}{n!}. \end{aligned} \tag{1.4}$$

In below, we are going to recall some very basic facts about Chebyshev polynomials of the first, second, third and fourth kinds, and Lucas polynomials. The Chebyshev polynomials belong to the family of classical orthogonal polynomials. For full accounts of this fascinating area of mathematics, we let the reader refer to [2,3,17].

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In terms of generating functions, the Lucas polynomials and Chebyshev polynomials of the first, second, third and fourth kinds are respectively given by

$$F(t,x)=\frac{2-xt}{1-xt-t^2}=\sum_{n=0}^{\infty}L_n(x)t^n, \tag{1.5}$$

$$G(t,x)=\frac{1-xt}{1-2xt+t^2}=\sum_{n=0}^{\infty}T_n(x)t^n, \tag{1.6}$$

$$\frac{1}{1-2xt+t^2}=\sum_{n=0}^{\infty}U_n(x)t^n, \tag{1.7}$$

$$\frac{1-t}{1-2xt+t^2}=\sum_{n=0}^{\infty}V_n(x)t^n, \tag{1.8}$$

$$\frac{1+t}{1-2xt+t^2}=\sum_{n=0}^{\infty}W_n(x)t^n. \tag{1.9}$$

They are also given by the following explicit expressions:

$$L_n(x)=n\sum_{l=0}^{[\frac{n}{2}]}\frac{1}{n-l}\binom{n-l}{l}x^{n-2l},\ (n\ge 1), \tag{1.10}$$

$$\begin{aligned}T_n(x)&={}_2F_1(-n,n;\tfrac{1}{2};\tfrac{1-x}{2})\\&=\frac{n}{2}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\frac{1}{n-l}\binom{n-l}{l}(2x)^{n-2l},\ (n\ge 1),\end{aligned} \tag{1.11}$$

$$\begin{aligned}U_n(x)&=(n+1){}_2F_1(-n,n+2;\tfrac{3}{2};\tfrac{1-x}{2})\\&=\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{n-l}{l}(2x)^{n-2l},\ (n\ge 0),\end{aligned} \tag{1.12}$$

$$\begin{aligned}V_n(x)&={}_2F_1(-n,n+1;\tfrac{1}{2};\tfrac{1-x}{2})\\&=\sum_{l=0}^{n}\binom{2n-l}{l}2^{n-l}(x-1)^{n-l},\ (n\ge 0),\end{aligned} \tag{1.13}$$

$$\begin{aligned}W_n(x)&=(2n+1){}_2F_1(-n,n+1;\tfrac{3}{2};\tfrac{1-x}{2})\\&=(2n+1)\sum_{l=0}^{n}\frac{2^{n-l}}{2n-2l+1}\binom{2n-l}{l}(x-1)^{n-l},\ (n\ge 0).\end{aligned} \tag{1.14}$$

The Chebyshev polynomials of the first, second, third and fourth kinds are given by Rodrigues' formulas.

$$T_n(x)=\frac{(-1)^n2^nn!}{(2n)!}(1-x^2)^{\frac{1}{2}}\frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}, \tag{1.15}$$

$$U_n(x)=\frac{(-1)^n2^n(n+1)!}{(2n+1)!}(1-x^2)^{-\frac{1}{2}}\frac{d^n}{dx^n}(1-x^2)^{n+\frac{1}{2}}, \tag{1.16}$$

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$$(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}V_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}\frac{d^n}{dx^n}(1-x)^{n-\frac{1}{2}}(1+x)^{n+\frac{1}{2}}, \tag{1.17}$$

$$(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}W_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}\frac{d^n}{dx^n}(1-x)^{n+\frac{1}{2}}(1+x)^{n-\frac{1}{2}}. \tag{1.18}$$

As is well known, the Chebyshev polynomials satisfy orthogonalities with respect to various weight functions as in the following.

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_n(x)T_m(x)\,dx = \frac{\pi}{\mathcal{E}_n}\delta_{n,m}, \tag{1.19}$$

$$where\ \delta_{n,m} = \begin{cases} 0, \text{if } n \neq m, \\ 1, \text{if } n = m, \end{cases} \quad \mathcal{E}_n = \begin{cases} 1, \text{if } n = 0, \\ 2, \text{if } n \geq 1, \end{cases} \tag{1.20}$$

$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}U_n(x)U_m(x)\,dx = \frac{\pi}{2}\delta_{n,m}, \tag{1.21}$$

$$\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}V_n(x)V_m\,dx = \pi\delta_{n,m}, \tag{1.22}$$

$$\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}W_n(x)W_m\,dx = \pi\delta_{n,m}. \tag{1.23}$$

For convenience, let us put

$$\begin{aligned} \alpha_{m,r}(x) = &\sum_{l=0}^{m}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l}\binom{r+l}{r}x^l T_{i_1}(x)T_{i_2}(x)\cdots T_{i_{r+1}}(x) \\ &-\sum_{l=0}^{m-2}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2}\binom{r+l}{r}x^l T_{i_1}(x)T_{i_2}(x)\cdots T_{i_{r+1}}(x), \\ &(m \geq 2, r \geq 1), \end{aligned} \tag{1.24}$$

$$\begin{aligned} \beta_{m,r}(x) = &\sum_{l=0}^{m}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l}\binom{r+l}{r}\left(\frac{x}{2}\right)^l L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x) \\ &+\sum_{l=0}^{m-2}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2}\binom{r+l}{r}\left(\frac{x}{2}\right)^l L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x), \\ &(m \geq 2, r \geq 1). \end{aligned} \tag{1.25}$$

We observe here that both $\alpha_{m,r}(x)$ and $\beta_{m,r}(x)$ are polynomials of degree m.

Here we are going to investigate the sums of finite products of Chebyshev polynomials of the first kind in (1.24) and those of Lucas polynomials in (1.25). Then we will represent $\alpha_{m,r}(x)$ and $\beta_{m,r}(x)$ in terms of Chebyshev polynomials of the four kinds $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$. These will be done by explicit computations, using the general formulas in **Proposition 2.1** and **Proposition 2.2**. We note here that the results in **Proposition 2.1** can be derived by making use of orthogonalities, Rodrigues' formulas and integration by parts.

The next two theorems are the main results of this paper.

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Theorem 1.1. *Let m, r be any integers with $m \geq 2, r \geq 1$. Then we have the following.*

$$\begin{aligned}
&\sum_{l=0}^{m} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l} \binom{r+l}{r} x^l T_{i_1}(x) T_{i_2}(x) \cdots T_{i_{r+1}}(x) \\
&- \sum_{l=0}^{m-2} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2} \binom{r+l}{r} x^l T_{i_1}(x) T_{i_2}(x) \cdots T_{i_{r+1}}(x) \\
&= \binom{m+r}{r} \sum_{j=0}^{[\frac{m}{2}]} \binom{m}{j} \mathcal{E}_{m-2j} {}_2F_1(-j, j-m; 1-m-r; 1) T_{m-2j}(x) \qquad (1.26) \\
&= \frac{(m+r)!}{(m+1)!} \sum_{j=0}^{[\frac{m}{2}]} \binom{m+1}{j} (m-2j+1) {}_2F_1(-j, j-m-1; 1-m-r; 1) U_{m-2j}(x) \qquad (1.27) \\
&= \binom{m+r}{r} \sum_{j=0}^{m} \binom{m}{[\frac{j}{2}]} {}_2F_1(-[\frac{j}{2}], [\frac{j}{2}]-m; 1-m-r; 1) V_{m-j}(x) \qquad (1.28) \\
&= \binom{m+r}{r} \sum_{j=0}^{m} (-1)^j \binom{m}{[\frac{j}{2}]} {}_2F_1(-[\frac{j}{2}], [\frac{j}{2}]-m; 1-m-r; 1) W_{m-j}(x). \qquad (1.29)
\end{aligned}$$

Here $[x]$ denotes the greatest integer $\leq x$.

Theorem 1.2. *Let m, r be integers with $m \geq 2$, $r \geq 1$. Then we have the following identities.*

$$\begin{aligned}
&\sum_{l=0}^{m} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l} \binom{r+l}{r} (\frac{x}{2})^l L_{i_1}(x) L_{i_2}(x) \cdots L_{i_{r+1}}(x) \\
&+ \sum_{l=0}^{m-2} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2} \binom{r+l}{r} (\frac{x}{2})^l L_{i_1}(x) L_{i_2}(x) \cdots L_{i_{r+1}}(x) \\
&= 2^{r+1-m} \binom{m+r}{r} \sum_{j=0}^{[\frac{m}{2}]} \binom{m}{j} \mathcal{E}_{m-2j} \\
&\quad \times {}_2F_1(-j, j-m; 1-m-r; -4) T_{m-2j}(x) \qquad (1.30) \\
&= \frac{2^{r+1-m}}{r} \binom{m+r}{r-1} \sum_{j=0}^{[\frac{m}{2}]} (m-2j+1) \binom{m+1}{j} \\
&\quad \times {}_2F_1(-j, j-m-1; 1-m-r; -4) U_{m-2j}(x) \qquad (1.31) \\
&= 2^{r+1-m} \binom{m+r}{r} \sum_{j=0}^{m} \binom{m}{[\frac{j}{2}]} \\
&\quad \times {}_2F_1(-[\frac{j}{2}], [\frac{j}{2}]-m; 1-m-r; -4) V_{m-j}(x) \qquad (1.32)
\end{aligned}$$

$$= 2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{m}(-1)^j\binom{m}{[\frac{j}{2}]}$$
$$\times {}_2F_1(-[\frac{j}{2}],[\frac{j}{2}]-m;1-m-r;-4)W_{m-j}(x). \tag{1.33}$$

Along the same line as the present paper, sums of finite products of several special polynomials, namely Chebyshev polynomials of the second, third and fourth kinds and Fibonacci, Legendre, Laguerre polynomials had been represented by Chebyshev polynomials of all kinds (see [7,11,14]). For sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials, they are also represented by Hermite, generalized Laguerre, Legendre, Gegenbauer and Jacobi polynomials in [15].

In [1,12], sums of finite products of Bernoulli and Euler polynomials were studied. Consequently, those sums of finite products of such polynomials are expressed as linear combinations of Bernoulli polynomials. These were done by deriving Fourier series expansions for functions closely related to those sums of finite products. The same had been done also for quite a few non-Appell polynomials in [8-10,13], namely Chebyshev polynomials of the first, second, third and fourth kinds, and Legendre, Laguerre, Fibonacci and Lucas polynomials. For related papers on Chebyshev polynomials, we let the reader refer to [4,16].

2. Proof of Theorem 1.1

Here we will prove only (1.26) and (1.28) in **Theorem 1.1**, leaving (1.27) and (1.29) as an exercise for the reader. For this purpose, we first state two results that are needed in showing **Theorem 1.1** and **Theorem 1.2**.

The formulas (a) and (b) in **Proposition 2.1** are respectively from the equations (24) and (36) of [6], while (c) and (d) are respectively from (23) and (38) of [5]. All of them follow easily from the Rodrigues' formulas (1.15) - (1.18), and the orthogonalities in (1.19) and (1.21) - (1.23).

Proposition 2.1. Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then we have the following.

$$(a) \quad q(x) = \sum_{k=0}^{n} C_{k,1}T_k(x), \text{ where}$$
$$C_{k,1} = \frac{(-1)^k 2^k k! \mathcal{E}_k}{(2k)!\pi}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}}dx,$$
$$(b) \quad q(x) = \sum_{k=0}^{n} C_{k,2}U_k(x), \text{ where}$$
$$C_{k,2} = \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi}\int_{-1}^{1} q(x)\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}dx,$$

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$$(c)\quad q(x)=\sum_{k=0}^{n}C_{k,3}V_k(x),\ \text{where}$$
$$C_{k,3}=\frac{(-1)^k k!2^k}{(2k)!\pi}\int_{-1}^{1}q(x)\frac{d^k}{dx^k}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}dx,$$
$$(d)\quad q(x)=\sum_{k=0}^{n}C_{k,4}W_k(x),\ \text{where}$$
$$C_{k,4}=\frac{(-1)^k k!2^k}{(2k)!\pi}\int_{-1}^{1}q(x)\frac{d^k}{dx^k}(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}}dx.$$

The next **Proposition** is stated and proved in [7].

Proposition 2.2. Let m,k be nonnegative integers. Then we have the following.

$$(a)\quad \int_{-1}^{1}(1-x^2)^{k-\frac{1}{2}}x^m dx=\begin{cases}0, & \text{if } m\equiv 1 \pmod 2,\\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m\equiv 0 \pmod 2,\end{cases}$$
$$(b)\quad \int_{-1}^{1}(1-x^2)^{k+\frac{1}{2}}x^m dx=\begin{cases}0, & \text{if } m\equiv 1 \pmod 2,\\ \frac{m!(2k+2)!\pi}{2^{m+2k+2}(\frac{m}{2}+k+1)!(\frac{m}{2})!(k+1)!}, & \text{if } m\equiv 0 \pmod 2,\end{cases}$$

$$(c)\quad \int_{-1}^{1}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}x^m dx$$
$$=\begin{cases}\frac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & \text{if } m\equiv 1 \pmod 2,\\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m\equiv 0 \pmod 2,\end{cases}$$
$$(d)\quad \int_{-1}^{1}(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}}x^m dx$$
$$=\begin{cases}-\frac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & \text{if } m\equiv 1 \pmod 2,\\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m\equiv 0 \pmod 2.\end{cases}$$

The following lemma was stated and proved in [13].

Lemma 2.3. Let m,r be integers with $m\geq 2, r\geq 1$. Then we have the following identity.

$$\begin{aligned}&\sum_{l=0}^{m}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l}\binom{r+l}{r}x^l T_{i_1}(x)T_{i_2}(x)\cdots T_{i_{r+1}}(x)\\ &-\sum_{l=0}^{m-2}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2}\binom{r+l}{r}x^l T_{i_1}(x)T_{i_2}(x)\cdots T_{i_{r+1}}(x)\\ &=\frac{1}{2^{r-1}r!}T_{m+r}^{(r)}(x),\end{aligned}\tag{2.1}$$

where the first and second inner sums on the left hand side are respectively over all nonnegative integers $i_1,i_2,\cdots i_{r+1}$, with $i_1+i_2+\cdots+i_{r+1}=m-l$ and $i_1+i_2+\cdots+i_{r+1}=m-l-2$.

From (1.11), the rth derivative of $T_n(x)$ is given by

$$T_n^{(r)}(x) = \frac{n}{2}\sum_{l=0}^{[\frac{n-r}{2}]}(-1)^l\frac{1}{n-l}\binom{n-l}{l}2^{n-2l}(n-2l)_r x^{n-2l-r}. \tag{2.2}$$

Thus, in particular, we have

$$\begin{aligned}&T_{m+r}^{(r+k)}(x)\\&=\frac{m+r}{2}\sum_{l=0}^{[\frac{m-k}{2}]}(-1)^l\frac{1}{m+r-l}\binom{m+r-l}{l}2^{m+r-2l}(m+r-2l)_{r+k}x^{m-k-2l}.\end{aligned} \tag{2.3}$$

With $\alpha_{m,r}(x)$ as in (1.24), we let

$$\alpha_{m,r}(x) = \sum_{k=0}^{m} C_{k,1}T_k(x). \tag{2.4}$$

Then, from (a) of **Proposition 2.1**, (2.1), (2.3), and integration by parts k times, we have

$$\begin{aligned}C_{k,1} &= \frac{(-1)^k 2^k k!\mathcal{E}_k}{(2k)!\pi}\int_{-1}^{1}\alpha_{m,r}(x)\frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}}dx\\&=\frac{(-1)^k 2^k k!\mathcal{E}_k}{(2k)!\pi 2^{r-1}r!}\int_{-1}^{1}T_{m+r}^{(r)}(x)\frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}}dx\\&=\frac{2^k k!\mathcal{E}_k}{(2k)!\pi 2^{r-1}r!}\int_{-1}^{1}T_{m+r}^{(r+k)}(x)(1-x^2)^{k-\frac{1}{2}}dx\\&=\frac{2^k k!\mathcal{E}_k}{(2k)!\pi 2^{r-1}r!}\frac{m+r}{2}\sum_{l=0}^{[\frac{m-k}{2}]}(-1)^l\frac{1}{m+r-l}\binom{m+r-l}{l}\\&\quad\times 2^{m+r-2l}(m+r-2l)_{r+k}\int_{-1}^{1}x^{m-k-2l}(1-x^2)^{k-\frac{1}{2}}dx.\end{aligned} \tag{2.5}$$

We note from (a) in **Proposition 2.2** that

$$\begin{aligned}&\int_{-1}^{1}x^{m-k-2l}(1-x^2)^{k-\frac{1}{2}}dx\\&=\begin{cases}0, & \text{if } k\not\equiv m \pmod 2,\\ \frac{(m-k-2l)!(2k)!\pi}{2^{m+k-2l}(\frac{m+k}{2}-l)!(\frac{m-k}{2}-l)!k!}, & \text{if } k\equiv m \pmod 2,\end{cases}\end{aligned} \tag{2.6}$$

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From (2.4) - (2.6), and after some simplifications, we get

$$\begin{aligned}\alpha_{m,r}(x) &= \sum_{\substack{0\le k\le m\\ k\equiv m \pmod 2}} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{\mathcal{E}_k(m+r)(-1)^l(m+r-l)!}{r!(m+r-l)l!(\frac{m+k}{2}-l)!(\frac{m-k}{2}-l)!}T_k(x)\\ &= \sum_{j=0}^{[\frac{m}{2}]} \frac{T_{m-2j}(x)\mathcal{E}_{m-2j}(m+r)}{r!} \sum_{l=0}^{j} \frac{(-1)^l(m+r-1-l)!}{l!(m-l-j)!(j-l)!}\\ &= \sum_{j=0}^{[\frac{m}{2}]} \frac{T_{m-2j}(x)\mathcal{E}_{m-2j}(m+r)!}{r!(m-j)!j!} \sum_{l=0}^{j} \frac{<-j>_l<j-m>_l}{l!<1-m-r>_l}\\ &= \binom{m+r}{r}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{j}\mathcal{E}_{m-2j}\,{}_2F_1(-j,j-m;1-m-r;1)T_{m-2j}(x).\end{aligned} \tag{2.7}$$

This completes the proof for (1.26) in **Theorem 1.1**.

Next, we let

$$\alpha_{m,r}(x) = \sum_{k=0}^{m} C_{k,3}V_k(x). \tag{2.8}$$

Then, from (c) of **Proposition 2.1**, (2.1), (2.3) and integration by parts k times , we obtain

$$\begin{aligned}C_{k,3} = &\frac{k!2^k}{(2k)!\pi 2^{r-1}r!}\frac{m+r}{2}\sum_{l=0}^{[\frac{m-k}{2}]}(-1)^l\frac{1}{m+r-l}\binom{m+r-l}{l}\\ &\times 2^{m+r-2l}(m+r-2l)_{r+k}\int_{-1}^{1}x^{m-k-2l}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}dx.\end{aligned} \tag{2.9}$$

From (c) of **Proposition 2.2**, we observe that

$$\begin{aligned}&\int_{-1}^{1}x^{m-k-2l}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}dx\\ &=\begin{cases}\frac{(m-k-2l+1)!(2k)!\pi}{2^{m+k-2l+1}(\frac{m+k+1}{2}-l)!(\frac{m-k+1}{2}-l)!k!}, & \text{if },k\not\equiv m \pmod 2),\\ \frac{(m-k-2l)!(2k)!\pi}{2^{m+k-2l}(\frac{m+k}{2}-l)!(\frac{m-k}{2}-l)!k!}, & \text{if } k\equiv m \pmod 2).\end{cases}\end{aligned} \tag{2.10}$$

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By (2.8)-(2.10), and after some simplifications, we get

$$\begin{aligned}\alpha_{m,r}(x) &= \frac{(m+r)}{2r!}\sum_{\substack{0\le k\le m\\ k\not\equiv m \pmod 2}}\sum_{l=0}^{[\frac{m-k}{2}]} V_k(x)\frac{(-1)^l(m+r-1-l)!(m-k-2l+1)}{l!(\frac{m+k+1}{2}-l)!(\frac{m-k+1}{2}-l)!}\\ &+\frac{(m+r)}{r!}\sum_{\substack{0\le k\le m\\ k\equiv m \pmod 2}}\sum_{l=0}^{[\frac{m-k}{2}]} V_k(x)\frac{(-1)^l(m+r-1-l)!}{l!(\frac{m+k}{2}-l)!(\frac{m-k}{2}-l)!}\\ &=\frac{(m+r)}{r!}\sum_{j=0}^{[\frac{m-1}{2}]}\sum_{l=0}^{j} V_{m-2j-1}(x)\frac{(-1)^l(m+r-1-l)!}{l!(m-j-l)!(j-l)!}\\ &+\frac{(m+r)}{r!}\sum_{j=0}^{[\frac{m}{2}]}\sum_{l=0}^{j} V_{m-2j}(x)\frac{(-1)^l(m+r-1-l)!}{l!(m-j-l)!(j-l)!}.\end{aligned} \tag{2.11}$$

Further modifications of (2.11) give us

$$\begin{aligned}\alpha_{m,r}(x) &= \frac{(m+r)}{r!}\sum_{j=0}^{[\frac{m-1}{2}]}\sum_{l=0}^{j}\frac{1}{(m-j)!j!}V_{m-2j-1}(x)\frac{<-j>_l<j-m>_l}{l!<1-m-r>_l}\\ &+\frac{(m+r)!}{r!}\sum_{j=0}^{[\frac{m}{2}]}\sum_{l=0}^{j}\frac{1}{(m-j)!j!}V_{m-2j}(x)\frac{<-j>_l<j-m>_l}{l!<1-m-r>_l}\\ &=\binom{m+r}{r}\sum_{j=0}^{[\frac{m-1}{2}]}\binom{m}{j}{}_2F_1(-j,j-m;1-m-r;1)V_{m-2j-1}(x)\\ &+\binom{m+r}{r}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{j}{}_2F_1(-j,j-m;1-m-r;1)V_{m-2j}(x)\\ &=\binom{m+r}{r}\sum_{j=0}^{m}\binom{m}{[\frac{j}{2}]}{}_2F_1(-[\frac{j}{2}],[\frac{j}{2}]-m;1-m-r;1)V_{m-j}(x).\end{aligned} \tag{2.12}$$

This finishes up the proof for (1.28) in **Theorem 1.1**.

3. Proof of Theorem 1.2

Here we will show only (1.31) and (1.33) in **Theorem 1.2**, leaving the proofs for (1.30) and (1.32) as an exercise to the reader. The following lemma is crucial to our discussion in this section. As it is stated in [13] but not proved, we are going to show this.

Lemma 3.1 Let m, r be integers with $m \ge 2, r \ge 1$. Then the following identity holds.

$$\begin{aligned}
&\sum_{l=0}^{m}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l}\binom{r+l}{r}(\frac{x}{2})^l L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)\\
&+\sum_{l=0}^{m-2}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2}\binom{r+l}{r}(\frac{x}{2})^l L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)\\
&=\frac{2^{r+1}}{r!}L_{m+r}^{(r)}(x),
\end{aligned} \tag{3.1}$$

where the first and second inner sums on the left hand side are respectively over all nonnegative integers $i_1, i_2, \cdots i_{r+1}$, with $i_1 + i_2 + \cdots + i_{r+1} = m - l$ and $i_1 + i_2 + \cdots + i_{r+1} = m - l - 2$.

Proof. By differentiating (1.5) r times, we have

$$\frac{\partial^r}{\partial x^r}F(t,x) = t^r(1+t^2)r!(1-xt-t^2)^{-(r+1)},\ (r \geq 1), \tag{3.2}$$

$$\frac{\partial^r}{\partial x^r}F(t,x) = \sum_{m=r}^{\infty} L_m^{(r)}(x)t^m = \sum_{m=0}^{\infty} L_{m+r}^{(r)}(x)t^{m+r}. \tag{3.3}$$

Equating (3.2) and (3.3), we obtain

$$\left(\frac{1}{1-xt-t^2}\right)^{r+1} = \frac{1}{r!(1+t^2)}\sum_{m=0}^{\infty} L_{m+r}^{(r)}(x)t^m. \tag{3.4}$$

On the other hand, from (1.5) and (3.4) we note that

$$\begin{aligned}
&\sum_{l=0}^{\infty}\sum_{i_1+i_2+\cdots+i_{r+1}=l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^l\\
=&\left(\sum_{l=0}^{\infty}L_l(x)t^l\right)^{r+1}\\
=&(2-xt)^{r+1}\left(\frac{1}{1-xt-t^2}\right)^{r+1}\\
=&(2-xt)^{r+1}(1+t^2)^{-1}\frac{1}{r!}\sum_{m=0}^{\infty}L_{m+r}^{(r)}(x)t^m.
\end{aligned} \tag{3.5}$$

Thus, from (3.5), we have

$$\begin{aligned}
&\frac{1}{r!}\sum_{m=0}^{\infty}L_{m+r}^{(r)}(x)t^m\\
&=2^{-(r+1)}(1+t^2)(1-\frac{xt}{2})^{-(r+1)}\sum_{l=0}^{\infty}\sum_{i_1+i_2+\cdots+i_{r+1}=l}L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^l\\
&=2^{-(r+1)}(1+t^2)\sum_{j=0}^{\infty}\binom{r+j}{r}(\frac{x}{2})^j t^j\sum_{l=0}^{\infty}\sum_{i_1+i_2+\cdots+i_{r+1}=l}L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^l.
\end{aligned} \tag{3.6}$$

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In turn, from (3.6), we get

$$\begin{aligned}
&\frac{2^{r+1}}{r!}\sum_{m=0}^{\infty} L_{m+r}^{(r)}(x)t^m \\
&= \Big(\sum_{j=0}^{\infty}\binom{r+j}{r}(\frac{x}{2})^j t^j + \sum_{j=2}^{\infty}\binom{r+j-2}{r}(\frac{x}{2})^{j-2}t^j\Big) \\
&\times \sum_{l=0}^{\infty}\sum_{i_1+i_2+\cdots+i_{r+1}=l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^l \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{r+m-l}{r}(\frac{x}{2})^{m-l}\sum_{i_1+i_2+\cdots+i_{r+1}=l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^m \\
&+ \sum_{m=2}^{\infty}\sum_{l=0}^{m-2}\binom{r+m-l-2}{r}(\frac{x}{2})^{m-l-2}\sum_{i_1+i_2+\cdots+i_{r+1}=l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^m \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{r+l}{r}(\frac{x}{2})^{l}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^m \\
&+ \sum_{m=2}^{\infty}\sum_{l=0}^{m-2}\binom{r+l}{r}(\frac{x}{2})^{l}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^m
\end{aligned} \tag{3.7}$$

Comparing the coefficients on both sides of (3.7), we get the desired result. □

From (1.10), we note that the rth derivative of $L_n(x)$ is given by

$$L_n^{(r)}(x) = n\sum_{l=0}^{[\frac{n-r}{2}]}\frac{1}{n-l}\binom{n-l}{l}(n-2l)_r x^{n-2l-r}. \tag{3.8}$$

In particular, we have

$$L_{m+r}^{(r+k)}(x) = (m+r)\sum_{l=0}^{[\frac{m-k}{2}]}\frac{1}{m+r-l}\binom{m+r-l}{l}(m+r-2l)_{r+k}x^{m-k-2l}. \tag{3.9}$$

With $\beta_{m,r}(x)$ as in (1.25), we put

$$\beta_{m,r}(x) = \sum_{k=0}^{m} C_{k,2}U_k(x). \tag{3.10}$$

Then, from (b) of **Proposition 2.2**, (3.1), (3.9), and integration by parts k times, we have

$$\begin{aligned}C_{k,2} &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi}\int_{-1}^{1}\beta_{m,r}(x)\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}dx\\ &= \frac{(-1)^k 2^{k+1}(k+1)!2^{r+1}}{(2k+1)!\pi r!}\int_{-1}^{1}L_{m+r}^{(r)}(x)\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}dx\\ &= \frac{2^{k+1}(k+1)!2^{r+1}}{(2k+1)!\pi r!}\int_{-1}^{1}L_{m+r}^{(r+k)}(x)(1-x^2)^{k+\frac{1}{2}}dx\\ &= \frac{2^{k+1}(k+1)!2^{r+1}(m+r)}{(2k+1)!\pi r!}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{1}{m+r-l}\binom{m+r-l}{l}(m+r-2l)_{r+k}\\ &\quad\times\int_{-1}^{1}x^{m-k-2l}(1-x^2)^{k+\frac{1}{2}}dx.\end{aligned} \tag{3.11}$$

We observe from (b) in **Proposition 2.2** that

$$\begin{aligned}&\int_{-1}^{1}x^{m-k-2l}(1-x^2)^{k+\frac{1}{2}}dx\\ &= \begin{cases}0, & \text{if } k \not\equiv m \pmod 2,\\ \frac{(m-k-2l)!(2k+2)!\pi}{2^{m+k-2l+2}(\frac{m+k}{2}-l+1)!(\frac{m-k}{2}-l)!(k+1)!}, & \text{if } k \equiv m \pmod 2,\end{cases}\end{aligned} \tag{3.12}$$

Now, from (3.10) - (3.12), and after some simplifications, we get

$$\begin{aligned}&\beta_{m,r}(x)\\ &= \frac{2^{r+1-m}(m+r)}{r!}\sum_{\substack{0\le k\le m\\ k\equiv m \pmod 2}}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(k+1)4^l(m+r-1-l)!}{l!(\frac{m+k}{2}-l+1)!(\frac{m-k}{2}-l)!}U_k(x)\\ &= \frac{2^{r+1-m}(m+r)}{r!}\sum_{j=0}^{[\frac{m}{2}]}\sum_{l=0}^{j}(m-2j+1)U_{m-2j}(x)\frac{4^l(m+r-1-l)!}{l!(m-j-l+1)!(j-l)!}\\ &= \frac{2^{r+1-m}(m+r)!}{r!}\sum_{j=0}^{[\frac{m}{2}]}\sum_{l=0}^{j}\frac{(m-2j+1)U_{m-2j}(x)}{(m-j+1)!j!}\frac{(-4)^l<-j>_l<j-m-1>_l}{l!<1-m-r>_l}\\ &= \frac{2^{r+1-m}}{r}\binom{m+r}{r-1}\sum_{j=0}^{[\frac{m}{2}]}(m-2j+1)\binom{m+1}{j}\\ &\times {}_2F_1(-j,j-m-1;1-m-r;-4)U_{m-2j}(x).\end{aligned} \tag{3.13}$$

This completes the proof for (1.31) in **Theorem 1.2**.

Next, we let

$$\beta_{m,r}(x) = \sum_{k=0}^{m} C_{k,4}W_k(x). \tag{3.14}$$

Then, from (d) of **Proposition 2.1**, (3.1), (3.9) and integration by parts k times, we have

$$\begin{aligned} C_{k,4} = & \frac{k!2^{k+r+1}(m+r)}{(2k)!\pi r!} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{1}{m+r-l}\binom{m+r-l}{l} \\ & \times (m+r-2l)_{r+k} \int_{-1}^{1} x^{m-k-2l}(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}}dx. \end{aligned} \tag{3.15}$$

From (d) of **Proposition 2.2**, we note that

$$\begin{aligned} & \int_{-1}^{1} x^{m-k-2l}(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}}dx \\ = & \begin{cases} -\frac{(m-k-2l+1)!(2k)!\pi}{2^{m+k-2l+1}(\frac{m+k+1}{2}-l)!(\frac{m-k+1}{2}-l)!k!}, & \text{if } , k \not\equiv m \pmod 2, \\ \frac{(m-k-2l)!(2k)!\pi}{2^{m+k-2l}(\frac{m+k}{2}-l)!(\frac{m-k}{2}-l)!k!}, & \text{if } k \equiv m \pmod 2. \end{cases} \end{aligned} \tag{3.16}$$

By (3.14)-(3.16), and after some simplifications, we obtain

$$\begin{aligned} & \beta_{m,r}(x) \\ = & -\frac{2^{r-m}(m+r)}{r!} \sum_{\substack{0 \le k \le m \\ k \not\equiv m \pmod 2}} \sum_{l=0}^{[\frac{m-k}{2}]} W_k(x) \frac{4^l(m+r-1-l)!(m-k-2l+1)}{l!(\frac{m+k+1}{2}-l)!(\frac{m-k+1}{2}-l)!} \\ & + \frac{2^{r+1-m}(m+r)}{r!} \sum_{\substack{0 \le k \le m \\ k \equiv m \pmod 2}} \sum_{l=0}^{[\frac{m-k}{2}]} W_k(x) \frac{4^l(m+r-1-l)!}{l!(\frac{m+k}{2}-l)!(\frac{m-k}{2}-l)!} \\ = & -\frac{2^{r-m}(m+r)}{r!} \sum_{j=0}^{[\frac{m-1}{2}]} \sum_{l=0}^{j} W_{m-2j-1}(x) \frac{4^l(m+r-1-l)!(2j-2l+2)}{l!(m-j-l)!(j-l+1)!} \\ & + \frac{2^{r+1-m}(m+r)}{r!} \sum_{j=0}^{[\frac{m}{2}]} \sum_{l=0}^{j} W_{m-2j}(x) \frac{4^l(m+r-1-l)!}{l!(m-j-l)!(j-l)!}. \end{aligned} \tag{3.17}$$

After further modifications of (3.17), we get

$$\begin{aligned} & \beta_{m,r}(x) \\ = & -\frac{2^{r+1-m}(m+r)!}{r!} \sum_{j=0}^{[\frac{m-1}{2}]} \frac{W_{m-2j-1}(x)}{(m-j)!j!} \sum_{l=0}^{j} \frac{(-4)^l <-j>_l <j-m>_l}{l! <1-m-r>_l} \\ & + \frac{2^{r+1-m}(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{W_{m-2j}(x)}{(m-j)!j!} \sum_{l=0}^{j} \frac{(-4)^l <-j>_l <j-m>_l}{l! <1-m-r>_l} \end{aligned}$$

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$$= -2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{[\frac{m-1}{2}]}\binom{m}{j}W_{m-2j-1}(x)_2F_1(-j,j-m;1-m-r;-4)$$

$$+ 2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{j}W_{m-2j}(x)_2F_1(-j,j-m;1-m-r;-4)$$

$$= 2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{m}(-1)^j\binom{m}{[\frac{j}{2}]}{}_2F_1(-[\frac{j}{2}],[\frac{j}{2}]-m;1-m-r;-4)W_{m-j}(x). \tag{3.18}$$

This finishes up the proof for (1.33) in **Theorem 1.2**.

4. Further remark

It is well known that the Lucas polynomials $L_n(x)$ and the Chebyshev polynomials of the first kind $T_n(x)$ are related by

$$L_n(x) = 2i^{-n}T_n(\frac{ix}{2}). \tag{4.1}$$

Then it is immediate to see from (1.24),(1.25) and (4.1) that the following identity holds.

$$2^{r+1}i^{-m}\alpha_{m,r}(\frac{ix}{2}) = \beta_{m,r}(x). \tag{4.2}$$

Now, the following **Theorem** follows from **Theorem 1.1**, **Theorem 1.2**, and (4.2).

Theorem 4.1. *Let m, r be integers with $m \geq 2, r \geq 1$. Then the following identities hold true.*

$$i^{-m}2^{r+1}\binom{m+r}{r}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{j}\mathcal{E}_{m-2j}{}_2F_1(-j,j-m;1-m-r;1)T_{m-2j}(\frac{ix}{2})$$

$$= i^{-m}2^{r+1}\frac{(m+r)!}{(m+1)!}\sum_{j=0}^{[\frac{m}{2}]}\binom{m+1}{j}(m-2j+1)$$

$$\times {}_2F_1(-j,j-m-1;1-m-r;1)U_{m-2j}(\frac{ix}{2})$$

$$= i^{-m}2^{r+1}\binom{m+r}{r}\sum_{j=0}^{m}\binom{m}{[\frac{j}{2}]}{}_2F_1(-[\frac{j}{2}],[\frac{j}{2}]-m;1-m-r;1)V_{m-j}(\frac{ix}{2})$$

$$= i^{-m}2^{r+1}\binom{m+r}{r}\sum_{j=0}^{m}(-1)^j\binom{m}{[\frac{j}{2}]}{}_2F_1(-[\frac{j}{2}],[\frac{j}{2}]-m;1-m-r;1)W_{m-j}(\frac{ix}{2})$$

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$$= 2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{j}\mathcal{E}_{m-2j}\,{}_2F_1(-j, j-m; 1-m-r; -4)T_{m-2j}(x)$$

$$= \frac{2^{r+1-m}}{r}\binom{m+r}{r-1}\sum_{j=0}^{[\frac{m}{2}]}(m-2j+1)\binom{m+1}{j}$$

$$\times\,{}_2F_1(-j, j-m-1; 1-m-r; -4)U_{m-2j}(x)$$

$$= 2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{m}\binom{m}{[\frac{j}{2}]}{}_2F_1(-[\frac{j}{2}], [\frac{j}{2}]-m; 1-m-r; -4)V_{m-j}(x)$$

$$= 2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{m}(-1)^j\binom{m}{[\frac{j}{2}]}{}_2F_1(-[\frac{j}{2}], [\frac{j}{2}]-m; 1-m-r; -4)W_{m-j}(x). \tag{4.3}$$

5. Conclusion

The classical connection problem concerns with determining the coefficients $c_{nm}(k)$ in the expansion of the product of two polynomials $a_n(x)$ and $b_m(x)$ in terms of an arbitrary polynomial sequence $\{q_k(x)\}_{k\geq 0}$. Namely,

$$a_n(x)b_m(x) = \sum_{k=0}^{n+m} c_{nm}(k)q_k(x).$$

In the present paper, we considered the sums of finite products of Chebyshev polynomials of the first kind in (1.24) and of Lucas polynomials in (1.25), and expanded each of them in terms of all kinds of Chebyshev polynomials to find that all the coefficients involve terminating hypergeometric functions ${}_2F_1$. Consequently, we were able to discover the amusing identities in (4.3) among all kinds of Chebyshev polynomials. Clearly, this may be viewed as a generalization of the above-mentioned connection problem.

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