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Some Bounds on Eigenvalues of the Hadamard Product and the Fan Product of Matrices

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Abstract: In this paper, some mixed type bounds on the spectral radius $\rho(A \circ B)$ for the Hadamard product of two nonnegative matrices (A and B) and the minimum eigenvalue $\tau(C \star D)$ of the Fan product of two M-matrices (C and D) are researched. These bounds complement some corresponding results on the simple type bounds. In addition, a new lower bound on the minimum eigenvalue of the Fan product of several M-matrices is also presented:

$$\tau(A_1 \star A_2 \cdots \star A_m) \geq \min_{1 \leq i \leq n} \{\prod_{k=1}^{m} A_k(i,i) - \prod_{k=1}^{m} [A_k(i,i)^{P_k} - \tau(A_k^{(P_k)})]^{\frac{1}{P_k}}\},$$

where $A_1, \ldots, A_k$ are $n \times n$ M-matrices and $P_1, \ldots, P_k > 0$ satisfy $\sum_{k=1}^{m} \frac{1}{P_k} \geq 1$. Some special cases of the above result and numerical examples show that this new bound improves some existing results.

Keywords: Hadamard product; Nonnegative matrices; Spectral radius; Fan product; M-matrix; Inverse M-matrix; Minimum eigenvalue

1. Introduction

As is well known, the Hadamard, Fan and Kronecker products play an important role in matrix methods for statistics and econometrics [16,17]. The research on eigenvalues of the Hadamard and Fan products of matrices is always one of the hot topics in matrix theory see [2,3,5–9,11,13,14,18]. In this paper, we will continue to research this topic and present some new results.

First, we introduce some notations, see [1]. For two real $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we write $A \geq B$ ($A > B$) if $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$) for all $i, j \in N$. If $A \geq 0$ ($A > 0$), we say that A is a nonnegative (positive) matrix. The spectral radius of A is denoted by $\rho(A)$. If A is a nonnegative matrix, the Perron-Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$, where $\sigma(A)$ is the set of all eigenvalues of A. Moreover, a matrix A is called reducible if there exists a nonempty proper subset $I \subset N$ such that $a_{ij} = 0, \forall i \in I, \forall j \notin I$. If A is not reducible, then we call A irreducible (see [7]).

In addition, we denote by $\mathcal{Z}_n$ the class of all $n \times n$ real matrices of whose off-diagonal entries are non-positive. If $A \in \mathcal{Z}_n$, then the minimum eigenvalue of A is defined by $\tau(A) := \min\{\text{Re}(\lambda) | \lambda \in \sigma(A)\}$. As a special case of $\mathcal{Z}_n$, a matrix $A = (a_{ij}) \in \mathcal{Z}_n$ is called a nonsingular M-matrix if A is nonsingular and $A^{-1} \geq 0$ (see [8]). Denote by $\mathcal{M}_n$ the set of all nonsingular M-matrices. Generally speaking, the following simple facts are well known (see Problems 16, 19 and 28 in Section 2.5 of [8]):

1. If $A \in \mathcal{Z}_n$, then $\tau(A) \in \sigma(A)$;
2. If $A, B \in \mathcal{M}_n$, and $A \geq B$, then $\tau(A) \geq \tau(B)$, moreover, $\tau(A) \leq \min\{a_{ii}\}$;
3. If $A \in \mathcal{M}_n$, then there exists a positive eigenvalue of A equal to $\tau(A) = [\rho(A^{-1})]^{-1}$, where $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}.

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Finally, for convenience, N denotes the set $\{1, 2, \cdots, n\}$ throughout. The sets of all $n \times n$ real and complex matrices are denoted by $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$, respectively. The Hadamard product of $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and $B = (b_{ij}) \in \mathbb{C}^{n\times n}$ is defined by $A \circ B = (a_{ij}b_{ij}) \in \mathbb{C}^{n\times n}$. As a variant of the Hadamard product, the Fan product of two real matrices $A, B \in \mathcal{Z}_n$ is denoted by $A \star B = C = (c_{ij}) \in \mathcal{Z}_n$, where

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & \text{if } i \neq j, \\ a_{ii}b_{ii}, & \text{if } i = j. \end{cases}$$

The remainder of the paper is organized as follows. In Section 2, we mainly exhibit a new 'mixed' type bound for eigenvalues of the Hadamard and Fan products of two matrices to complement the corresponding results. In addition, a comparison between the simple type and mixed type bounds is given in Section 3. In Section 4, we generalize the result of the Fan product of two matrices to the case of several matrices. Finally, some concluding remarks on further research are given in Section 5.

2. A mixed type bound for eigenvalues of the Hadamard and Fan products of two matrices

In recent years, on the problem of $\rho(A \circ B)$ of two nonnegative matrices A and B, there exist some rich results based on the $\rho(A)$ and $\rho(B)$.

- In ([8], p. 358), $\rho(A \circ B) \leq \rho(A)\rho(B)$.
- Fang [6] gave an upper bound for $\rho(A \circ B)$, *i.e.*,

$$\rho(A \circ B) \leq \max_{1\leq i\leq n} \left\{2a_{ii}b_{ii} + \rho(A)\rho(B) - b_{ii}\rho(A) - a_{ii}\rho(B)\right\}. \tag{1}$$

- Liu *et al.* [14] further improved the above results and obtained the following bound

$$\begin{aligned}\rho(A \circ B) \leq \max_{i\neq j} \tfrac{1}{2}\Big\{&a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ &+ 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})(\rho(A) - a_{jj})(\rho(B) - b_{jj})]^{\frac{1}{2}}\Big\}.\end{aligned} \tag{2}$$

Recently, Cheng [5] also obtained the following results[1] based on the row maximum non-diagonal elements and the commutative property of Hadamard product.

Theorem 1. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be nonnegative matrices, $s_i = \max_{j\neq i}\{a_{ij}\}$, $t_i = \max_{j\neq i}\{b_{ij}\}$, then*

$$\rho(A \circ B) \leq \\ min \left\{ \begin{array}{l} \max_{i\neq j} \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_it_j(\rho(A) - a_{ii})(\rho(A) - a_{jj})]^{1/2}\right\}, \\ \max_{i\neq j} \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4s_is_j(\rho(B) - b_{ii})(\rho(B) - b_{jj})]^{1/2}\right\}. \end{array} \right\}. \tag{3}$$

This expression is interesting, which is called as a 'simple' type bound. Motivated by the above result, one may propose the following result. Here, we call it the 'mixed' type one.

Theorem 2. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be nonnegative matrices, $s_i = \max_{j\neq i}\{a_{ij}\}$, $t_i = \max_{j\neq i}\{b_{ij}\}$, then*

$$\begin{aligned}\rho(A \circ B) \leq \max_{i\neq j} \tfrac{1}{2}\Big\{&a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ &+ 4t_is_j(\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{\frac{1}{2}}\Big\}.\end{aligned} \tag{4}$$

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[1] Note that there exists a little formula spelling mistake in [5].

Proof. It is evident that the inequality (4) holds with the equality for $n = 1$. Therefore, we assume that $n \geq 2$ and divide two cases to prove this problem.

Case 1. Suppose that both A and B are irreducible. By Perron-Frobenius theorem in [8], there exists a positive right Perron vector $u = (u_1, u_2, \cdots, u_n)$ for any $D = diag(d_1, d_2, \cdots, d_n), d_i > 0$ such that

$$(D^{-1}AD)u = \rho(D^{-1}AD)u = \rho(A)u,$$

i.e.,

$$\sum_{j\neq i} \frac{a_{ij}d_j u_j}{d_i u_i} = \rho(A) - a_{ii}.$$

In addition, if $D = diag(d_1, d_2, \cdots, d_n)$ is just the right Perron eigenvector of the irreducible nonnegative matrix B, then we obtain similarly that

$$\sum_{j\neq i} \frac{b_{ij}d_j u_j}{d_i u_i} = \rho(B) - b_{ii}.$$

Define $U = diag(u_1, u_2, \cdots, u_n)$, $C = (DU)^{-1}A(DU)$, then we see that

$$C = \begin{pmatrix} a_{11} & \frac{d_2u_2}{d_1u_1}a_{12} & \cdots & \frac{d_nu_n}{d_1u_1}a_{1n} \\ \frac{d_1u_1}{d_2u_2}a_{21} & a_{22} & \cdots & \frac{d_nu_n}{d_2u_2}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d_1u_1}{d_nu_n}a_{n1} & \frac{d_2u_2}{d_nu_n}a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is an irreducible nonnegative matrix and

$$C \circ B = (m_{ij}) = \begin{pmatrix} a_{11}b_{11} & \frac{d_2u_2}{d_1u_1}a_{12}b_{12} & \cdots & \frac{d_nu_n}{d_1u_1}a_{1n}b_{1n} \\ \frac{d_1u_1}{d_2u_2}a_{21}b_{21} & a_{22}b_{22} & \cdots & \frac{d_nu_n}{d_2u_2}a_{2n}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d_1u_1}{d_nu_n}a_{n1}b_{n1} & \frac{d_2u_2}{d_nu_n}a_{n2}b_{n2} & \cdots & a_{nn}b_{nn} \end{pmatrix}.$$

Since (see [8]) $(DU)^{-1}(A \circ B)(DU) = (DU)^{-1}A(DU) \circ B = C \circ B$, therefore, $\rho(A \circ B) = \rho(C \circ B)$. By Brauer's theorem (see [5]) and $\rho(A \circ B) \geq a_{ii}b_{ii}$ (see [1]), for any $j \neq i \in N$, we have

$$\begin{aligned} (\rho(A \circ B) - a_{ii}b_{ii})(\rho(A \circ B)) - a_{jj}b_{jj}) &\leq \sum_{k\neq i}|m_{ik}| \sum_{l\neq j}|m_{jl}| \\ &= \sum_{k\neq i}\frac{d_ku_ka_{ik}b_{ik}}{d_iu_i} \sum_{l\neq j}\frac{d_lu_la_{jl}b_{jl}}{d_ju_j} \\ &\leq \left(\max_{k\neq i}\{b_{ik}\} \sum_{k\neq i}\frac{d_ku_ka_{ik}}{d_iu_i}\right)\left(\max_{l\neq j}\{a_{jl}\} \sum_{l\neq j}\frac{d_lu_lb_{jl}}{d_ju_j}\right) \\ &\leq \max_{k\neq i}\{b_{ik}\}(\rho(A) - a_{ii}) \max_{l\neq j}\{a_{jl}\}(\rho(B) - b_{jj}) \\ &= t_is_j\Big(\rho(A) - a_{ii}\Big)\Big(\rho(B) - b_{jj}\Big). \end{aligned} \tag{5}$$

Thus, by solving the quadratic inequality (5), we get that

$$\begin{aligned} \rho(A \circ B) &\leq \frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_is_j(\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{\frac{1}{2}}\Big\} \\ &\leq \max_{i\neq j}\frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_is_j(\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{\frac{1}{2}}\Big\}. \end{aligned}$$

i.e., the conclusion (4) holds.

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Case 2. If one of A and B is reducible. We may denote by $P = (p_{ij})$ the $n \times n$ permutation matrix with

$$p_{12} = p_{23} = \cdots = p_{n-1,n} = p_{n,1} = 1,$$

the remaining p_{ij} zero, then both $A + \varepsilon P$ and $B + \varepsilon P$ are irreducible nonnegative matrices for any chosen sufficiently small positive real number ε. Next we substitute $A + \varepsilon P$ and $B + \varepsilon P$ for A and B, respectively in the previous Case 1, and then letting $\varepsilon \to 0$, the result (4) follows by continuity. □

Remark 1. *Now, we give a comparison between the inequalities (2) and (4). According to the definitions of t_i and s_j, if $t_i + b_{ii} \le \rho(B)$ and $s_j + a_{jj} \le \rho(A)$ $(i, j = 1, \cdots, n)$, then $t_i s_j \le (\rho(B) - b_{ii})(\rho(A) - a_{jj})$. Thus, the inequality (4) is better than the inequality (2).*

Example 1. *([14]). Let A and B be the following two nonnegative matrices:*

$$A := (a_{ij}) = \begin{pmatrix} 4 & 1 & 0 & 2 \\ 1 & 0.05 & 1 & 1 \\ 0 & 1 & 4 & 0.5 \\ 1 & 0.5 & 0 & 4 \end{pmatrix}, \quad B := (b_{ij}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

By calculation, $\rho(A \circ B) = \rho(A) = 5.7339$ and $\rho(B) = 4.0$. Thus the result of Ref.[8] (see p.358) is that

$$\rho(A \circ B) \le \rho(A)\rho(B) = 22.9336.$$

If we respectively apply (1) and (2) to them, according to [14], then

$$\rho(A \circ B) \le \max_{1 \le i \le 4} \left\{ 2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A) \right\} = 17.1017,$$

and

$$\begin{aligned} \rho(A \circ B) \le \max_{i \ne j} \tfrac{1}{2} \Big\{ & a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ & + 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})(\rho(A) - a_{jj})(\rho(B) - b_{jj})]^{\frac{1}{2}} \Big\} = 11.6478. \end{aligned}$$

However, if Theorem 2 is used, then the following inequality can be obtained

$$\begin{aligned} \rho(A \circ B) \le \max_{i \ne j} \tfrac{1}{2} \Big\{ & a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ & + 4t_i s_j(\rho(A) - a_{ii})(\rho(B) - b_{jj})]^{\frac{1}{2}} \Big\} = 8.1897, \end{aligned}$$

which shows that the bound in Theorem 2 is the best among the above bounds.

Corollary 1. *If A and B are two stochastic matrices (i.e., probability matrices, transition matrices, or Markov matrices), then*

$$\begin{aligned} \rho(A \circ B) &\le \max_{i \ne j} \tfrac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_i s_j(1 - a_{ii})(1 - b_{jj})]^{\frac{1}{2}} \right\} \\ &\le \max_{i \ne j} \tfrac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(1 - a_{ii})^2(1 - b_{ii})^2 \right]^{1/2} \right\}. \end{aligned}$$

Since the Fan product of two M-matrices has a lot of similar properties with the Hadamard product of two nonnegative matrices see [2,5–9,13,14]. Note that if A is an irreducible nonsingular M-matrix, then there also exist two positive left and right Perron eigenvectors u and v such that $v^T A = \tau(A)v^T$ and $Au = \tau(A)u$, respectively. Therefore, we may similarly extend the above result (4) to the case of the Fan product of two M-matrices.

Theorem 3. *If $A = (a_{ij})$ and $B = (b_{ij})$ are nonsingular M-matrices, $s_i = \max_{j \neq i} |a_{ij}|$, $t_i = \max_{j \neq i} |b_{ij}|$, then*

$$\tau(A \star B) \geq \min_{i \neq j} \frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4t_is_j(a_{ii} - \tau(A))(b_{jj} - \tau(B))]^{\frac{1}{2}}\Big\}. \quad (6)$$

Proof. This proof is completely similar with the previous Theorem 2. □

3. Comparisons of the simple and mixed type bounds

Though Theorem 1 and Theorem 2 are similar, they are different in form. Next, we give a simple comparison.

Theorem 4. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two nonnegative matrices, if for any $i \neq j$, $t_is_j(\rho(A) - a_{ii})(\rho(B) - b_{jj}) \neq t_js_i(\rho(A) - a_{jj})(\rho(B) - b_{ii})$, then Theorem 1 is better than Theorem 2.*

Proof. According to (3) and (4), for any $i \neq j$, we need only compare

$$\max\left\{t_is_j(\rho(A) - a_{ii})(\rho(B) - b_{jj}), t_js_i(\rho(A) - a_{jj})(\rho(B) - b_{ii})\right\} := M_{ij}$$

with

$$\min\left\{t_it_j(\rho(A) - a_{ii})(\rho(A) - a_{jj}), s_is_j(\rho(B) - b_{ii})(\rho(B) - b_{jj})\right\} := S_{ij}.$$

Without loss of generality, let

$$M_{ij} = t_is_j(\rho(A) - a_{ii})(\rho(B) - b_{jj}). \quad (7)$$

We assume that $M_{ij} < S_{ij}$, *i.e.*,

$$t_is_j(\rho(A) - a_{ii})(\rho(B) - b_{jj}) < \min\left\{t_it_j(\rho(A) - a_{ii})(\rho(A) - a_{jj}), s_is_j(\rho(B) - b_{ii})(\rho(B) - b_{jj})\right\}.$$

Then,

$$0 \leq s_j(\rho(B) - b_{jj}) < t_j(\rho(A) - a_{jj}), \quad (8)$$

and

$$0 \leq t_i(\rho(A) - a_{ii}) < s_i(\rho(B) - b_{ii}). \quad (9)$$

Therefore,

$$0 \leq t_is_j(\rho(A) - a_{ii})(\rho(B) - b_{jj}) < s_it_j(\rho(B) - b_{ii})(\rho(A) - a_{jj}), \quad (10)$$

which is in conflict with the previous condition (7). Thus we see that $M_{ij} \geq S_{ij}$ for any $i \neq j$. This proof is completed. □

By the above discussions, we see that, generally speaking, $M_{ij} \geq S_{ij}$ for any $i \neq j$ when $t_is_j(\rho(A) - a_{ii})(\rho(B) - b_{jj}) \neq s_it_j(\rho(B) - b_{ii})(\rho(A) - a_{jj})$. Next, let us consider the case of $t_is_j(\rho(A) - a_{ii})(\rho(B) - b_{jj}) = s_it_j(\rho(B) - b_{ii})(\rho(A) - a_{jj})$.

Theorem 5. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two nonnegative matrices, if for any $i \neq j$, $t_is_j(\rho(A) - a_{ii})(\rho(B) - b_{jj}) = t_js_i(\rho(A) - a_{jj})(\rho(B) - b_{ii})$ and $\frac{t_j}{s_j} = \frac{\rho(B) - b_{jj}}{\rho(A) - a_{jj}}$, then Theorem 2 is equivalent to Theorem 1.*

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Proof. According to the above Theorem 4, let us reconsider the relationship between M_{ij} and S_{ij} for any $i \neq j$. For convenience, we define

$$\alpha_j = \frac{s_j(\rho(B) - b_{jj})}{t_j(\rho(A) - a_{jj})}.$$

Obviously,

$$t_i t_j(\rho(A) - a_{ii})(\rho(A) - a_{jj}) = s_i t_j(\rho(B) - b_{ii})(\rho(A) - a_{jj})\frac{1}{\alpha_j},$$

and

$$s_i s_j(\rho(B) - b_{ii})(\rho(B) - b_{jj}) = t_i s_j(\rho(B) - b_{jj})(\rho(A) - a_{ii})\alpha_j.$$

Therefore, if $\alpha_j \neq 1$, then we have always $M_{ij} > S_{ij}$ for any $i \neq j$ under the conditions of this theorem. However, when $\alpha_j = 1$, *i.e.*, $\frac{t_j}{s_j} = \frac{\rho(B)-b_{jj}}{\rho(A)-a_{jj}}$, we have $M_{ij} = S_{ij}$ for any $i \neq j$. Therefore, Theorem 2 is equivalent to Theorem 1. Thus, the proof is completed. □

4. Inequalities for the Fan product of several M-matrices

In the previous sections, we mainly consider the Hadamard product of two matrices. In fact, there exist also many of similar inequalities for the minimum eigenvalue of Fan product of two M-matrices:

- In (p.359, [8]), R.A. Horn and C.R. Johnson pointed out that

$$\tau(A \star B) \geq \tau(A)\tau(B). \tag{11}$$

- In 2007, Fang gave another lower bound in the Remark 3 of Ref. [6]

$$\tau(A \star B) \geq \min_{1 \leq i \leq n} \Big\{ b_{ii}\tau(A) + a_{ii}\tau(B) - \tau(A)\tau(B) \Big\}. \tag{12}$$

- In 2009, Liu *et al.*[14] gave a sharper bound than (12), *i.e.*,

$$\begin{aligned} \tau(A \star B) \geq \tfrac{1}{2} \min_{i \neq j} \Big\{ & a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ & + 4(b_{ii} - \tau(B))(a_{ii} - \tau(A))(b_{jj} - \tau(B))(a_{jj} - \tau(A))]^{\frac{1}{2}} \Big\}. \end{aligned} \tag{13}$$

Note that the classes of M-matrices and H-matrices are both closed under the Fan product (see Observation 5.7.2 in [8]). Therefore, we may consider the case of the product of several matrices. For convenience, we shall continue to use notation employed previously. But, according to Ref. [8], the definition of the function $\tau(\cdot)$ should be extended to general matrices via the comparison matrix. The comparison matrix $M(A) = (m_{ij})$ of a given matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ is defined by

$$m_{ij} = \begin{cases} -|a_{ij}|, & \text{if } i \neq j, \\ |a_{ii}|, & \text{if } i = j. \end{cases}$$

Definition 1. *([8]). For any $A \in \mathbb{R}^{n \times n}$, $\tau(A) := \tau(M(A))$, where $M(A)$ is the comparison matrix of A.*

In addition, if $A = (a_{i,j}) \in \mathbb{R}^{m \times n}$ has nonnegative entries and $\alpha \geq 0$, we write $A^{(\alpha)} \equiv (a_{i,j}^{\alpha})$ for the αth Hadamard power of A. Moreover, we use the convention $0^0 \equiv 0$ to ensure continuity in a for $\alpha \geq 0$, see [8].

In [8], it is shown by Theorem 5.7.15 that if $A_1, \ldots A_m$ are $n \times n$ H-matrices and $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$ satisfy $\sum_{k=1}^{m} \alpha_k \geq 1$, then

$$\tau(A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)}) \geq \prod_{k=1}^{m} [\tau(A_k)]^{\alpha_k}, \tag{14}$$

where $A^{(\alpha)}$ is again defined as entrywise and any scalar definition of a^α such that $|a^\alpha| = |a|^\alpha$ is allowed (see [8]). Next, for convenience, we define

$$a^\alpha := \begin{cases} a^\alpha, & \text{if } a \geq 0, \\ -|a|^\alpha, & \text{if } a < 0. \end{cases} \tag{15}$$

The above theorem (14) provides a beautiful result, which encourages us to continue researching this problem. Since for any H-matrix A, according to Definition 1, $\tau(A) = \tau(M(A))$. Therefore, we need only consider the M-matrix case.

First, let us recall the following lemmas.

Lemma 1. *([11]). Let A be an irreducible nonsingular M-matrix, if $AZ \geq kZ$ for a nonnegative nonzero vector Z, then $k \leq \tau(A)$.*

Lemma 2. *([10]). Let $x_j = (x_j(1), \cdots, x_j(n))^T \geq 0$, $j \in \{1, 2\cdots, m\}$, if $P_j > 0$ and $\sum_{k=1}^m \frac{1}{P_k} \geq 1$, then we have*

$$\sum_{i=1}^n \prod_{j=1}^m x_j(i) \leq \prod_{j=1}^m \Big\{ \sum_{i=1}^n [x_j(i)]^{P_j} \Big\}^{\frac{1}{P_j}}. \tag{16}$$

Next, according to these lemmas, we generalize the inequality (12) of the Fan product of two matrices to the Fan product of several matrices.

Theorem 6. *For any positive integer P_k with $\sum_{k=1}^m \frac{1}{P_k} \geq 1$, if $A_k \in M_n$ for all $k \in \{1, 2, \cdots, m\}$, then*

$$\tau(A_1 \star A_2 \cdots \star A_m) \geq \min_{1\leq i \leq n} \Big\{ \prod_{k=1}^m A_k(i,i) - \prod_{k=1}^m [A_k(i,i)^{P_k} - \tau(A_k^{(P_k)})]^{\frac{1}{P_k}} \Big\}. \tag{17}$$

Proof. It is quite evident that the (17) holds with the equality for $n = 1$. Below we assume that $n \geq 2$.

Case 1. Suppose that A_k ($k \in \{1, 2, \cdots, m\}$) is irreducible, then $A_k^{(P_k)}$ is also irreducible. Let $u_k^{(P_k)} = (u_k(1)^{P_k}, \cdots, u_k(n)^{P_k})^T > 0$ be a right Perron eigenvector of $A_k^{(P_k)}$, and $u_k = (u_k(1), \cdots, u_k(n))^T > 0$, thus for any $i \in N$, we have

$$A_k^{(P_k)} u_k^{(P_k)} = \tau(A_k^{(P_k)}) u_k^{(P_k)},$$

$$A_k(i,i)^{P_k} u_k(i)^{P_k} - \sum_{j \neq i} |A_k(i,j)^{P_k}| u_k(j)^{P_k} = \tau(A_k^{(P_k)}) u_k(i)^{(P_k)},$$

i.e.,

$$\sum_{j \neq i} |A_k(i,j)^{P_k}| u_k(j)^{P_k} = \Big(A_k(i,i)^{P_k} - \tau(A_k^{(P_k)}) \Big) u_k(i)^{P_k}. \tag{18}$$

Denote $C = A_1 \star A_2 \cdots \star A_m$, $Z = u_1 \star u_2 \cdots \star u_m = (Z(1), \cdots, Z(n))^T > 0$, where $Z(i) = \prod_{k=1}^m u_k(i)$. By the Lemma 2 and (18), we get that

$$\begin{aligned}
(CZ)_i &= \Big(\prod_{k=1}^m A_k(i,i) \Big) Z(i) - \Big(\sum_{j \neq i} \prod_{k=1}^m |A_k(i,j)| \Big) Z(j) \\
&= \Big(\prod_{k=1}^m A_k(i,i) \Big) Z(i) - \sum_{j \neq i} \prod_{k=1}^m \Big(|A_k(i,j)| u_k(j) \Big) \\
&\geq \Big(\prod_{k=1}^m A_k(i,i) \Big) Z(i) - \prod_{k=1}^m \Big\{ \sum_{j \neq i} [|A_k(i,j)| u_k(j)]^{(P_k)} \Big\}^{\frac{1}{P_k}} \quad \text{(by the equality (18))} \\
&= \Big(\prod_{k=1}^m A_k(i,i) \Big) Z(i) - \prod_{k=1}^m \Big\{ [A_k(i,i)^{P_k} - \tau(A_k^{(P_k)})] u_k(i)^{P_k} \Big\}^{\frac{1}{P_k}} \\
&= \Big\{ \prod_{k=1}^m A_k(i,i) - \prod_{k=1}^m [A_k(i,i)^{P_k} - \tau(A_k^{(P_k)})] \Big\}^{\frac{1}{P_k}} Z(i).
\end{aligned}$$

According to the Lemma 1, we obtain that

$$\tau(A_1 \star A_2 \cdots \star A_m) \geq \min_{1 \leq i \leq n} \left\{ \prod_{k=1}^{m} A_k(i,i) - \prod_{k=1}^{m} [A_k(i,i)^{P_k} - \tau(A_k^{(P_k)})]^{\frac{1}{P_k}} \right\}.$$

Case 2. If one of A_k $(i = 1, 2, \cdots, m)$ is reducible. Similar to the Case 2 of the previous Theorem 2, let $P = (p_{ij})$ be the $n \times n$ permutation matrix with $p_{12} = p_{23} = \cdots = p_{n-1,n} = p_{n,1} = 1$, the remaining p_{ij} zero, then $A_k - \varepsilon P$ is an irreducible nonsingular M-matrix for any chosen positive real number ε. Now we substitute $A_k - \varepsilon P$ for A_k, in the previous Case 1, and then letting $\varepsilon \to 0$, the result (17) follows by continuity. □

Remark 2. *If we take $m = 2$ in Theorem 6, one can obtain the following results:*

- *If $p_1 = p_2 = 1$, $A_1 = A = (a_{ij})$, $A_2 = B = (b_{ij})$, we have*

$$\tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ a_{ii}b_{ii} - (a_{ii} - \tau(A))(b_{ii} - \tau(B)) \right\},$$

  which is just the inequality (12).
- *If $p_1 = p_2 = 2$, $A_1 = A = (a_{ij})$, $A_2 = B = (b_{ij})$, then*

$$\tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ a_{ii}b_{ii} - [a_{ii}^2 - \tau(A \star A)]^{\frac{1}{2}} [b_{ii}^2 - \tau(B \star B)]^{\frac{1}{2}} \right\}. \tag{19}$$

  In addition, by using the inequalities of arithmetic and geometric means, we know that

$$a_{ii}^2 \tau(B \star B) + b_{ii}^2 \tau(A \star A) \geq 2a_{ii}b_{ii}[\tau(A \star A)\tau(B \star B)]^{\frac{1}{2}},$$

  so

$$(a_{ii}^2 - \tau(A \star A))(b_{ii}^2 - \tau(B \star B)) \leq \left\{ a_{ii}b_{ii} - [\tau(A \star A)\tau(B \star B)]^{\frac{1}{2}} \right\}^2. \tag{20}$$

  Since for any $A, B \in M_n$, $\tau(A \star B) \geq \tau(A)\tau(B)$ (see [14] or (11)), then, by (20), we know that

$$a_{ii}b_{ii} - \left[(a_{ii}^2 - \tau(A \star A))(b_{ii}^2 - \tau(B \star B)) \right]^{\frac{1}{2}} \geq [\tau(A \star A)\tau(B \star B)]^{\frac{1}{2}} \geq \tau(A)\tau(B).$$

  That is, the inequality (19) is better than the inequality (11). In addition, the following example shows that the inequality (19) is also better than the inequality (13).

Example 2. *([14]). Consider the following two 3×3 M-matrices:*

$$A := (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -0.5 \\ -0.5 & -1 & 2 \end{pmatrix}, \quad B := (b_{ij}) = \begin{pmatrix} 1 & -0.25 & -0.25 \\ -0.5 & 1 & -0.25 \\ -0.25 & -0.5 & 1 \end{pmatrix}.$$

By direct calculation, $\tau(A) = 0.5402$, $\tau(B) = 0.3432$ and $\tau(A \star B) = 0.9377$. According to Ref. [14], the inequality (13) shows that

$$\tau(A \star B) \geq 0.7655.$$

If we apply (19) to them, then

$$\tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ a_{ii}b_{ii} - [a_{ii}^2 - \tau(A \star A)]^{\frac{1}{2}} [b_{ii}^2 - \tau(B \star B)]^{\frac{1}{2}} \right\} = 0.8579,$$

which shows that our result is much closer to the exact value 0.9377.

- *If* $p_1 = 1, p_2 = 2, A_1 = A = (a_{ij}), A_2 = B = (b_{ij})$*, then we get*

$$\tau(A \star B) \geq \min_{1 \leq i \leq n} \left\{ a_{ii} b_{ii} - [a_{ii} - \tau(A)][b_{ii}^2 - \tau(B \star B)]^{\frac{1}{2}} \right\}. \tag{21}$$

5. Concluding remarks

This paper mainly presents the relationship between the simple type [5] and the mixed type (4), which perfects the corresponding theory. In addition, we also research the problem on the minimum eigenvalue of the Fan product of more M-matrices and obtain several interesting results, see the inequalities (17), (20) and (21). Since for any $A \in \mathcal{M}_n$, $\tau(A \star A) \geq (\tau(A))^2$*, numerical examples and some analyses show that the special cases (e.g., (20) and (21)) of the inequality (17) improve some known results stated in this paper.*

Finally, it is worthy mentioning that there also exist other products in statistics or econometrics, such as the block Hadamard product [12], Khatri-Rao and Tracy-Singh products [15]. Are there similar results on these products? It may be still an interesting problem.

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