

Article

Lie Symmetry and Painlevé test for the SIRD model

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Abstract: In this paper, Lie symmetry and Painlevé Techniques are applied to the SIRD (Susceptible, Infected, Recovered and Dead) model. A demonstration of the integrability of the model is provided to present an exact solution. The study revealed that nonlinear system passes Painlevé test and does not possess complex chaotic behaviour. However, the system fails to pass the Painlevé test while constraints reach values equivalent to the corresponding complex chaotic behaviour. The two-dimensional Lie symmetry algebra and the commutator table of the infinitesimal generators are obtained. Lie symmetry analysis serves to linearize the nonlinear system and find the corresponding invariant solution.

Keywords: Painlevé Analysis; Lie symmetry Analysis; SIR Model

1. Introduction

In this paper, we modified the classical SIR model of Kermack and McKendrick [5] by assuming that an individual can be born infected. We assumed that after the recovery process, the disease person becomes resistant and the number of individuals died from the disease are counted. The model divides the total population into three different classes: The susceptible class, S , are those who can get infected with the disease; the infective class, I , are those who can spread the disease after getting infected; the removed class, R , are those individuals who recovered from the disease, resistant or sequestered while waiting for recovery; and the death class, D , are those who die from the disease. Most of the infantile viruses, essentially measles, have a removed and death class [14]. The model flow diagram which represented the disease is given by

$$S \rightarrow I \rightarrow R \rightarrow D.$$

The model is governed by the following nonlinear system of first order ordinary differential equations [8]

$$\dot{S} = -\beta SI + \gamma I - \mu_1 S + \nu_1 S \quad (1)$$

$$\dot{I} = \beta SI - (\alpha + \gamma)I - \mu_2 I + \nu_2 I \quad (2)$$

$$\dot{R} = \alpha I - \mu_3 R + \nu_3 R \quad (3)$$

$$\dot{D} = \mu_1 S + \mu_2 I + \mu_3 R \quad (4)$$

the dot represents differentiation with respect to time, t , the vulnerable (or susceptible) population is denoted by $S(t)$, $I(t)$ represents the infected population by the disease, the population group cured by the disease are represented by $R(t)$, $D(t)$ denotes dead population due to infectious disease, μ_1 represents the natural death rate of vulnerable group of population, a death rate due to infectious disease is represented by μ_2 , the rate at which a rescue (or recovered) individual may die is denoted by μ_3 , the natural birth rate is represented by ν_1 , an individual may be born infected at rate ν_2 , the proportionate birth rate of the recovered individual is denoted by ν_3 , the rate in which a recovered individual becomes immune is denoted by α , the infection rate is represented by γ , while β denotes the rate of infected individual becoming susceptible, after efficient treatment.

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The discussion in the present paper is organised as follows. In Section 2, we reduced the four-dimensional system (1)-(4) into a single ordinary differential equation of second-order. In Section 3, the painlevé-analysis was performed for the solutions of nonlinear second order differential equation; parameters values in which nonlinear system (1)-(4) passes painlevé test is found. In Section 4, we performed a Lie symmetry method of the reduced equations to obtain invariant solutions. The explicit solutions and discussion were established in Section 5 and 6 respectively.

2. The Reduced form of the nonlinear system (1)-(2)

In this Section we reduce the four-dimensional system (1)-(4) to a one-dimension second order ordinary differential equation. Since equations (1) and (2) does not dependent on R and D , therefore, we can find the number of individual who are recovered, R once we know the infected individual I , hereafter we can excluded R in any consequent analysis of the system.

From (2) we have

$$S = \frac{1}{\beta} \dot{I} + \frac{\alpha + \gamma + \mu_2 - \nu_2}{\beta}. \quad (5)$$

The derivative of (5) gives

$$\dot{S} = \frac{1}{\beta} \left(\frac{I\ddot{I} - \dot{I}^2}{I^2} \right). \quad (6)$$

The substitution of (5) and (6) into (1) gives

$$\begin{aligned} I\ddot{I} - \dot{I}^2 &= -\beta\dot{I}I^2 - \beta(\alpha + \gamma + \mu_2 - \nu_2)I^3 + \gamma\beta I^3 - \mu_1\dot{I}I \\ &- \mu_1(\alpha + \gamma + \mu_2 - \nu_2)I^2 + \nu_1\dot{I}I + \nu_1(\alpha + \gamma + \mu_2 - \nu_2)I^2. \end{aligned} \quad (7)$$

We have after some arrangement

$$I\ddot{I} - \dot{I}^2 + \beta\dot{I}I^2 = -\beta(\alpha + 2\gamma + \mu_2 - \nu_2)I^3 + (\nu_1 - \mu_1)\dot{I}I + (\nu_1 - \mu_1)(\alpha + \gamma + \nu_2 - \mu_2)I^2. \quad (8)$$

We may attain the following simplification by means of the given change of variable:

$$I = \frac{u}{\beta}. \quad (9)$$

The substitution of (9) to (8) gives

$$u\ddot{u} - \dot{u}^2 + \dot{u}u^2 = a\dot{u}u - (b + \gamma)u^3 + abu^2 \quad (10)$$

with

$$\nu_1 - \mu_1 = a \quad (11)$$

$$\alpha + \gamma + \mu_2 - \nu_2 = b. \quad (12)$$

3. Painlevé Analysis

The method of Painlevé Analysis was found by a Russian mathematician, Kowalevski [6]. She was very determined to know about the integrability conditions of the Euler equations which was in great importance throughout the historical time of La Belle Époque [16]. The technique of Painlevé Analysis have been contributed significantly in solving nonlinear differential equations. Concerning the methodology, We referred the interested reader to the book written by Tabor in [18] and later on explain by Ramani et al. in [17]. The quintessence of investigating an ordinary differential equation as well as a system of nonlinear ordinary differential equations from the view point of singularity analysis is the Willpower of the existence of isolated movable singularities whereby one may obtained a Laurent series expansion containing arbitrary coefficients which will be the same with the order of

the corresponding differential equations [7]. However, the initial conditions of the system are the main aspect as far as the location of the singularity is concern. Nevertheless, a more complex equation (or system of equations) involving multifaceted arrangement retained more than one polelike singularity. Conversely, when it comes to the system of differential equations with many singularities, one need to assure the existence of a Laurent expansion containing an essential amount of arbitrary constant. Nevertheless, a counter example to this can be find in [7] Whereby the behaviour of the first form of singularity preserves a Laurent series expansion containing a precise amount of arbitrary parameters while the second does not preserve, nonetheless contain an uneven solution [13] of the category discussed by Ince [4]. Though, the general solution of the nonlinear system is remarkably explicit.

Ablowitz [1], developed a standard algorithm in order to analyse a differential equation from the view point of Painlevé. Even if there are some illustrations of specific significance of the Painlevé method in analysing a system of nonlinear first order differential equations which are common in mathematical modelling of epidemics, such that the tactical method approach promoted in [2] is considered. In this section, we will primarily, summarise the typical algorithm due to Ablowitz [1]. Furthermore, different approach will be provided. We will start by considering the following autonomous system of first-order ordinary differential equations

$$\Phi_i(x, \dot{x}, \sigma) = 0, \quad i = 1, n \quad (13)$$

with n dependent variables represented by x , the independent variable is denoted by t while σ is the conservative of constraints which constantly appears to increase a system commonly used in mathematical modelling of natural phenomena. The assumption made here is that the first derivative of the n functions Φ_i are linear and the later are polynomials functions in the dependent variables x . We have to bear in mind that those assumptions are not entirely necessary, nevertheless they do simplify the complexity of the model and explain the reality.

3.1. The Painlevé Test

In [15], Ove proved that if a differential equation admits solutions which are single value in a neighborhood of non characteristic movable singularity manifolds, then this equation is integrable and therefore possesses the Painlevé property. The author also claimed that the method described by Weiss and Carnevale [19] proposes a necessary condition of integrability, also known as the Painlevé test. However, while computing the Painlevé test, one seeks solution of a given rational differential equation in the form of a Laurent series (also known as the Painlevé expansion) [15].

The execution of Painlevé test suggests that solution of the following differential equation

$$F(x, u, u_{x_1}, u_{x_2}, \dots) = 0 \quad (14)$$

with independent variable $x = (x_0, x_1, x_2, \dots, x_{n-1})$ has the form below

$$u(x) = \phi^{-m}(x) \sum_{j=0}^{\infty} u_j(x) \phi^j(x) \quad (15)$$

where the functions $\phi, u_j (u_0 \neq 0)$ are analytic of x around the region of $\phi(x) = 0$.

After substituting equation (15) into (14), if one obtains the correct number of arbitrary functions which are required by the Cauchy-Kovalevskaya expression, then the corresponding differential equation (14) passes the Painlevé test. The Cauchy-Kovalevskaya expression is represented by the expansion coefficients in (15), whereby ϕ is counted as one of the arbitrary functions. The exponents in the Painlevé expansion, where the arbitrary functions are to appear, are known as the resonances. If a

73 given differential equation possesses a Painlevé test, then the construction of Bäcklund transformations
 74 to linear equations or other known integrable equations becomes possible. In this regards, the
 75 sufficient condition of integrability and the Painlevé property of the given equation is evident.

76

77 In the following subsection, Ablowitz's algorithm will be used to find the leading-order behaviour
 78 of the nonlinear system (1)-(4) and the reduced equation (10). We will start by substituting $x_i =$
 79 $\sigma_i \tau^{\nu_i}$, $i = 1, n$, with $\tau = t - t_0$ and t_0 the assumed location of the movable singularity, into the system
 80 (13) and compare the resulting power.

81 3.2. Painlevé Analysis of the nonlinear system (1)-(4)

As mentioned in Section 2, the Painlevé analysis of nonlinear equations (1)-(4) start by analyzing
 the first two equations (1)-(2). In this regard, we commence in the customary manner by substituting

$$S = \sum_{i=0}^{\infty} c_i \tau^{i-1}, \quad I = \sum_{i=0}^{\infty} d_i \tau^{i-1} \quad (16)$$

82 into (1)-(2) and obtain the following

$$\begin{aligned} \sum_{i,j=0}^{\infty} c_i (i-1) \tau^{i-1} + \beta c_i d_j \tau^{i+j-2} - \gamma d_i \tau^{i-1} (\mu_1 - \nu_1) c_i \tau^{i-1} &= 0, \\ \sum_{i,j=0}^{\infty} d_i (i-1) \tau^{i-1} - \beta c_i d_j \tau^{i+j-2} + (\nu_2 - \alpha - \gamma - \mu_2) d_i \tau^{i-1} &= 0. \end{aligned}$$

At τ^{-2} , we obtain

$$c_0 = -\frac{1}{\beta}, \quad d_0 = \frac{1}{\beta} \quad (17)$$

At τ^{-1} , we obtain

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} \frac{\mu_1 - \nu_1}{\beta} \\ \frac{\alpha + \mu_2 - \gamma - \nu_2}{\beta} \end{pmatrix}. \quad (18)$$

The determination of the resonances is find by substituting [9]

$$S = -\frac{1}{\beta} \tau^{-1} + k_1 \tau^{r-1}, \quad I = \frac{1}{\beta} \tau^{-1} + k_2 \tau^{r-1} \quad (19)$$

into (1)-(2) such that arbitrary constants of integration are obtained. We therefore find that nontrivial
 solution of the system

$$\begin{pmatrix} r & -1 \\ -1 & r \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (20)$$

83 exists if the resonance is $r = \pm 1$. There is no single arbitrary constant which is introduced at the
 84 resonance $r = \pm 1$. Therefore, the solution of system (1)-(2) is presented as a general solution of the
 85 nonlinear system. This solution is obtained in Section 5. However, it does not required an introduction
 86 of logarithmic term that destroys the analytical nature of the solution. Hence, system (1)-(4) passes
 87 Painlevé test.

88 3.3. Painlevé Analysis of the reduced equation (10)

89 **Theorem 1.** Equation (10) passes the Painlevé test under parameter values $a = \gamma + b$ and $\nu_1 - \mu_1 =$
 90 $\nu_2 - \mu_2 + \alpha + 2\gamma$.

Proof of Theorem 1. In order to obtain the formal Laurent series expansion, we substitute equation

$$u = \sum_{i=0}^{\infty} \sigma_i \tau^{i-1} \quad (21)$$

into equation (10) which gives the following equation:

$$\begin{aligned} & \sigma_i \sigma_j (i-1)(i-2) \tau^{i+j-4} - (i-1)(i-1) \sigma_i \sigma_j \tau^{i+j-4} + (i-1) \sigma_i \sigma_j \sigma_k \tau^{i+j+k-4} \\ = & a(i-1) \sigma_i \sigma_j \tau^{i+j-3} - (b+\gamma) \sigma_i \sigma_j \sigma_k \tau^{i+j+k-3} + ab \sigma_i \sigma_j \tau^{i+j-2} \end{aligned} \quad (22)$$

for $i = j = k = 0, 1, 2, \dots$. At τ^{-4} we require

$$2\sigma_0^2 - \sigma_0^2 - \sigma_0^3 = 0.$$

Therefore

$$\sigma_0 = 1.$$

We move to the next power, τ^{-3} , and find

$$2\sigma_0 \sigma_1 - \sigma_0 \sigma_1 - \sigma_0^2 \sigma_1 = -a\sigma_0^2 - (b+\gamma)\sigma_0^3.$$

Since $\sigma_0 = 1$, this gives an arbitrary σ_1 only if

$$a = \gamma + b. \quad (23)$$

From (11) and (23) we have:

$$\nu_1 - \mu_1 = \nu_2 - \mu_2 + \alpha + 2\gamma. \quad (24)$$

Hence, the reduced equation (10) passes the Painlevé test under parameter values $a = \gamma + b$ and

$\nu_1 - \mu_1 = \nu_2 - \mu_2 + \alpha + 2\gamma$ and does not possess chaotic behaviour. \square

4. Lie Symmetry Analysis

A second order ordinary differential equation

$$u_t - F(t, u, u_{(1)}) = 0 \quad (25)$$

admits a one-parameter Lie group of transformations

$$\begin{aligned} \bar{t} & \approx t + a\zeta^0(t, u) \\ \bar{u} & \approx u + a\eta(t, u) \end{aligned} \quad (26)$$

with infinitesimal generator

$$X = \zeta^0(t, u) \frac{\partial}{\partial t} + \eta(t, u) \frac{\partial}{\partial u} \quad (27)$$

if

$$\bar{u}_{\bar{t}} - F(\bar{t}, \bar{u}, \bar{u}_{(1)}) = 0 \quad (28)$$

96 The group transformations \bar{t} and \bar{u} are obtained by solving the following Lie equations [3,10]

$$\begin{aligned}\frac{d\bar{t}}{da} &= \zeta^0(\bar{t}, \bar{u}) \\ \frac{d\bar{u}}{da} &= \eta(\bar{t}, \bar{u})\end{aligned}\quad (29)$$

with initial conditions

$$\bar{t}|_{a=0} = t, \bar{u}|_{a=0} = u. \quad (30)$$

97 The infinitesimal form of $\bar{u}_{\bar{t}}, \bar{u}_{(1)}$ are found by the given formulas [3,11]:

$$\begin{aligned}\bar{u}_{\bar{t}} &\approx u_t + a\zeta_0(t, u, u_t, u_{(1)}) \\ \bar{u}_{\bar{x}^i} &\approx u_{x^i} + a\zeta_i(t, u, u_t, u_{(1)})\end{aligned}\quad (31)$$

98 The functions ζ_0 and ζ_i are found by using the prolongation formulas below [12]

$$\begin{aligned}\zeta_0 &= D_t(\eta) - u_t D_t(\zeta^0) \\ \zeta_i &= D_i(\eta) - u_t D_i(\zeta^0)\end{aligned}\quad (32)$$

In [9], Matadi claimed that the equation (25) possesses symmetry (group generator)

$$X = \zeta^0(t, x, u) \frac{\partial}{\partial t} + \zeta^i(t, x, u) \frac{\partial}{\partial x^i} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (33)$$

iff

$$X^{[N]}N|_{N=0} = 0 \quad (34)$$

99 with $X^{[N]}$ the n-th extension of G .

100

Equation (10) is equivalent to [8]

$$yy'' - y'^2 + y'y^2 + y'y + \frac{b+\gamma}{a}y^3 + \frac{b}{a}y^2 = 0. \quad (35)$$

with y and t the new dependent and independent variables respectively such that

$$u = ay \text{ and } t = \frac{x}{a}. \quad (36)$$

101 Hence, we have the following

- 102 • Equation (35) under parameters values $a \neq b$ possesses chaotic behavior and does not pass the
- 103 Painlevé test.
- 104 • Equation (35) passes Painlevé test under parameters values $a = b$ and $\gamma = 0$ does not possess
- 105 chaotic behavior.

106 Since equation (10) under parameters values $a = b$ and $\gamma = 0$ does not possess chaotic behavior and

107 pass the Painlevé test, the infinitesimal symmetry of equation (35) has coefficient functions of the form

$$\zeta(x, y) = c_1 + c_2 e^x \quad (37)$$

$$\eta(x, y) = -c_2 y e^x \quad (38)$$

108 where c_1 and c_2 are arbitrary constants. Thus the Lie algebra of equation (35) is spanned by the
 109 following two infinitesimal generator:

$$X_1 = \partial_x \quad (39)$$

$$X_2 = e^x(\partial_x - y\partial_y). \quad (40)$$

Computing the Lie bracket we obtain the given commutator table:

Table 1. The commutator table of the infinitesimal generator

	X_1	X_2
X_1	0	X_2
X_2	0	0

110

From the commutator table, we conclude that the reduction of (35) can be made by X_2 only. The Lagrange's system associated to X_2 is given by

$$\frac{dx}{1} = \frac{dy}{-y} = \frac{dy'}{-2y' - y} \quad (41)$$

Solving equation (41) we obtain the new dependent variable, X , and independent variable, Y , namely:

$$X = x + \log y, \quad Y = \frac{y'}{y^2} + \frac{1}{y} \quad (42)$$

Therefore equation (35) becomes

$$\frac{dY}{dX} + Y + 1 = 0 \quad (43)$$

the integration of equation (43) gives

$$(Y + 1) \exp[X] = A. \quad (44)$$

The substitution of (42) into (44) gives

$$\frac{y'}{y} + y + 1 = A \exp[-x]. \quad (45)$$

The integration of (45) gives

$$y = \frac{C \exp[-x]}{D \exp[A \exp[-x]] + B}. \quad (46)$$

Substituting (11) and (46) into (36) we have

$$u(t) = \frac{(\mu_1 - \nu_1)C \exp[-(\mu_1 - \nu_1)t]}{D \exp[A \exp[-(\mu_1 - \nu_1)t]] + B}. \quad (47)$$

The number of infected population is obtain by substituting (47) into (9)

$$I(t) = \frac{(\mu_1 - \nu_1)C \exp[-(\mu_1 - \nu_1)t]}{\beta[D \exp[A \exp[-(\mu_1 - \nu_1)t]] + B]}. \quad (48)$$

The substitution of (48) into (5) gives

$$S(t) = \frac{1}{\beta}(\mu_1 - \nu_1) + \frac{1}{\beta}(\mu_1 - \nu_1) \exp[-(\mu_1 - \nu_1)t] + \frac{\alpha + \mu_2 - \nu_2}{\beta}. \quad (49)$$

The substitution of (48) and (49) into (3)-(4) gives

$$R(t) = \exp[(v_3 - \mu_3)t] \left[B + \int \frac{(\mu_1 - v_1)C \exp[-(\mu_1 - v_1)t]}{\beta[D \exp[A \exp[-(\mu_1 - v_1)t]] + B]} dt \right] \quad (50)$$

111 and

$$\begin{aligned} D(t) &= \frac{\mu_1}{\beta} \int [(\mu_1 - v_1) + (\mu_1 - v_1) \exp[-(\mu_1 - v_1)t] + (\alpha + \mu_2 - v_2)] dt \\ &+ \frac{\mu_2(\mu_1 - v_1)}{\beta} \int \frac{\exp[-(\mu_1 - v_1)t]}{[D \exp[A \exp[-(\mu_1 - v_1)t]] + B]} dt \\ &+ \mu_3 \left[\int \exp[(v_3 - \mu_3)t] \left[B + \int \frac{(\mu_1 - v_1)C \exp[-(\mu_1 - v_1)t]}{\beta[D \exp[A \exp[-(\mu_1 - v_1)t]] + B]} dt \right] dt \right]. \end{aligned}$$

112 5. The General Solutions

113 In this Section we obtain the analytical solution of the system (1)-(4) by combining the first two
114 equations.

$$\begin{aligned} \dot{S} + \dot{I} &= (v_1 - \mu_1)S + (v_2 - \mu_2 - \alpha)I \\ &= (v_1 - \mu_1)(S + I). \end{aligned} \quad (51)$$

115 Let

$$N = S + I. \quad (52)$$

116 Equation (51) becomes

$$\dot{N} = (v_1 - \mu_1)N. \quad (53)$$

117 The solution of (53) is

$$N(t) = N(0) \exp[(v_1 - \mu_1)t]. \quad (54)$$

118 From (52) and (54) we have

$$S = N(0) \exp[(v_1 - \mu_1)t] - I. \quad (55)$$

119 Equation (6) becomes

$$\begin{aligned} \dot{I} &= -\beta I^2 + \beta I N(0) \exp[(v_1 - \mu_1)t] - (\alpha + \gamma + \mu_2 - v_2)I \\ \frac{\dot{I}}{I^2} &= -\beta + \frac{\beta}{I} N(0) \exp[(v_1 - \mu_1)t] - \frac{(\alpha + \gamma + \mu_2 - v_2)}{I}. \end{aligned} \quad (56)$$

120 With the use of the transformation

$$u = \frac{1}{I} \quad (57)$$

121 equation (56) becomes:

$$\dot{u} = \beta - \beta N(0) \exp[(v_1 - \mu_1)t]u + (\alpha + \gamma + \mu_2 - v_2)u. \quad (58)$$

Since

$$v_1 - \mu_1 = a$$

and

$$\alpha + \gamma + \mu_2 - \nu_2 = b,$$

equation (58) gives

$$\dot{u} + (\beta N(0) \exp[at] - b) u = \beta \quad (59)$$

which has the integrating factor

$$\exp \left[\int (\beta N(0) \exp[at] - b) dt \right].$$

122 The solution of (59) is

$$u = A \exp \left[- \int (\beta N(0) \exp[at] - b) dt \right] + \exp \left[- \int (\beta N(0) \exp[at] - b) dt \right] \int \beta \exp \left[\int (\beta N(0) \exp[at] - b) dt \right] dt.$$

From (57) we have

$$I(t) = \frac{\exp \left[\int (\beta N(0) \exp[at] - b) dt \right]}{A + \int \beta \exp \left[\int (\beta N(0) \exp[at] - b) dt \right]} \quad (60)$$

and from (55), we have

$$S(t) = N(0) \exp[(\nu_1 - \mu_1)t] - \frac{\exp \left[\int (\beta N(0) \exp[at] - b) dt \right]}{A + \int \beta \exp \left[\int (\beta N(0) \exp[at] - b) dt \right]}. \quad (61)$$

123 From (3) we obtain

$$\dot{R} - (\nu_3 - \mu_3)R = \alpha I. \quad (62)$$

Equation (62) has the integrating factor $\exp[(\nu_3 - \mu_3)t]$. Therefore

$$R(t) = \exp[(\nu_3 - \mu_3)t] \left[B + \int \frac{\alpha \exp[-(\nu_3 - \mu_3)t] \exp \left[\int (\beta N(0) \exp[at] - b) dt \right]}{A + \int \beta \exp \left[\int (\beta N(0) \exp[at] - b) dt \right]} dt \right] \quad (63)$$

124 and from (4) we obtain the death component of the population to be

$$D(t) = \mu_1 N(0) \exp[(\nu_1 - \mu_1)t] + (\mu_2 - \mu_1) \frac{\exp \left[\int (\beta N(0) \exp[at] - b) dt \right]}{A + \int \beta \exp \left[\int (\beta N(0) \exp[at] - b) dt \right]} + \mu_3 \exp[(\nu_3 - \mu_3)t] \left[B + \int \frac{\alpha \exp[-(\nu_3 - \mu_3)t] \exp \left[\int (\beta N(0) \exp[at] - b) dt \right]}{A + \int \beta \exp \left[\int (\beta N(0) \exp[at] - b) dt \right]} dt \right].$$

125 6. Discussion

126 In this section we give a numerical result based on the Susceptibles and Infected component of
 127 the population. The parameters are chosen as $\nu_1 = 0.0003$, $\nu_2 = 0.0001$, $\nu_3 = 0.0003$, $\mu_1 = 0.0002$,
 128 $\mu_2 = 0.0003$, $\mu_3 = 0.0002$, $\alpha = 0.01$, $\beta = 0.04$, $\gamma = 0.04$. Figure 1 suggests that the solution is globally
 129 asymptotically stable.

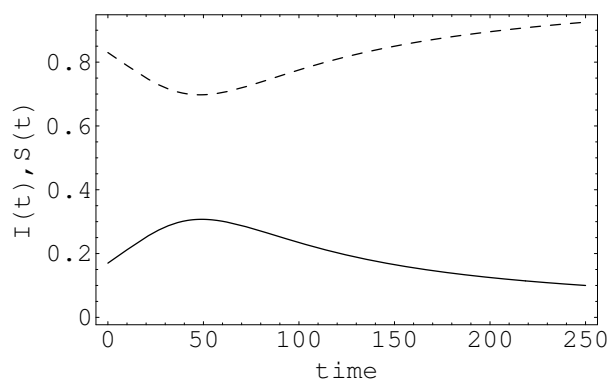


Figure 1. Susceptibles (dashed line), Infected (solid line). The computer packages Mathematica is used with the initial value are $I(0) = 0.17$ and $S(0) = 0.83$

130 7. Conclusion

131 In order to understand physical model, the analysis of a nonlinear differential play an essential
 132 role. Ove [15] stated that the by finding a closed form solution of a nonlinear differential, one can arrive
 133 at a complete understanding of the phenomena which are modeled. In this paper, four dimensional
 134 system of the SIRD epidemial model is reduced into a one dimensional second order differential
 135 equation. The Painlevé-analysis was performed for solutions of nonlinear second order differential
 136 equation. When pameters attain the values corresponding to complex chaotic behavior, equation (35)
 137 possesses chaotic behavior if $a \neq b$, consequently it does not pass the Painlevé test. The result revealed
 138 that under parameters values $a = b$ and $\gamma = 0$, equation (35) possesses chaotic behavior and does pass
 139 the Painlevé test. The techniques of Symmetry Analysis is performed to reduce equation and obtain
 140 the combinations of parameters which lead to the possibility of the linearisation of the system and
 141 provide the corresponding solutions.

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144 Abbreviations

145 The following abbreviations are used in this manuscript:

146 SIR Model Susceptible, Infected, Recovered Model
 147 SIRD Model Susceptible, Infected, Recovered and Dead Model

148 References

- 149 1. Ablowitz, MJ., Ramani, A. and Segur, H. Nonlinear evolution equations and ordinary differential equations
 150 of Painlevé Type. *Lettere al Nuovo Cimento* **1978**, *23*, 333-337
- 151 2. Hua, DD., Cairó, L., Feix, MR., Govinder, KS. and Leach, PGL. Connection between the existence of first
 152 integrals and the Painlevé property in two-dimensional Lotka-Volterra and Quadratic Systems. *Proceedings*
 153 *of the Royal Society* **1996**, *452*, 859-880
- 154 3. Gazizov, RK. and Ibragimov, NH. Lie symmetry analysis of differential equations in finance. *Nonlinear*
 155 *Dynamics* **1998**, *17*, 387-407
- 156 4. Ince, EL. Ordinary Differential Equations. London: Longmans, Green and Co, 1927
- 157 5. Kermack, WO. and McKendrick, AG. A contribution to the mathematical theory of epidemics. *Proc Roy Soc*
 158 *London* **1927**, *115*, 700-721
- 159 6. Kowalewski, S. Sur le problème de la rotation d'un corps solide autour d'un point fixé. *Acta Math* **1880**, *12*,
 160 117-232
- 161 7. Leach, PGL., Cotsakis, S. and Flessas, GP. Symmetry, singularity and integrability in complex dynamical
 162 systems I: The reduction problem. *J Nonlinear Math Phys* **2001**, *7* 445-479

- 163 8. Matadi, MB. The SIRD epidemial model. *Far East Journal of Applied Mathematics* **2014**, *89*, 1-14
- 164 9. Matadi, MB. Singularity and Lie group Analyses for Tuberculosis with Exogenous Reinfection. *International*
165 *Journal of Biomathematics* **2015**, *8*, 1-12
- 166 10. Matadi, MB. Lie Symmetry Analysis Of Early Carcinogenesis Model. *Applied Mathematics E-Notes* **2018**, *18*,
167 238-249
- 168 11. Matadi, MB. Symmetry and conservation laws for tuberculosis model. *International Journal of Biomathematics*,
169 **2017**, *10*, 1750042
- 170 12. Matadi, MB. The Conservative Form of Tuberculosis Model with Demography. *Far East Journal of Mathematical*
171 *Sciences*, **2017**, *102*, 2403-2416
- 172 13. Miritzis, J., Leach, PGL. and Cotsakis, S. Symmetry, singularities and integrability in complex dynamical
173 systems IV: Painleve integrability of isotropic cosmologies, *Gravit. Cosmol*, **2000**, *6* 282-290
- 174 14. Oke, S., Matadi, MB. and Xulu, SS. Optimal Control Analysis of a Mathematical Model for Breast Cancer.
175 *Mathematical and Computational Applications* **2018**, *23*
- 176 15. Ove, L. Painlevé Analysis and Transformations Nonlinear Partial Differential Equations. *PhD Thesis*,
177 *Department of Mathematics Lulea University of Technology*, Sweden, 2001
- 178 16. Painlevé, P. Mémoire sur les équations différentielles dont l'intégral est uniforme. *Bull Soc Mat France* **1900**,
179 *28*, 201-261
- 180 17. Ramani, B., Grammaticos, and Bountis, T. The Painlevé property and singularity analysis of integrable and
181 nonintegrable systems. *Phys. Rep.* **1989**, *180*, 159-245
- 182 18. Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An introduction*; John Wiley & Sons, Inc., New
183 York, 2006
- 184 19. Weiss, J., Tabor, M. and Carnevale, G. The Painlevé Property for Partial Differential Equation I. *Math. Phys.*
185 **1983**, *24*, 522-526

186 **Sample Availability:** Samples of the compounds are available from the authors.