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Type I Almost-Homogeneous Manifolds of Cohomogeneity One—IV

Daniel Guan, Pilar Orellana, Anthony Van
Department of Mathematics,
The University of California at Riverside,
Riverside, CA 92521 U. S. A.

Abstract

This is the fourth part of [6] on the existence of Kähler Einstein metrics of the general type I almost homogeneous manifolds of cohomogeneity one. We actually carry out all the results in [8] to the type I cases. In part II [14], we obtained a lot of new Kähler-Einstein manifolds as well as Fano manifolds without Kähler-Einstein metrics. In particular, by applying Theorem 15 therein, we have complete results in the Theorems 3 and 4 in that paper. However, we only have some partial results in Theorem 5 there. In this note, we shall give a report of recent progress on the Fano manifolds $N_{n,m}$ when $n > 15$ and $N'_{n,m}$ when $n > 4$. We actually give two nice pictures for these two classes of manifolds. See our Theorems 1 and 2 in the last section. Moreover, we post two conjectures. Once we could solve these two conjectures, the question for these two classes of manifolds would be completely solved. With applying our results to the canonical circle bundles we also obtain Sasakian manifolds with or without Sasakian-Einstein metrics. That also give some open Calabi-Yau manifolds.

Keywords: Kähler manifolds, Einstein metrics, Ricci curvature, fibration, almost-homogeneous, cohomogeneity one, semisimple Lie group, Sasakian Einstein, Calabi-Yau metrics.

AMS Subject Classification: 53C10, 53C25, 53C55, 14J45.

1. Introduction

This paper is the fourth part of [6]. In [6], we prove that:

Proposition 1. *For any simply connected type I compact Kähler complex almost homogeneous manifolds of cohomogeneity one with a hypersurface end, there is an extremal metrics in a given Kähler class if and only if the condition (7) in [6] holds.*

The condition (7) therein was a sign for a topological integral as it was shown in **6.1** of the part III of this paper [15].

We finished the similar results for the higher condimensional end case and the general case, in our sections 3 and 4 of the part II. See also Theorem 15 in the last section for the Kähler-Einstein case therein.

As an application, considering the canonical circle bundle, we also obtained Sasakian manifolds with and without Sasaki-Einstein metrics (with the same Reeb vector field and CR structure, see [3] Theorem 2.4 (iv), also [18], [28]).

We shall prove the converse for the type II cases in [9].

Therefore, we finished all the possible cases in which the existence of the extremal metrics could be reduced to an ordinary differential equation problem. We also give many examples for both the stable and the unstable cases, as we promised in [8]. It is difficult for us to find any example which is semistable but not stable. We note that since the automorphism group is semisimple, the original Futaki invariants are zero for the most manifolds we considered in this paper. Therefore, we give many and more classical examples than the example in [27].

[24] studied a few of the first case in [6], i.e., the cases with $F = F(OP_n)$ in which $S = SO(4, \mathbf{C}), SO(6, \mathbf{C}), SO(8, \mathbf{C}), SO(10, \mathbf{C})$, where S is the induced group action on the fibers.

A classification which we refer to in this paper can be found in [8] section 12. We put these manifolds into three types there, type I, II and III.

In [6] we used very explicit and elementary calculation to avoid the Cauchy-Riemann structure and other very abstract tools in [25] and [24] which also cumulating on results from other papers of Spiro.

We dealt with the uniqueness in the second section therein, where we proved the existence of smooth geodesics in the Mabuchi moduli spaces of the Kähler metrics.

Proposition 2. *For any two smooth Kähler metrics which are equivariant under the maximal compact subgroup in a given Kähler class on a type I compact complex almost homogeneous manifold of cohomogeneity one, there is a smooth geodesic connecting them in the Mabuchi moduli space of Kähler metrics.*

The same result was proven for the toric manifolds and type III manifolds in [12] and for some type II manifolds in [8], then for all type II manifolds in [9].

We also obtained many Kähler Einstein manifolds as well as Fano manifolds without Kähler Einstein metric of type I in the fifth section in part II. The Futaki invariants are zero in our case since the automorphism groups are semisimple. It turns out that our method is also easier than that in [20] for finding a Kähler-Einstein metrics since they depended on the zero of a Futaki invariant which might be very rare. Therefore, their method is more suitable for finding manifolds without Kähler Einstein metric. In our case, both the Kähler Einstein manifolds and manifolds without Kähler Einstein metrics might be dense in certain Zariski sense. It seems to me that I hardly see any example with vanishing generalized Futaki invariant. Also, most of our manifolds are Fano, it is not that simple in [20]. Moreover, it is very easy to check that [20] can be a Corollary of [22] and does not involve much stability.

In the part II, Theorems 3 and 4, we completely solved the existence of Kähler-Einstein metric for the two classes of manifolds $M_{n,m}$ and $M'_{n.m}$ therein.

To understand better our examples, M_n in our section 5 in the part II, or $M_{n,1}$ in our Theorem 3 there, it is not difficult to see that M_n is a Q^n bundle over

$$Q^{n+1} = \{[z_0, z_1, \cdots, z_n, z_{n+1}, z_{n+2}] \in \mathbf{C}P^{n+2}|_{z_0^2+\cdots+z_{n+2}^2=0}\}.$$

Let $N_n = \mathbf{P}(T^*_{Q^{n+1}})$, then it is just our manifold with $F = \mathbf{C}P^n$. Therefore, N_n is never Fano (see [29, 21]). To construct M_n we notice that by [19] p.590–593 there is a holomorphic conformal structure on Q^{n+1} with respect to the line bundle $N = \mathcal{O}(2)$. Therefore, $N_n = \mathbf{P}(T^*_{Q^{n+1}}) = \mathbf{P}(N \otimes T^*_{Q^{n+1}}) = \mathbf{P}(T_{Q^{n+1}})$. The exceptional divisor comes from the zero set of the corresponding holomorphic symmetric 2-tensor in N_n. Therefore, the vector bundle $\mathcal{O}(1) \oplus T_{Q^{n+1}}$ also has a conformal structure with respect to N. M_n is just the zero set of the corresponding holomorphic symmetric 2-tensor in $\mathbf{P}(\mathcal{O}(1) \oplus T_{Q^{n+1}})$. The branch double covering map $M_n \to N_n$ is just introduced by the projection $\mathcal{O}(1) \oplus T_{Q^{n+1}} \to T_{Q^{n+1}}$.

In the part II Theorem 5, we completely solved the integral problem for $N''_{n,m}$. Actually, they are all homogeneous and hence Kähler-Einstein. However, for $N_{n,m}$ and $N'_{n,m}$ therein, we only had some partial results.

From part II, the picture in the case of $C_k = Sp(k, \mathbf{C})$ fiber structure (the paragraph after Theorem 9, comparing with Theorems 3, 4, 5 and the results of this note, for example) is quite different from that in the case of $SO(n+1, \mathbf{C})$ or $Spin(7, \mathbf{C})$ structures. The results of Theorem 6 to 12 there is quite complete.

In the next two sections, we shall give some more details. Especially, in section 3, we shall given two new Theorems and some conjectures, which might lead to a complete solution of the two classes of manifolds $N_{n,m}$ and $N'_{n,m}$.

2. The Preliminary

Now, we apply our arguments in [8, 6, 14, 15]. In this note, we only consider the case in which $S = SO(n+1, \mathbf{C})$. In part II, we first consider the case in which $G = S = SO(n+1, \mathbf{C})$. Let m be the codimension, then as before we have the integral in the Theorem 15 of part II:

$$\int_0^{-K(F)+m-1} (-K(F) - D(F) - x)Q(x)dx.$$

Here $K(F)$ is related to the canonical line bundle. We know that $K(F) = -n-1$ if $F = \mathbf{C}P^n$ and $-4k$ if $F = Gr(2k, 2)$, and $K(F) = -n$ if $F = Q^n$. See the paragraph right before Theorem 15 in part II. In the case of $F = \mathbf{C}P^n$, $D = Q^{n-1}$ and therefore, $D(F) = 2$. In the case of $F = Q^n$, $D = Q^{n-1}$, which is the intersection of $Q^n \subset \mathbf{C}P^{n+1}$ with a hyperplane. Therefore $D(Q^n) = 1$. We also have $D(Gr(2k,\ 2)) = 2$. See the paragraph after Theorem 1 in part II. In the both cases of $N_{n,m}$ and $N'_{n,m}$, the divisors are of codimension 1. That is, $m - 1 = 0$. The upper limits of our integrals are $-K(F)$. In both cases, we have;

$$-K(F) - D(F) = n - 1.$$

From the proof of the Lemma 6 in [6], we see that the eigenvalues of the Ricci curvature of the exceptional divisor in the direction other than the 1-strings of roots with zero eigenvalues is represented as

$$Ric_s = n - 1 + m_2, c^{-1}a_{\rho,i} \pm (n - 1 + m_2),$$

where s are indices, similar to i. In the same way, the eigenvalues of the restriction of the Ricci curvature of the whole manifold in those direction is represented by

$$\tilde{Ric}_s = -l_\rho, c^{-1}a_{\rho,i} \mp l_\rho.$$

When $S = SO(n+1, \mathbf{C})$, by the fourth paragraph after Theorem 6 in part I, we have $m_2 = 0$. By the paragraph after the Theorem 4 in part I, we have $c = 1$. By the second paragrah of the section 3 in part II, we have $l_\rho = -(n+m+m_2)c = -n-1$ if $F = \mathbf{C}P^n$ and $l_\rho = -(n-1+m)c = -n$ for $F = Q^n$.

Therefore, when $F = \mathbf{C}P^n$, we have

$$\tilde{Ric}_s = n+1, a_{\rho,i} \mp (n+1),$$

$$Ric_s = n-1, a_{\rho,i} \pm (n-1).$$

When $F = Q^n$, Ric_s are the same and

$$\tilde{Ric}_s = n, a_{\rho,i} \pm n.$$

Therefore, according to the calculation of part III in 6.1 or Theorem 2 in part I, we have:

$$Q(x) = x^{n-1} \prod_i (a_{\rho,i}^2 - x^2).$$

Another way to get the $a_{\rho,i}$ is to look at the fiberation

$$D \to G/P.$$

We see that $a_{\rho,i}$ are just the eigenvalues of the Ricci curvatures of G/P. However, $a_{\rho,i}$ only occur in our volume if e_1 acts nontrivially on the corresponding root. The concrete $a_{\rho,i}$ can be calculated by our Theorem 2 in part I. There, we have the Ricci curvature formula for G/P:

$$q_{G/P} = \sum_{\alpha \in \Delta^+ - \Delta_P} H_\alpha.$$

There, the sum is over all the positive roots contribute to the tangent space of G/P, or the positive roots which are not in P.

For example, when $S = SO(8, \mathbf{C})$ and $G = SO(10, \mathbf{C})$ as in the item 4 in [24], the Ricci curvature of G/P is decided by $\sum_{\alpha \in \Delta^+ - \Delta_{P_{2,4}}} H_\alpha$. Here we assume that S is generated by e_i $i > 1$. So is the parabolic subgroup P being determinated, that is why we put the first index 2 to imply e_2 as our e_1 earlier and the second index 4 as $S = D_4$. We see that $q_{G/P} = 8e_1$ and the only positive roots which are not in P with nontrivial e_2 actions are $e_1 \pm e_2$. Therefore, $a_{\rho,1} = (q_{G/P}, e_1 \pm e_2) = 8$. Therefore, it is not Fano for the $\mathbf{C}P^7$ case and the coefficient in [24] item 4 was wrong(in which it was 16 but not 8). Moreover, G/P has a complex dimension 8. These come from $e_1 \pm e_i$. But of these, only $e_1 \pm e_2$ are acted nontrivially by e_2. This was why we have a power 1 in part II right before Theorem 3 but in [24] they had a power 4. The one with a $F = Q^n$ fiber is our $M_{7,1}$ in our part II.

One series of examples which we already knew are the product P_n of two copies of $\mathbf{C}P^n$ and M_n, N_n in [5, 7, 8]. Since M_n are Kähler Einstein, so are P_n and N_n, whenever it is Fano.

3. New Kähler-Einstein Metrics

If the Kähler class is the Ricci class, we have, as in part II section 5, $\alpha = \frac{u}{c}$, $l = l_\rho$,

$$m(u) = 2Q_1(u).$$

Therefore,

$$f_l = [n - 1 + m_2 - c^{-1}\sqrt{U}]U^{\frac{n-2}{2}}Q_1(U).$$

In this section, we shall check case by case on the type of the groups (S, G).

We say that a manifold is *nef* if the anti-canonical line bundle is nef. We say that a manifold is *Fano* if the anti-canonical line bundle is positive.

First, if $S = G = B_k$ $k \geq 2$, we have $n = 2k$, Q_1 is a constant, $l_\rho = -(n+1)$ if $F = \mathbf{C}P^n$ or $-n$ if $F = Q^n$. Then,

$$\begin{aligned} C_n &= \int_0^{l_\rho^2} f_{l_\rho} dU = \int_0^{(n+1)^2} (n - 1 - \sqrt{U})U^{\frac{n-2}{2}} dU \\ &= \frac{2(n-1)}{n}(n+1)^n - 2(n+1)^n = -\frac{2(n+1)^n}{n} < 0 \end{aligned}$$

for the case in which $F = \mathbf{C}P^n$ and

$$C_n' = -\frac{n^{n-1}}{n+1} < 0$$

for the case in which $F = Q^n$. Therefore, there is a Kähler Einstein metric. Again, this is actually known since the manifolds are homogeneous. The same formula is true for $G = S = D_k, n = 2k - 1$.

Now, we consider the situation in which $G = B_{k+1}$ and $S = B_k$. We have

$$a_{\rho,1} = 1 + 2k = n + 1 = -l_\rho$$

if $F = \mathbf{C}P^n$ or

$$a_{\rho,1} = n + 1 = -l_\rho + 1$$

if $F = Q^n$. The manifolds are nef but not Fano (or are Fano). The same thing is true for $G = D_{k+1}$ and $S = D_k$. We only need to consider the case in which $F = Q^n$. The integral is

$$\begin{aligned} I_n &= \int_0^n (n - 1 - v)v^{n-1}((n+1)^2 - v^2)dv \\ &= \frac{n^{n-1}}{(n+2)(n+3)}(-(n+1)(n+2)(n+3) + 3n^2) \\ &= \frac{n^{n-1}}{(n+2)(n+3)}(2n^3 - 6n^2 - 11n - 6) > 0 \end{aligned}$$

if $n \geq 5$. Otherwise, $I_n < 0$. Therefore, the corresponding manifolds M_n are Kähler Einstein for $n \leq 4$. Others are non-Kähler Einstein Fano manifolds. Each M_n is a Q^n bundle over Q^{n+1}.

Now, we consider the general situation in which $S = SO(n+1, \mathbf{C})$ and $G = SO(2m+n+1, \mathbf{C})$, P be the smallest parabolic subgroup of G containing S as a simple factor. In this case,

$$Q_1(v) = \prod_{j=0}^{m-1}((n+2j+1)^2 - v^2).$$

They are nef but not Fano (or are Fano). We have the integrals:

$$I_{n,m} = \int_0^n (n-1-v)v^{n-1}\prod_0^{m-1}((n+2j+1)^2 - v^2)dv.$$

For $m = 2$ we can use Mathematica to check $I_{n,2}$ with

`Integrate[(n-1-v)v^(n-1)`$((n+1)^2 - v^2)((n+3)^2 - v^2)$`, {v, 0, n}]`

and have $I_{4,2} > 0$ but $I_{3,2} < 0$. In the same way, we can use Mathematica and check that $I_{3,m} < 0$ for $m \leq 7$ but $I_{3,8} > 0$.

We denote the corresponding manifolds by $M_{n,m}$. Therefore, we have in part II:

Proposition 3. *$M_{n,0}$ are homogeneous Kähler Einstein manifolds.*

$$M_{3,1}, M_{4,1}, M_{3,2}, M_{3,3}, M_{3,4}, M_{3,5}, M_{3,6}, M_{3,7}$$

are nonhomogeneous Kähler Einstein manifolds. Other $M_{n,m}$ are Fano manifolds without Kähler Einstein metric.

Next, we consider the case in which $S = SO(n+1, \mathbf{C}), G = SO(2m+n+1, \mathbf{C})$ and S_1 in the section 3 of [6] is maximal. In this case, we have positive roots in $\Delta^+ - \Delta_{P_{m+1,k}}$, e.g., when $n = 2k-1$, $e_i \pm e_j, i \leq m < j, e_i + e_k, i < k \leq m$. For each i, there are only $e_i \pm e_{m+1}$ which are acted nontrivially by e_{m+1}. $(q_{G/P}, e_i \pm e_{m+1}) = 2k + m - i + i - 1 = 2k + m - 1 = n + 1 + m - 1 = n + m$. Altogether there are m choices of i. Therefore,

$$Q_1(v) = ((n+m)^2 - v^2)^m$$

with $m > 1$. When $m > 1$, they are all Fano. The integral is

$$J_{n,m} = \int_0^{n+1}(n-1-v)v^{n-1}((n+m)^2 - v^2)^m dv$$

if $F = \mathbf{C}P^n$ or

$$J'_{n,m} = \int_0^n (n-1-v)v^{n-1}((n+m)^2 - v^2)^m dv$$

if $F = Q^n$, and

$$m^{-2m}J_{n,m} \to e^{2(n-1)}C_n < 0$$

or

$$m^{-2m}J'_{n,m} \to e^{2(n-1)}C'_n.$$

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Therefore, $J_{n.m} < 0$ (or $J'_{n,m} < 0$) when m is big enough.

Also, we can compare the change rate of the factor

$$h(v) = ((n+m)^2 - v^2)^m$$

for different m and n. We let

$$t(m) = (\log h)' = m(\frac{1}{n+m+v} - \frac{1}{n+m-v}) = \frac{-2mv}{(n+m)^2 - v^2}.$$

Then,

$$t(m+1) - t(m) = \frac{-2v[n^2 - m(m+1) - v^2]}{((n+m)^2 - v^2)((n+m+1)^2 - v^2)} > 0$$

if $m \geq n$. Therefore, if $J_{n,m} \leq 0$ with $m \geq n$ then $J_{n,m+1} < 0$. The same thing is also true for $J'_{n,m}$.

Now, we can use Mathematica to check $J'_{n,n}$ with

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Integrate[(n-1-v) v^(n-1) ((2n)^2 -v^2 )^(n) , {v, 0, n}]
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we get $J'_{n,n} < 0$ when $n = 3, 4$ but $J'_{5,5} > 0$.

We then use Mathematica to check $J'_{5,10}$ with

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Integrate[(5-1-v)v^4 (225 -v^2 )^(10) , {v, 0, 5}]
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and have $J'_{5,10} > 0$.

Similarly, by using Mathematica we have $J'_{5,20} < 0$ and $J'_{5,m} > 0$ if $2 \leq m \leq 13$ and $J'_{5,14} < 0$. Therefore, when $m \geq 14$ $J'_{5,m} < 0$, otherwise $J'_{5,m} > 0$. We can also check that $J_{5,m} < 0$ for $m > 1$.

Similarly, we use Mathematica to check $J'_{4,m}$ for $m = 2, 3$ and $J'_{3,m}$ for $m = 2$. We find that all of them < 0. Therefore, $J'_{3,k}, J'_{4,k} < 0$ if $2 \leq k$. So are $J_{3,k}, J_{4,k}$.

In general, we expect that if

$$m > \frac{n^2(n-1)}{e},$$

then $J'_{n,m} < 0$. For example, if $n = 6$ we expect $J'_{6,60} < 0$. We check it with Mathematica and get

$$J'_{6,60}, J'_{6,30}, J'_{6,27} < 0, \quad J'_{6,20}, J'_{6,25}, J'_{6,26} > 0.$$

In the same way, we find $J'_{6,k} > 0$ for $2 \leq k \leq 5$. Therefore, $J'_{6,k} > 0$ if $2 \leq 26$, otherwise $J'_{6,k} < 0$. We can also check that $J_{6,k} < 0$ for all $k > 1$. One might expect that $J_{n,m} < 0$ always. However, we have $J_{n,n} > 0$ for $n = 101, 51, 26, 25$, $J_{n,n} < 0$ for $n = 11, 21, 24$. Then, one might expect that $J_{n,m} < 0$ for $n \leq 24$. However, we have $J_{n,m} > 0$ for $(n,m) = (24, 12), (18, 9), (16, 8)$ etc.. One can check that $J_{n,m} < 0$ for $n \leq 15$. And $J_{16,m} > 0$ if $5 \leq m \leq 8$, otherwise $J_{16,m} < 0$. We can also check that $J_{n,2} < 0$. And

$$\begin{aligned} J_{n,3} &= C(n)(8n^8 + 6n^7 - 1534n^6 - 16019n^5 \\ &- 75163n^4 - 194786n^3 - 263486n^2 - 216981n - 76545) < 0 \end{aligned}$$

if and only if $n \leq 17$. We can check that $J_{17,m} < 0$ if and only if $m \leq 3$ or ≥ 11.

Therefore, we obtained in part II that if we denote the corresponding manifolds by $N_{n,m}$ (or $N'_{n.m}$, then:

Proposition 4. *$N_{n,m}$ $3 \leq n \leq 15$, and $N'_{3,m}, N'_{4,m}$ admit Kähler-Einstein metric for all $m > 1$. $N'_{5,m}$ admit Kähler-Einstein metric if and only if $m > 13$. $N'_{6,m}$ admit Kähler Einstein metric if and only if $m > 26$. $N_{16,m}$ admit Kähler Einstein metric if and only if $m > 8$ or $2 \leq m < 5$. $N_{17,m}$ admit Kähler Einstein metric if and only if $m > 10$ or $2 \leq m < 4$. $N_{n,2}$ admit Kähler Einstein metric for any n. $N_{n,3}$ admit Kähler Einstein metric if and only if $n \leq 17$. In general, $N_{n,m}$ (or $N'_{n,m}$) admit Kähler-Einstein metric when m big enough, i.e., there is an integer $N(n)$ (or $N'(n)$) such that if $m > N(n)$ (or $> N'(n)$) then $N_{n,m}$ (or $N'_{n,m}$) admit Kähler-Einstein metric. Moreover, if $m \geq n$ and $N_{n,m}$ (or $N'_{n,m}$) admit a Kähler-Einstein metric, so does $N_{n,m+1}$ (or so does $N'_{n,m+1}$).*

Now, what would happen if n is big? We used computer to test the possible smallest $N(n)$ and $N'(n)$ in Proposition 4. We obtained following new results:

Theorem 1. *$N(n)$ is:*
1. $2n - 24$ *if* $15 < n < 24$;
2. $2n - 25$ *if* $23 < n < 31$;
3. $2n - 26$ *if* $30 < n < 42$;
4. $2n - 27$ *if* $41 < n < 57$;
5. $2n - 28$ *if* $56 < n < 82$;
6. $2n - 29$ *if* $81 < n < 133$;
7. $2n - 30$ *if* $132 < n < 287$;
8. $2n - 31$ *if* $286 < n < 2601$ *(at least)*
We also have:
1. $M_{n,2}$ *is Einstein iff* $n < 27$;
2. $M_{n,3}$ *is Einstein iff* $n < 18$;
3. $M_{n,4}$ *is Einstein iff* $n < 17$.

We conjecture that $N(n)$ has an asymptotic line $2n - a$. Since all the function points and values are integers, if there is truely an asymptotic line $2n - a$, then we have $N(n) = 2n - a$ eventually.

One might think that the same would be true for $N'(n)$. However, we have:

Theorem 2. *$N'(n) = 2n^2 - 10n + b(n)$, with $b(n) =$:*
1. 13 if $n = 5$;
2. 14 if $n = 6$;
3. 15 if $6 < n < 9$;
4. 16 if $8 < n < 11$;
5. 17 if $10 < n < 17$;
6. 18 if $16 < n < 32$;
7. 19 if $31 < n < 101$ *(at least).*

Although $N'(n)$ will not have an asymptotic when n is big enough, if $b(n)$ has a limit, then again because everything are integers, we would get $N'(n)$ when n is big enough.

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