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CLASSIFICATION THEOREMS OF RULED SURFACES IN MINKOWSKI 3-SPACE

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Abstract. By generalizing the notion of pointwise 1-type Gauss map, the generalized 1-type Gauss map has been recently introduced. Without any assumption, we classified all possible ruled surfaces with generalized 1-type Gauss map in a 3-dimensional Minkowski space. In particular, null scrolls do not have the proper generalized 1-type Gauss map. In fact, it is harmonic.

1. Introduction

A Riemannian manifold can be imbedded in a Euclidean space by Nash's imbedding theorem. That enables us to study Riemannian manifolds as submanifolds of a Euclidean space. In the late 1970's, B.-Y. Chen introduced the notion of finite-type immersion of Riemannian manifolds into Euclidean space by generalizing the eigenvalue problem of the immersion ([1]). An isometric immersion x of a Riemannian manifold M into a Euclidean space $\mathbb{E}^m$ is said to be of *finite-type* if it has the spectral decomposition as

$$x = x_0 + x_1 + \cdots + x_k,$$

where x_0 is a constant vector and $\Delta x_i = \lambda_i x_i$ for some positive integer k and $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. Here, Δ denotes the Laplacian operator defined on M. If $\lambda_1, \ldots, \lambda_k$ are mutually different, M is said to be of k-type. By putting together the eigenvectors of the same eigenvalue, we may assume that a finite-type immersion x of a Riemannian manifold into a Euclidean space is of k-type for some positive integer k.

The notion of finite-type immersion of submanifold into Euclidean space was extended to the study of finite-type immersion or smooth maps defined on submanifolds of a pseudo-Euclidean space $\mathbb{E}^m_s$ with the indefinite metric of index $s \geq 1$. In this sense, it is very natural for geometers to have interest in finite-type Gauss map of submanifolds of a pseudo-Euclidean space ([9, 11, 12]).

We now focus on surfaces of the Minkowski space $\mathbb{E}^3_1$. Let M be a surface in the 3-dimensional Minkowski space $\mathbb{E}^3_1$ with non-degenerate induced metric. From now on,

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a surface M in $\mathbb{E}^3_1$ means non-degenerate, i.e., its induced metric is non-degenerate without otherwise stated. The map $G: M \to Q^2(\epsilon) \subset \mathbb{E}^3_1$ which maps each point of M to a unit normal vector to M at the point is called the *Gauss map* of M, where $\epsilon\ (=\pm 1)$ denotes the sign of the vector field G and $Q^2(\epsilon)$ is a 2-dimensional space form with constant sectional curvature ϵ. A helicoid or a right cone in $\mathbb{E}^3$ has the unique form of Gauss map G which looks like 1-type Gauss map in the usual sense. However, it is quite different and thus the authors et al. defined the following definition.

Definition 1.1. ([2]) A surface M in $\mathbb{E}^3_1$ is said to have *pointwise* 1*-type Gauss map* G or the Gauss map G is of pointwise 1-type if the Gauss map G of M satisfies

$$\Delta G = f(G + \mathbb{C})$$

for some non-zero smooth function f and a constant vector $\mathbb{C}$. In particular, if $\mathbb{C}$ is zero, then the Gauss map is said to be of pointwise 1-type of the first kind. Otherwise, it is said to be of pointwise 1-type of the second kind.

Some other surfaces of $\mathbb{E}^3$ such as conical surfaces have an interesting type of Gauss map. A surface in $\mathbb{E}^3_1$ parameterized by

$$x(s,t) = p + t\beta(s),$$

where p is a point and $\beta(s)$ a unit speed curve is called a *conical surface*. The typical conical surfaces are a right (circular) cone and a plane.

Example 1.2. ([13]) Let M be a surface in $\mathbb{E}^3$ parameterized by

$$x(s,t) = (t\cos^2 s, t\sin s\cos s, t\sin s).$$

Then, the Gauss map G can be obtained by

$$G = \frac{1}{\sqrt{1+\cos^2 s}}(-\sin^3 s, (2-\cos^2 s)\cos s, -\cos^2 s).$$

Its Laplacian turns out to be

$$\Delta G = fG + g\mathbb{C}$$

for some non-zero smooth functions f, g and a constant vector $\mathbb{C}$. The surface M is a kind of conical surfaces generated by a spherical curve $\beta(s) = (\cos^2 s, \sin s\cos s, \sin s)$ on the unit sphere $\mathbb{S}^2(1)$ centered at the origin.

Based on such an example, by generalizing the notion of pointwise 1-type Gauss map, the so-called generalized 1-type Gauss map was introduced.

Definition 1.3. ([13]) A surface M in $\mathbb{E}^3_1$ is said to have *generalized 1-type Gauss map* G or the Gauss map G is of generalized 1-type if the Gauss map G of M satisfies

$$\Delta G = fG + g\mathbb{C} \tag{1.1}$$

for some non-zero smooth functions f, g and a constant vector $\mathbb{C}$. In particular, If the generalized 1-type Gauss map G is not of pointwise 1-type, it is said to be *proper*.

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Definition 1.4. A conical surface with generalized 1-type Gauss map is called a *conical surface of G-type.*

Remark 1.5. ([13]) A conical surface of G-type can be constructed by the functions f, g and the constant vector $\mathbb{C}$ by solving the differential equations generated by (1.1).

Here, we provide an example of a cylindrical ruled surface in the 3-dimensional Minkowski space $\mathbb{E}^3_1$ with generalized 1-type Gauss map.

Example 1.6. Let M be a ruled surface in the Minkowski 3-space $\mathbb{E}^3_1$ parameterized by

$$x(s,t) = \left(\frac{1}{2}\left(s\sqrt{s^2-1} - \ln(s+\sqrt{s^2-1})\right), \frac{1}{2}s^2, t\right), \quad s \geq 1.$$

Then, the Gauss map G is given by

$$G = (-s, -\sqrt{s^2-1}, 0).$$

By a direct computation, we see that its Laplacian satisfies

$$\Delta G = \frac{s-\sqrt{s^2-1}}{(s^2-1)^{\frac{3}{2}}} G + \frac{s(s-\sqrt{s^2-1})}{(s^2-1)^{\frac{3}{2}}}(1,-1,0),$$

which indicates that M has generalized 1-type Gauss map.

2. Preliminaries

Let $\mathbb{E}^3_1$ be a Minkowski 3-space with the Lorentz metric $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) denotes the standard coordinate system in $\mathbb{E}^3_1$. Let M be a non-degenerate surface in $\mathbb{E}^3_1$. A curve in $\mathbb{E}^3_1$ is said to be *space-like*, *time-like* or *null* if its tangent vector field is space-like, time-like or null, respectively. It is well known that in terms of the local coordinates $\{\bar{x}_i\}$ of M the Laplacian Δ is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial \bar{x}_i}\left(\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial \bar{x}_j}\right),$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $\mathcal{G}$ is the determinant of the matrix (g_{ij}) consisting of the components of the first fundamental form.

Now, we define a ruled surface M in the Minkowski 3-space $\mathbb{E}^3_1$. Let I and J be some open intervals in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve in $\mathbb{E}^3_1$ defined on I and $\beta = \beta(s)$ a transversal vector field with $\alpha'(s)$ along α. From now on, $'$ denotes the differentiation with respect to the parameter s unless otherwise stated. Then, a parametrization of a ruled surface M is given by

$$x(s,t) = \alpha(s) + t\beta(s), \quad s \in I, \quad t \in J.$$

The curve $\alpha = \alpha(s)$ is called a *base curve* and $\beta = \beta(s)$ a *director vector field* or a *ruling*. In particular, if β is constant, M is said to be *cylindrical*. Otherwise, it is said to be *non-cylindrical*.

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Ruled surfaces in $\mathbb{E}^3_1$ with non-null base curve may have different types according to their causal character of the base curve and the director vector field. If the base curve α is space-like or time-like, the director vector field β can be chosen to be orthogonal to α that is normalized. The ruled surface M is said to be of type M_+ or M_-, respectively if α is spacelike or timelike, respectively. Also, the ruled surface of type M_+ can be divided into three types. If β is space-like, it is said to be of type M^1_+ or M^2_+ if β' is non-null or null, respectively. When β is time-like, β' is space-like because of the character of the causal vectors, which is said to be of type M^3_+. On the other hand, when α is time-like, β is always space-like. Accordingly, it is also said to be of type M^1_- or M^2_- if β' is non-null or null, respectively. The ruled surface of type M^1_+ or M^2_+ (resp. M^3_+, M^1_- or M^2_-) is clearly space-like (resp. time-like).

If the base curve α is null, the director vector field β along α must be chosen to be null since the ruled surface is non-degenerate. Such a ruled surface M is called a *null scroll*. One of such is a B-scroll ([7], [9]). Other cases such as α is non-null and β is null, or α is null and β is non-null are reduced to one of the types $M^1_\pm$, $M^2_\pm$ and M^3_+, or a null scroll by an appropriate change of the base curve ([10]). Among null scrolls, a B-scroll has an interesting geometric property such as it has constant mean curvature and constant Gaussian curvature. Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{E}^3_1$ with Cartan frame $\{A, B, C\}$, that is, A, B, C are vector fields along α in $\mathbb{E}^3_1$ satisfying the following conditions:

$$\langle A, A\rangle = \langle B, B\rangle = 0, \quad \langle A, B\rangle = 1, \quad \langle A, C\rangle = \langle B, C\rangle = 0, \quad \langle C, C\rangle = 1,$$

$$\alpha' = A, \quad C' = -aA - k(s)B,$$

where a is a constant and $k(s)$ a nowhere vanishing function. A null scroll parametrized by $x = x(s,t) = \alpha(s) + tB(s)$ is called a B-scroll which has mean curvature $H = a$ and Gaussian curvature $K = a^2$. Furthermore, its Laplacian ΔG of the Gauss map G is given by

$$\Delta G = -2a^2 G,$$

from which, we see that a B-scroll is minimal if and only if it is flat.

Throughout the paper, all surfaces in $\mathbb{E}^3_1$ are smooth and connected unless otherwise stated.

3. Cylindrical ruled surfaces in $\mathbb{E}^3_1$ with generalized 1-type Gauss map

In this section, we study the cylindrical ruled surfaces with generalized 1-type Gauss map in the Minkowski 3-space $\mathbb{E}^3_1$.

Let M be a cylindrical ruled surface of type M^1_+, M^1_- or M^3_+ in $\mathbb{E}^3_1$. Then M is parameterized by a base curve α and a unit constant vector β such that

$$x(s,t) = \alpha(s) + t\beta$$

satisfying $\langle \alpha', \alpha'\rangle = \varepsilon_1$ $(= \pm 1)$, $\langle \alpha', \beta\rangle = 0$ and $\langle \beta, \beta\rangle = \varepsilon_2$ $(= \pm 1)$.

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We now suppose that M has generalized 1-type Gauss map G. Then the Gauss map G satisfies the condition (1.1). We put the constant vector $\mathbb{C} = (c_1, c_2, c_3)$ in (1.1) for some constants c_1, c_2 and c_3.

Suppose that $f = g$. Then the Gauss map G is nothing but of pointwise 1-type. A classification of cylindrical ruled surfaces with pointwise 1-type Gauss map in $\mathbb{E}^3_1$ was described in [5].

If M is of type M^1_+, then M is an open part of a Euclidean plane or a cylinder over a curve of infinite-type satisfying

$$c^2 f^{-\frac{1}{3}} - \ln |c^2 f^{-\frac{1}{3}} + 1| = \pm c^3 (s + k) \tag{3.1}$$

if $\mathbb{C}$ is null, or

$$\begin{aligned}\sqrt{\left(c^2 f^{-\frac{1}{3}} + 1\right)^2 + (-c_1^2 + c_2^2)} - \ln \left(c^2 f^{-\frac{1}{3}} + 1 + \sqrt{\left(c^2 f^{-\frac{1}{3}} + 1\right)^2 + (-c_1^2 + c_2^2)} \right) \\ + \ln \sqrt{|-c_1^2 + c_2^2|} = \pm c^3 (s+k)\end{aligned} \tag{3.2}$$

if $\mathbb{C}$ is non-null, where c is some non-zero constant and k is a constant.

If M is of type M^1_-, M is an open part of a Minkowski plane or a cylinder over a curve of infinite-type satisfying

$$c^2 f^{-\frac{1}{3}} + \ln |c^2 f^{-\frac{1}{3}} - 1| = \pm c^3 (s + k) \tag{3.3}$$

or

$$\begin{aligned}\sqrt{\left(c^2 f^{-\frac{1}{3}} - 1\right)^2 - (-c_1^2 + c_2^2)} + \ln \left(c^2 f^{-\frac{1}{3}} - 1 + \sqrt{\left(c^2 f^{-\frac{1}{3}} - 1\right)^2 + |-c_1^2 + c_2^2|} \right) \\ - \ln \sqrt{|-c_1^2 + c_2^2|} = \pm c^3 (s+k)\end{aligned} \tag{3.4}$$

depending on the constant vector $\mathbb{C}$ is null or non-null, respectively, for some non-zero constant c and some constant k.

If M is of type M^3_+, M is an open part of either a Minkowski plane or a cylinder over a curve of infinite-type satisfying

$$\sqrt{c_2^2 + c_3^2 - \left(c^2 f^{-\frac{1}{3}} - 1\right)^2} - \sin^{-1} \left(\frac{c^2 f^{-\frac{1}{3}} - 1}{\sqrt{c_2^2 + c_3^2}} \right) = \pm c^3 (s + k), \tag{3.5}$$

where c is a non-zero constant and k a constant.

We now assume that $f \neq g$. Here, we consider two cases.

Case 1. Let M be a cylindrical ruled surface of type M^1_+ or M^1_-, i.e., $\varepsilon_2 = 1$. Without loss of generality, we may assume that $\alpha(s) = (\alpha_1(s), \alpha_2(s), 0)$ is a plane curve parameterized by an arc length s and β is chosen as $\beta = (0, 0, 1)$. Then the

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Gauss map G of M and the Laplacian ΔG of the Gauss map are respectively obtained by

$$G = (-\alpha_2'(s), -\alpha_1'(s), 0) \quad \text{and} \quad \Delta G = (\varepsilon_1 \alpha_2'''(s), \varepsilon_1 \alpha_1'''(s), 0). \tag{3.6}$$

With the help of (1.1) and (3.6), it immediately follows

$$\mathbb{C} = (c_1, c_2, 0)$$

for some constants c_1 and c_2. And we also have

$$\begin{aligned} \varepsilon_1 \alpha_2''' &= -f\alpha_2' + gc_1, \\ \varepsilon_1 \alpha_1''' &= -f\alpha_1' + gc_2. \end{aligned} \tag{3.7}$$

Firstly, we consider the case that M is of type M_+^1. Since α is space-like, we may put

$$\alpha_1'(s) = \sinh\theta(s) \quad \text{and} \quad \alpha_2'(s) = \cosh\theta(s)$$

for some function $\theta(s)$ of s. Then (3.7) can be written in the form

$$\begin{aligned} (\theta')^2 \cosh\theta + \theta'' \sinh\theta &= -f\cosh\theta + gc_1, \\ (\theta')^2 \sinh\theta + \theta'' \cosh\theta &= -f\sinh\theta + gc_2. \end{aligned}$$

It implies that

$$(\theta')^2 = -f + g(c_1 \cosh\theta - c_2 \sinh\theta) \tag{3.8}$$

and

$$\theta'' = g(-c_1 \sinh\theta + c_2 \cosh\theta). \tag{3.9}$$

In fact, θ' is the signed curvature of the base curve $\alpha = \alpha(s)$.

Suppose θ is a constant, i.e., $\theta' = 0$. Then α is part of a straight line. In this case, M is an open part of a Euclidean plane.

Now we suppose that $\theta' \neq 0$. From (3.7), we see that the functions f and g depend only on the parameter s, i.e., $f(s,t) = f(s)$ and $g(s,t) = g(s)$. Taking the derivative of equation (3.8) and using (3.9), we get

$$3\theta'\theta'' = -f' + g'(c_1 \cosh\theta - c_2 \sinh\theta).$$

With the help of (3.8), it follows that

$$\frac{3}{2}\left((\theta')^2\right)' = -f' + \frac{g'}{g}\left((\theta')^2 + f\right).$$

Solving the above differential equation, we have

$$\theta'(s)^2 = k_1 g^{\frac{2}{3}} + \frac{2}{3} g^{\frac{2}{3}} \int g^{-\frac{2}{3}} f \left(-\frac{f'}{f} + \frac{g'}{g}\right) ds, \quad k_1 \ (\neq 0) \in \mathbb{R}. \tag{3.10}$$

We put

$$\theta'(s) = \pm\sqrt{p(s)},$$

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where $p(s) = |k_1 g^{\frac{2}{3}} + \frac{2}{3} g^{\frac{2}{3}} \int g^{-\frac{2}{3}} f \left(-\frac{f'}{f} + \frac{g'}{g}\right) ds|$. It means that the function θ is determined by the functions f, g and a constant vector satisfying (1.1). Therefore, the cylindrical ruled surface M satisfying (1.1) is determined by a base curve α such that

$$\alpha(s) = \left(\int \sinh\theta(s)ds, \int \cosh\theta(s)ds, 0\right)$$

and the director vector field $\beta(s) = (0, 0, 1)$.

In this case, if f and g are constant, the signed curvature θ' of a base curve α is non-zero constant and the Gauss map G is of usual 1-type. Hence, M is an open part of a hyperbolic cylinder or a circular cylinder ([6]).

Suppose that one of the functions f and g is not constant. Then M is an open part of a cylinder over the base curve of infinite-type satisfying (3.10). For a curve of finite-type in a plane of $\mathbb{E}^3_1$, see [6] in details.

Next we consider the case that M is of type M^1_-. Since α is time-like, we may put

$$\alpha_1'(s) = \cosh\theta(s) \quad \text{and} \quad \alpha_2'(s) = \sinh\theta(s)$$

for some function $\theta(s)$ of s.

As was given in the previous case of type M^1_+, if the signed curvature θ' of the base curve α is zero, M is part of a Minkowski plane.

We now assume that $\theta' \neq 0$. Quite similarly as above, we have

$$\theta'(s)^2 = k_2 g^{\frac{2}{3}} + \frac{2}{3} g^{\frac{2}{3}} \int g^{-\frac{2}{3}} f \left(\frac{f'}{f} - \frac{g'}{g}\right) ds, \quad k_2\ (\neq 0) \in \mathbb{R}, \tag{3.11}$$

or, we put

$$\theta'(s) = \pm\sqrt{q(s)},$$

where $q(s) = |k_2 g^{\frac{2}{3}} + \frac{2}{3} g^{\frac{2}{3}} \int g^{-\frac{2}{3}} f \left(\frac{f'}{f} - \frac{g'}{g}\right) ds|$.

Case 2. Let M be a cylindrical ruled surface of type M^3_+. In this case, without loss of generality we may assume that $\alpha(s) = (0, \alpha_2(s), \alpha_3(s))$ is a plane curve parameterized by the arc length s and β is chosen as $\beta = (1, 0, 0)$. Then the Gauss map G of M and the Laplacian ΔG of the Gauss map are obtained by

$$G = (0, \alpha_3', -\alpha_2') \quad \text{and} \quad \Delta G = (0, -\alpha_3''', \alpha_2'''). \tag{3.12}$$

The relationship (3.12) and the condition (1.1) imply that the constant vector $\mathbb{C}$ has the form

$$\mathbb{C} = (0, c_2, c_3)$$

for some constants c_2 and c_3.

If f and g are both constant, the Gauss map is of 1-type in the usual sense and thus M is an open part of a circular cylinder ([1]).

We now assume that the functions f and g are not both constant. Then, with the help of (1.1) and (3.12), we get

$$\begin{aligned} -\alpha_3''' &= f\alpha_3' + gc_2, \\ \alpha_2''' &= -f\alpha_2' + gc_3. \end{aligned} \tag{3.13}$$

Since α is parameterized by the arc length s, we may put

$$\alpha_2'(s) = \cos\theta(s) \quad \text{and} \quad \alpha_3'(s) = \sin\theta(s)$$

for some function $\theta(s)$ of s. Hence, (3.13) can be expressed as

$$\begin{aligned} (\theta')^2 \sin\theta - \theta'' \cos\theta &= f\sin\theta + gc_2, \\ (\theta')^2 \cos\theta + \theta'' \sin\theta &= f\cos\theta - gc_3. \end{aligned}$$

It follows

$$(\theta')^2 = f + g(c_2 \sin\theta - c_3 \cos\theta). \tag{3.14}$$

Thus, M is a cylinder over the base curve α given by

$$\alpha(s) = \left(0, \int \cos\left(\int \sqrt{r(s)}ds\right) ds, \int \sin\left(\int \sqrt{r(s)}ds\right) ds\right)$$

and the ruling $\beta(s) = (1, 0, 0)$, where $r(s) = |f(s) + g(s)\,(c_2 \sin\theta(s) - c_3 \cos\theta(s))\,|$.

Consequently, we have

Theorem 3.1 (Classification of cylindrical ruled surfaces in $\mathbb{E}^3_1$)**.** *Let M be a cylindrical ruled surface with generalized 1-type Gauss map in the Minkowski 3-space $\mathbb{E}^3_1$. Then, M is an open part of a Euclidean plane, a Minkowski plane, a circular cylinder, a hyperbolic cylinder or a cylinder over a base curve of infinite-type satisfying* (3.1), (3.2), (3.3), (3.4), (3.5), (3.10), (3.11) *or* (3.14).

4. Non-cylindrical ruled surfaces with generalized 1-type Gauss map

In this section, we classify the non-cylindrical ruled surfaces with generalized 1-type Gauss map in $\mathbb{E}^3_1$.

Case 1. Let M be a non-cylindrical ruled surface of type M^1_+, M^3_+ or M^1_-. Then M is parameterized by, up to a rigid motion,

$$x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \alpha', \beta\rangle = 0$, $\langle \beta, \beta\rangle = \varepsilon_2\ (= \pm 1)$ and $\langle \beta', \beta'\rangle = \varepsilon_3\ (= \pm 1)$. Then, $\{\beta, \beta', \beta\times\beta'\}$ is an orthonormal frame along the base curve α. For later use, we define the smooth functions q, u, Q and R as follows:

$$q = \|x_s\|^2 = \varepsilon_4\langle x_s, x_s\rangle, \quad u = \langle \alpha', \beta'\rangle, \quad Q = \langle \alpha', \beta\times\beta'\rangle, \quad R = \langle \beta'', \beta\times\beta'\rangle,$$

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where ε_4 is the sign of the coordinate vector field $x_s = \partial x/\partial s$. The vector fields α', β'', $\alpha' \times \beta$ and $\beta \times \beta''$ are represented in terms of the orthonormal frame $\{\beta, \beta', \beta \times \beta'\}$ along the base curve α as

$$\begin{aligned} \alpha' &= \varepsilon_3 u\beta' - \varepsilon_2\varepsilon_3 Q\beta \times \beta', \\ \beta'' &= -\varepsilon_2\varepsilon_3\beta - \varepsilon_2\varepsilon_3 R\beta \times \beta', \\ \alpha' \times \beta &= \varepsilon_3 Q\beta' - \varepsilon_3 u\beta \times \beta', \\ \beta \times \beta'' &= -\varepsilon_3 R\beta'. \end{aligned} \tag{4.1}$$

Therefore, the smooth function q is given by

$$q = \varepsilon_4(\varepsilon_3 t^2 + 2ut + \varepsilon_3 u^2 - \varepsilon_2\varepsilon_3 Q^2).$$

Note that t is chosen so that q takes positive values.

Furthermore, the Gauss map G of M is given by

$$G = q^{-1/2}\left(\varepsilon_3 Q\beta' - (\varepsilon_3 u + t)\beta \times \beta'\right). \tag{4.2}$$

By using the determinants of the first fundamental form and the second fundamental form, the mean curvature H and the Gaussian curvature K of M are obtained by, respectively,

$$\begin{aligned} H &= \frac{1}{2}\varepsilon_2 q^{-3/2}\left(Rt^2 + (2\varepsilon_3 uR + Q')t + u^2 R + \varepsilon_3 uQ' - \varepsilon_3 u'Q - \varepsilon_2 Q^2 R\right), \\ K &= q^{-2}Q^2. \end{aligned} \tag{4.3}$$

Applying the Gauss and Weingarten formulas, the Laplacian of the Gauss map G of M in $\mathbb{E}^3_1$ is expressed by

$$\Delta G = 2\mathrm{grad}H + \langle G, G\rangle(\mathrm{tr}A_G^2)G, \tag{4.4}$$

where A_G denotes the shape operator of the surface M in $\mathbb{E}^3_1$ and grad$H$ is the gradient of H. Using (4.3), we get

$$\begin{aligned} 2\mathrm{grad}H &= 2\langle e_1, e_1\rangle e_1(H)e_1 + 2\langle e_2, e_2\rangle e_2(H)e_2 \\ &= 2\varepsilon_4 e_1(H)e_1 + 2\varepsilon_2 e_2(H)e_2 \\ &= q^{-7/2}\{-\varepsilon_2(\varepsilon_3 u + t)A_1\beta' - \varepsilon_4 qB_1\beta + \varepsilon_3 QA_1\beta \times \beta'\}, \end{aligned}$$

where $e_1 = \frac{x_s}{||x_s||}$, $\quad e_2 = \frac{x_t}{||x_t||}$,

$$\begin{aligned} A_1 =& 3(u't + \varepsilon_3 uu' - \varepsilon_2\varepsilon_3 QQ')\{Rt^2 + (2\varepsilon_3 uR + Q')t + u^2R + \varepsilon_3 uQ' - \varepsilon_3 u'Q - \varepsilon_2 Q^2 R\} \\ &- (\varepsilon_3 t^2 + 2ut + \varepsilon_3 u^2 - \varepsilon_2\varepsilon_3 Q^2)\{R't^2 + (2\varepsilon_3 u'R + 2\varepsilon_3 uR' + Q'')t + 2uu'R + u^2R' \\ &+ \varepsilon_3 uQ'' - \varepsilon_3 u''Q - 2\varepsilon_2 QQ'R - \varepsilon_2 Q^2 R'\}, \\ B_1 =& \varepsilon_3 Rt^3 + (3uR + 2\varepsilon_3 Q')t^2 + (3\varepsilon_3 u^2 R + 4uQ' - 3u'Q - \varepsilon_2\varepsilon_3 Q^2 R)t + u^3 R + 2\varepsilon_3 u^2 Q' \\ &- \varepsilon_2 uQ^2 R - 3\varepsilon_3 uu'Q + \varepsilon_2\varepsilon_3 Q^2 Q'. \end{aligned}$$

The straightforward computation gives

$$\mathrm{tr}A_G^2 = -\varepsilon_2\varepsilon_4 q^{-3} D_1,$$

where

$$D_1 = -\varepsilon_4(u't+\varepsilon_3 uu'-\varepsilon_2\varepsilon_3 QQ')^2+\varepsilon_3 q\{(\varepsilon_2 QR+\varepsilon_3 u')^2-\varepsilon_2(Q'+\varepsilon_3 uR+Rt)^2-2\varepsilon_3 Q^2\}.$$

Thus, the Laplacian ΔG of the Gauss map G of M is obtained by

$$\Delta G = q^{-7/2}[-\varepsilon_4 qB_1\beta + \{-\varepsilon_2(\varepsilon_3 u+t)A_1+\varepsilon_3 QD_1\}\beta' + \{\varepsilon_3 QA_1-(\varepsilon_3 u+t)D_1\}\beta\times\beta']. \tag{4.5}$$

Now, suppose that the Gauss map G of M is of generalized 1-type. Hence, from (1.1), (4.2) and (4.5), we get

$$\begin{aligned} q^{-7/2}[-\varepsilon_4 qB_1\beta + \{-\varepsilon_2(\varepsilon_3 u+t)A_1+\varepsilon_3 QD_1\}\beta' + \{(\varepsilon_3 QA_1-(\varepsilon_3 u+t)D_1\}\beta\times\beta'] \\ = fq^{-1/2}\left(\varepsilon_3 Q\beta'-(\varepsilon_3 u+t)\beta\times\beta'\right)+g\mathbb{C}. \end{aligned} \tag{4.6}$$

If we take the indefinite scalar product to equation (4.6) with β,β' and $\beta\times\beta'$, respectively, then we obtain respectively,

$$-\varepsilon_2\varepsilon_4 q^{-5/2}B_1 = g\,\langle\mathbb{C},\beta\rangle, \tag{4.7}$$

$$q^{-7/2}\{-\varepsilon_2\varepsilon_3(\varepsilon_3 u+t)A_1+QD_1\} = fq^{-1/2}Q+g\,\langle\mathbb{C},\beta'\rangle, \tag{4.8}$$

$$q^{-7/2}\{-\varepsilon_2 QA_1+\varepsilon_2\varepsilon_3(\varepsilon_3 u+t)D_1\} = fq^{-1/2}\varepsilon_2\varepsilon_3(\varepsilon_3 u+t)+g\,\langle\mathbb{C},\beta\times\beta'\rangle. \tag{4.9}$$

On the other hand, the constant vector $\mathbb{C}$ can be written as

$$\mathbb{C} = c_1\beta+c_2\beta'+c_3\beta\times\beta',$$

where $c_1 = \varepsilon_2\langle\mathbb{C},\beta\rangle$, $c_2 = \varepsilon_3\langle\mathbb{C},\beta'\rangle$ and $c_3 = -\varepsilon_2\varepsilon_3\langle\mathbb{C},\beta\times\beta'\rangle$. Differentiating the functions c_1, c_2 and c_3 with respect to s, we have

$$\begin{aligned} &c_1'-\varepsilon_2\varepsilon_3 c_2 = 0,\\ &c_1+c_2'-\varepsilon_3 Rc_3 = 0,\\ &\varepsilon_2\varepsilon_3 Rc_2-c_3' = 0. \end{aligned} \tag{4.10}$$

Also, equations (4.7), (4.8) and (4.9) are expressed as follows:

$$-\varepsilon_4 q^{-5/2}B_1 = gc_1, \tag{4.11}$$

$$q^{-7/2}\{-\varepsilon_2(\varepsilon_3 u+t)A_1+\varepsilon_3 QD_1\} = fq^{-1/2}\varepsilon_3 Q+gc_2, \tag{4.12}$$

$$q^{-7/2}\{-\varepsilon_3 QA_1+(\varepsilon_3 u+t)D_1\} = fq^{-1/2}(\varepsilon_3 u+t)-gc_3. \tag{4.13}$$

Combining equations (4.11), (4.12) and (4.13), we have

$$\{-\varepsilon_2(\varepsilon_3 u+t)A_1+\varepsilon_3 QD_1\}c_1+q\varepsilon_4 B_1 c_2 = q^3 f\varepsilon_3 Qc_1, \tag{4.14}$$

$$\{-\varepsilon_3 QA_1+(\varepsilon_3 u+t)D_1\}c_1-q\varepsilon_4 B_1 c_3 = q^3 f(\varepsilon_3 u+t)c_1. \tag{4.15}$$

Hence, equations (4.14) and (4.15) yield that

$$-\varepsilon_2\varepsilon_3 A_1 c_1+B_1\{c_2(\varepsilon_3 u+t)+\varepsilon_3 Qc_3\} = 0. \tag{4.16}$$

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First of all, we prove

Theorem 4.1. *Let M be a non-cylindrical ruled surface of type M_+^1, M_+^3 or M_-^1 parameterized by the base curve α and the director vector field β in $\mathbb{E}_1^3$ with generalized 1-type Gauss map. If β, β' and β'' are coplanar along α, then M is an open part of a plane, the helicoid of the first kind, the helicoid of the second kind or the helicoid of the third kind.*

Proof. If the constant vector $\mathbb{C}$ is zero in definition (1.1), then the Gauss map is nothing but of pointwise 1-type of the first kind. Thus, according to Classification Theorem of ruled surfaces in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the first kind in [12], M is an open part of the helicoid of the first kind, the helicoid of the second kind or the helicoid of the third kind.

Now we assume that the constant vector $\mathbb{C}$ is non-zero. In this case, if the function Q is identically zero on M, then M is an open part of a plane because of (4.3).

Suppose that an open subset $U = \{s \in \mathrm{dom}(\alpha) | Q(s) \neq 0\}$ of $\mathrm{dom}(\alpha)$ is not empty. Since β, β' and β'' are coplanar along α, R vanishes. Thus, c_3 is a constant and $c_1'' = -\varepsilon_2\varepsilon_3 c_1$ from (4.10). Since the left hand side of (4.16) is a polynomial in t with functions of s as the coefficients, all of the coefficients which are of functions of s must be zero. From the leading coefficient, we have

$$\varepsilon_2\varepsilon_3 c_1 Q'' + 2c_2 Q' = 0. \tag{4.17}$$

Observing the coefficient of the term involving t^2 of (4.16) with the help of (4.17), we get

$$\varepsilon_2\varepsilon_3 c_1(3u'Q' + u''Q) + 3c_2 u'Q - 2c_3 QQ' = 0. \tag{4.18}$$

Examining the coefficient of the linear term in t of (4.16) and using (4.17) and (4.18), we also get

$$Q\{c_1\left(\varepsilon_2(u')^2 + (Q')^2\right) + \varepsilon_2\varepsilon_3 c_2 QQ' - \varepsilon_3 c_3 u'Q\} = 0.$$

On U,

$$c_1\left(\varepsilon_2(u')^2 + (Q')^2\right) + \varepsilon_2\varepsilon_3 c_2 QQ' - \varepsilon_3 c_3 u'Q = 0. \tag{4.19}$$

Similarly, from the constant term with respect to t of (4.16), we have

$$\varepsilon_3 c_1(-3u'Q' + u''Q) + \varepsilon_2 c_3 QQ' = 0 \tag{4.20}$$

by using (4.17), (4.18) and (4.19). Combining (4.18) and (4.20), we obtain

$$2\varepsilon_3 c_1 u'Q' + \varepsilon_2 c_2 u'Q - \varepsilon_2 c_3 QQ' = 0. \tag{4.21}$$

Now suppose that $u'(s) \neq 0$ at some point $s \in U$ and then $u' \neq 0$ on an open interval $U_1 \subset U$. Equation (4.19) yields

$$\varepsilon_3 c_3 Q = \frac{1}{u'}\{c_1\left(\varepsilon_2(u')^2 + (Q')^2\right) + \varepsilon_2\varepsilon_3 c_2 QQ'\}. \tag{4.22}$$

Substituting (4.22) into (4.21), we get

$$\{(u')^2 - \varepsilon_2(Q')^2\}(\varepsilon_3 c_1 Q' + \varepsilon_2 c_2 Q) = 0,$$

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or, using $c_2 = \varepsilon_2\varepsilon_3 c_1'$ in (4.10),

$$\{(u')^2 - \varepsilon_2(Q')^2\}(c_1Q)' = 0.$$

Suppose that $\big((u')^2 - \varepsilon_2(Q')^2\big)(s_0) \neq 0$ for some $s_0 \in U_1$. Then c_1Q is constant on a component U_2 containing s_0 of U_1.

If $c_1 = 0$ on U_2, we easily see that $c_2 = 0$ by (4.10). Hence, (4.19) yields that $c_3u'Q = 0$ and so $c_3 = 0$. Since $\mathbb{C}$ is a constant vector, $\mathbb{C}$ is zero on M. It contradicts our assumption. Thus, $c_1 \neq 0$ on U_2. From the equation $c_1'' + \varepsilon_2\varepsilon_3c_1 = 0$, we get

$$c_1 = k_1\cos(s + s_1) \quad \text{or} \quad c_1 = k_2\cosh(s + s_2)$$

for some non-zero constants k_i and $s_i \in \mathbb{R}$ $(i = 1,\ 2)$. Since c_1Q is constant, k_1 and k_2 must be zero. Hence $c_1 = 0$, a contradiction. Thus, $(u')^2 - \varepsilon_2(Q')^2 = 0$ on U_1, from which, we get $\varepsilon_2 = 1$ and $u' = \pm Q'$. If $u' \neq -Q'$, then $u' = Q'$ on an open subset U_3 in U_1. Hence (4.19) implies that $Q'(2\varepsilon_3c_1Q' + c_2Q - c_3Q) = 0$. On U_3, we get $c_3Q = 2\varepsilon_3c_1Q' + c_2Q$. Putting it into (4.20), we have

$$\varepsilon_3c_1(Q')^2 - \varepsilon_3c_1QQ'' - c_2QQ' = 0. \tag{4.23}$$

Combining (4.17) and (4.23), c_1Q is constant on U_3. Similarly as above, we can derive that $\mathbb{C}$ is zero on M, which is a contradiction. Therefore, we have $u' = -Q'$ on U_1. Similarly as we just did to the case under the assumption $u' \neq -Q'$, it is also proved that the constant vector $\mathbb{C}$ becomes zero. It is also a contradiction and so $U_1 = \emptyset$. Thus, $u' = 0$ and $Q' = 0$. From (4.3), the mean curvature H vanishes. In this case, the Gauss map G is of pointwise 1-type of the first kind. Hence, the open set U is empty. Therefore $Q = 0$ on M. Because of (4.3), M is an open part of a plane. □

From now on, we assume that R is non-vanishing, i.e., $\beta \wedge \beta' \wedge \beta'' \neq 0$ everywhere on M.

If $f = g$, the Gauss map of the non-cylindrical ruled surface of type M_+^1, M_-^1 or M_+^3 in $\mathbb{E}_1^3$ is of pointwise 1-type. According to Classification Theorem given in [8], M is part of a circular cone or a hyperbolic cone.

Now, we suppose that $f \neq g$ and the constant vector $\mathbb{C}$ is non-zero unless otherwise stated. Similarly as before, we develop our argument with (4.16). The left hand side of (4.16) is a polynomial in t with functions of s as the coefficients and thus they are zero. From the leading coefficient of the left hand side of (4.16), we obtain

$$\varepsilon_2c_1R' + \varepsilon_3c_2R = 0. \tag{4.24}$$

With the help of (4.10), c_1R is constant. If we examine the coefficient of the term of t^3 of the left hand side of (4.16), we get

$$c_1(-\varepsilon_2\varepsilon_3u'R + \varepsilon_2Q'') + 2c_2\varepsilon_3Q' + c_3QR = 0. \tag{4.25}$$

From the coefficient of the term involving t^2 in (4.16), using (4.10) and (4.25), we also get

$$c_1(-3\varepsilon_2\varepsilon_3u'Q' + QQ'R - \varepsilon_2\varepsilon_3u''Q - Q^2R') - 3c_2u'Q + 2c_3QQ' = 0. \tag{4.26}$$

Furthermore, considering the coefficient of the linear term in t of (4.16) and making use of equations (4.10), (4.25) and (4.26), we obtain

$$Q\{c_1(\varepsilon_2(u')^2 + (Q')^2) + c_2\varepsilon_2\varepsilon_3 QQ' - c_3\varepsilon_3 u'Q\} = 0. \tag{4.27}$$

Now, we consider the open set $V = \{s \in \mathrm{dom}(\alpha) | Q(s) \neq 0\}$. Suppose $V \neq \emptyset$. From (4.27),

$$c_1(\varepsilon_2(u')^2 + (Q')^2) + c_2\varepsilon_2\varepsilon_3 QQ' - c_3\varepsilon_3 u'Q = 0. \tag{4.28}$$

Similarly as above, observing the constant term in t of the left hand side of (4.16) with the help of (4.10) and (4.24), and using (4.25), (4.26) and (4.28), we have

$$Q^2(2c_1\varepsilon_3 u'Q' + c_2\varepsilon_2 u'Q - c_3\varepsilon_2 QQ') = 0.$$

Since $Q \neq 0$ on V, one can have

$$2c_1\varepsilon_3 u'Q' + c_2\varepsilon_2 u'Q - c_3\varepsilon_2 QQ' = 0. \tag{4.29}$$

Our making use of the first and the second equations in (4.10), (4.25) reduces to

$$c_1\varepsilon_2 u'R - \varepsilon_2\varepsilon_3(c_1 Q)'' - c_1 Q = 0. \tag{4.30}$$

Suppose that $u'(s) \neq 0$ for some $s \in V$. Then, $u' \neq 0$ on an open subset $V_1 \subset V$. From (4.28), on V_1

$$c_3 Q = \frac{1}{u'}\{\varepsilon_2\varepsilon_3 c_1(u')^2 + \varepsilon_3 c_1(Q')^2 + \varepsilon_2 c_2 QQ'\}. \tag{4.31}$$

Putting (4.31) into (4.29), we have $\{(u')^2 - \varepsilon_2(Q')^2\}(\varepsilon_3 c_1 Q' + \varepsilon_2 c_2 Q) = 0$. With the help of $c_1' = \varepsilon_2\varepsilon_3 c_2$, it becomes

$$\{(u')^2 - \varepsilon_2(Q')^2\}(c_1 Q)' = 0.$$

Suppose that $\left((u')^2 - \varepsilon_2(Q')^2\right)(s) \neq 0$ on V_1. Then $c_1 Q$ is constant on a component V_2 of V_1. Hence, (4.30) yields that

$$c_1 Q = \varepsilon_2 c_1 u'R. \tag{4.32}$$

If $c_1 \equiv 0$ on V_2, (4.10) gives that $c_2 = 0$ and $c_3 R = 0$. Since $R \neq 0$, $c_3 = 0$. Hence, the constant vector $\mathbb{C}$ is zero, a contradiction. Therefore, $c_1 \neq 0$ on V_2. From (4.32), $Q = \varepsilon_2 u'R$. Moreover, u' is a non-zero constant because $c_1 Q$ and $c_1 R$ are constants. Thus, (4.26) and (4.29) can be reduced to as follows

$$c_1 Q'R - c_1 QR' + 2c_3 Q' = 0, \tag{4.33}$$

$$\varepsilon_3 c_1 u'Q' - \varepsilon_2 c_3 QQ' = 0. \tag{4.34}$$

Our putting $Q = \varepsilon_2 u'R$ into (4.33), $c_3 Q' = 0$ is derived. By (4.34), $c_1 u'Q' = 0$. Hence, $Q' = 0$. It follows that Q and R are non-zero constants on V_2.

On the other hand, since the torsion of the director vector field β viewed as a curve in $\mathbb{E}^3_1$ is zero, β is part of a plane curve. Moreover, β has constant curvature $\sqrt{\varepsilon_2 - \varepsilon_2\varepsilon_3 R^2}$.

Hence, β is a circle or a hyperbola on the unit pseudo-sphere or the hyperbolic space of radius 1 in $\mathbb{E}^3_1$. Without loss of generality, we may put

$$\beta(s) = \frac{1}{p}(R, \cos ps, \sin ps) \quad \text{or} \quad \beta(s) = \frac{1}{p}(\sinh ps, \cosh ps, R),$$

where $p^2 = \varepsilon_2(1 - \varepsilon_3 R^2)$ and $p > 0$. Then the function $u = \langle \alpha', \beta' \rangle$ is given by

$$u = -\alpha_2'(s) \sin ps + \alpha_3'(s) \cos ps \quad \text{or} \quad u = -\alpha_1'(s) \cosh ps + \alpha_2'(s) \sinh ps,$$

where $\alpha'(s) = (\alpha_1'(s), \alpha_2'(s), \alpha_3'(s))$. Therefore we have

$$u' = -(\alpha_2'' + p\alpha_3') \sin ps - (p\alpha_2' - \alpha_3'') \cos ps \quad \text{or} \quad u' = (-\alpha_1'' + p\alpha_2') \cosh ps - (p\alpha_1' - \alpha_2'') \sinh ps.$$

Since u' is a constant, u' must be zero. It is a contradiction on V_1 and so

$$(u')^2 = \varepsilon_2 (Q')^2$$

on V_1. It immediately follows

$$\varepsilon_2 = 1$$

on V_1. Therefore, we get $u' = \pm Q'$. Suppose $u' \neq -Q'$ on V_1. Then $u' = Q'$ and (4.28) can be written as

$$Q'(2\varepsilon_3 c_1 Q' + c_2 Q - c_3 Q) = 0.$$

Since $Q' \neq 0$ on V,

$$c_3 Q = 2\varepsilon_3 c_1 Q' + c_2 Q. \tag{4.35}$$

Putting (4.35) into (4.25) and (4.26), respectively, we obtain

$$\varepsilon_3 c_1 Q' R + c_2 Q R + 2\varepsilon_3 c_2 Q' + c_1 Q'' = 0, \tag{4.36}$$

$$\varepsilon_3 c_1 (Q')^2 + c_1 Q Q' R - \varepsilon_3 c_1 Q Q'' - c_1 Q^2 R' - c_2 Q Q' = 0. \tag{4.37}$$

Putting together equations (4.36) and (4.37) with the help of (4.24), we get

$$(\varepsilon_3 c_1 Q' + c_2 Q)(Q' + 2\varepsilon_3 Q R) = 0.$$

Suppose $(\varepsilon_3 c_1 Q' + c_2 Q)(s) \neq 0$ on V_1. Then $Q' = -2\varepsilon_3 Q R$. If we make use of it, we can derive $R(\varepsilon_3 c_1 Q' + c_2 Q) = 0$ from (4.36). Since R is non-vanishing, $\varepsilon_3 c_1 Q' + c_2 Q = 0$, a contradiction. Thus

$$\varepsilon_3 c_1 Q' + c_2 Q = 0, \tag{4.38}$$

that is, $c_1 Q$ is constant on each component of V_1. From (4.30), $c_1 Q = c_1 u' R$. Similarly as before, it is seen that $c_1 \neq 0$ and u' is a non-zero constant. Hence, $Q = u' R$. If we use the fact that $c_1 Q$ and Q' are constant, $c_2 Q' = 0$ is derived from (4.36). Therefore $c_2 = 0$ on each component of V_1. By (4.38), $c_1 = 0$ on each component of V_1. Hence, (4.35) implies that $c_3 = 0$ on each component of V_1. Since $\mathbb{C}$ is a constant vector, $\mathbb{C}$ is zero on M, a contradiction. Thus, we obtain $u' = -Q'$ on V_1. Equation (4.28) with $u' = -Q'$ gives that

$$c_3 Q = -2\varepsilon_3 c_1 Q' - c_2 Q. \tag{4.39}$$

Putting (4.39) together with $u' = -Q'$ into (4.25), we have

$$c_1 Q'' = \varepsilon_3 c_1 Q' R + c_2 Q R - 2\varepsilon_3 c_2 Q'. \tag{4.40}$$

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Also, equations (4.24), (4.26), (4.39) and (4.40) give

$$(\varepsilon_3 c_1 Q' + c_2 Q)(Q' - 2\varepsilon_3 QR) = 0$$

on V_1.

Suppose $\varepsilon_3 c_1 Q' + c_2 Q \neq 0$. Then, $Q' = 2\varepsilon_3 QR$ and thus $Q'' = 2\varepsilon_3 Q'R + 2\varepsilon_3 QR'$. Putting it into (4.40) with the help of (4.24), we get

$$R(\varepsilon_3 c_1 Q' + c_2 Q) = 0,$$

from which, $\varepsilon_3 c_1 Q' + c_2 Q = 0$, a contradiction. Therefore, we get

$$\varepsilon_3 c_1 Q' + c_2 Q = 0$$

on V_1. Thus, $c_1 Q$ is constant on each component of V_1. Similarly developing the argument as before, we see that the constant vector $\mathbb{C}$ is zero which contradicts our assumption. Consequently, the open subset V_1 is empty, i.e., the functions u and Q are constant on each component of V. Since $Q = u'R$, Q vanishes on V. Thus, the open subset V is empty and hence Q vanishes on M. Thus, (4.3) shows that the Gaussian curvature K automatically vanishes on M.

Thus, we obtain

Theorem 4.2. *Let M be a non-cylindrical ruled surface of type M_+^1, M_+^3 or M_-^1 parameterized by the non-null base curve α and the director vector field β in $\mathbb{E}_1^3$ with generalized 1-type Gauss map. If β, β' and β'' are not coplanar along α, then M is flat.*

Combining Definition 1.4, Theorem 4.1, Theorem 4.2 and Classification Theorem of flat surfaces with generalized 1-type Gauss map in Minkowski 3-space in [13], we have the following

Theorem 4.3. *Let M be a non-cylindrical ruled surface of type M_+^1, M_+^3 or M_-^1 in $\mathbb{E}_1^3$ with generalized 1-type Gauss map. Then M is locally part of a plane, the helicoid of the first kind, the helicoid of the second kind, the helicoid of the third kind, a circular cone, a hyperbolic cone or a conical surface of G-type.*

Case 2. Let M be a non-cylindrical ruled surface of type M_+^2, M_-^2. Then, up to a rigid motion, a parametrization of M is given by

$$x(s,t) = \alpha(s) + t\beta(s)$$

satisfying $\langle \alpha', \beta \rangle = 0$, $\langle \alpha', \alpha' \rangle = \varepsilon_1(=\pm 1)$, $\langle \beta, \beta \rangle = 1$ and $\langle \beta', \beta' \rangle = 0$ with $\beta' \neq 0$.

Again, we put the smooth functions q and u as follows:

$$q = \|x_s\|^2 = |\langle x_s, x_s \rangle|, \quad u = \langle \alpha', \beta' \rangle.$$

We see that the null vector fields β' and $\beta \times \beta'$ are orthogonal and they are parallel. It is easily derived as $\beta' = \beta \times \beta'$. Moreover, we may assume that $\beta(0) = (0, 0, 1)$ and

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β can be taken by

$$\beta(s) = (as, as, 1)$$

for a non-zero constant a. Then $\{\alpha', \beta, \alpha' \times \beta\}$ forms an orthonormal frame along the base curve α. With respect to this frame, we can put

$$\beta' = \varepsilon_1 u(\alpha' - \alpha' \times \beta) \quad \text{and} \quad \alpha'' = -u\beta + \frac{u'}{u}\alpha' \times \beta. \tag{4.41}$$

Note that the function u is non-vanishing.

On the other hand, we can compute the Gauss map G of M such as

$$G = q^{-1/2}(\alpha' \times \beta - t\beta'). \tag{4.42}$$

And the mean curvature H and the Gaussian curvature K of M are obtained by, respectively,

$$H = \frac{1}{2}q^{-3/2}\left(u't - \varepsilon_1\frac{u'}{u}\right) \quad \text{and} \quad K = q^{-2}u^2. \tag{4.43}$$

Our using (4.4), the Laplacian of the Gauss map G of M is expressed as

$$\Delta G = q^{-7/2}\left(A_2\alpha' + B_2\beta + D_2\alpha' \times \beta\right) \tag{4.44}$$

with respect to the orthonormal frame $\{\alpha', \beta, \alpha' \times \beta\}$, where we put

$$\begin{aligned}
A_2 =& 3\varepsilon_1\frac{(u')^2}{u}t + \varepsilon_4\varepsilon_1 q\left(-\frac{u''}{u} + \frac{(u')^2}{u^2} + uu''t^2 + \varepsilon_1\frac{(u')^2}{u}t\right) + q\frac{(u')^2}{u}t - 3\varepsilon_1 u(u')^2t^3 \\
&+ \varepsilon_4\varepsilon_1 u(u')^2t^3 + 2\varepsilon_4\varepsilon_1 qu^3t, \\
B_2 =& \varepsilon_4 qu'(4\varepsilon_1 - ut), \\
D_2 =& 3\varepsilon_1 u(u')^2t^3 - 3(u')^2t^2 - \varepsilon_4 q\left(\varepsilon_1 uu''t^2 - u''t + \frac{(u')^2}{u}t\right) - \varepsilon_1 q\frac{(u')^2}{u^2} - q\frac{(u')^2}{u}t \\
&- \varepsilon_4(u')^2t^2 - 2\varepsilon_4 qu^2 - \varepsilon_4\varepsilon_1 u(u')^2t^3 - 2\varepsilon_4\varepsilon_1 qu^3t.
\end{aligned}$$

We now suppose that the Gauss map G of M is of generalized 1-type satisfying the condition (1.1). Then, from (4.41), (4.42) and (4.44), we get

$$q^{-7/2}\left(A_2\alpha' + B_2\beta + D_2\alpha' \times \beta\right) = fq^{-1/2}\{(1 + \varepsilon_1 ut)\alpha' \times \beta - \varepsilon_1 ut\alpha'\} + g\mathbb{C}. \tag{4.45}$$

If the constant vector $\mathbb{C}$ is zero, the Gauss map G is nothing but of pointwise 1-type of the first kind. By a result of [12], M is part of the conjugate of Enneper's surface of the second kind.

From now on for a while, we assume that $\mathbb{C}$ is a non-zero constant vector.

Taking the indefinite scalar product to equation (4.45) with the orthonormal vector fields α', β and $\alpha' \times \beta$, respectively, we obtain

$$\varepsilon_1 q^{-7/2}A_2 = -fq^{-1/2}ut + g\langle\mathbb{C}, \alpha'\rangle, \tag{4.46}$$

$$q^{-7/2}B_2 = g\langle\mathbb{C}, \beta\rangle, \tag{4.47}$$

$$\varepsilon_1 q^{-7/2}D_2 = fq^{-1/2}(\varepsilon_1 + ut) - g\langle\mathbb{C}, \alpha' \times \beta\rangle. \tag{4.48}$$

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On the other hand, in terms of the orthonormal frame $\{\alpha', \beta, \alpha' \times \beta\}$, the constant vector $\mathbb{C}$ can be written as

$$\mathbb{C} = c_1\alpha' + c_2\beta + c_3\alpha' \times \beta,$$

where we have put $c_1 = \varepsilon_1\langle\mathbb{C}, \alpha'\rangle$, $c_2 = \langle\mathbb{C}, \beta\rangle$ and $c_3 = -\varepsilon_1\langle\mathbb{C}, \alpha' \times \beta\rangle$. Then equations (4.46), (4.47) and (4.48) are expressed as follows:

$$\varepsilon_1 q^{-7/2} A_2 = -f q^{-1/2} ut + \varepsilon_1 g c_1, \tag{4.49}$$

$$q^{-7/2} B_2 = g\ c_2, \tag{4.50}$$

$$\varepsilon_1 q^{-7/2} D_2 = f q^{-1/2}(\varepsilon_1 + ut) + \varepsilon_1 g c_3. \tag{4.51}$$

Differentiating the functions c_1, c_2 and c_3 with respect to the parameter s, we get

$$\begin{aligned} c_1' &= -\varepsilon_1 u c_2 - \frac{u'}{u} c_3, \\ c_2' &= u c_1 + u c_3, \\ c_3' &= -\frac{u'}{u} c_1 + \varepsilon_1 u c_2. \end{aligned} \tag{4.52}$$

Combining equations (4.49), (4.50) and (4.51), we obtain

$$c_2(\varepsilon_1 + ut)A_2 - \{\varepsilon_1 c_1 + (c_1 + c_3)ut\}B_2 + c_2 ut D_2 = 0. \tag{4.53}$$

As before, from (4.53), we obtain the following

$$c_2(2uu'' - 3(u')^2) + (c_1 + c_3)u^2u' = 0, \tag{4.54}$$

$$7c_2(u')^2 - 5c_1u^2u' - 7c_3u^2u' = 0, \tag{4.55}$$

$$c_2(7(u')^2 - 3uu'') - 11c_1u^2u' - 4c_3u^2u' = 0, \tag{4.56}$$

$$c_2(uu'' - (u')^2) + 4c_1u^2u' = 0. \tag{4.57}$$

Combining equations (4.54) and (4.56), we get

$$5c_2(uu'' - (u')^2) - 7c_1u^2u' = 0. \tag{4.58}$$

From (4.57) and (4.58), we get $c_1u' = 0$. Hence, equations (4.55) and (4.57) become

$$u'(c_2u' - c_3u^2) = 0, \tag{4.59}$$

$$c_2(uu'' - (u')^2) = 0. \tag{4.60}$$

Now suppose that $u'(s_0) \neq 0$ at some point $s_0 \in \mathrm{dom}(\alpha)$. Then, there exists an open interval J such that $u' \neq 0$ on J. Then $c_1 = 0$ on J. Hence, (4.52) reduces to as follows

$$\begin{aligned} \varepsilon_1 u^2 c_2 + u' c_3 &= 0, \\ c_2' &= u c_3, \\ c_3' &= \varepsilon_1 u c_2. \end{aligned} \tag{4.61}$$

From the above relationships, we see that c_2' is constant on J. In this case, if $c_2 = 0$, then $c_3 = 0$. Hence $\mathbb{C}$ is zero on J. Since $\mathbb{C}$ is a constant vector, $\mathbb{C}$ is zero on M. It is a contradiction. Therefore, c_2 is non-zero. Solving the differential equation (4.59) with the help of $c_2' = uc_3$ in (4.61), we get $u = kc_2$ for some non-zero constant k. Moreover, since c_2' is constant, $u'' = 0$. Thus equation (4.60) implies that $u' = 0$, which is a contradiction. Therefore, there does not exist such a point $s_0 \in \mathrm{dom}(\alpha)$ such that $u'(s_0) \neq 0$. Hence, u is constant on M. With the help of (4.43), the mean curvature H of M vanishes on M. It is easily seen from (4.4) that the Gauss map G of M is of pointwise 1-type of the first kind which means (1.1) is satisfied with $\mathbb{C} = 0$. Thus, this case does not occur.

As a consequence, we give the following classification:

Theorem 4.4. *Let M be a non-cylindrical ruled surface of type M_+^2 or M_-^2 in $\mathbb{E}_1^3$ with generalized 1-type Gauss map G. Then the Gauss map G is of pointwise 1-type of the first kind and M is an open part of the conjugate of Enneper's surface of the second kind.*

5. Null scrolls with generalized 1-type Gauss map

In this section, we examine the null scrolls with generalized 1-type Gauss map in Minkowski 3-space $\mathbb{E}_1^3$. In particular, we focus on proving the following theorem.

Theorem 5.1. *Let M be a null scroll in Minkowski 3-space $\mathbb{E}_1^3$. Then M has generalized 1-type Gauss map G if and only if M is part of a Minkowski plane or a B-scroll.*

Proof. Suppose that a null scroll M has generalized 1-type Gauss map. Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{E}_1^3$ and $\beta = \beta(s)$ a null vector field along α such that $\langle \alpha', \beta \rangle = 1$. Then the null scroll M is parameterized by

$$x(s,t) = \alpha(s) + t\beta(s)$$

and we have the natural frame $\{x_s, x_t\}$ given by

$$x_s = \alpha' + t\beta' \quad \text{and} \quad x_t = \beta.$$

We put the smooth functions u, v, Q and R by

$$u = \langle \alpha', \beta' \rangle, \quad v = \langle \beta', \beta' \rangle, \quad Q = \langle \alpha', \beta' \times \beta \rangle, \quad R = \langle \alpha', \beta'' \times \beta \rangle. \tag{5.1}$$

Then, $\{\alpha', \beta, \alpha' \times \beta\}$ is a pseudo-orthonormal frame along α.

Straightforward computation gives the Gauss map G of M and the Laplacian ΔG of G by

$$G = \alpha' \times \beta + t\beta' \times \beta \quad \text{and} \quad \Delta G = -2\beta'' \times \beta + 2(u + tv)\beta' \times \beta.$$

With respect to the pseudo-orthonormal frame $\{\alpha', \beta, \alpha' \times \beta\}$, the vector fields β', $\beta' \times \beta$ and $\beta'' \times \beta$ are represented as

$$\beta' = u\beta - Q\alpha' \times \beta, \quad \beta' \times \beta = Q\beta \quad \text{and} \quad \beta'' \times \beta = R\beta - v\alpha' \times \beta. \tag{5.2}$$

Thus, the Gauss map G and its Laplacian ΔG are expressed by

$$G = \alpha' \times \beta + tQ\beta \quad \text{and} \quad \Delta G = -2(R - uQ - tvQ)\beta + 2v\alpha' \times \beta. \tag{5.3}$$

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Since M has generalized 1-type Gauss map, $\Delta G = fG + g\mathbb{C}$ is satisfied for some non-zero smooth functions f, g and a constant vector $\mathbb{C}$. From (5.3), we get

$$-2(R - uQ - tvQ)\beta + 2v\alpha' \times \beta = f(\alpha' \times \beta + tQ\beta) + g\mathbb{C}. \tag{5.4}$$

If the constant vector $\mathbb{C}$ is zero, the Gauss map G is of pointwise 1-type of the first kind. According to Classification Theorem in [12], M is an open part of a Minkowski plane or a B-scroll.

Suppose that $\mathbb{C}$ is non-zero.

If we take the indefinite inner product to equation (5.4) with α', β and $\alpha' \times \beta$, respectively, we get

$$-2(R - uQ - tvQ) = ftQ + gc_2, \quad gc_1 = 0, \quad 2v = f + gc_3, \tag{5.5}$$

where we have put

$$c_1 = \langle \mathbb{C}, \beta\rangle, \; c_2 = \langle \mathbb{C}, \alpha'\rangle \quad \text{and} \quad c_3 = \langle \mathbb{C}, \alpha' \times \beta\rangle.$$

Since $g \neq 0$, equation (5.5) gives $\langle \mathbb{C}, \beta'\rangle = 0$. Together with (5.2), we see that $c_3Q = 0$.

Suppose that $Q(s) \neq 0$ on an open interval $\tilde{I} \subset \mathrm{dom}(\alpha)$. Then $c_3 = 0$ on $\tilde{I}$. So the constant vector $\mathbb{C}$ can be written as $\mathbb{C} = c_2\beta$ on $\tilde{I}$. If we differentiate $\mathbb{C} = c_2\beta$ with respect to s, $c_2'\beta + c_2\beta' = 0$ and thus $c_2v = 0$. On the other hand, from (5.1) and (5.2), we have $v = Q^2$. Hence v is non-zero on $\tilde{I}$ and so $c_2 = 0$. It contradicts that $\mathbb{C}$ is a non-zero vector. In the sequel, Q vanishes identically. Then, $\beta' = u\beta$, which implies $R = 0$. Thus, the Gauss map G is reduced to $G = \alpha' \times \beta$ which depends only on the parameter s, from which, the shape operator S of M is easily derived as

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad S = \begin{pmatrix} 0 & 0 \\ k(s) & 0 \end{pmatrix}$$

for some non-vanishing function k. Therefore, the null scroll M is part of a Minkowski plane or a flat B-scroll described in Section 2 determined by $A = \alpha'$, $B = \beta$, $C = G$ satisfying $C' = -k(s)B$. The converse is obvious. It completes the proof. □

Corollary 5.2. *There do not exist null scrolls in $\mathbb{E}^3_1$ with proper generalized 1-type Gauss map.*

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