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On the evolutionary form of the constraints in electrodynamics

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Abstract

The constraint equations in Maxwell theory are investigated. In analogy with some recent results on the constraints of general relativity it is shown, regardless of the signature and dimension of the ambient space, that the "divergence of a vector field" type constraints can always be put into linear first order hyperbolic form for which global existence and uniqueness of solutions to an initial-boundary value problem is guaranteed.

1 Introduction

The Maxwell equations, as we know them since the seminal addition of Ampere's law by Maxwell in 1865, are [3]

$$\boldsymbol{\nabla}\times\mathbf{H}=\mathbf{J}+\partial_t\mathbf{D} \qquad \boldsymbol{\nabla}\times\mathbf{E}+\partial_t\mathbf{B}=0 \tag{1}$$

$$\boldsymbol{\nabla}\cdot\mathbf{D}=\rho \qquad \boldsymbol{\nabla}\cdot\mathbf{B}=0\,, \tag{2}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the macroscopic electric and magnetic field variables, which in vacuum are related to $\mathbf{D}$ and $\mathbf{H}$ by the relations $\mathbf{D}=\boldsymbol{\epsilon_0}\,\mathbf{E}$ and $\mathbf{H}=\boldsymbol{\mu_0^{-1}}\,\mathbf{B}$, where ϵ_0 and $\boldsymbol{\mu}_0$ are the dielectric constant and magnetic permeability, and where ρ and $\mathbf{J}$ stand for charge and current densities, respectively.

The top two equations in (1) express that the time dependent magnetic field induces an electric field and also that the changing electric field induces a magnetic

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field even if there are no electric currents. Obviously there have been plenty of brilliant theoretical, experimental and technological developments based on the use of these equations. Nevertheless, from time to time some new developments (for a recent examples see, for instance, [2, 4]) have stimulated reconsideration of claims which previously were treated as text-book material in Maxwell theory.

In this short note the pair of simple constraint equations on the bottom line in (2) are the centre of interest. These relations for the divergence of a vector field are customarily treateds as elliptic equations. The main purpose of this letter is to show that by choosing basic variables in a geometrically preferred way the constraints in (2) can also be solved as evolutionary equations. This also happens in the more complicated case of the constraints in general relativity [5].

Once the Maxwell equations (1) and (2) are given it is needless to explain in details what is meant to be the ambient spacetime (tacitly it is assumed to be the Minkowski spacetime) or the initial data surface (usually chosen to be a "$t = const$" hypersurface in Minkowski spacetime). As seen below the entire argument, outlined in more details in the succeeding sections, is very simple. In addition, it applies with almost no cost to a generic ambient space (M, g_{ab}), with a generic three-dimensional initial data surface Σ. We shall treat the generic case. This allows us to apply our new results to the solution of "divergence of a vector field type constraints",

$$\boldsymbol{\nabla} \cdot \mathbf{L} = \ell\,, \quad \text{(in index notation)} \quad D_i L^i = \ell\,, \tag{3}$$

for a vector field $\mathbf{L}$ or (in index notation) L^i with a generic source ℓ, on an arbitrary fixed curved background, which may have various applications.

Note that for the Maxwell system, given by (1) and (2), the two divergence of a vector field constraints decouple so it suffices to solve them independently. Note also that it is easy to see that all the arguments presented in the succeeding subsections generalize to an arbitrary $n \geq 3$ dimension of Σ. Nevertheless, for the sake of simplicity, our consideration here will be restricted to the case of three-dimensional initial data surfaces.

Since the constraints are almost exclusively referred to as elliptic equations in text-books, one may question the point of putting them into evolutionary form. We believe that the appearance of time evolution in a Riemannian space could itself be of interest on its own right. Nevertheless, it is important to emphasize that there are valuable applications of the proposed new method. For instance, it may offer solutions to problems which are hard to solve properly in the standard elliptic approach. An immediate example of this sort arisises in the initialization of the time evolution of point charges governed by the coupled Maxwell-Lorentz equations. As pointed out recently in [2], unless suitable additional conditions are applied in addition to the Maxwell constraints, the electromagnetic field develops singularities along the light

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cones emanating from the original positions of the point charges. It is important to be mentioned here that analogous problems arise in the context of binary black hole configurations which can, however, be properly initialized by applying the superposed Kerr-Schild metric and the evolutionary form of the constraints in general relativity [6]. Based on this observation we expect that the problem raised in [2] in the context of Maxwell-Lorentz systems could also be treated using the hyperbolic form of the constraints.

An additional, and not the least important, potential advantage of the proposed new method is that it offers an unprecedented flexibility in solving the constraint equations. This originates from the fact that neither the choice of the underlying foliations of the three-dimensional initial data surface Σ nor the choice of the evolutionary flow have any limitations. This makes the proposed method applicable to a high variety of problems that might benefit from this new approach to solving the constraints.

Another advantage of this new approach constraints is that, regardless of the choice of foliation and flow, the geometrically preferred set of variables constructed in carrying out the main steps of the procedure always satisfy a *linear first order symmetric hyperbolic equation*. Considering the robustness of the approach, it is remarkable that, starting with the "divergence of a vector field constraint", the global existence of a unique smooth solution for the geometrically preferred dependent variables (under suitable regularity conditions on the coefficients and source terms) is guaranteed for the linear first order symmetric hyperbolic equation (see, e.g. subsection VIII.12.1 in [1]).

2 Preliminaries

The construction starts by choosing a three-dimensional initial data surface Σ with an induced Riemannian metric h_{ij} and its associated torsion free covariant derivative operator D_i. Σ may be assumed to lie in an ambient space (M, g_{ab}) whose metric could have either Lorentzian or Euclidean signature. More importantly, Σ will be assumed to be a topological product

$$\Sigma \approx \mathbb{R} \times \mathscr{S} \,, \tag{4}$$

where $\mathscr{S}$ could be of a two-surface with arbitrary topology. In the simplest practical case, however, $\mathscr{S}$ would have either planar, cylindrical, toroidal or spherical topology. In these cases, we may assume that there exists a smooth real function $\rho : \Sigma \to \mathbb{R}$ whose $\rho = const$ level sets give the $\mathscr{S}_\rho$ leaves of the foliation and that its gradient $\partial_i \rho$ does not vanish, apart from some isolated locations where the foliation

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may degenerate.[1]

The above condition guarantees (as indicated in Fig.1) that locally Σ is smoothly foliated by a one-parameter family of $\rho = const$ level two-surfaces $\mathscr{S}_\rho$. Given these

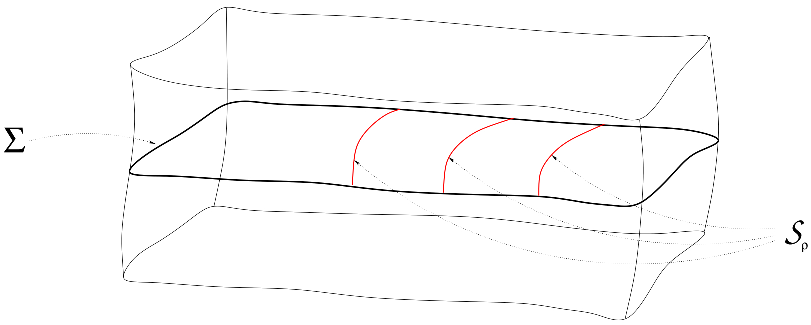

Figure 1: The initial data surface Σ foliated by a one-parameter family of two-surfaces $\mathscr{S}_\rho$ is indicated.

leaves, the non-vanishing gradient $\partial_i \rho$ can be normalized to a unit normal $\widehat{n}_i = \partial_i\rho/\sqrt{h^{ij}(\partial_i\rho)(\partial_j\rho)}$, using the Riemannian metric h_{ij}. Raising the index according to $\widehat{n}^i = h^{ij}\widehat{n}_j$ gives the unit vector field normal to $\mathscr{S}_\rho$. The operator $\widehat{\gamma}^i{}_j$ formed form the combination of $\widehat{n}_i$ and $\widehat{n}^i$ and the identity operator $\delta^i{}_j$,

$$\widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i\widehat{n}_j \tag{5}$$

projects fields on Σ to the tangent space of the $\mathscr{S}_\rho$ leaves.

We also apply flows interrelating the fields defined on the successive $\mathscr{S}_\rho$ leaves. A vector field ρ^i on Σ is called a flow if its integral curves intersect each of the leaves precisely once and it is normalized such that $\rho^i\partial_i\rho = 1$ holds everywhere on Σ. The contraction $\widehat{N} = \rho^j\widehat{n}_j$ of ρ^i with $\widehat{n}_i$ and its projection $\widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j$ of ρ^i to the leaves are referred to as the "lapse" and "shift" of the flow and we have

$$\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i\,. \tag{6}$$

The inner geometry of the $\mathscr{S}_\rho$ leaves can be characterized by the metric

$$\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i\,\widehat{\gamma}^l{}_j\,h_{kl} \tag{7}$$

induced on the $\rho = const$ level surfaces. It is also known that a unique torsion free covariant derivative operator $\widehat{D}_i$ associated with the metric $\widehat{\gamma}_{ij}$ acts on fields intrinsic

[1] If, for instance, Σ has the topology $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{S}^2\times\mathbb{R}$ or $\mathbb{S}^2\times\mathbb{S}^1$ and it is foliated by topological two-spheres then there exists one, two or, in the later two cases, no points of degeneracy at all. If point charges are involved it may be preferable to place the associated physical singularities at the location of these degeneracy. Note also that we often shorthand partial derivatives $\partial/\partial x^i$ by ∂_i.

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to the $\mathscr{S}_\rho$ leaves, e.g. acting on the field $\mathbf{N}_l = \widehat\gamma^p{}_l\, N_p$ obtained by the projection of N_p according to

$$\widehat D_i \mathbf{N}_j = \widehat\gamma^k{}_i\, \widehat\gamma^l{}_j D_k \,[\, \widehat\gamma^p{}_l\, N_p\,]\;. \tag{8}$$

It is straightforward to check that $\widehat D_i$ is indeed metric compatible in the sense that $\widehat D_k \widehat\gamma_{ij}$ vanishes.

Note also that the exterior geometry of the $\mathscr{S}_\rho$ leaves can be characterized by the extrinsic curvature tensor $\widehat K_{ij}$ and the acceleration $\dot{\widehat n}_i$ of the unit normal, given by

$$\widehat K_{ij} = \frac{1}{2}\,\mathscr{L}_{\widehat n} \widehat\gamma_{ij} \quad \text{and} \quad \dot{\widehat n}_i = \widehat n^l D_l \widehat n_i = -\widehat D_i \ln \widehat N\,, \tag{9}$$

where $\mathscr{L}_{\widehat n}$ is the Lie derivative operator with respect to the vector field $\widehat n^i$ and $\widehat N$ is the lapse of the flow.

3 The evolutionary form the constraints

This section is to put the divergence type constraint (3) into evolutionary form. This is achieved by applying a $2+1$ decompositions where, as we see below, the main conclusion is completely insensitive to the choice of the foliation and of the flow.

Consider first an arbitrary co-vector field L_i on Σ. By making use of the projector $\widehat\gamma^i{}_j$ defined in the previous section we obtain

$$L_i = \delta^j{}_i\, L_j = (\widehat\gamma^j{}_i + \widehat n^j \widehat n_i)\, L_j = \boldsymbol\lambda\, \widehat n_i + \mathbf{L}_i\,, \tag{10}$$

where the boldfaced variables $\boldsymbol\lambda$ and $\mathbf{L}_i$ are fields intrinsic to the individual $\mathscr{S}_\rho$ leaves of the foliation of Σ. They are defined via the contractions

$$\boldsymbol\lambda = \widehat n^l\, L_l \quad \text{and} \quad \mathbf{L}_i = \widehat\gamma^j{}_i\, L_j\,. \tag{11}$$

By applying an analogous decomposition of $D_i L_j$ we obtain

$$D_i L_j = \delta^k{}_i \delta^l{}_j D_k \,[\, \delta^p{}_l\, L_p\,] = (\widehat\gamma^k{}_i + \widehat n^k \widehat n_i)(\widehat\gamma^l{}_j + \widehat n^l \widehat n_j) D_k \,[\, (\widehat\gamma^p{}_l + \widehat n^p \widehat n_l)\, L_p\,]\;, \tag{12}$$

which, in terms of the induced metric (7), the associated covariant derivative operator, the extrinsic curvature and the acceleration (9), can be written as

$$\begin{aligned} D_i L_j = {} & \big[\, \widehat D_i \boldsymbol\lambda + \widehat n_i\, \mathscr{L}_{\widehat n} \boldsymbol\lambda \,\big] \widehat n_j + \boldsymbol\lambda\, (\widehat K_{ij} + \widehat n_i \dot{\widehat n}_j) + \widehat D_i \mathbf{L}_j - \widehat n_i \widehat n_j\, (\dot{\widehat n}^l \mathbf{L}_l) \\ & + \big\{ \widehat n_i\, \mathscr{L}_{\widehat n} \mathbf{L}_j - \widehat n_i\, \mathbf{L}_l \widehat K^l{}_j - \widehat n_j\, \mathbf{L}_l \widehat K^l{}_i \big\}\,. \end{aligned} \tag{13}$$

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By contracting the last equation with the inverse $h^{ij} = \widehat{\gamma}^{ij} + \widehat{n}^i\widehat{n}^j$ of the three-metric h_{ij} on Σ, we obtain

$$D^l L_l = h^{ij} D_i L_j = (\widehat{\gamma}^{ij} + \widehat{n}^i\widehat{n}^j)\, D_i L_j = \mathscr{L}_{\widehat{n}}\boldsymbol{\lambda} + \boldsymbol{\lambda}\,(\widehat{K}^l{}_l) + \widehat{D}^l\mathbf{L}_l + \dot{\widehat{n}}^l\mathbf{L}_l\,. \tag{14}$$

In virtue of (3) and in accord with the last equation, it is straightforward to see that the divergence of a vector field constraint can be put into the form

$$\mathscr{L}_{\widehat{n}}\boldsymbol{\lambda} + \boldsymbol{\lambda}\,(\widehat{K}^l{}_l) + \widehat{D}^l\mathbf{L}_l + \dot{\widehat{n}}^l\mathbf{L}_l = \ell\,. \tag{15}$$

Now, by choosing arbitrary coordinates (x^2, x^3) on the $\rho = const$ leaves and by Lie dragging them along the chosen flow ρ^i, coordinates (ρ, x^2, x^3) adapted to both the foliation $\mathscr{S}_\rho$ and the flow $\rho^i = (\partial_\rho)^i$ can be introduced on Σ. In these coordinates, (15) takes the strikingly simple form in terms of the lapse and shift of the flow,

$$\partial_\rho\boldsymbol{\lambda} - \widehat{N}^K\partial_K\boldsymbol{\lambda} + \boldsymbol{\lambda}\,\widehat{N}\,(\widehat{K}^L{}_L) + \widehat{N}\big[\widehat{D}_L\mathbf{L}^L + \dot{\widehat{n}}_L\mathbf{L}^L\big] = \ell\,. \tag{16}$$

Some remarks are now in order. First, (16) is a scalar equation whereby it is natural to view it as an equation for the scalar part $\boldsymbol{\lambda} = \widehat{n}_i L^i$ of the vector field L^i on Σ and to solve it for $\boldsymbol{\lambda}$. All the coefficients and source terms in (16) are determined explicitly by freely specifying the fields $\mathbf{L}^L$ and ℓ, whereas the metric h_{ij} and its decomposition in terms of the variables $\widehat{N}, \widehat{N}^I, \widehat{\gamma}_{IJ}$, is also known throughout Σ. Thus (16) can be solved for $\boldsymbol{\lambda}$. Note that (16), is manifestly independent of the choice made for the foliation and flow, and (16) is always a linear hyperbolic equation for $\boldsymbol{\lambda}$, with ρ "playing the role of time". As it was emphasized in the introduction, the global existence of unique smooth solutions (under suitable regularity conditions on the coefficients and source terms) is always guaranteed to such linear first order symmetric hyperbolic equations. This makes it tempting to consider applications of this evolutionary equation for the geometrically distinguished scalar field $\boldsymbol{\lambda}$, which has been motivated by the constraints of Maxwell theory.

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