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Article

Modular uniform convexity of Lebesgue spaces of variable integrability.

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Abstract: We analyze the modular geometry of the variable exponent Lebesgue space $L^{p(\cdot)}$. We show that $L^{p(\cdot)}$ possesses a modular uniform convexity property. Part of the novelty is that the property holds even in the case $\sup_{x\in\Omega} p(x) = \infty$. We present specific applications to fixed point theory.

Keywords: Fixed point theorem, modular uniform convexity, modular vector spaces, Nakano spaces, uniform convexity, variable exponent spaces.

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0. Introduction

In this work we prove a hitherto unknown modular convexity property of the variable exponent Lebesgue spaces $L^{p(\cdot)}$, which has far reaching applications in fixed point theory, remarkably even in the case in with the exponent $p(\cdot)$ is unbounded.

Lebesgue spaces of variable-exponent, $L^{p(\cdot)}$ were first mentioned in [15]. In the late 19^{th} century these spaces were brought into the center stage of mathematical research as they were realized to be the natural solution space for partial differential equations with non-standard growth. The first systematic treatment of variable exponent spaces was given in [9]. In 1997, while studying differential equations in electromagnetism, V. Zhikov's work [20] conduced to the minimization of integrals of the type

$$\int_\Omega |\nabla f(x)|^{p(x)} dx,$$

which in turn leads to the corresponding Lagrange-Euler equation:

$$\Delta_{p(\cdot)} u := \operatorname{div}\left(|\nabla u|^{p(\cdot)-2} \nabla u\right) = 0. \tag{1}$$

Because of the variability of $p(x)$, (1) is said to have non-standard growth. The natural space for the solutions of such differential equations must take into consideration the dependence of $p(x)$ on the space variable x. It is at this point obvious that the classical L^p theory is not sufficient in this situation and that a condition such as

$$\int_\Omega |\nabla u(x)|^{p(x)} dx < \infty$$

should be imposed as an a-priori requirement.
Similar considerations arise in the study of the hydrodynamic equations governing non-Newtonian fluids ([16,17]). These equations have non-standard growth and model, in particular, electrorheological fluids, i.e., fluids whose viscosity can be changed dramatically and in a few mili-seconds when exposed

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to a magnetic or an electric field. Electrorheological fluids are currently the object of intense research activity in both, theoretical and applied fields. Their applications include medicine, civil engineering and military science, [1–3,18]

Through these applications, then, there inexorably emerged the need for a deeper understanding of these generalized functional spaces with variable integrability.
The article is structured as follows: In Section 2 we give the definition of a convex modular and introduce the definition of the $UUC2$ condition. In Section 3 we lay the ground for our main result by properly defining the variable exponent Lebesgue spaces. In Section 3 we prove Theorem 3, which constitutes the main result of this work and in Section 4 we present applications.

1. Modular spaces

In this section we introduce the basic definitions and terminology on modular spaces to be used in the sequel. We also state the concept of modular uniform convexity. for a detailed account of the ideas expounded in this section, we refer the interested reader to the monograph [8]. Let V be a real or complex vector space. We denote the scalar field with $\mathbb{K}$.

Definition 1. *An s-convex modular* ($0 < s \leq 1$) *on a real or complex vector space V is a function*

$$\rho : X \longrightarrow [0, \infty]$$

that satisfies the following conditions:

1. $\rho(x) = 0 \iff x = 0$
2. $\rho(\alpha x) = |\alpha|\rho(x)$ *for any* $x \in V$, $|\alpha| = 1$
3. $\rho(\alpha x + (1-\alpha)y) \leq \alpha^s \rho(x) + (1-\alpha)^s \rho(y)$ *for all* $x, y \in V$ *and* $\alpha \in (0, 1]$.

In particular, if $s = 1$, *the modular is said to be convex. A convex modular* ρ *on a vector space V is left-(right-) continuous if for any* $x \in V$ *the map*

$$\alpha \longrightarrow \rho(\alpha x)$$

is left-(right-) continuous on $[0, \infty)$ *(on* $(0, \infty)$ *for left continuity); if* ρ *is both left- and right-continuous we refer to it as a continuous modular.*

If ρ has properties (i) and (ii) but fails property (i), it is said to be a semimodular on V. By reason of its relevance to the present work, the following standard example is noted: Let $\Omega \subseteq \mathbb{R}^n$ be a domain and let $\mathcal{M}$ stand for the vector space of all Borel-measurable real-valued functions on Ω. Then the functional

$$u \longrightarrow \rho_\infty(u) = \begin{cases} 0 & \text{if } u \text{ is bounded a.e.} \\ \infty & \text{otherwise.} \end{cases} \tag{2}$$

is a semimodular on $\mathcal{M}$. The following definition is standard ([11], [14]):

Definition 2. *A convex modular* ρ *on a vector space V is said to be uniformly convex if for every* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that for every* $u \in V$ *and* $v \in V$ *with* $\rho(u) = 1$ *and* $\rho(v) = 1$ *and* $\rho(u - v) > \varepsilon$ *it holds that* $\rho\left(\frac{u+v}{2}\right) < 1 - \delta(\varepsilon)$.

1.1. Modular uniform convexity

We refer the reader to [8] for the following related concepts:

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Definition 3. *Let ρ be a convex modular on a real vector space V, $r > 0$, $\varepsilon > 0$. Set*

$$D(r,\varepsilon) = \left\{(u,v) \in V \times V : \rho(u) \le r\,,\, \rho(v) \le r\,,\, \rho\left(\frac{u-v}{2}\right) \ge \varepsilon r\right\} \tag{3}$$

and

$$\delta(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{u+v}{2}\right) : (u,v) \in D(r,\varepsilon)\right\}. \tag{4}$$

If $D(r,\varepsilon) = \emptyset$ we define $\delta(r,\varepsilon) = 1$. Notice that for $\varepsilon > 0$ small enough, $D(r,\varepsilon) \neq \emptyset$.

Definition 4. *The modular ρ is said to be type 2-uniformly convex UUC2 (see [8]) if for each $s \ge 0$, $\varepsilon > 0$, there exists $\eta(s,\varepsilon) > 0$ such that for any $r > s > 0$*

$$\delta(r,\varepsilon) > \eta(s,\varepsilon).$$

2. Variable exponent Lebesgue spaces

In this section we delve into the uniform convexity of the Lebesgue spaces of variable exponent. We start by stating the basic definitions ([5,6,9,12]). Given a domain $\Omega \subset \mathbb{R}^n$, $\mathcal{M}(\Omega)$ will stand for the vector space of all real-valued, Borel-measurable functions defined on Ω. The subset of $\mathcal{M}$ consisting of functions

$$p : \Omega \longrightarrow [1,\infty]$$

will be denoted by $\mathcal{P}(\Omega)$. As usual, the Lebesgue measure of a subset $A \subset \mathbb{R}^n$ will be denoted by $|A|$. For each such p define the sets:

$$\begin{aligned} \Omega_0 &= \{x \in \Omega : 1 < p(x) < \infty\} \\ \Omega_1 &= \{x \in \Omega : p(x) = 1\} \\ \Omega_\infty &= \{x \in \Omega : p(x) = \infty\} \end{aligned}$$

and set

$$p_- = \operatorname*{ess\,inf}_{x\in\Omega_0} p(x) \text{ and } p_+ = \operatorname*{ess\,sup}_{x\in\Omega_0} p(x) \text{ if } |\Omega_0| > 0.$$

Theorem 1. *The function*

$$\begin{aligned} \rho_p &: \mathcal{M}(\Omega) \longrightarrow [0,\infty] \\ \rho_p(u) &= \int\limits_{\Omega_0\cup\Omega_1} |u(x)|^{p(x)} d\mu + \sup_{x\in\Omega_\infty} |u(x)| \end{aligned}$$

defines a convex, continuous modular on $\mathcal{M}(\Omega)$.

Proof of Theorem 1. See [6,9]. $\square$

On the subspace V of $\mathcal{M}(\Omega)$ defined as

$$V = \{\rho_p(\lambda v) < \infty \text{ for some } \lambda > 0\},$$

the functional

$$\|u\|_p = \inf\left\{\lambda > 0 : \rho_p(\lambda^{-1}u) \le 1\right\} \tag{5}$$

is a norm; it is called the Luxemburg norm. Furnished with the Luxemburg norm, V becomes a Banach space, which coincides with the classical Lebesgue space L^p if p is constant on Ω. For this reason, V is

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referred to as the variable exponent Lebesgue space, $L^{p(\cdot)}(\Omega)$.
To the author's best knowledge, the first reference to the modular given in theorem 1 is to be found in the work by Orlicz [15]. We refer the reader to [5,6,9] for a systematic study of the variable exponent Lebesgue spaces. Notice that if $|\Omega_0| = |\Omega_1| = 0$, then $\rho_p = \rho_\infty$.
We point out in passing that $L^{p(\cdot)}(\Omega)$ is the Musielak-Orlicz space corresponding to the Musielak-Orlicz function

$$\begin{aligned} \varphi : \Omega \times [0,\infty) &\longrightarrow [0,\infty) \\ \varphi(x,t) &= t^{p(x)}. \end{aligned}$$

These spaces were introduced by Nakano in 1950 [13]; we refer to the surveys [5,11,12] for more detailed information on this vast topic.
If p is constant in Ω, the modular ρ_p is simply the p^{th} power of the Luxemburg norm (5). For this reason, when working whether with the norm or with the modular, one faces essentially the same technicalities. If p is non-constant, however, the situation changes radically. In this case, the handling of the norm presents technical challenges and its often desirable to work with the modular whenever possible. This is especially true when dealing with uniform convexity.

3. Uniform convexity

Recall that a normed space $(X, \|\cdot\|)$ is said to be uniformly convex iff for any $\varepsilon : 0 < \varepsilon \le 2$ one has

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1\,,\ \|y\| \le 1\,,\ \|x-y\| \ge \varepsilon\right\} > 0.$$

The number $\delta_X(\varepsilon)$ is known as the modulus of uniform convexity of X (see, for example, [4] and [7]). For the variable exponent spaces $L^{p(\cdot)}(\Omega)$, uniform convexity is fully characterized: We refer the reader to [10] for the proof of the following theorem, from which it follows that the uniform convexity of the Luxemburg norm (5) is equivalent to the Δ_2-condition.

Theorem 2. *Let $p \in \mathcal{P}(\Omega)$. The following statements are equivalent:*
(i) $L^{p(\cdot)}(\Omega)$ is uniformly convex;
(ii) $1 < p_- \le p_+ < \infty$;
(iii) The modular ρ_p satisfies the Δ_2-condition, i.e., there exists a positive constant C such that for any $u \in L^{p(\cdot)}(\Omega)$ it holds that $\rho_p(2u) \le C\rho_p(u)$.

4. Modular uniform convexity

Though it follows from Theorem 2 that there is no hope for norm-uniform convexity of $L^{p(\cdot)}(\Omega)$ if the exponent p is unbounded, we will show in this section that even when $p_+ = \infty$, the modular ρ_p still exhibits the uniform-convexity property $UUC2$ introduced in Definition 3. As will be discussed in Section 5 this property has far-reaching implications.
To tackle the modular uniform convexity property in aim, te following auxiliaries inequalities are necessary:

Lemma 1. *The following inequalities are valid:*

(i) [4] If $p \ge 2$, then we have

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \le \frac{1}{2}\Big(|a|^p + |b|^p\Big),$$

for any $a, b \in \mathbb{R}$.

(ii) *[19] If* $1 < p \leq 2$, *then we have*

$$\left|\frac{a+b}{2}\right|^p + \frac{p(p-1)}{2}\left|\frac{a-b}{|a|+|b|}\right|^{2-p}\left|\frac{a-b}{2}\right|^p \leq \frac{1}{2}\Big(|a|^p + |b|^p\Big),$$

for any $a, b \in \mathbb{R}$ *such that* $|a| + |b| \neq 0$.

A detailed proof of (ii) is given in [12].
We next set out to state and prove Theorem 3, which is the central result of this work.

Theorem 3. *Let* $\Omega \subseteq \mathbb{R}^n$ *be open and* $p \in \mathcal{P}(\Omega)$. *If* $|\Omega_\infty| = 0$ *and* $p_- > 1$ *then the modular*

$$\rho_p : L^p(\Omega) \longrightarrow [0, \infty)$$
$$\rho_p(u) = \int_\Omega |u(x)|^{p(x)} dx$$

satisfies the UCC2 condition.

Remark 1. The condition $|\Omega_\infty| = 0$ cannot be removed, as it is easily shown that $L^\infty(\Omega)$ does not have the $UUC2$ property if $|\Omega| > 0$.

Proof of Theorem 3. Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $p \in \mathcal{P}(\Omega)$; let ρ_p be as in Theorem 1.
Let $r > 0$, $\varepsilon > 0$ and consider $u, v \in D(r, \varepsilon)$, that is, assume that

$$\rho_p(u) \leq r\,,\ \rho_p(v) \leq r\,,\ \rho_p\left(\frac{u-v}{2}\right) \geq \varepsilon r.$$

On account of the convexity of ρ_p we have $\varepsilon \leq 1$: indeed,

$$r\varepsilon \leq \rho_p\left(\frac{u-v}{2}\right) \leq r.$$

Let $\Omega_1 := \{x \in \Omega : p(x) \geq 2\}$. Then either

$$\int_{\Omega_1} \left|\frac{u(x)-v(x)}{2}\right|^{p(x)} dx \geq \frac{r\varepsilon}{2} \tag{6}$$

or

$$\int_{\Omega\setminus\Omega_1} \left|\frac{u(x)-v(x)}{2}\right|^{p(x)} dx \geq \frac{r\varepsilon}{2}. \tag{7}$$

If (6) holds, one has, by virtue of inequality (i) in Lemma 1:

$$\begin{aligned}\int_{\Omega_1} \left|\frac{u(x)-v(x)}{2}\right|^{p(x)} dx + \int_{\Omega_1} \left|\frac{u(x)+v(x)}{2}\right|^{p(x)} dx \\ \leq \frac{1}{2}\left(\int_{\Omega_1} |u(x)|^{p(x)} dx + \int_{\Omega_1} |v(x)|^{p(x)} dx\right).\end{aligned}$$

It is thus concluded that in this case,

$$\int_{\Omega_1} \left|\frac{u(x)+v(x)}{2}\right|^{p(x)} dx \leq \frac{1}{2}\left(\int_{\Omega_1} |u(x)|^{p(x)} dx + \int_{\Omega_1} |v(x)|^{p(x)} dx\right) - \frac{r\varepsilon}{2}.$$

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Thus,

$$
\begin{aligned}
\rho_p\left(\frac{u+v}{2}\right) &= \int\limits_{\Omega_1}\left|\frac{u(x)+v(x)}{2}\right|^{p(x)}dx + \int\limits_{\Omega\setminus\Omega_1}\left|\frac{u(x)+v(x)}{2}\right|^{p(x)}dx \\
&\le \frac{1}{2}\left(\int\limits_{\Omega_1}|u(x)|^{p(x)}dx + \int\limits_{\Omega_1}|v(x)|^{p(x)}dx\right) - \frac{r\varepsilon}{2} \\
&\quad + \frac{1}{2}\left(\int\limits_{\Omega\setminus\Omega_1}|u(x)|^{p(x)}dx + \int\limits_{\Omega\setminus\Omega_1}|v(x)|^{p(x)}dx\right) \\
&= \frac{1}{2}\left(\rho_p(u)+\rho_p(v)\right) - \frac{r\varepsilon}{2} \\
&\le r\left(1-\frac{\varepsilon}{2}\right).
\end{aligned}
$$

On the other hand, if (7) holds, we define

$$
\Omega_2 := \left\{x \in \Omega\setminus\Omega_1 : |u(x)-v(x)| \le \frac{\varepsilon}{4}(|u(x)|+|v(x)|)\right\}.
$$

With this notation, it follows that

$$
\begin{aligned}
\int\limits_{\Omega_2}\left|\frac{u(x)-v(x)}{2}\right|^{p(x)}dx &\le \frac{\varepsilon}{8}\left(\int\limits_{\Omega_2}|u(x)|^{p(x)}dx + \int\limits_{\Omega_2}|v(x)|^{p(x)}\right) \\
&\le \frac{\varepsilon}{8}\left(\rho_p(u)+\rho_p(u)\right) \le \frac{r\varepsilon}{4}.
\end{aligned}
$$

The validity of (7) implies in particular that

$$
\begin{aligned}
\int\limits_{\Omega\setminus(\Omega_1\cup\Omega_2)}\left|\frac{u(x)-v(x)}{2}\right|^{p(x)} &= \int\limits_{\Omega\setminus\Omega_1}\left|\frac{u(x)-v(x)}{2}\right|^{p(x)} - \int\limits_{\Omega_2}\left|\frac{u(x)-v(x)}{2}\right|^{p(x)} \\
&\ge \frac{r\varepsilon}{2} - \frac{r\varepsilon}{4} = \frac{r\varepsilon}{4}.
\end{aligned}
$$

It follows from inequality (ii) in Lemma 1 that if $x \in \Omega\setminus(\Omega_1\cup\Omega_2)$, one has

$$
\begin{aligned}
&\left|\frac{u(x)+v(x)}{2}\right|^{p(x)} + (p_- - 1)\frac{\varepsilon}{8}\left|\frac{u(x)-v(x)}{2}\right|^{p(x)} \\
&\le \left|\frac{u(x)+v(x)}{2}\right|^{p(x)} + \frac{p(x)(p(x)-1)}{2}\left(\frac{\varepsilon}{4}\right)^{2-p(x)}\left|\frac{u(x)-v(x)}{2}\right|^{p(x)} \\
&\le \left|\frac{u(x)+v(x)}{2}\right|^{p(x)} + \frac{p(x)(p(x)-1)}{2}\left|\frac{u(x)-v(x)}{|u(x)|+|v(x)|}\right|^{2-p(x)}\left|\frac{u(x)-v(x)}{2}\right|^{p(x)} \\
&\le \frac{1}{2}(|u(x)|^{p(x)}+|v(x)|^{p(x)}).
\end{aligned}
$$

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Integrating the last inequality over $\Omega \setminus (\Omega_1 \cup \Omega_2)$ it is easily concluded that

$$\int\limits_{\Omega\setminus(\Omega_1\cup\Omega_2)} \left|\frac{u(x)+v(x)}{2}\right|^{p(x)} dx + (p_- - 1)\,\frac{\varepsilon}{8} \int\limits_{\Omega\setminus(\Omega_1\cup\Omega_2)} \left|\frac{u(x)-v(x)}{2}\right|^{p(x)} dx$$
$$\le \frac{1}{2}\left(\int\limits_{\Omega\setminus(\Omega_1\cup\Omega_2)} |u(x)|^{p(x)} dx + \int\limits_{\Omega\setminus(\Omega_1\cup\Omega_2)} |v(x)|^{p(x)} dx \right).$$

We arrive thus at

$$\int\limits_{\Omega\setminus(\Omega_1\cup\Omega_2)} \left|\frac{u(x)+v(x)}{2}\right|^{p(x)} dx$$
$$\le \frac{1}{2}\left(\int\limits_{\Omega\setminus(\Omega_1\cup\Omega_2)} |u(x)|^{p(x)} dx + \int\limits_{\Omega\setminus(\Omega_1\cup\Omega_2)} |v(x)|^{p(x)} dx \right) - (p_- - 1)\,\frac{\varepsilon^2}{32} r.$$

In all

$$\begin{aligned}
\rho_p\left(\frac{u+v}{2}\right) &= \int\limits_{\Omega_1\cup\Omega_2} \left|\frac{u(x)+v(x)}{2}\right|^{p(x)} dx + \int\limits_{\Omega\setminus(\Omega_1\cup\Omega_2)} \left|\frac{u(x)+v(x)}{2}\right|^{p(x)} dx \\
&\le \frac{1}{2}\left[\int\limits_{\Omega_1\cup\Omega_2} |u(x)|^{p(x)} dx + \int\limits_{\Omega_1\cup\Omega_2} |v(x)|^{p(x)} dx \right] \\
&+ \frac{1}{2}\left[\int_{\Omega\setminus(\Omega_1\cup\Omega_2)} |u(x)|^{p(x)} dx + \int\limits_{\Omega\setminus\Omega_1\cup\Omega_2} |v(x)|^{p(x)} dx \right] \\
&- (p_- - 1)\,\frac{\varepsilon^2}{32} r \\
&\le r - (p_- - 1)\,\frac{\varepsilon^2}{32} r = r\left(1 - (p_- - 1)\,\frac{\varepsilon^2}{32}\right).
\end{aligned}$$

We conclude that for any $r > 0$, $\varepsilon > 0$ and arbitrary $u, v \in D(r, \varepsilon)$ as specified in Definition 3, it holds that

$$1 - \frac{1}{r}\rho_p\left(\frac{u+v}{2}\right) \ge \min\left\{\frac{\varepsilon}{2}, (p_- - 1)\,\frac{\varepsilon^2}{32}\right\} > 0,$$

and it is concluded by definition that $L^{p(\cdot)}(\Omega)$ is $UUC2$. $\square$

5. Applications

A remarkable fact about the above discussion is that the $UUC2$ property holds even if $p_+ = \sup_{x\in\Omega} p(x) = \infty$, that is, in the absence of the Δ_2 condition. This observation makes the $UUC2$ condition a valuable tool for dealing with certain applications that have been hitherto heavily Δ_2-dependent. For an exhaustive treatment of the interplay between the theory of modular spaces and the theory of fixed point of mappings we refer the reader to the monograph [8].

Norm convergence is equivalent to modular convergence in $L^{p(\cdot)}(\Omega)$ if and only if the modular ρ_p satisfies the Δ_2 condition ([5,12]). Bearing this fact in mind, we introduce some terminology before proceeding any further: A subset $W \in L^{p(\cdot)}(\Omega)$ will be called ρ_p**-bounded** if there exists a constant $C \ge 0$ such that the inequality

$$\rho_p(u) \le C$$

holds for any $u \in W$. W is said to be ρ_p-closed if whenever

$$u_n \overset{\rho_p}{\to} u$$

one has $u \in W$. Notice that if $p_+ = \infty$, then ρ-closedness and ρ-boundedness are strictly weaker than norm-closedness and norm-boundedness, respectively.
The next observation is of particular importance in the sequel: Let $u \in L^{p(\cdot)}(\Omega)$, $v \in L^{p(\cdot)}(\Omega)$ and let (v_n) be ρ_p-convergent to v. Fatou's Lemma yields the following inequality

$$\rho_p(u-v) \leq \liminf_{n\to\infty} \rho_p(u-v_n).$$

For obvious reasons, the above is known as the Fatou property of the modular ρ_p.

Theorem 4. *Let $p : \Omega \longrightarrow (1,\infty)$; assume $p_- = \inf_{x\in\Omega} p(x) > 1$. Let $W \subset L^{p(\cdot)}(\Omega)$ be convex and ρ_p-closed and $u \in L^{p(\cdot)}(\Omega)$ satisfy*

$$d_{\rho_p}(u,W) = \inf\left\{\rho_p(u-v) : v \in W\right\} < \infty. \tag{8}$$

Then there exists a unique $v_0 \in W$ for which

$$d_{\rho_p}(u,W) = \rho_p(u-v_0).$$

Proof of Theorem 4. One can clearly assume that $u \notin W$, otherwise there is nothing to prove. Under this assumption, one must have $d(u,W) > 0$, due to the ρ_p-closedness of W. Let $(v_n) \subseteq W$ be such that

$$\rho_p(u-v_n) < d(u,W)\left(1+\frac{1}{n}\right).$$

Then the sequence $\left(\frac{v_n}{2}\right)$ must be ρ_p-Cauchy, i.e., it must necessarily hold that $\rho_p(2^{-1}(v_n - v_m)) \to 0$ as $m,n \to \infty$. The latter follows by contradiction. Indeed, if otherwise, there would exist $\delta > 0$ and strictly increasing subsequences $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ with $n_k > m_k$ for every k such that

$$\rho_p\left(\frac{v_{n_k} - v_{m_k}}{2}\right) \geq \delta \tag{9}$$

for each $k \in \mathbb{N}$. Since $n_k > m_k$, it would then hold that

$$\max\{\rho_p(u-v_{n_k}), \rho_p(u-v_{m_k})\} \leq d(u,W)\left(1+\frac{1}{m_k}\right) := r_k.$$

Together with (9) and in by virtue of Definitions (3) and (4) and Theorem 1, there exists $\eta > 0$

$$1 - \frac{1}{r}\rho_p\left(u - \frac{(v_{m_k}+v_{n_k})}{2}\right) \geq \eta > 0,$$

for any $k \in \mathbb{N}$. Though not mentioned explicitly there, it is apparent from the proof of Theorem 1 that η is independent of r_k. Since W is convex by assumption, the last inequality above yields

$$\begin{aligned} d(u,W) &\leq \rho_p\left(u - \frac{(v_{m_k}+v_{n_k})}{2}\right) \leq r(1-\eta) \\ &= d(u,W)\left(1+\frac{1}{m_k}\right)(1-\eta). \end{aligned}$$

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Letting k tend to ∞ one clearly reaches a contradiction: in conclusion, the sequence $\left(\frac{v_n}{2}\right)$ is ρ_p-Cauchy, as claimed. Since $L^{p(\cdot)}(\Omega)$ is ρ_p-complete, we define v as

$$\lim_{n\to\infty} \rho_p(v - 2^{-1}v_n) = 0.$$

Notice that

$$\rho_p\left(2v - \left(v + \frac{v_k}{2}\right)\right) \to 0 \text{ as } k \to \infty;$$

and that for fixed $k \in \mathbb{N}$, $\left(\frac{v_k+v_n}{2}\right)_n$ converges to $\frac{v_k}{2} + v$. The convexity and ρ_p-closedness of W imply then that $\frac{v_k}{2} + v \in W$ for each k and invoking again the closedness of W we conclude that $2v \in W$. On account of the Fatou's property for the modular ρ_p one concludes that

$$\begin{aligned} d(u, W) &\le \rho_p(u - 2v) \le \liminf_{k\to\infty} \rho_p\left(u - \left(v + \frac{v_k}{2}\right)\right) \\ &\le \liminf_{n\to\infty} \liminf_{k\to\infty} \rho_p\left(u - \left(\frac{v_n + v_k}{2}\right)\right) \\ &\le \liminf_{n\to\infty} \liminf_{k\to\infty} \frac{1}{2}\left(\rho_p(u - v_n) + \rho_p(u - v_k)\right) \\ &= d(u, W). \end{aligned}$$

It follows that

$$d(u, W) = \rho_p(u - 2v).$$

If $w \in W$ and $d(u, W) = \rho_p(u - w)$ it is therefore concluded that

$$d \le \rho_p\left(u - \frac{2v + w}{2}\right) \le \frac{1}{2}\left(\rho_p(u - 2v) + \rho_p(u - w)\right) = d.$$

Since ρ_p has the $UUC2$ property, it is strictly convex. Hence $w = 2v$, which yields the uniqueness statement. $\square$

It should be emphasized at this point that Theorem 4 can be restated as the following minimization result:

Theorem 5. *In the notation and under the hypotheses of Theorem 4, there is a unique solution $u_0 \in W$ for the minimization problem*

$$\inf_{u-w\in W} \int_\Omega |w(x)|^{p(x)} dx. \tag{10}$$

Proof of Theorem 5. It is immediate from Theorem 4 that the unique solution is given by $w = u - v_0$. $\square$

Aiming at presenting further applications of the $UUC2$ property for $L^{p(\cdot)}(\Omega)$ we state and prove the following :

Theorem 6. *Let $(C_n)_n$ be a non-increasing sequence of ρ_p-closed, convex, nonempty subsets of $L^{p(\cdot)}(\Omega)$ and assume that*

$$p_- > 1.$$

Assume that there exists $v \in L^{p(\cdot)}(\Omega)$ such that $\sup_{n\ge1} d(v, C_n) < \infty$. Then

$$\bigcap_{n=1}^{\infty} C_n \ne \emptyset.$$

Proof of Theorem 6. It suffices to assume that for some $n_0 \in \mathbb{N}$ it holds $v \notin C_{n_0}$, for otherwise there would be nothing to prove. From the ρ_p-closedness of C_{n_0} it is easily derived that $d(v, C_{n_0}) > 0$. Since the sequence $(C_n)_n$ is non-increasing by assumption, the inequalities

$$\infty > \sup_{n\geq 1} d(v, C_n) \geq d(v, C_n) = \inf_{u \in C_n} d(v, u) \geq \inf_{u \in C_{n-1}} d(v, u) = d(v, C_{n-1})$$

are clear for any $n > 1$. Thus, the sequence $d(v, C_n)$ is non-decreasing and bounded. Let $L = \lim_{n\to\infty} d(v, C_n) < \infty$; clearly $L > 0$. For each $n \in \mathbb{N}$ let $u_n \in C_n$ be chosen so that $\rho_p(v - u_n) = d(v, C_n)$ as in Theorem 4 it follows that the sequence $\left(\frac{u_n}{2}\right)_n$ is ρ_p-Cauchy in $L^{p(\cdot)}(\Omega)$ and hence it ρ_p-converges to, say, $u/2 \in L^{p(\cdot)}(\Omega)$. Fix $k \in \mathbb{N}$. Then the sequence $\left(\frac{u_n}{2}\right)_{n\geq k}$ is contained in C_k and ρ_p-converges to $\frac{u}{2}$, which implies that $\frac{u}{2} \in C_k$, since C_k is ρ_p-closed. In conclusion,

$$\frac{u}{2} \in \bigcap_{n=1}^{\infty} C_n,$$

i.e. $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$, as claimed. □

Theorem 7. *Let $p_- > 1$ and assume that $\emptyset \neq C \subset L^{p(\cdot)}(\Omega)$ is a ρ_p-closed, ρ_p-bounded, convex set; let $(C_i)_{i\in I} \subset 2^C$ be a family of subsets of C having the finite intersection property, i.e., such that for every finite subset $\{i_1, ... i_k\} \subset I$ it holds that $\bigcap_{j=1}^{k} C_{i_j} \neq \emptyset$. Then*

$$\bigcap_{i\in I} C_i \neq \emptyset.$$

Proof of Theorem 7. C is ρ_p-bounded, it is therefore immediate that for any $u \in C$ and $i \in I$,

$$d(u, C_i) = \inf_{v\in C_i} \rho_p(u - v) \leq \sup_{v\in C} \rho_p(u - v) < \infty.$$

For any finite subset $A \subset I$ let

$$d_A = d\left(u, \bigcap_{j\in A} C_j\right).$$

If A and B be finite subsets of I, $A \subseteq B$, then $\bigcap_{j\in B} C_j \subseteq \bigcap_{j\in A} C_j$ and consequently,

$$d\left(u, \bigcap_{j\in A} C_j\right) = \inf_{v\in\bigcap_{j\in A} C_j} \rho_p(u, v) \leq \inf_{v\in\bigcap_{j\in B} C_j} \rho_p(u, v),$$

i.e. $d_A \leq d_B$. Write

$$d_I = \sup\left\{ d\left(u, \bigcap_{i\in J} C_i\right) \quad J \subset I: \quad \text{and} \quad \bigcap_{i\in J} C_i \neq \emptyset \right\}.$$

Let (A_n) be the sequence defined by

$$d_I - \frac{1}{n} < d_{A_n} \leq d_I,$$

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write $B_n = \bigcup_{k=1}^{n} A_k$ and $J = \bigcup_{n=1}^{\infty} B_n$. It is clear then that for each $n \in \mathbb{N}$, the set $\bigcap_{i \in B_n} C_i$ is ρ_p-closed, convex and non-empty and that the sequence $\left(\bigcap_{i \in B_n} C_i \right)$ is non-increasing. Hence, Theorem 6 applies and we have

$$S = \bigcap_{i \in J} C_i \neq \emptyset.$$

By definition, for each $n \in \mathbb{N}$, it holds that

$$\bigcap_{i \in J} C_i \subseteq \bigcap_{i \in A_n} C_i,$$

and it follows that for each n one has

$$d_I - \frac{1}{n} < d_{A_n} \leq d(u, S) \leq d_I.$$

Thus $d(u, S) = d_I$. On account of Theorem 4, there exists a unique $z \in S$ which satisfies $\rho_p(u - z) = d_I$ and therefore, for any index $i_0 \in I$ one has

$$S \supseteq S \cap C_{i_0} = \bigcap_{i \in J \cup \{i_0\}} C_i \neq \emptyset;$$

it is seen immediately that $d_I \leq d(u, S) \leq d(u, S \cap C_{i_0}) \leq d_I$. In all,

$$d(u, S) = d(u, S \cap C_{i_0})$$

and by Theorem 4 there exists a unique $w \in S \cap C_{i_0}$ for which

$$\rho_p(u - w) = d(u, S \cap C_{i_0}) = d_I.$$

In particular, $w \in S$, thus, invoking the uniqueness part of Theorem 4, one must necessarily have $w = z$. Since i_0 is arbitrary, it is concluded that $z \in \bigcap_{i \in I} C_i$ and hence, the latter intersection is non-empty, as claimed. $\square$

The following theorem is another consequence of the $UUC2$ property for $L^{p(\cdot)}(\Omega)$.

Theorem 8. *Let $p_- > 1, \emptyset \neq C \subset L^{p(\cdot)}(\Omega)$ be a convex, ρ_p-closed, ρ_p bounded and assume that C is not a singleton (i.e., C at least two distinct points). Then there exists $u \in C$ for which*

$$\sup_{v \in C} \rho_p(u - v) < diam(C),$$

where as usual $diam(C) = \sup_{a,b \in C} \rho_p(a - b)$ stands for the ρ_p-diameter of C.

The property established in Theorem 8 is commonly referred to as the ρ_p-normal structure property. Theorem 8 can thus be rephrased as asserting that if $p_- > 1$, then $L^{p(\cdot)}(\Omega)$ has ρ_p-normal structure.

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Proof of Theorem 8. The assumptions imply that $\delta(C) > 0$ and that there exist two distinct points $u \in C, v \in C, u \neq v$. For any $w \in C$, invoking the $UUC2$ property it follows at once that, for δ as in the definition of $UUC2$, (3),

$$\rho_p\left(\frac{u+v}{2} - w\right) = \rho_p\left(\frac{u-w+v-w}{2}\right)$$
$$\leq \operatorname{diam}(C)\left(1 - \delta\left(\operatorname{diam}(C), \frac{\varepsilon}{\operatorname{diam}(C)}\right)\right).$$

The arbitrariness of w in concert with the convexity of C yields the claim. □

Theorem 9. *If $p_- > 1$ and $\emptyset \neq C \subset L^{p(\cdot)}(\Omega)$ is convex, ρ_p-closed and ρ_p-bounded, then any map*

$$T : C \longrightarrow C$$

for which the bound

$$\rho_p\left(T(u) - T(v)\right) \leq \rho_p\left(u - v\right)$$

holds for any $u \in C, v \in C$, has a fixed point, that is to say, there exists $w \in C$ such that

$$T(w) = w.$$

Proof of Theorem 9. It is obvious that the theorem is true if C is a singleton. Thus, it can be assumed that the cardinality of C is at least 2. Let

$$\mathcal{F} = \left\{\emptyset \neq K \subset C : K \text{ is } \rho_p\text{-closed and } T(K) \subseteq K\right\}.$$

Since $C \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Moreover, $\mathcal{F}$ is partially ordered by the order relation

$$X \leq Y \iff Y \subseteq X.$$

If $\mathcal{G}$ is a totally order subfamily of $\mathcal{F}$, then $\mathcal{G}$ possesses the finite intersection property and on account of Theorem 7 it follows that

$$\bigcap_{X \in \mathcal{G}} X \neq \emptyset;$$

this clearly implies that $\bigcap_{X \in \mathcal{G}} X \in \mathcal{F}$ and therefore that $\bigcap\limits_{X \in \mathcal{G}} X$ is an upper bound for $\mathcal{G}$.
Zorn's Lemma yields the existence of a maximal element $X_0 \in \mathcal{F}$. We set about to prove that X_0 contains exactly one point. Denote the intersection of all ρ_p-closed, convex subsets of C that contain $T(X_0)$ by $\overline{conv}^{\rho_p}\left(T(X_0)\right)$. In particular, since $X_0 \in \mathcal{F}$,

$$\overline{conv}^{\rho_p}\left(T(X_0)\right) \subseteq X_0.$$

On the other hand, $\overline{conv}^{\rho_p}\left(T(X_0)\right) \in \mathcal{F}$, for it is convex, ρ_p-closed and

$$T\left(\overline{conv}^{\rho_p}\left(T(X_0)\right)\right) \subseteq T(X_0) \subseteq \overline{conv}^{\rho_p}\left(T(X_0)\right).$$

As a consequence of the maximality of X_0 with respect to the indicated inclusion, one has

$$\overline{conv}^{\rho_p}\left(T(X_0)\right) = X_0. \tag{11}$$

Theorem 8 yields the existence of an element $x_0 \in X_0$ such that

$$r_0 = \sup_{u \in X_0} \rho_p(x_0 - u) < \operatorname{diam}(X_0). \tag{12}$$

Let $B_{\rho_p}(a,s)$ denote the ρ_p-ball of radius s centered at a; we remark the obvious fact that the convexity and the Fatou property of the modular ρ_p imply that $B_{\rho_p}(a,s)$ is ρ_p-closed and convex. Set

$$M = \bigcap_{v\in X_0} B_{\rho_p}(v,r_0) \cap X_0 = \left\{ u \in X_0 : \sup_{v\in X_0} \rho_p(u-v) \leq r_0 \right\};$$

then M is ρ_p-closed and convex and $M \subset X_0$. Moreover, if $x \in M$, then for any $v \in X_0$

$$\rho_p\left(T(x) - T(v)\right) \leq \rho_p\left(x - v\right) \leq r_0.$$

In other words, if $v \in X_0$, $\rho_p(T(v) - T(x)) \leq r_0$, i.e., $T(X_0) \subseteq B_{\rho_p}(T(x), r_0)$. By definition of $\overline{conv}^{\rho_p}\left(T(X_0)\right)$ it is plain that:

$$\overline{conv}^{\rho_p}\left(T(X_0)\right) \subseteq B_{\rho_p}(T(x), r_0);$$

and from (11) it follows that

$$X_0 = \overline{conv}^{\rho_p}\left(T(X_0)\right) \subseteq B_{\rho_p}(T(x), r_0);$$

that is, for any $v \in X_0$, $\rho_p(T(x) - v) \leq r_0$, i.e., $T(x) \in B_{\rho_p}(v, r_0)$. It follows from the definition of M that

$$T(M) \subseteq M,$$

so that $M \in \mathcal{F}$. and since $M \subseteq X_0$ and X_0 is maximal, one has *a fortiori*:

$$X_0 = M.$$

By definition, then, if $w \in X_0$,

$$\rho_p(w - x_0) \leq r_0;$$

this forces the inequality $\mathrm{diam}(X_0) \leq r_0$, which contradicts (12) unless $\mathrm{diam}(X_0) = 0$. Hence, $\mathrm{diam}(X_0) = 0$ and $X_0 = \{a\}$ is a singleton. Since also $T(X_0) \subseteq X_0$, necessarily

$$T(a) = a,$$

we conclude that T has a fixed point, as claimed. □

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