ULω and IULω are substructural fuzzy logics

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Abstract Two representable substructural logics ULω and IULω are logics for finite UL and IUL-algebras, respectively. In this paper, the standard completeness of ULω and IULω is proved by the method developed by Jenei, Montagna, Esteva, Gispert, Godo and Wang. This shows that ULω and IULω are substructural fuzzy logics.

Keywords Substructural fuzzy logics · Residuated lattices · Semilinear substructural logics · Standard completeness · Fuzzy logic

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1 Introduction

In [10], we constructed three representable substructural logics ULω, IULω and HpsULω∗ by adding one simple axiom

\[(\text{FIN}) \; (A \to e) \leftrightarrow (A \odot A \to e)\]

to Metcalfe and Montagna’s uninorm logic UL, involutive uninorm logic IUL [6], and a suitable extension HpsUL∗ [7] of Metcalfe, Olivetti and Gabbay’s pseudo-uninorm logic HpsUL [5], respectively. Especially, we showed that ULω and IULω are logics for finite UL and IUL-algebras, respectively.

In this paper, we prove that ULω and IULω are standard complete by Wang’s constructions in [8] and [9], which are some generalizations of Jenei and Montagna-style approach for proving standard completeness for monoidal t-norm based logic MTL [4] and the proof of the standard completeness for IMTL given by Esteva, Gispert, Godo and Montagna in [2]. This shows that ULω and IULω are substructural fuzzy logics.

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We have proved that $\text{HpsUL}^*$ is standard complete in [11]. However, we are unable to prove whether $\text{HpsUL}^\omega$ is standard complete or, complete with respect to finite $\text{HpsUL}^*$-algebras and left them as open problems.

2 $\text{HpsUL}^\omega$, $\text{UL}_\omega$, $\text{IUL}_\omega$ and algebras involved

The Hilbert system $\text{HpsUL}$ is the logic of bounded representable residuated lattices, which is based on a countable propositional language with formulas built inductively as usual from a set of propositional variables, binary connectives $\odot, \rightarrow, \sim, \land, \lor$ and constants $e, f, \bot, \top$, with definable connectives:

$$\neg \varphi := \varphi \rightarrow f,$$
$$\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi),$$
$$\lambda_x(\varphi) := (\chi \rightarrow \varphi \odot \chi) \land e,$$
$$\rho_x(\varphi) := (\chi \sim \chi \odot \varphi) \land e.$$

Definition 1 $\text{HpsUL}$ consists of the following axioms and rules [5]:

$$(A_1) \vdash \varphi \rightarrow \varphi$$
$$(A_2) \vdash (\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$$
$$(A_3) \vdash \varphi \rightarrow ((\varphi \sim \psi) \rightarrow \psi)$$
$$(A_4) \vdash (\varphi \sim (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \sim \chi))$$
$$(A_5) \vdash \psi \rightarrow (\varphi \rightarrow \varphi \odot \psi)$$
$$(A_6) \vdash (\psi \rightarrow (\varphi \rightarrow \chi)) \rightarrow (\varphi \odot \psi \rightarrow \chi)$$
$$(A_7) \vdash (\psi \sim \psi \odot (\psi \rightarrow \varphi)) \rightarrow (\psi \sim \varphi)$$
$$(A_8) \vdash (\varphi \land t) \odot (\psi \land t) \rightarrow \varphi \land \psi$$
$$(A_9) \vdash \varphi \land \psi \rightarrow \psi$$
$$(A_{10}) \vdash \varphi \land \psi \rightarrow \varphi$$
$$(A_{11}) \vdash (\chi \rightarrow \varphi) \land (\chi \rightarrow \psi) \rightarrow (\chi \rightarrow \varphi \land \psi)$$
$$(A_{12}) \vdash \varphi \rightarrow \varphi \lor \psi$$
$$(A_{13}) \vdash \psi \rightarrow \varphi \lor \psi$$
$$(A_{14}) \vdash (\varphi \rightarrow \chi) \land (\psi \rightarrow \chi) \rightarrow (\varphi \lor \psi \rightarrow \chi)$$
$$(A_{15}) \vdash e$$
$$(A_{16}) \vdash \varphi \rightarrow (e \rightarrow \varphi)$$
$$(A_{17}) \vdash \varphi \rightarrow \top$$
$$(A_{18}) \vdash \bot \rightarrow \varphi$$

(PRL) $\vdash (\lambda_x(\varphi \lor \psi \rightarrow \varphi)) \lor (\rho_x(\varphi \lor \psi \rightarrow \varphi))$

(MP) $\varphi, \varphi \rightarrow \psi \vdash \psi$

(ADJ\textsubscript{UL}) $\varphi \vdash \varphi \land e$

(PN\rightarrow) $\varphi \vdash \varphi \rightarrow \varphi \odot \psi$

(PN\leftarrow) $\varphi \vdash \varphi \sim \psi \odot \varphi$

Definition 2 [6, 7] A logic is a schematic extension (extension for short) of $\text{HpsUL}$ if it results from $\text{HpsUL}$ by adding axioms in the same language. In particular,

- $\text{HpsUL}^*$ is $\text{HpsUL}$ plus (WCM) $\vdash (\varphi \sim e) \rightarrow (\varphi \rightarrow e)$;
- $\text{UL}$ is $\text{HpsUL}$ plus $\vdash \varphi \odot \psi \rightarrow \psi \odot \varphi$;
- $\text{IUL}$ is $\text{UL}$ plus $\vdash \sim \varphi \rightarrow \varphi$.  

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Definition 3 New extensions of HpsUL are defined as follows.
- HpsUL⁺ is HpsUL* plus (FIN) ⊢ (φ → e) ↔ (φ ⊗ φ → e);
- UL₂ and IUL₂ are UL and IUL plus (FIN), respectively.

Let L ∈ {HpsUL⁺, UL, IUL, HpsUL*, UL₂, IUL₂} in the remainder of this section. A proof in L of a formula φ from a set Γ of formulas is defined as usual. We write Γ ⊢_L φ if such a proof exists.

Definition 4 [5] An HpsUL-algebra is a bounded residuated lattice A = ⟨A, ∧, ∨, ·, →, ⊥, ⊤⟩ with universe A, binary operations ∧, ∨, ·, →, ⊥, ⊤, and constants e, f, ⊥, ⊤ such that:
(i) ⟨A, ∧, ∨, ⊥, ⊤⟩ is a bounded lattice with top element ⊤ and bottom element ⊥;
(ii) ⟨A, ·, e⟩ is a monoid;
(iii) ∀x, y, z ∈ A, x · y ≤ z iff x ≤ y → z and y ≤ x → z;
(iv) ∀x, y, u, v ∈ A, (λ_u(x ∨ y → x)) ∨ (ρ_v(x ∨ y → y)) = e, where, for any a, b ∈ A, λ_a(b) := (a → b) · a and ρ_a(b) := (a · b) → a.

We use the convention that · binds stronger than other binary operations and we shall often omit ·; we will thus write xy instead of x · y, for example. Suitable classes of algebras of extensions of HpsUL are defined as follows.

Definition 5 [5,7,10] Let A = ⟨A, ∧, ∨, →, ⊥, ⊤⟩ be an HpsUL-algebra. For L an extension of HpsUL, A is an L-algebra if all axioms of L are valid in A. An L-chain is an L-algebra that is linearly ordered. In particular:
- A is an HpsUL*^-algebra if the weak commutativity (Wcm) holds: xy ≤ e iff yx ≤ e for all x, y ∈ A;
- A is an UL-algebra if xy = yx for all x, y ∈ A;
- A is an IUL-algebra if it is an UL-algebra such that ¬¬x = x for all x ∈ A;
- A is an HpsUL^-algebra (UL₂ or IUL₂-algebra) if it is an HpsUL*-algebra (UL or IUL-algebra) such that the following identity (Fin) holds:
  x → e = x² → e for all x ∈ A.

Definition 6 [5] Let A = ⟨A, ∧, ∨, →, ⊥, ⊤⟩ be an L-algebra. (i) An L-valuation v is a homomorphism from the term algebra determined by formulas in L to A; (ii) A formula φ is valid in A if v(φ) ≥ e holds for any A-valuation v; (iii) The relation of semantic consequence Γ ⊨_A φ holds if each A-valuation that validates all formulae in a theory Γ validates φ as well.

Theorem 1 [5] Γ ⊨_L φ iff Γ ⊨_A φ for every L-chain A, i.e., L is a presentable substructural logic.

Lemma 1 Let A be an HpsUL⁺₂-chain and, s, t, u ∈ A. Then
(i) st ≤ e iff st² ≤ e;
(ii) stu = s implies st = s and su = s;
(iii) stu = u implies su = u and tu = u;
(iv) st = e implies s = t = e.

Proof Only (ii) is proved as follows and, others see [10]. If tu ≤ e then tut ≤ e and utu ≤ e by (1) and (Wcm). Thus stat ≤ s and stut ≤ st. Hence st ≤ s and s ≤ st. Therefore st = s. The case of tu > e can be proved in the same way.

Clearly, Lemma 1 holds for all UL₂ and IUL₂-algebras.
3 Wang’s Construction and Standard completeness

In this section, let \( L_\omega \in \{ UL_\omega, IUL_\omega \} \), \( \mathcal{A} = \{ A, \land, \lor, \to, \neg, e, f, \bot, \top \} \) be a finite or countable linearly ordered \( L_\omega \)-algebra and \( s, t, u \) be arbitrary elements of \( A \).

**Definition 7** [7,8] Let \( A \) be an \( UL_\omega \)-algebra. For each \( s \in A \), \( t \) is the immediate predecessor of \( s \) in \( A \) if (i) \( t \notin A \), \( t < s \); (ii) \( \forall u \in A , u < s \) implies \( u \leq t \). For each \( s \in A \), let \( s^- \) denote the immediate predecessor of \( s \) in \( A \) if it exists, otherwise take \( s^- = s \).

Let \( X = \{(s,1) : s \in A \} \cup \{(s,q) : s \in A, s > s'^-, q \in Q \cap (0,1)\} \), we define:

\[
(s,q) \leq (t,r) \text{ iff either } s <_{S} t \text{ or } s = t \text{ and } q \leq r \text{ and,}
\]

\[
\begin{align*}
I_1 &= \{(s,t) : s,t \in A, s \neq t, s > s^- \} \\
I_2 &= \{(s,t) : s,t \in A, s \neq t, t > s^- \} \\
I_3 &= \{(s,t) : s,t \in A, s = t, s > s^- \} \\
I_4 &= \{(s,t) : s,t \in A, \text{ or countable linearly ordered } \text{ or } s = s^- \text{ or } s = 1 \}.
\end{align*}
\]

Now define, for \((s,q),(t,r) \in X:\)

\[
(s,q) \odot (t,r) = \begin{cases} (s,q) & (s,t) \in I_1, \\
(t,r) & (s,t) \in I_2, \\
(s,q) \land_X (t,r) & (s,t) \in I_3, \\
(st,1) & (s,t) \in I_4,
\end{cases}
\]

where \( \land_X \) and \( \lor_X \) is meant \( \min_X \) and \( \max_X \) with respect to \( \leq_X \), respectively. We will omit index if it does not cause confusion.

**Lemma 2** Let \( \mathcal{A} \) be an \( UL_\omega \)-algebra. Then \((s,q) \odot (t,r) \leq (e,1) \iff (s,q) \odot (t,r) \odot (t,r) \leq (e,1) \) for all \((s,q),(t,r) \) in \( X \).

**Proof** Let \((s,q) \odot (t,r) \leq (e,1) \). Since \((s,q) \odot (t,r) = (st,\Diamond) \) for some \( \Diamond \in \{q,r\} \) by Definition 7, then \( st \leq e \). Thus \( st \) \( e \) by (Fin). Hence \((s,q) \odot (t,r) \odot (t,r) \leq (e,1) \). The sufficiency part of the lemma can be proved in the same way.

**Definition 8** [2,9] Let \( \mathcal{A} \) be an \( IUL_\omega \)-algebra. Let

\[
I^* := \{(s,t) : s,t \in A, s^- < s, t^- < t, t = s^- \},
\]

\[
I^{**} := \{(s,t) : s,t \in A, ss = s^- s = s = t \}.
\]

\( \forall(s,q),(t,r) \in X, \) define

\[
(s,q) \triangle (t,r) = \begin{cases} (s,q) \odot (t^-,1) \lor (s^-,1) \odot (t,r) & \text{if } (s,t) \in I^*, q + r \leq 1, \\
(s,q) \lor (s^-,1) \odot (t^-,1) & \text{if } (s,t) \in I^{**}, \\
(s,q) \odot (t,r) & \text{otherwise}.
\end{cases}
\]

**Lemma 3** Let \( \mathcal{A} \) be an \( IUL_\omega \)-chain and \( s,t \in A \). (i) If \( st^- \neq s, st^- \leq e \), \( s^- t \leq e \) then \( st^- t \leq e \); (ii) If \( st^- = s^- t \) and \( s^- t \leq e \) then \( st^- t \leq e \); (iii) \((s,q) \triangle (t,r) \leq (s,q) \odot (t,r) \).
Proof (i) If \( st \leq e \) then \( stt \leq e \) by Lemma 1(i) and thus \( st \leq e \). If \( t \leq e \) then \( stt \leq t \leq e \) by \( st \leq e \). Thus, let \( st > e \) and \( t > e \) in the following.

Thus \( t' \geq e \) by \( t' \neq e \) by \( st' \neq s \). Then \( t' > e \). Thus \( st' > s \) by \( st' \neq s \). Hence \( st' > s \). Therefore \( st' < s \) by \( st' \neq s \). Then \( st < s < t' \). Thus \( st < t' \). Hence \( s < e \).

Suppose that \( st \leq t' \). Then \( st < st' \leq e \). Thus \( st < e \) by Lemma 1(i), a contradiction and hence \( st > t' \). Therefore \( st \geq t' \). Then \( st < t' \). Hence \( s < e \).

Suppose that \( st \geq t' \). Then \( st > st' \). Thus \( st < e \) by Lemma 1(i), a contradiction and hence \( st < t' \). Thus \( st \leq t' \). Hence \( s \leq e \).

Therefore \( s \leq e \). Hence \( s \leq e \). Then \( st = s \). Thus \( st \leq s \) by Lemma 1(ii).

Thus \( s = s \). Hence \( s \). Then \( st \leq s \). Thus \( st \leq s \) and \( st = t \). Thus \( st = s \) by Lemma 1(ii).

By a similar procedure, we prove that \( st' = s \). Thus \( st' \leq e \) by \( st \leq e \) and \( st' \leq e \) does not exist. This completes the proof of (i).

(ii) It follows from \( st < e \) that \( stt < e \) by Lemma 1(i). Then \( stt < e \) and \( st < e \). Then \( stt < e \) and \( st < e \).

(iii) See Proposition 3.7 (2) of [9].

Lemma 4 Let \( A \) be a finite \( IUL_{\omega} \)-algebra. Then \( (s, q) \triangle (t, r) \leq (e, 1) \) if and only if \( (s, q) \triangle (t, r) \leq (e, 1) \) for all \( (s, q), (t, r) \) in \( X \).

Proof Let \( (s, q) \triangle (t, r) \leq (e, 1) \). There are three cases to be considered.

Case 1. \( (s, t) \in I^* \) and \( q + r \leq 1 \). Then \( (s, q) \triangle (t, r) = (s, q) \circ (t, r) \leq (s, 1) \circ (t, 1) \leq (e, 1) \). Thus \( s \leq e \) and \( s < t \). Then \( s \leq e \) by Lemma 1(i). If \( (s, q) \triangle (t, r) = (s, 1) \circ (t, r) \) then \( (s, q) \triangle (t, r) = (s, 1) \circ (t, r) \leq (s, 1) \circ (t, r) \leq (s, 1) \circ (t, r) \) by Lemma 3(iii). Let \( (s, q) \triangle (t, r) = (s, q) \circ (t, r) \) \( (s, q) \circ (t, r) \) in the following. If \( (s, q) \circ (t, r) = (s, q) \circ (t, r) \), then \( (s, q) \circ (t, r) = (s, q) \circ (t, r) = (s, q) \circ (t, r) \). Otherwise \( st \neq s \) or \( st = s \). Then \( st \leq e \) by Lemma 1(i) and 3(ii). Thus \( s \leq e \). Then \( st \leq e \) by Lemma 1(i) and 3(ii). Thus \( (s, q) \triangle (t, r) = (1, 1) \circ (t, r) \leq (s, q) \circ (t, r) \) and \( (s, q) \circ (t, r) \). Thus \( st \leq e \) by Lemma 1(i).

By a similar procedure, we prove that \( (s, q) \triangle (t, r) \leq (e, 1) \) if \( (s, q) \triangle (t, r) \leq (e, 1) \).

Lemma 5 Let \( A \) be an \( HpsUL_{\omega} \)-algebra, \( X \) and the binary operation \( \circ \) on \( X \) be as in Definition 7. The following conditions hold:

- (a) \( X \) is densely ordered, and has a maximum \( \top_X = (T, 1) \) and a minimum \( \bot_X = (\bot, 1) \).
- (b) \( (X, \circ, \leq_X, e_X) \) is a linearly ordered monoid, where \( e_X = (e, 1) \).
- (c) \( \circ \) is left-continuous with respect to the order topology on \( (X, \leq_X) \).
(d) There is a map $\Phi$ from $A$ into $X$ such that $\Phi$ is an embedding of the structure $(A, \wedge, \lor, \cdot, 1, T)$ into $(X, \wedge_X, \vee_X, \circ, e_X, \bot_X, \top_X)$, and for all $s, t \in A, \Phi(s \to t)$ is the residuum of $\Phi(s)$ and $\Phi(t)$ in $(X, \wedge_X, \vee_X, \circ, e_X, \bot_X, \top_X)$, respectively.

(e) $\forall (s, q), (t, r) \in X, (s, q) \circ (t, r) \leq (e, 1)$ iff $(s, q) \circ (t, r) \circ (t, r) \leq (e, 1)$.

Proof Claim (e) has been proved by Lemma 2. As pointed out in [7], the associativity of $\circ$ is mainly dependent on Lemma 1(ii) and 1(iii). Other claims can be proved in the same way as that of [7, Theorem 4.5].

**Lemma 6** Every countable linearly ordered UL*-algebra can be embedded into a standard UL*-algebra.

Proof Let $X, A$, etc. be as in Definition 7. We can assume, without loss of generality, that $X = Q \cap [0, 1]$. Now define for $\alpha, \beta \in [0, 1], \alpha \ast \beta = \sup \{x \circ y : x, y \in X, x \leq \alpha, y \leq \beta\}$. The proof of the weak commutativity, the monotonicity, associativity, left-continuity, etc. of $\ast$ is the same as that of [7, Theorem 4.6]. The neutral element of $\ast$ is $e_X$ in $Q \cap [0, 1]$. By the left-continuity of $\ast$, the following property holds.

(F) $\alpha, \beta, \gamma \in [0, 1], \alpha \ast \beta \ast \gamma = \sup \{x \circ y \circ z : x, y, z \in X, x \leq \alpha, y \leq \beta, z \leq \gamma\}$.

We prove that $\alpha \ast \beta \leq e_X$ iff $\alpha \ast \beta \ast \beta \leq e_X$ for any $\alpha, \beta$ in $[0, 1]$. Given $\alpha \ast \beta \leq e_X$ then $x \circ y \leq e_X$ for all $x, y \in X, x \leq \alpha, y \leq \beta$. Let $x, y, z \in X, x \leq \alpha, y \leq \beta, z \leq \beta$. Then $x \circ y \leq e_X, x \circ z \leq e_X$. Thus $x \circ y \circ z \leq e_X$. Hence $x \circ y \circ z \leq e_X, x \circ z \circ z = e_X$ by Lemma 5(e). Therefore $\alpha \ast \beta \ast \beta \leq e_X$ by (F). The sufficient part of the claim can be proved in the similar way.

By Lemma 1, Definition 8, Lemma 4, we can prove the claims similar to Lemma 5 and 6 for IUL*-algebras. As a consequence of these lemmas, and extending [4, Theorem 3.3] in the obvious way, we obtain the following standard completeness.

**Theorem 2** UL*= and IUL*= are complete with respect to the class of standard algebras involved.

4 Concluding remarks

Roughly speaking, the methodological significance of Jenei and Montagna’s proof is that it does not require a complete understanding of the structure of the MTL-algebras by embedding a countable MTL-algebras into a dense one. It is indeed different from the proof of the BL’s standard completeness given by Hajek, Cignoli, Esteva, Godo, Torrens et al in [1,3]. The validation of the structure $X$ in Definitions 7, 8 and Theorems 3.6, 3.7 is dependent on Lemma 1(ii) which claims that $stu = s$ implies $st = s$. However, we are unable to prove the condition that $stu = t$ implies $st = t$ in HpsUL*$. It seems that we need to introduce some more strong axioms into HpsUL* to guarantee its completeness with respect to finite (or standard) HpsUL*-algebras.

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Compliance with ethical standards

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