CLASSIFICATION THEOREMS OF RULED SURFACES IN
MINKOWSKI 3-SPACE

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ABSTRACT. By generalizing the notion of pointwise 1-type Gauss map, the generalized 1-type Gauss map has been recently introduced. Without any assumption, we classified all possible ruled surfaces with generalized 1-type Gauss map in a 3-dimensional Minkowski space. In particular, null scrolls do not have the proper generalized 1-type Gauss map. In fact, it is harmonic.

1. Introduction

A Riemannian manifold can be imbedded in a Euclidean space by Nash’s imbedding theorem. That enables us to study Riemannian manifolds as submanifolds of a Euclidean space. In the late 1970’s, B.-Y. Chen introduced the notion of finite-type immersion of Riemannian manifolds into Euclidean space by generalizing the eigenvalue problem of the immersion ([1]). An isometric immersion $x$ of a Riemannian manifold $M$ into a Euclidean space $\mathbb{E}^m$ is said to be of finite-type if it has the spectral decomposition as

$$x = x_0 + x_1 + \cdots + x_k,$$

where $x_0$ is a constant vector and $\Delta x_i = \lambda_i x_i$ for some positive integer $k$ and $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. Here, $\Delta$ denotes the Laplacian operator defined on $M$. If $\lambda_1, \ldots, \lambda_k$ are mutually different, $M$ is said to be of $k$-type. By putting together the eigenvectors of the same eigenvalue, we may assume that a finite-type immersion $x$ of a Riemannian manifold into a Euclidean space is of $k$-type for some positive integer $k$.

The notion of finite-type immersion of submanifold into Euclidean space was extended to the study of finite-type immersion or smooth maps defined on submanifolds of a pseudo-Euclidean space $\mathbb{E}^m_s$ with the indefinite metric of index $s \geq 1$. In this sense, it is very natural for geometers to have interest in finite-type Gauss map of submanifolds of a pseudo-Euclidean space ([9, 11, 12]).

We now focus on surfaces of the Minkowski space $\mathbb{E}^3_1$. Let $M$ be a surface in the 3-dimensional Minkowski space $\mathbb{E}^3_1$ with non-degenerate induced metric. From now on,
a surface $M$ in $\mathbb{E}^3_1$ means non-degenerate, i.e., its induced metric is non-degenerate without otherwise stated. The map $G : M \to Q^2(\epsilon) \subset \mathbb{E}^3_1$ which maps each point of $M$ to a unit normal vector to $M$ at the point is called the Gauss map of $M$, where $\epsilon$ ($= \pm 1$) denotes the sign of the vector field $G$ and $Q^2(\epsilon)$ is a 2-dimensional space form with constant sectional curvature $\epsilon$. A helicoid or a right cone in $\mathbb{E}^3$ has the unique form of Gauss map $G$ which looks like 1-type Gauss map in the usual sense. However, it is quite different and thus the authors et al. defined the following definition.

**Definition 1.1.** ([2]) A surface $M$ in $\mathbb{E}^3_1$ is said to have pointwise 1-type Gauss map $G$ or the Gauss map $G$ is of pointwise 1-type if the Gauss map $G$ of $M$ satisfies
\[
\Delta G = f(G + C)
\]
for some non-zero smooth function $f$ and a constant vector $C$. In particular, if $C$ is zero, then the Gauss map is said to be of pointwise 1-type of the first kind. Otherwise, it is said to be of pointwise 1-type of the second kind.

Some other surfaces of $\mathbb{E}^3$ such as conical surfaces have an interesting type of Gauss map. A surface in $\mathbb{E}^3_1$ parameterized by
\[
x(s, t) = p + t\beta(s),
\]
where $p$ is a point and $\beta(s)$ a unit speed curve is called a conical surface. The typical conical surfaces are a right (circular) cone and a plane.

**Example 1.2.** ([13]) Let $M$ be a surface in $\mathbb{E}^3$ parameterized by
\[
x(s, t) = (t \cos^2 s, t \sin s \cos s, t \sin s).
\]
Then, the Gauss map $G$ can be obtained by
\[
G = \frac{1}{\sqrt{1 + \cos^2 s}}(-\sin^3 s, (2 - \cos^2 s) \cos s, -\cos^2 s).
\]
Its Laplacian turns out to be
\[
\Delta G = fG + gC
\]
for some non-zero smooth functions $f$, $g$ and a constant vector $C$. The surface $M$ is a kind of conical surfaces generated by a spherical curve $\beta(s) = (\cos^2 s, \sin s \cos s, \sin s)$ on the unit sphere $\mathbb{S}^2(1)$ centered at the origin.

Based on such an example, by generalizing the notion of pointwise 1-type Gauss map, the so-called generalized 1-type Gauss map was introduced.

**Definition 1.3.** ([13]) A surface $M$ in $\mathbb{E}^3_1$ is said to have generalized 1-type Gauss map $G$ or the Gauss map $G$ is of generalized 1-type if the Gauss map $G$ of $M$ satisfies
\[
\Delta G = fG + gC
\]
for some non-zero smooth functions $f$, $g$ and a constant vector $C$. In particular, If the generalized 1-type Gauss map $G$ is not of pointwise 1-type, it is said to be proper.
Definition 1.4. A conical surface with generalized 1-type Gauss map is called a \emph{conical surface of $G$-type}.

Remark 1.5. ([13]) A conical surface of $G$-type can be constructed by the functions $f$, $g$ and the constant vector \( C \) by solving the differential equations generated by (1.1).

Here, we provide an example of a cylindrical ruled surface in the 3-dimensional Minkowski space \( \mathbb{E}^3_1 \) with generalized 1-type Gauss map.

Example 1.6. Let \( M \) be a ruled surface in the Minkowski 3-space \( \mathbb{E}^3_1 \) parameterized by
\[
x(s, t) = \left( \frac{1}{2} \left( s\sqrt{s^2 - 1} - \ln(s + \sqrt{s^2 - 1}) \right), \frac{1}{2}s^2, t \right), \quad s \geq 1.
\]
Then, the Gauss map \( G \) is given by
\[
G = (-s, -\sqrt{s^2 - 1}, 0).
\]
By a direct computation, we see that its Laplacian satisfies
\[
\Delta G = \frac{s - \sqrt{s^2 - 1}}{(s^2 - 1)^{3/2}} G + \frac{s(s - \sqrt{s^2 - 1})}{(s^2 - 1)^{3/2}} (1, -1, 0),
\]
which indicates that \( M \) has generalized 1-type Gauss map.

2. Preliminaries

Let \( \mathbb{E}^3_1 \) be a Minkowski 3-space with the Lorentz metric \( ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 \), where \((x_1, x_2, x_3)\) denotes the standard coordinate system in \( \mathbb{E}^3_1 \). Let \( M \) be a non-degenerate surface in \( \mathbb{E}^3_1 \). A curve in \( \mathbb{E}^3_1 \) is said to be \emph{space-like}, \emph{time-like} or \emph{null} if its tangent vector field is space-like, time-like or null, respectively. It is well known that in terms of the local coordinates \( \{\bar{x}_i\} \) of \( M \) the Laplacian \( \Delta \) is given by
\[
\Delta = -\frac{1}{\sqrt{|G|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial \bar{x}_i} \left( \sqrt{|G|} g^{ij} \frac{\partial}{\partial \bar{x}_j} \right),
\]
where \((g^{ij}) = (g_{ij})^{-1}\) and \( G \) is the determinant of the matrix \((g_{ij})\) consisting of the components of the first fundamental form.

Now, we define a ruled surface \( M \) in the Minkowski 3-space \( \mathbb{E}^3_1 \). Let \( I \) and \( J \) be some open intervals in the real line \( \mathbb{R} \). Let \( \alpha = \alpha(s) \) be a curve in \( \mathbb{E}^3_1 \) defined on \( I \) and \( \beta = \beta(s) \) a transversal vector field with \( \alpha'(s) \) along \( \alpha \). From now on, \( \, ' \) denotes the differentiation with respect to the parameter \( s \) unless otherwise stated. Then, a parametrization of a ruled surface \( M \) is given by
\[
x(s, t) = \alpha(s) + t\beta(s), \quad s \in I, \quad t \in J.
\]
The curve \( \alpha = \alpha(s) \) is called a \emph{base curve} and \( \beta = \beta(s) \) a \emph{director vector field} or a \emph{ruling}. In particular, if \( \beta \) is constant, \( M \) is said to be \emph{cylindrical}. Otherwise, it is said to be \emph{non-cylindrical}. 
Ruled surfaces in $E^3_1$ with non-null base curve may have different types according to their causal character of the base curve and the director vector field. If the base curve $\alpha$ is space-like or time-like, the director vector field $\beta$ can be chosen to be orthogonal to $\alpha$ that is normalized. The ruled surface $M$ is said to be of type $M_+^1$ or $M_-^1$, respectively if $\alpha$ is spacelike or timelike, respectively. Also, the ruled surface of type $M_+^1$ can be divided into three types. If $\beta$ is space-like, it is said to be of type $M_1^1$ or $M_2^1$ if $\beta'$ is non-null or null, respectively. When $\beta$ is time-like, $\beta'$ is space-like because of the character of the causal vectors, which is said to be of type $M_3^3$. On the other hand, when $\alpha$ is time-like, $\beta$ is always space-like. Accordingly, it is also said to be of type $M_1^2$ or $M_2^2$ if $\beta'$ is non-null or null, respectively. The ruled surface of type $M_1^1$ (resp. $M_2^1$, $M_1^2$ or $M_2^2$) is clearly space-like (resp. time-like).

If the base curve $\alpha$ is null, the director vector field $\beta$ along $\alpha$ must be chosen to be null since the ruled surface is non-degenerate. Such a ruled surface $M$ is called a null scroll. One of such is a $B$-scroll ([7], [9]). Other cases such as $\alpha$ is non-null and $\beta$ is null, or $\alpha$ is null and $\beta$ is non-null are reduced to one of the types $M_1^1$, $M_2^1$, or a null scroll by an appropriate change of the base curve ([10]). Among null scrolls, a $B$-scroll has an interesting geometric property such as it has constant mean curvature and constant Gaussian curvature. Let $\alpha = \alpha(s)$ be a null curve in $E^3_1$ with Cartan frame $\{A, B, C\}$, that is, $A, B, C$ are vector fields along $\alpha$ in $E^3_1$ satisfying the following conditions:

\[
\langle A, A \rangle = \langle B, B \rangle = 0, \quad \langle A, B \rangle = 1, \quad \langle A, C \rangle = \langle B, C \rangle = 0, \quad \langle C, C \rangle = 1,
\]

\[
\alpha' = A, \quad C' = -aA - k(s)B,
\]

where $a$ is a constant and $k(s)$ a nowhere vanishing function. A null scroll parametrized by $x = x(s, t) = \alpha(s) + tB(s)$ is called a $B$-scroll which has mean curvature $H = a$ and Gaussian curvature $K = a^2$. Furthermore, its Laplacian $\Delta G$ of the Gauss map $G$ is given by

\[
\Delta G = -2a^2G,
\]

from which, we see that a $B$-scroll is minimal if and only if it is flat.

Throughout the paper, all surfaces in $E^3_1$ are smooth and connected unless otherwise stated.

3. CYLINDRICAL RULED SURFACES IN $E^3_1$ WITH GENERALIZED 1-TYPE GAUSS MAP

In this section, we study the cylindrical ruled surfaces with generalized 1-type Gauss map in the Minkowski 3-space $E^3_1$.

Let $M$ be a cylindrical ruled surface of type $M_1^1$, $M_1^3$ or $M_3^3$ in $E^3_1$. Then $M$ is parameterized by a base curve $\alpha$ and a unit constant vector $\beta$ such that

\[
x(s, t) = \alpha(s) + t\beta
\]

satisfying $\langle \alpha', \alpha' \rangle = \varepsilon_1 (= \pm 1), \langle \alpha', \beta \rangle = 0$ and $\langle \beta, \beta \rangle = \varepsilon_2 (= \pm 1)$. 
We now suppose that $M$ has generalized 1-type Gauss map $G$. Then the Gauss map $G$ satisfies the condition (1.1). We put the constant vector $\mathbb{C} = (c_1, c_2, c_3)$ in (1.1) for some constants $c_1$, $c_2$ and $c_3$.

Suppose that $f = g$. Then the Gauss map $G$ is nothing but of pointwise 1-type. A classification of cylindrical ruled surfaces with pointwise 1-type Gauss map in $\mathbb{E}^3_1$ was described in [5].

If $M$ is of type $M_{1}^{+}$, then $M$ is an open part of a Euclidean plane or a cylinder over a curve of infinite-type satisfying
\begin{equation}
\frac{c^2}{2}f^{-\frac{1}{3}} - \ln |c^2f^{-\frac{1}{3}} + 1| = \pm c^3(s + k)
\end{equation}
if $\mathbb{C}$ is null, or
\begin{equation}
\sqrt{\left(c^2f^{-\frac{1}{3}} + 1\right)^2 + (-c^2_1 + c^2_2)} - \ln \left(c^2f^{-\frac{1}{3}} + 1 + \sqrt{\left(c^2f^{-\frac{1}{3}} + 1\right)^2 + (-c^2_1 + c^2_2)}\right)
\end{equation}
\begin{equation}
+ \ln \sqrt{|c^2_1 + c^2_2|} = \pm c^3(s + k)
\end{equation}
if $\mathbb{C}$ is non-null, where $c$ is some non-zero constant and $k$ is a constant.

If $M$ is of type $M_{1}^{-}$, $M$ is an open part of a Minkowski plane or a cylinder over a curve of infinite-type satisfying
\begin{equation}
\frac{c^2}{2}f^{-\frac{1}{3}} + \ln |c^2f^{-\frac{1}{3}} - 1| = \pm c^3(s + k)
\end{equation}
and
\begin{equation}
\sqrt{\left(c^2f^{-\frac{1}{3}} - 1\right)^2 - (-c^2_1 + c^2_2)} + \ln \left(c^2f^{-\frac{1}{3}} - 1 + \sqrt{\left(c^2f^{-\frac{1}{3}} - 1\right)^2 + (-c^2_1 + c^2_2)}\right)
\end{equation}
\begin{equation}
- \ln \sqrt{|c^2_1 + c^2_2|} = \pm c^3(s + k)
\end{equation}
depending on the constant vector $\mathbb{C}$ is null or non-null, respectively, for some non-zero constant $c$ and some constant $k$.

If $M$ is of type $M_{3}^{+}$, $M$ is an open part of either a Minkowski plane or a cylinder over a curve of infinite-type satisfying
\begin{equation}
\sqrt{c^2_1 + c^2_2} - \left(c^2f^{-\frac{1}{3}} - 1\right)^2 - \sin^{-1}\left(\frac{c^2f^{-\frac{1}{3}} - 1}{\sqrt{c^2_1 + c^2_2}}\right) = \pm c^3(s + k),
\end{equation}
where $c$ is a non-zero constant and $k$ a constant.

We now assume that $f \neq g$. Here, we consider two cases.

Case 1. Let $M$ be a cylindrical ruled surface of type $M_{1}^{+}$ or $M_{1}^{-}$, i.e., $\varepsilon_2 = 1$. Without loss of generality, we may assume that $\alpha(s) = (\alpha_1(s), \alpha_2(s), 0)$ is a plane curve parameterized by an arc length $s$ and $\beta$ is chosen as $\beta = (0, 0, 1)$. Then the
Gauss map $G$ of $M$ and the Laplacian $\Delta G$ of the Gauss map are respectively obtained by

$$G = (-\alpha_2'(s), -\alpha_1'(s), 0) \quad \text{and} \quad \Delta G = (\varepsilon_1\alpha_2'''(s), \varepsilon_1\alpha_1'''(s), 0).$$

(3.6)

With the help of (1.1) and (3.6), it immediately follows

$$C = (c_1, c_2, 0)$$

for some constants $c_1$ and $c_2$. And we also have

$$\varepsilon_1\alpha_2''' = -f\alpha_2' + gc_1,$$

$$\varepsilon_1\alpha_1''' = -f\alpha_1' + gc_2.$$  

(3.7)

Firstly, we consider the case that $M$ is of type $M^1_+$. Since $\alpha$ is space-like, we may put $\alpha_1'(s) = \sinh \theta(s)$ and $\alpha_2'(s) = \cosh \theta(s)$ for some function $\theta(s)$ of $s$. Then (3.7) can be written in the form

$$\theta'(s)^2 \cosh \theta + \theta''(s) \sinh \theta = -f \cosh \theta + gc_1,$$

$$\theta'(s)^2 \sinh \theta + \theta''(s) \cosh \theta = -f \sinh \theta + gc_2.$$  

It implies that

$$\theta'(s)^2 = -f + g(c_1 \cosh \theta - c_2 \sinh \theta)$$  

(3.8)

and

$$\theta''(s) = g(-c_1 \sinh \theta + c_2 \cosh \theta).$$  

(3.9)

In fact, $\theta'$ is the signed curvature of the base curve $\alpha = \alpha(s)$.

Suppose $\theta$ is a constant, i.e., $\theta' = 0$. Then $\alpha$ is part of a straight line. In this case, $M$ is an open part of a Euclidean plane.

Now we suppose that $\theta' \neq 0$. From (3.7), we see that the functions $f$ and $g$ depend only on the parameter $s$, i.e., $f(s,t) = f(s)$ and $g(s,t) = g(s)$. Taking the derivative of equation (3.8) and using (3.9), we get

$$3\theta''\theta''' = -f' + g'(c_1 \cosh \theta - c_2 \sinh \theta).$$

With the help of (3.8), it follows that

$$\frac{3}{2} ((\theta')^2)' = -f' + g'(\theta')^2 + f.$$

Solving the above differential equation, we have

$$\theta'(s)^2 = k_1 g^{\frac{2}{3}} + \frac{2}{3} k_1^{\frac{2}{3}} \int g^{-\frac{2}{3}} f \left( -\frac{f'}{f} + \frac{g'}{g} \right) ds, \quad k_1 (\neq 0) \in \mathbb{R}.$$  

(3.10)

We put

$$\theta'(s) = \pm \sqrt{p(s)},$$
where \( p(s) = |k_1 g^\frac{2}{3} + \frac{2}{3} g^\frac{2}{3} f g^{-\frac{2}{3}} f \left( -\frac{f'}{f} + \frac{g'}{g} \right) ds| \). It means that the function \( \theta \) is determined by the functions \( f, g \) and a constant vector satisfying (1.1). Therefore, the cylindrical ruled surface \( M \) satisfying (1.1) is determined by a base curve \( \alpha \) such that
\[
\alpha(s) = \left( \int \sinh \theta(s) ds, \int \cosh \theta(s) ds, 0 \right)
\]
and the director vector field \( \beta(s) = (0, 0, 1) \).

In this case, if \( f \) and \( g \) are constant, the signed curvature \( \theta' \) of a base curve \( \alpha \) is non-zero constant and the Gauss map \( G \) is of usual 1-type. Hence, \( M \) is an open part of a hyperbolic cylinder or a circular cylinder ([6]).

Suppose that one of the functions \( f \) and \( g \) is not constant. Then \( M \) is an open part of a cylinder over the base curve of infinite-type satisfying (3.10). For a curve of finite-type in a plane of \( E_1^3 \), see [6] in details.

Next we consider the case that \( M \) is of type \( M_1^{-} \). Since \( \alpha \) is time-like, we may put
\[
\alpha'(s) = \cosh \theta(s) \quad \text{and} \quad \alpha_2'(s) = \sinh \theta(s)
\]
for some function \( \theta(s) \) of \( s \).

As was given in the previous case of type \( M_1^{+} \), if the signed curvature \( \theta' \) of the base curve \( \alpha \) is zero, \( M \) is part of a Minkowski plane.

We now assume that \( \theta' \neq 0 \). Quite similarly as above, we have
\[
\theta'(s)^2 = k_2 g^\frac{2}{3} + \frac{2}{3} g^\frac{2}{3} f g^{-\frac{2}{3}} f \left( \frac{f'}{f} - \frac{g'}{g} \right) ds, \quad k_2 \neq 0 \in \mathbb{R},
\]
(3.11)
or, we put
\[
\theta'(s) = \pm \sqrt{q(s)},
\]
where \( q(s) = |k_2 g^\frac{2}{3} + \frac{2}{3} g^\frac{2}{3} f g^{-\frac{2}{3}} f \left( \frac{f'}{f} - \frac{g'}{g} \right) ds| \).

Case 2. Let \( M \) be a cylindrical ruled surface of type \( M_3^{+} \). In this case, without loss of generality we may assume that \( \alpha(s) = (0, \alpha_2(s), \alpha_3(s)) \) is a plane curve parameterized by the arc length \( s \) and \( \beta \) is chosen as \( \beta = (1, 0, 0) \). Then the Gauss map \( G \) of \( M \) and the Laplacian \( \Delta G \) of the Gauss map are obtained by
\[
G = (0, \alpha_3', -\alpha_2') \quad \text{and} \quad \Delta G = (0, -\alpha_3'', \alpha_2'').
\]
(3.12)
The relationship (3.12) and the condition (1.1) imply that the constant vector \( \mathbb{C} \) has the form
\[
\mathbb{C} = (0, c_2, c_3)
\]
for some constants \( c_2 \) and \( c_3 \).

If \( f \) and \( g \) are both constant, the Gauss map is of 1-type in the usual sense and thus \( M \) is an open part of a circular cylinder ([1]).
We now assume that the functions \( f \) and \( g \) are not both constant. Then, with the help of (1.1) and (3.12), we get
\[
\begin{align*}
-\alpha''''_3 &= f\alpha_3' + gc_2, \\
\alpha''''_2 &= -f\alpha_2' + gc_3.
\end{align*}
\]
(3.13)

Since \( \alpha \) is parameterized by the arc length \( s \), we may put
\[
\begin{align*}
\alpha_2'(s) &= \cos \theta(s) \quad \text{and} \quad \alpha_3'(s) = \sin \theta(s)
\end{align*}
\]
for some function \( \theta(s) \) of \( s \). Hence, (3.13) can be expressed as
\[
\begin{align*}
(\theta')^2 \sin \theta - \theta'' \cos \theta &= f \sin \theta + gc_2, \\
(\theta')^2 \cos \theta + \theta'' \sin \theta &= f \cos \theta - gc_3.
\end{align*}
\]
It follows
\[
(\theta')^2 = f + g(c_2 \sin \theta - c_3 \cos \theta).
\]
(3.14)

Thus, \( M \) is a cylinder over the base curve \( \alpha \) given by
\[
\alpha(s) = \left( 0, \int \cos \left( \int \sqrt{r(s)} ds \right) ds, \int \sin \left( \int \sqrt{r(s)} ds \right) ds \right)
\]
and the ruling \( \beta(s) = (1, 0, 0) \), where \( r(s) = |f(s) + g(s) (c_2 \sin \theta(s) - c_3 \cos \theta(s))| \).

Consequently, we have

**Theorem 3.1 (Classification of cylindrical ruled surfaces in \( \mathbb{E}^3 \)).** Let \( M \) be a cylindrical ruled surface with generalized 1-type Gauss map in the Minkowski 3-space \( \mathbb{E}^3 \). Then, \( M \) is an open part of a Euclidean plane, a Minkowski plane, a circular cylinder, a hyperbolic cylinder or a cylinder over a base curve of infinite-type satisfying (3.1), (3.2), (3.3), (3.4), (3.5), (3.10), (3.11) or (3.14).

4. **Non-cylindrical ruled surfaces with generalized 1-type Gauss map**

In this section, we classify the non-cylindrical ruled surfaces with generalized 1-type Gauss map in \( \mathbb{E}^3 \).

Case 1. Let \( M \) be a non-cylindrical ruled surface of type \( \mathbb{M}^1, \mathbb{M}^3 \) or \( \mathbb{M}^- \). Then \( M \) is parameterized by, up to a rigid motion,
\[
x(s, t) = \alpha(s) + t\beta(s)
\]
such that \( \langle \alpha', \beta \rangle = 0 \), \( \langle \beta, \beta \rangle = \varepsilon_2 (= \pm 1) \) and \( \langle \beta', \beta' \rangle = \varepsilon_3 (= \pm 1) \). Then, \( \{\beta, \beta', \beta \times \beta'\} \) is an orthonormal frame along the base curve \( \alpha \). For later use, we define the smooth functions \( q, u, Q \) and \( R \) as follows:
\[
q = \|x_s\|^2 = \varepsilon_4 \langle x_s, x_s \rangle, \quad u = \langle \alpha', \beta' \rangle, \quad Q = \langle \alpha', \beta \times \beta' \rangle, \quad R = \langle \beta'', \beta \times \beta' \rangle,
\]
where \( \varepsilon_4 \) is the sign of the coordinate vector field \( x_s = \partial x/\partial s \). The vector fields \( \alpha' \), \( \beta'' \), \( \alpha' \times \beta \) and \( \beta \times \beta'' \) are represented in terms of the orthonormal frame \( \{ \beta, \beta', \beta \times \beta' \} \) along the base curve \( \alpha \) as

\[
\begin{align*}
\alpha' &= \varepsilon_3 u\beta' - \varepsilon_2 \varepsilon_3 Q \beta \times \beta', \\
\beta'' &= -\varepsilon_2 \varepsilon_3 \beta - \varepsilon_2 \varepsilon_3 R \beta \times \beta', \\
\alpha' \times \beta &= \varepsilon_3 Q \beta' - \varepsilon_3 u \beta \times \beta', \\
\beta \times \beta'' &= -\varepsilon_3 R \beta'.
\end{align*}
\]  

(4.1)

Therefore, the smooth function \( q \) is given by

\[
q = \varepsilon_4 (\varepsilon_3 t^2 + 2ut + \varepsilon_3 u^2 - \varepsilon_2 \varepsilon_3 Q^2).
\]

Note that \( t \) is chosen so that \( q \) takes positive values.

Furthermore, the Gauss map \( G \) of \( M \) is given by

\[
G = q^{-1/2} (\varepsilon_3 Q \beta' - (\varepsilon_3 u + t) \beta \times \beta').
\]

(4.2)

By using the determinants of the first fundamental form and the second fundamental form, the mean curvature \( H \) and the Gaussian curvature \( K \) of \( M \) are obtained by, respectively,

\[
\begin{align*}
H &= \frac{1}{2} \varepsilon_2 q^{3/2} \left( R t^2 + (2 \varepsilon_3 u R + Q') t + u^2 R + \varepsilon_3 u Q' - \varepsilon_3 u' Q - \varepsilon_2 Q^2 R \right), \\
K &= q^{-2} Q^2.
\end{align*}
\]

(4.3)

Applying the Gauss and Weingarten formulas, the Laplacian of the Gauss map \( G \) of \( M \) in \( \mathbb{E}_1^3 \) is expressed by

\[
\Delta G = 2 \text{grad} H + \langle G, G \rangle (\text{tr} A_G^2) G,
\]

(4.4)

where \( A_G \) denotes the shape operator of the surface \( M \) in \( \mathbb{E}_1^3 \) and \( \text{grad} H \) is the gradient of \( H \). Using (4.3), we get

\[
2 \text{grad} H = 2 \langle e_1, e_1 \rangle e_1 (H) e_1 + 2 \langle e_2, e_2 \rangle e_2 (H) e_2
\]

\[
= 2 \varepsilon_4 e_1 (H) e_1 + 2 \varepsilon_2 e_2 (H) e_2
\]

\[
= q^{-7/2} \{-\varepsilon_2 (\varepsilon_3 u + t) A_1 \beta' - \varepsilon_4 q B_1 \beta + \varepsilon_3 Q A_1 \beta \times \beta' \},
\]

where \( e_1 = \frac{x_1}{\| x_1 \|}, \quad e_2 = \frac{x_2}{\| x_2 \|} \),

\[
A_1 = 3 (u' t + \varepsilon_3 uu' - \varepsilon_2 \varepsilon_3 Q Q') \{ R t^2 + (2 \varepsilon_3 u R + Q') t + u^2 R + \varepsilon_3 u Q' - \varepsilon_3 u' Q - \varepsilon_2 Q^2 R \}
\]

\[
- (\varepsilon_3 t^2 + 2ut + \varepsilon_3 u^2 - \varepsilon_2 \varepsilon_3 Q^2) \{ R t^2 + (2 \varepsilon_3 u R + 2 \varepsilon_3 u R' + Q') t + 2 uu' R + u^2 R'
\]

\[
+ \varepsilon_3 uu' - \varepsilon_3 u Q - \varepsilon_2 QQ' R - \varepsilon_2 Q^2 R' \},
\]

\[
B_1 = \varepsilon_3 R t^3 + (3u R + 2 \varepsilon_3 Q') t^2 + (3 \varepsilon_3 u^2 R + 4u Q' - 3u' Q - \varepsilon_2 \varepsilon_3 Q^2 R) t + u^3 R + 2 \varepsilon_3 u^2 Q'
\]

\[
- \varepsilon_2 u Q^2 R - 3 \varepsilon_3 uu' Q + \varepsilon_2 \varepsilon_3 Q^2 Q'.
\]
The straightforward computation gives
\[ \text{tr} A_G^2 = -\varepsilon q^{-3} D_1, \]
where
\[ D_1 = -\varepsilon_4 (u't + \varepsilon_3 u'u - \varepsilon_2 \varepsilon_3 Q Q')^2 + \varepsilon_3 q \{ (\varepsilon_2 R + \varepsilon_3 u')^2 - \varepsilon_2 (Q' + \varepsilon_3 uR + R t)^2 - 2 \varepsilon_3 Q^2 \}. \]
Thus, the Laplacian \( \Delta G \) of the Gauss map \( G \) of \( M \) is obtained by
\[ \Delta G = q^{-7/2} [-\varepsilon q B_1 \beta + \{ -\varepsilon_2 (\varepsilon_3 u + t) A_1 + \varepsilon_3 Q D_1 \} \beta' + \{ \varepsilon_3 Q A_1 - (\varepsilon_3 u + t) D_1 \} \beta \times \beta']. \]

Now, suppose that the Gauss map \( G \) of \( M \) is of generalized 1-type. Hence, from (1.1), (4.2) and (4.5), we get
\[ q^{-7/2} [-\varepsilon q B_1 \beta + \{ -\varepsilon_2 (\varepsilon_3 u + t) A_1 + \varepsilon_3 Q D_1 \} \beta' + \{ \varepsilon_3 Q A_1 - (\varepsilon_3 u + t) D_1 \} \beta \times \beta']
\[ = f q^{-1/2} (\varepsilon_3 Q \beta' - (\varepsilon_3 u + t) \beta \times \beta') + g \mathbb{C}. \]

If we take the indefinite scalar product to equation (4.6) with \( \beta, \beta' \) and \( \beta \times \beta' \), respectively, then we obtain respectively,
\[ -\varepsilon q^{-5/2} B_1 = g \langle \mathbb{C}, \beta \rangle, \]
\[ q^{-7/2} [-\varepsilon_2 (\varepsilon_3 u + t) A_1 + \varepsilon_3 Q D_1 \beta] + q^{-1/2} Q + g \langle \mathbb{C}, \beta' \rangle, \]
\[ q^{-7/2} [-\varepsilon_2 Q A_1 + \varepsilon_2 (\varepsilon_3 u + t) D_1 \beta] = f q^{-1/2} (\varepsilon_3 u + t) + g \langle \mathbb{C}, \beta \times \beta' \rangle. \]

On the other hand, the constant vector \( \mathbb{C} \) can be written as
\[ \mathbb{C} = c_1 \beta + c_2 \beta' + c_3 \beta \times \beta', \]
where \( c_1 = \varepsilon_2 \langle \mathbb{C}, \beta \rangle, \ c_2 = \varepsilon_3 \langle \mathbb{C}, \beta' \rangle \) and \( c_3 = -\varepsilon_2 \varepsilon_3 \langle \mathbb{C}, \beta \times \beta' \rangle \). Differentiating the functions \( c_1, c_2 \) and \( c_3 \) with respect to \( s \), we have
\[ c_1' - \varepsilon_2 \varepsilon_3 c_2 = 0, \]
\[ c_1 + c_2' - \varepsilon_3 R c_3 = 0, \]
\[ \varepsilon_2 \varepsilon_3 R c_2 - c_3' = 0. \]

Also, equations (4.7), (4.8) and (4.9) are expressed as follows:
\[ -\varepsilon_4 q^{-5/2} B_1 = g c_1, \]
\[ q^{-7/2} [-\varepsilon_2 (\varepsilon_3 u + t) A_1 + \varepsilon_3 Q D_1] = f q^{-1/2} \varepsilon_3 Q + g c_2, \]
\[ q^{-7/2} [-\varepsilon_2 Q A_1 + (\varepsilon_3 u + t) D_1] = f q^{-1/2} (\varepsilon_3 u + t) - g c_3. \]

Combining equations (4.11), (4.12) and (4.13), we have
\[ \{-\varepsilon_2 (\varepsilon_3 u + t) A_1 + \varepsilon_3 Q D_1\} c_1 + q \varepsilon q B_1 c_2 = q^3 f \varepsilon_3 Q c_1, \]
\[ \{-\varepsilon_3 Q A_1 + (\varepsilon_3 u + t) D_1\} c_1 - q \varepsilon q B_1 c_3 = q^3 f (\varepsilon_3 u + t) c_1. \]

Hence, equations (4.14) and (4.15) yield that
\[ -\varepsilon_2 \varepsilon_3 A_1 c_1 + B_1 \{ c_2 (\varepsilon_3 u + t) + \varepsilon_3 Q c_3 \} = 0. \]
First of all, we prove

**Theorem 4.1.** Let $M$ be a non-cylindrical ruled surface of type $M_1^1$, $M_2^2$ or $M_3^3$ parameterized by the base curve $\alpha$ and the director vector field $\beta$ in $E^3$ with generalized 1-type Gauss map. If $\beta$, $\beta'$ and $\beta''$ are coplanar along $\alpha$, then $M$ is an open part of a plane, the helicoid of the first kind, the helicoid of the second kind or the helicoid of the third kind.

**Proof.** If the constant vector $C$ is zero in definition (1.1), then the Gauss map is nothing but of pointwise 1-type of the first kind. Thus, according to Classification Theorem of ruled surfaces in $E^3$ with pointwise 1-type Gauss map of the first kind in [12], $M$ is an open part of the helicoid of the first kind, the helicoid of the second kind or the helicoid of the third kind.

Now we assume that the constant vector $C$ is non-zero. In this case, if the function $Q$ is identically zero on $M$, then $M$ is an open part of a plane because of (4.3).

Suppose that an open subset $U = \{ s \in \text{dom}(\alpha)|Q(s) \neq 0 \}$ of $\text{dom}(\alpha)$ is not empty. Since $\beta$, $\beta'$ and $\beta''$ are coplanar along $\alpha$, $R$ vanishes. Thus, $c_3$ is a constant and $c''_3 = -\varepsilon_2\varepsilon_3c_1$ from (4.10). Since the left hand side of (4.16) is a polynomial in $t$ with functions of $s$ as the coefficients, all of the coefficients which are of functions of $s$ must be zero. From the leading coefficient, we have

$$\varepsilon_2\varepsilon_3c_1Q'' + 2c_2Q' = 0. \quad (4.17)$$

Observing the coefficient of the term involving $t^2$ of (4.16) with the help of (4.17), we get

$$\varepsilon_2\varepsilon_3c_1(3u'Q' + u''Q) + 3c_2u'Q - 2c_3QQ' = 0. \quad (4.18)$$

Examining the coefficient of the linear term in $t$ of (4.16) and using (4.17) and (4.18), we also get

$$Q\{c_1(\varepsilon_2(u')^2 + (Q')^2) + \varepsilon_2\varepsilon_3c_2QQ' - \varepsilon_3c_3u'Q\} = 0.$$ 

On $U$,

$$c_1(\varepsilon_2(u')^2 + (Q')^2) + \varepsilon_2\varepsilon_3c_2QQ' - \varepsilon_3c_3u'Q = 0. \quad (4.19)$$

Similarly, from the constant term with respect to $t$ of (4.16), we have

$$\varepsilon_3c_1(-3u'Q' + u''Q) + \varepsilon_2\varepsilon_3c_3QQ' = 0 \quad (4.20)$$

by using (4.17), (4.18) and (4.19). Combining (4.18) and (4.20), we obtain

$$2\varepsilon_3c_1u'Q' + \varepsilon_2\varepsilon_3c_2u'Q - \varepsilon_2\varepsilon_3c_3QQ' = 0. \quad (4.21)$$

Now suppose that $u'(s) \neq 0$ at some point $s \in U$ and then $u' \neq 0$ on an open interval $U_1 \subset U$. Equation (4.19) yields

$$\varepsilon_3c_3Q = \frac{1}{u'}\{c_1(\varepsilon_2(u')^2 + (Q')^2) + \varepsilon_2\varepsilon_3c_2QQ'\}. \quad (4.22)$$

Substituting (4.22) into (4.21), we get

$$\{(u')^2 - \varepsilon_2(Q')^2\} (\varepsilon_3c_1Q' + \varepsilon_2c_2Q) = 0,$$
or, using $c_2 = \varepsilon_2 \varepsilon_3 c_4'$ in (4.10),

$$\{(u')^2 - \varepsilon_2 (Q')^2\}(c_1 Q)' = 0.$$  

Suppose that $((u')^2 - \varepsilon_2 (Q')^2)(s_0) \neq 0$ for some $s_0 \in U_1$. Then $c_1 Q$ is constant on a component $U_2$ containing $s_0$ of $U_1$.

If $c_1 = 0$ on $U_2$, we easily see that $c_2 = 0$ by (4.10). Hence, (4.19) yields that $c_3 u'Q = 0$ and so $c_3 = 0$. Since $C$ is a constant vector, $\mathbb{C}$ is zero on $M$. It contradicts our assumption. Thus, $c_1 \neq 0$ on $U_2$. From the equation $c_1' + \varepsilon_2 \varepsilon_3 c_1 = 0$, we get

$$c_1 = k_1 \cos(s + s_1) \text{ or } c_1 = k_2 \cosh(s + s_2)$$

for some non-zero constants $k_1$ and $s_i \in \mathbb{R} \ (i = 1, 2)$. Since $c_1 Q$ is constant, $k_1$ and $k_2$ must be zero. Hence $c_1 = 0$, a contradiction. Thus, $(u')^2 - \varepsilon_2 (Q')^2 = 0$ on $U_1$, from which, we get $\varepsilon_2 = 1$ and $u' = \pm Q'$. If $u' \neq -Q'$, then $u' = Q'$ on an open subset $U_3$ in $U_1$. Hence (4.19) implies that $Q'(2\varepsilon_3 c_1 Q' + c_2 Q - c_3 Q) = 0$. On $U_3$, we get $c_3 Q = 2\varepsilon_3 c_1 Q' + c_2 Q$. Putting it into (4.20), we have

$$\varepsilon_3 c_1 (Q')^2 - \varepsilon_3 c_1 QQ'' - c_2 QQ' = 0. \quad (4.23)$$

Combining (4.17) and (4.23), $c_1 Q$ is constant on $U_3$. Similarly as above, we can derive that $\mathbb{C}$ is zero on $M$, which is a contradiction. Therefore, we have $u' = -Q'$ on $U_1$. Similarly as we just did to the case under the assumption $u' \neq -Q'$, it is also proved that the constant vector $\mathbb{C}$ becomes zero. It is also a contradiction and so $U_1 = \emptyset$. Thus, $u' = 0$ and $Q' = 0$. From (4.3), the mean curvature $H$ vanishes. In this case, the Gauss map $G$ is of pointwise 1-type of the first kind. Hence, the open set $U$ is empty. Therefore $Q = 0$ on $M$. Because of (4.3), $M$ is an open part of a plane.

From now on, we assume that $R$ is non-vanishing, i.e., $\beta \wedge \beta' \wedge \beta'' \neq 0$ everywhere on $M$.

If $f = g$, the Gauss map of the non-cylindrical ruled surface of type $M^1_+$, $M^1_-$ or $M^3_+$ in $\mathbb{E}^3_1$ is of pointwise 1-type. According to Classification Theorem given in [8], $M$ is part of a circular cone or a hyperbolic cone.

Now, we suppose that $f \neq g$ and the constant vector $\mathbb{C}$ is non-zero unless otherwise stated. Similarly as before, we develop our argument with (4.16). The left hand side of (4.16) is a polynomial in $t$ with functions of $s$ as the coefficients and thus they are zero. From the leading coefficient of the left hand side of (4.16), we obtain

$$\varepsilon_2 c_1 R' + \varepsilon_3 c_2 R = 0. \quad (4.24)$$

With the help of (4.10), $c_1 R$ is constant. If we examine the coefficient of the term of $t^3$ of the left hand side of (4.16), we get

$$c_1 (-\varepsilon_2 \varepsilon_3 u'R + \varepsilon_2 QQ') + 2c_2 \varepsilon_3 Q' + c_3 QR = 0. \quad (4.25)$$

From the coefficient of the term involving $t^2$ in (4.16), using (4.10) and (4.25), we also get

$$c_1 (-3\varepsilon_2 \varepsilon_3 u'Q' + QQ'R - \varepsilon_2 \varepsilon_3 u''Q - Q^2 R') - 3c_2 u'Q + 2c_3 QQ' = 0. \quad (4.26)$$
Furthermore, considering the coefficient of the linear term in \( t \) of (4.16) and making use of equations (4.10), (4.25) and (4.26), we obtain
\[
Q \{ c_1(\varepsilon_2(u')^2 + (Q')^2) + c_2\varepsilon_2\varepsilon_3QQ' - c_3\varepsilon_3u'Q \} = 0. \tag{4.27}
\]

Now, we consider the open set \( V = \{ s \in \text{dom}(a) | Q(s) \neq 0 \} \). Suppose \( V \neq \emptyset \). From (4.27),
\[
c_1(\varepsilon_2(u')^2 + (Q')^2) + c_2\varepsilon_2\varepsilon_3QQ' - c_3\varepsilon_3u'Q = 0. \tag{4.28}
\]
Similarly as above, observing the constant term in \( t \) of the left hand side of (4.16) with the help of (4.10) and (4.24), and using (4.25), (4.26) and (4.28), we have
\[
Q^2(2c_1\varepsilon_3u'Q' + c_2\varepsilon_2u'Q - c_3\varepsilon_2QQ') = 0.
\]
Since \( Q \neq 0 \) on \( V \), one can have
\[
2c_1\varepsilon_3u'Q' + c_2\varepsilon_2u'Q - c_3\varepsilon_2QQ' = 0. \tag{4.29}
\]
Our making use of the first and the second equations in (4.10), (4.25) reduces to
\[
c_1\varepsilon_2u'R - \varepsilon_2\varepsilon_3(c_1Q)'' - c_1Q = 0. \tag{4.30}
\]
Suppose that \( u'(s) \neq 0 \) for some \( s \in V \). Then, \( u' \neq 0 \) on an open subset \( V_1 \subset V \). From (4.28), on \( V_1 \)
\[
c_3Q = \frac{1}{u'} \{ \varepsilon_2\varepsilon_3c_1(u')^2 + \varepsilon_3c_1(Q')^2 + \varepsilon_2c_2QQ' \}. \tag{4.31}
\]
Putting (4.31) into (4.29), we have \( \{ (u')^2 - \varepsilon_2(Q')^2 \} (\varepsilon_3c_1Q' + \varepsilon_2c_2Q) = 0 \). With the help of \( c'_1 = \varepsilon_2\varepsilon_3c_2 \), it becomes
\[
\{ (u')^2 - \varepsilon_2(Q')^2 \} (c_1Q)' = 0.
\]
Suppose that \( (u')^2 - \varepsilon_2(Q')^2 \)(s) \neq 0 on \( V_1 \). Then \( c_1Q \) is constant on a component \( V_2 \) of \( V_1 \). Hence, (4.30) yields that
\[
c_1Q = \varepsilon_2c_1u'R. \tag{4.32}
\]
If \( c_1 \equiv 0 \) on \( V_2 \), (4.10) gives that \( c_2 = 0 \) and \( c_3R = 0 \). Since \( R \neq 0 \), \( c_3 = 0 \). Hence, the constant vector \( \mathbb{C} \) is zero, a contradiction. Therefore, \( c_1 \neq 0 \) on \( V_2 \). From (4.32), \( Q = \varepsilon_2u'R \). Moreover, \( u' \) is a non-zero constant because \( c_1Q \) and \( c_1R \) are constants. Thus, (4.26) and (4.29) can be reduced to as follows
\[
c_1Q'R - c_1QR' + 2c_3Q' = 0, \tag{4.33}
\]
\[
\varepsilon_3c_1u'Q' - \varepsilon_2c_3QQ' = 0. \tag{4.34}
\]
Our putting \( Q = \varepsilon_2u'R \) into (4.33), \( c_3Q' = 0 \) is derived. By (4.34), \( c_1u'Q' = 0 \). Hence, \( Q' = 0 \). It follows that \( Q \) and \( R \) are non-zero constants on \( V_2 \).

On the other hand, since the torsion of the director vector field \( \beta \) viewed as a curve in \( \mathbb{E}^3 \) is zero, \( \beta \) is part of a plane curve. Moreover, \( \beta \) has constant curvature \( \sqrt{\varepsilon_2 - \varepsilon_2\varepsilon_3R^2} \).
Hence, $\beta$ is a circle or a hyperbola on the unit pseudo-sphere or the hyperbolic space of radius 1 in $\mathbb{E}_3^3$. Without loss of generality, we may put

$$\beta(s) = \frac{1}{p}(R, \cos ps, \sin ps) \quad \text{or} \quad \beta(s) = \frac{1}{p}(\sinh ps, \cosh ps, R),$$

where $p^2 = \varepsilon_2(1 - \varepsilon_3 R^2)$ and $p > 0$. Then the function $u = \langle \alpha', \beta' \rangle$ is given by

$$u = -\alpha'_2(s) \sin ps + \alpha'_3(s) \cos ps \quad \text{or} \quad u = -\alpha'_1(s) \cosh ps + \alpha'_2(s) \sinh ps,$$

where $\alpha'(s) = (\alpha'_1(s), \alpha'_2(s), \alpha'_3(s))$. Therefore we have

$$u' = -(\alpha''_2 + p\alpha'_3) \sin ps - (p\alpha'_2 - \alpha''_3) \cosh ps \quad \text{or} \quad u' = -\alpha''_1 + p\alpha'_2) \cosh ps - (p\alpha'_1 - \alpha''_2) \sinh ps.$$

Since $u'$ is a constant, $u'$ must be zero. It is a contradiction on $V_1$ and so

$$(u')^2 = \varepsilon_2(Q')^2$$
on $V_1$. It immediately follows

$$\varepsilon_2 = 1$$
on $V_1$. Therefore, we get $u' = \pm Q'$. Suppose $u' \neq -Q'$ on $V_1$. Then $u' = Q'$ and (4.28) can be written as

$$Q'(2\varepsilon_3 c_1 Q' + c_2 Q - c_3 Q) = 0.$$

Since $Q' \neq 0$ on $V$,

$$c_3 Q = 2\varepsilon_3 c_1 Q' + c_2 Q. \quad (4.35)$$

Putting (4.35) into (4.25) and (4.26), respectively, we obtain

$$\varepsilon_3 c_1 Q'R + c_2 Q R + 2\varepsilon_3 c_2 Q' + c_1 Q'' = 0, \quad (4.36)$$

$$\varepsilon_3 c_1 (Q')^2 + c_1 QQ'R - \varepsilon_3 c_1 QQ'' - c_1 Q^2 R' - c_2 QQ' = 0. \quad (4.37)$$

Putting together equations (4.36) and (4.37) with the help of (4.24), we get

$$(\varepsilon_3 c_1 Q' + c_2 Q)(Q' + 2\varepsilon_3 Q R) = 0.$$

Suppose $(\varepsilon_3 c_1 Q' + c_2 Q)(s) \neq 0$ on $V_1$. Then $Q' = -2\varepsilon_3 Q R$. If we make use of it, we can derive $R(\varepsilon_3 c_1 Q' + c_2 Q) = 0$ from (4.36). Since $R$ is non-vanishing, $\varepsilon_3 c_1 Q' + c_2 Q = 0$, a contradiction. Thus

$$\varepsilon_3 c_1 Q' + c_2 Q = 0, \quad (4.38)$$

that is, $c_1 Q$ is constant on each component of $V_1$. From (4.30), $c_1 Q = c_1 u'R$. Similarly as before, it is seen that $c_1 \neq 0$ and $u'$ is a non-zero constant. Hence, $Q = u'R$. If we use the fact that $c_1 Q$ and $Q'$ are constant, $c_2 Q' = 0$ is derived from (4.36). Therefore $c_2 = 0$ on each component of $V_1$. By (4.38), $c_1 = 0$ on each component of $V_1$. Hence, (4.35) implies that $c_3 = 0$ on each component of $V_1$. Since $C$ is a constant vector, $C$ is zero on $M$, a contradiction. Thus, we obtain $u' = -Q'$ on $V_1$. Equation (4.28) with $u' = -Q'$ gives that

$$c_3 Q = -2\varepsilon_3 c_1 Q' - c_2 Q. \quad (4.39)$$

Putting (4.39) together with $u' = -Q'$ into (4.25), we have

$$c_1 Q'' = \varepsilon_3 c_1 Q'R + c_2 Q R - 2\varepsilon_3 c_2 Q'. \quad (4.40)$$
Also, equations (4.24), (4.26), (4.39) and (4.40) give
\[(\varepsilon c_1 Q' + c_2 Q)(Q' - 2\varepsilon_3 QR) = 0\]
on \(V_1\).

Suppose \(\varepsilon c_1 Q' + c_2 Q \neq 0\). Then, \(Q' = 2\varepsilon_3 QR\) and thus \(Q'' = 2\varepsilon_3 Q'R + 2\varepsilon_3 QR'\). Putting it into (4.40) with the help of (4.24), we get
\[R(\varepsilon c_1 Q' + c_2 Q) = 0,\]
from which, \(\varepsilon c_1 Q' + c_2 Q = 0\), a contradiction. Therefore, we get
\[\varepsilon c_1 Q' + c_2 Q = 0\]
on \(V_1\). Thus, \(c_1 Q\) is constant on each component of \(V_1\). Similarly developing the argument as before, we see that the constant vector \(C\) is zero which contradicts our assumption. Consequently, the open subset \(V_1\) is empty, i.e., the functions \(u\) and \(Q\) are constant on each component of \(V\). Since \(Q = u'R\), \(Q\) vanishes on \(V\). Thus, the open subset \(V\) is empty and hence \(Q\) vanishes on \(M\). Thus, (4.3) shows that the Gaussian curvature \(K\) automatically vanishes on \(M\).

Thus, we obtain

**Theorem 4.2.** Let \(M\) be a non-cylindrical ruled surface of type \(M^1_+, M^3_+\) or \(M^1_-\) parameterized by the non-null base curve \(\alpha\) and the director vector field \(\beta\) in \(E^3_1\) with generalized 1-type Gauss map. If \(\beta, \beta'\) and \(\beta''\) are not coplanar along \(\alpha\), then \(M\) is flat.

Combining Definition 1.4, Theorem 4.1, Theorem 4.2 and Classification Theorem of flat surfaces with generalized 1-type Gauss map in Minkowski 3-space in [13], we have the following

**Theorem 4.3.** Let \(M\) be a non-cylindrical ruled surface of type \(M^1_+, M^3_+\) or \(M^1_-\) in \(E^3_1\) with generalized 1-type Gauss map. Then \(M\) is locally part of a plane, the helicoid of the first kind, the helicoid of the second kind, the helicoid of the third kind, a circular cone, a hyperbolic cone or a conical surface of \(G\)-type.

Case 2. Let \(M\) be a non-cylindrical ruled surface of type \(M^2_+, M^2_-\). Then, up to a rigid motion, a parametrization of \(M\) is given by
\[x(s, t) = \alpha(s) + t\beta(s)\]
satisfying \(\langle \alpha', \beta \rangle = 0, \langle \alpha', \alpha' \rangle = \varepsilon_1(= \pm 1), \langle \beta, \beta \rangle = 1\) and \(\langle \beta', \beta' \rangle = 0\) with \(\beta' \neq 0\).

Again, we put the smooth functions \(q\) and \(u\) as follows:
\[q = \|x_s\|^2 = |\langle x_s, x_s \rangle|, \quad u = \langle \alpha', \beta' \rangle.\]
We see that the null vector fields \(\beta'\) and \(\beta \times \beta'\) are orthogonal and they are parallel. It is easily derived as \(\beta' = \beta \times \beta'\). Moreover, we may assume that \(\beta(0) = (0, 0, 1)\) and
\[ \beta \text{ can be taken by } \beta(s) = (as, as, 1) \]

for a non-zero constant \( a \). Then \( \{ \alpha', \beta, \alpha' \times \beta \} \) forms an orthonormal frame along the base curve \( \alpha \). With respect to this frame, we can put

\[
\beta' = \varepsilon_1 u(\alpha' - \alpha' \times \beta) \quad \text{and} \quad \alpha'' = -u\beta + \frac{u'}{u}\alpha' \times \beta. \tag{4.41}
\]

Note that the function \( u \) is non-vanishing.

On the other hand, we can compute the Gauss map \( G \) of \( M \) such as

\[
G = q^{-1/2}(\alpha' \times \beta - t\beta'). \tag{4.42}
\]

And the mean curvature \( H \) and the Gaussian curvature \( K \) of \( M \) are obtained by, respectively,

\[
H = \frac{1}{2}q^{-3/2} \left( u't - \varepsilon_1 \frac{u'}{u} \right) \quad \text{and} \quad K = q^{-2}u^2. \tag{4.43}
\]

Our using (4.4), the Laplacian of the Gauss map \( G \) of \( M \) is expressed as

\[
\Delta G = q^{-7/2}(A_2\alpha' + B_2\beta + D_2\alpha' \times \beta) \tag{4.44}
\]

with respect to the orthonormal frame \( \{ \alpha', \beta, \alpha' \times \beta \} \), where we put

\[
A_2 = 3\varepsilon_1 \frac{(u')^2}{u} t + \varepsilon_4 \varepsilon_1 q \left( -u'' + \frac{(u')^2}{u^2} + uu''t^2 + \varepsilon_1 \frac{(u')^2}{u} t \right) + q \frac{(u')^2}{u} t - 3\varepsilon_1 u(u')^2 t^3 \\
+ \varepsilon_4 \varepsilon_1 u(u')^2 t^3 + 2\varepsilon_4 \varepsilon_1 qu^3 t,
\]

\[
B_2 = \varepsilon_4 qu'(4\varepsilon_1 - ut),
\]

\[
D_2 = 3\varepsilon_1 u(u')^2 t^3 - 3(u')^2 t^2 - \varepsilon_4 q \left( \varepsilon_1 uu''t^2 - u''t + \frac{(u')^2}{u} t \right) - \varepsilon_1 q \frac{(u')^2}{u^2} - q \frac{(u')^2}{u} t \\
- \varepsilon_4 (u')^2 t^2 - 2\varepsilon_4 qu^2 - \varepsilon_4 \varepsilon_1 u(u')^2 t^3 - 2\varepsilon_4 \varepsilon_1 qu^3 t.
\]

We now suppose that the Gauss map \( G \) of \( M \) is of generalized 1-type satisfying the condition (1.1). Then, from (4.41), (4.42) and (4.44), we get

\[
q^{-7/2}(A_2\alpha' + B_2\beta + D_2\alpha' \times \beta) = f q^{-1/2}(1 + \varepsilon_1 ut)\alpha' \times \beta - \varepsilon_1 uto \alpha' + gC. \tag{4.45}
\]

If the constant vector \( C \) is zero, the Gauss map \( G \) is nothing but of pointwise 1-type of the first kind. By a result of [12], \( M \) is part of the conjugate of Enneper’s surface of the second kind.

From now on for a while, we assume that \( C \) is a non-zero constant vector.

Taking the indefinite scalar product to equation (4.45) with the orthonormal vector fields \( \alpha' \), \( \beta \) and \( \alpha' \times \beta \), respectively, we obtain

\[
\varepsilon_1 q^{-7/2}A_2 = -f q^{-1/2}ut + g \langle C, \alpha' \rangle, \tag{4.46}
\]

\[
q^{-7/2}B_2 = g \langle C, \beta \rangle, \tag{4.47}
\]

\[
\varepsilon_1 q^{-7/2}D_2 = f q^{-1/2}(\varepsilon_1 + ut) - g \langle C, \alpha' \times \beta \rangle. \tag{4.48}
\]
On the other hand, in terms of the orthonormal frame \( \{ \alpha', \beta, \alpha' \times \beta \} \), the constant vector \( C \) can be written as
\[
C = c_1 \alpha' + c_2 \beta + c_3 \alpha' \times \beta,
\]
where we have put \( c_1 = \varepsilon_1 \langle C, \alpha' \rangle \), \( c_2 = \langle C, \beta \rangle \) and \( c_3 = -\varepsilon_1 \langle C, \alpha' \times \beta \rangle \). Then equations (4.46), (4.47) and (4.48) are expressed as follows:

\[
\varepsilon_1 q^{-7/2} A_2 = -f q^{-1/2} u + \varepsilon_1 g c_1, \tag{4.49}
\]
\[
q^{-7/2} B_2 = g c_2, \tag{4.50}
\]
\[
\varepsilon_1 q^{-7/2} D_2 = f q^{-1/2} (\varepsilon_1 + u) + \varepsilon_1 g c_3. \tag{4.51}
\]

Differentiating the functions \( c_1 \), \( c_2 \) and \( c_3 \) with respect to the parameter \( s \), we get
\[
c'_1 = -\varepsilon_1 u c_2 - \frac{u'}{u} c_3, \tag{4.52}
\]
\[
c'_2 = u c_1 + u c_3,
\]
\[
c'_3 = -\frac{u'}{u} c_1 + \varepsilon_1 u c_2.
\]

Combining equations (4.49), (4.50) and (4.51), we obtain
\[
c_2(\varepsilon_1 + u) A_2 - \{ \varepsilon_1 c_1 + (c_1 + c_3) u \} B_2 + c_2 u t D_2 = 0. \tag{4.53}
\]
As before, from (4.53), we obtain the following
\[
c_2(2 u w'' - 3 (u')^2) + (c_1 + c_3) u^2 u' = 0, \tag{4.54}
\]
\[
7 c_2 (u')^2 - 5 c_1 u^2 u' - 7 c_3 u^2 u' = 0, \tag{4.55}
\]
\[
c_2 (7 (u')^2 - 3 u u'' - 11 c_1 u^2 u' - 4 c_3 u^2 u') = 0, \tag{4.56}
\]
\[
c_2 (u u'' - (u')^2) + 4 c_1 u^2 u' = 0. \tag{4.57}
\]
Combining equations (4.54) and (4.56), we get
\[
5 c_2 (u u'' - (u')^2) - 7 c_1 u^2 u' = 0. \tag{4.58}
\]
From (4.57) and (4.58), we get \( c_1' u' = 0 \). Hence, equations (4.55) and (4.57) become
\[
u' (c_2 u' - c_3 u^2) = 0, \tag{4.59}
\]
\[
c_2 (u u'' - (u')^2) = 0. \tag{4.60}
\]
Now suppose that \( u'(s_0) \neq 0 \) at some point \( s_0 \in \text{dom}(\alpha) \). Then, there exists an open interval \( J \) such that \( u' \neq 0 \) on \( J \). Then \( c_1 = 0 \) on \( J \). Hence, (4.52) reduces to as follows
\[
\varepsilon_1 u^2 c_2 + u' c_3 = 0,
\]
\[
c'_2 = u c_3,
\]
\[
c'_3 = \varepsilon_1 u c_2. \tag{4.61}
\]
From the above relationships, we see that \( c'_2 \) is constant on \( J \). In this case, if \( c_2 = 0 \), then \( c_3 = 0 \). Hence \( C \) is zero on \( J \). Since \( C \) is a constant vector, \( C \) is zero on \( M \). It is a contradiction. Therefore, \( c_2 \) is non-zero. Solving the differential equation (4.59)
with the help of \( c'^2 = uc_3 \) in (4.61), we get \( u = kc_2 \) for some non-zero constant \( k \). Moreover, since \( c'^2 \) is constant, \( u'' = 0 \). Thus equation (4.60) implies that \( u' = 0 \), which is a contradiction. Therefore, there does not exist such a point \( s_0 \in \text{dom}(\alpha) \) such that \( u'(s_0) \neq 0 \). Hence, \( u \) is constant on \( M \). With the help of (4.43), the mean curvature \( H \) of \( M \) vanishes on \( M \). It is easily seen from (4.4) that the Gauss map \( G \) of \( M \) is of pointwise 1-type of the first kind which means (1.1) is satisfied with \( C = 0 \). Thus, this case does not occur.

As a consequence, we give the following classification:

**Theorem 4.4.** Let \( M \) be a non-cylindrical ruled surface of type \( M_2^2 \) or \( M_2^- \) in \( E_3^1 \) with generalized 1-type Gauss map \( G \). Then the Gauss map \( G \) is of pointwise 1-type of the first kind and \( M \) is an open part of the conjugate of Enneper’s surface of the second kind.

5. **Null scrolls with generalized 1-type Gauss map**

In this section, we examine the null scrolls with generalized 1-type Gauss map in Minkowski 3-space \( E_3^1 \). In particular, we focus on proving the following theorem.

**Theorem 5.1.** Let \( M \) be a null scroll in Minkowski 3-space \( E_3^1 \). Then \( M \) has generalized 1-type Gauss map if and only if \( M \) is part of a Minkowski plane or a \( B \)-scroll.

**Proof.** Suppose that a null scroll \( M \) has generalized 1-type Gauss map. Let \( \alpha = \alpha(s) \) be a null curve in \( E_3^1 \) and \( \beta = \beta(s) \) a null vector field along \( \alpha \) such that \( \langle \alpha', \beta \rangle = 1 \). Then the null scroll \( M \) is parameterized by

\[
x(s, t) = \alpha(s) + t\beta(s)
\]

and we have the natural frame \( \{x_s, x_t\} \) given by

\[
x_s = \alpha' + t\beta' \quad \text{and} \quad x_t = \beta.
\]

We put the smooth functions \( u, v, Q \) and \( R \) by

\[
u = \langle \alpha', \beta' \rangle, \quad v = \langle \beta', \beta' \rangle, \quad Q = \langle \alpha', \beta \times \beta \rangle, \quad R = \langle \alpha', \beta'' \times \beta \rangle.
\]

(5.1)

Then, \( \{\alpha', \beta, \alpha' \times \beta\} \) is a pseudo-orthonormal frame along \( \alpha \).

Straightforward computation gives the Gauss map \( G \) of \( M \) and the Laplacian \( \Delta G \) of \( G \) by

\[
G = \alpha' \times \beta + t\beta' \times \beta \quad \text{and} \quad \Delta G = -2\beta'' \times \beta + 2(u + tv)\beta' \times \beta.
\]

With respect to the pseudo-orthonormal frame \( \{\alpha', \beta, \alpha' \times \beta\} \), the vector fields \( \beta', \beta' \times \beta \) and \( \beta'' \times \beta \) are represented as

\[
\beta' = u\beta - Q\alpha' \times \beta, \quad \beta' \times \beta = Q\beta \quad \text{and} \quad \beta'' \times \beta = R\beta - v\alpha' \times \beta.
\]

(5.2)

Thus, the Gauss map \( G \) and its Laplacian \( \Delta G \) are expressed by

\[
G = \alpha' \times \beta + tQ\beta \quad \text{and} \quad \Delta G = -2(R - uQ - tvQ)\beta + 2v\alpha' \times \beta.
\]

(5.3)
Since $M$ has generalized 1-type Gauss map, $\Delta G = fG + gC$ is satisfied for some non-zero smooth functions $f$, $g$ and a constant vector $C$. From (5.3), we get

$$-2(R - uQ - tQ)\beta + 2v\alpha' \times \beta = f(\alpha' \times \beta + tQ\beta) + gC. \quad (5.4)$$

If the constant vector $C$ is zero, the Gauss map $G$ is of pointwise 1-type of the first kind. According to Classification Theorem in [12], $M$ is an open part of a Minkowski plane or a $B$-scroll.

Suppose that $C$ is non-zero.

If we take the indefinite inner product to equation (5.4) with $\alpha'$, $\beta$ and $\alpha' \times \beta$, respectively, we get

$$-2(R - uQ - tQ) = ftQ + gc_2, \quad gc_1 = 0, \quad 2v = f + gc_3, \quad (5.5)$$

where we have put $c_1 = \langle C, \beta \rangle$, $c_2 = \langle C, \alpha' \rangle$ and $c_3 = \langle C, \alpha' \times \beta \rangle$.

Since $g \neq 0$, equation (5.5) gives $\langle C, \beta' \rangle = 0$. Together with (5.2), we see that $c_3Q = 0$.

Suppose that $Q(s) \neq 0$ on an open interval $\tilde{I} \subset \text{dom}(\alpha)$. Then $c_3 = 0$ on $\tilde{I}$. So the constant vector $C$ can be written as $C = c_2\beta$ on $\tilde{I}$. If we differentiate $C = c_2\beta$ with respect to $s$, $c_2'\beta + c_2\beta' = 0$ and thus $c_2v = 0$. On the other hand, from (5.1) and (5.2), we have $v = Q^2$. Hence $v$ is non-zero on $\tilde{I}$ and so $c_2 = 0$. It contradicts that $C$ is a non-zero vector. In the sequel, $Q$ vanishes identically. Then, $\beta' = u\beta$, which implies $R = 0$. Thus, the Gauss map $G$ is reduced to $G = \alpha' \times \beta$ which depends only on the parameter $s$, from which, the shape operator $S$ of $M$ is easily derived as

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } S = \begin{pmatrix} 0 & 0 \\ k(s) & 0 \end{pmatrix}$$

for some non-vanishing function $k$. Therefore, the null scroll $M$ is part of a Minkowski plane or a flat $B$-scroll described in Section 2 determined by $A = \alpha'$, $B = \beta$, $C = G$ satisfying $C' = -k(s)B$. The converse is obvious. It completes the proof. □

**Corollary 5.2.** There do not exist null scrolls in $E^3_1$ with proper generalized 1-type Gauss map.

**References**


