On sets $X \subseteq \mathbb{N}$ for which we know an algorithm that computes a threshold number $t(X) \in \mathbb{N}$ such that $X$ is infinite if and only if $X$ contains an element greater than $t(X)$

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Abstract

Let $\Gamma_{[n]}(k)$ denote $(k-1)!$, where $n \in \{3, \ldots, 16\}$ and $k \in \{2\} \cup \{2^{2^{2^{n-3}}} + 1, 2^{2^{n-3}} + 2, 2^{2^{n-3}} + 3, \ldots\}$. For an integer $n \in \{3, \ldots, 16\}$, let $S_n$ denote the following statement: if a system of equations $S \subseteq \Gamma_{[n]}(x_i) = x_k : i, k \in \{1, \ldots, n\}$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq 2^{2^{n-2}}$. The statement $S_n$ proves the following implication: if the equation $x(x+1) = y!$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x, y)$ belongs to the set $\{(1, 2), (2, 3)\}$. The statement $S_6$ proves the following implication: if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x, y)$ belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$. The statement $S_\infty$ implies the infinitude of primes of the form $n^2 + 1$. The statement $S_9$ implies that any prime of the form $n! + 1$ with $n \geq 2^{2^{n-3}}$ proves the infinitude of primes of the form $n! + 1$. The statement $S_{14}$ implies the infinitude of twin primes. The statement $S_{16}$ implies the infinitude of Sophie Germain primes. A modified statement $S_\infty$ implies the infinitude of Wilson primes.

Key words and phrases: Brocard’s problem, Brocard-Ramanujan equation $x! + 1 = y^2$, composite Fermat numbers, Erdős’ equation $x(x+1) = y!$, prime numbers of the form $n^2 + 1$, prime numbers of the form $n! + 1$, Richert’s lemma, Sophie Germain primes, Wilson primes, twin prime conjecture.

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1 Introduction

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [18] p. 39. The following statement

(1) “For every non-negative integer $n$ there exist prime exist numbers $p$ and $q$

such that $p + 2 = q$ and $p \in \{10^q, 10^q + 1\}$

is a $\Pi_1$ statement which strengthens the twin prime conjecture, see [5] p. 43), cf. [7] pp. 337–338]. Statement (1) is equivalent to the non-halting of a Turing machine. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger $\Pi_1$ statements, see [1].

In this article, we study sets $X \subseteq \mathbb{N}$ for which we know an algorithm that computes a threshold number $t(X) \in \mathbb{N}$ such that $X$ is infinite if and only if $X$ contains an element greater than $t(X)$. If $X$ is computable, then this property implies that the infinity of $X$ is equivalent to the halting of a Turing machine. If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any non-negative integer $m$ is a threshold number of $X$. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $X$ form the set $\{\max(X), \max(X) + 1, \max(X) + 2, \ldots\}$.

The classes of the infinite recursively enumerable sets and of the infinite recursive sets are not recursively enumerable, see [19] p. 234).

Corollary 1. If an algorithm $Alg_1$ for every recursive set $R \subseteq \mathbb{N}$ finds a non-negative integer $Alg_1(R)$, then there exists a finite set $W \subseteq \mathbb{N}$ such that $W \cap [Alg_1(W) + 1, \infty) \neq \emptyset$. If an algorithm $Alg_2$ for
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every recursively enumerable set $R \subseteq \mathbb{N}$ finds a non-negative integer $\text{Alg}_2(R)$, then there exists a finite set $W \subseteq \mathbb{N}$ such that $W \cap \{\text{Alg}_2(W) + 1, \infty\} \neq \emptyset$.

2 A Diophantine equation whose non-solvability expresses the consistency of ZFC

Gödel’s second incompleteness theorem and the Davis-Putnam-Robinson-Matiyasevich theorem imply the following theorem.

**Theorem 1.** ([2] p. 35). There exists a polynomial $D(x_1, \ldots, x_m)$ with integer coefficients such that if ZFC is arithmetically consistent, then the sentences "The equation $D(x_1, \ldots, x_m) = 0$ is solvable in non-negative integers" and "The equation $D(x_1, \ldots, x_m) = 0$ is not solvable in non-negative integers" are not provable in ZFC.

Let $Y$ denote the set of all non-negative integers $k$ such that the equation $D(x_1, \ldots, x_m) = 0$ has no solutions in $[0, \ldots, k]^m$. Since the set $[0, \ldots, k]^m$ is finite, we know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in Y$. Let $\gamma : \mathbb{N}^{m+1} \to \mathbb{N}$ be a computable bijection, and let $E \subseteq \mathbb{N}^{m+1}$ be the solution set of the equation $D(x_1, \ldots, x_m) + 0 \cdot x_{m+1} = 0$. Theorem 1 implies Theorems 2 and 3.

**Theorem 2.** If ZFC is arithmetically consistent, then for every $n \in \mathbb{N}$ the sentences "$n$ is a threshold number of $Y$" and "$n$ is not a threshold number of $Y$" are not provable in ZFC.

**Theorem 3.** We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \gamma(E)$. The set $\gamma(E)$ is empty or infinite. In both cases, every non-negative integer $n$ is a threshold number of $\gamma(E)$. If ZFC is arithmetically consistent, then the sentences "$\gamma(E)$ is empty", "$\gamma(E)$ is not empty", "$\gamma(E)$ is finite", and "$\gamma(E)$ is infinite" are not provable in ZFC.

In Figure 1, $D(x_1, \ldots, x_m)$ stands for the polynomial described in Theorem 1. Let $K$ denote the set of all positive integers $k$ such that the algorithm in Figure 1 halts for $k$ on the input. If ZFC is consistent, then $K = \emptyset$. Otherwise, $\text{card}(K) = 1$.

![Fig. 1 The algorithm which may halt only when ZFC is inconsistent](image-url)

**Theorem 4.** If ZFC is consistent, then for every positive integer $n$, the inclusion $K \subseteq \{1, \ldots, n\}$ is not provable in ZFC.

**Proof.** It follows from Gödel’s second incompleteness theorem because the inclusion $K \subseteq \{1, \ldots, n\}$ implies $K = \emptyset$ and the consistency of ZFC.

**Theorem 5.** (cf. Theorem 28). If ZFC is consistent and a computer program halts for at most finitely many positive integers $k$ on the input, then not always we can write the decimal expansion of a positive integer $n$ which is not smaller than every such number $k$.
Proof. We write a computer program which implements the algorithm in Figure 1. This program halts exactly for elements of \(\mathcal{K}\) on the input. The set \(\mathcal{K}\) is finite as \(\text{card}(\mathcal{K}) \leq 1\). By Theorem 4, if ZFC is consistent, then for every positive integer \(n\), the inclusion \(\mathcal{K} \subseteq \{1, \ldots, n\}\) is not provable in ZFC. \(\square\)

3 Hypothetical statements \(\Psi_3, \ldots, \Psi_{16}\) and number-theoretic lemmas

For a positive integer \(n\), let \(\Gamma(n)\) denote \((n - 1)!\). Let \(f(1) = 2\), \(f(2) = 4\), and let \(f(n + 1) = f(n)!\) for every integer \(n \geq 2\). Let \(h(1) = 1\), and let \(h(n + 1) = 2^{2h(n)}\) for every positive integer \(n\). Let \(g(3) = 4\), and let \(g(n + 1) = g(n)!\) for every integer \(n \geq 3\). For an integer \(n \geq 3\), let \(\mathcal{U}_n\) denote the following system of equations:

\[
\begin{align*}
\forall i \in \{1, \ldots, n - 1\} \setminus \{2\} & \quad x_i! = x_{i+1} \\
x_1 \cdot x_2 & = x_3 \\
x_2 \cdot x_2 & = x_3
\end{align*}
\]

The diagram in Figure 2 illustrates the construction of the system \(\mathcal{U}_n\).

![Fig. 2 Construction of the system \(\mathcal{U}_n\)](image)

\textbf{Lemma 1.} For every integer \(n \geq 3\), the system \(\mathcal{U}_n\) has exactly two solutions in positive integers, namely \((1, 1, 1)\) and \((2, 2, g(3), \ldots, g(n))\).

Let \(B_n = \{x_i! = x_k \mid (i, k \in \{1, \ldots, n\}) \land (i \neq k)\} \cup \{x_i \cdot x_j = x_k \mid i, j, k \in \{1, \ldots, n\}\}\)

For an integer \(n \geq 3\), let \(\Psi_n\) denote the following statement: if a system \(S \subseteq B_n\) has only finitely many solutions in positive integers \(x_1, \ldots, x_n\), then each such solution \((x_1, \ldots, x_n)\) satisfies \(x_1, \ldots, x_n \leq g(n)\). The statement \(\Psi_n\) says that for subsystems of \(B_n\) the largest known solution is indeed the largest possible.

\textbf{Hypothesis 1.} The statements \(\Psi_3, \ldots, \Psi_{16}\) are true.

\textbf{Theorem 6.} Every statement \(\Psi_n\) is true with an unknown integer bound that depends on \(n\).

\textbf{Proof.} For every positive integer \(n\), the system \(B_n\) has a finite number of subsystems. \(\square\)

\textbf{Theorem 7.} For every statement \(\Psi_n\), the bound \(g(n)\) cannot be decreased.

\textbf{Proof.} It follows from Lemma 1 because \(\mathcal{U}_n \subseteq B_n\). \(\square\)

\textbf{Lemma 2.} For every positive integers \(x\) and \(y\), \(x! \cdot y! = y!\) if and only if \((x + 1 = y) \lor (x = y = 1)\)

\textbf{Lemma 3.} For every positive integers \(x\) and \(y\), \(x \cdot \Gamma(x) = \Gamma(y)\) if and only if \((x + 1 = y) \lor (x = y = 1)\)

\textbf{Lemma 4.} For every positive integers \(x\) and \(y\), \(x + 1 = y\) if and only if \((1 \neq y) \land (x! \cdot y! = y!)\)
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Lemma 5. For every non-negative integers $b$ and $c$, $b + 1 = c$ if and only if $2^{2b} \cdot 2^{2b} = 2^{2c}$.

Let $\mathcal{P}$ denote the set of prime numbers.

Lemma 6. (Wilson’s theorem, [9 p. 89]). For every positive integer $x$, $x$ divides $(x - 1)! + 1$ if and only if $x \in \{1\} \cup \mathcal{P}$.

4 Heuristic arguments against the statement $\forall n \in \mathbb{N} \setminus \{0, 1, 2\} \Psi_n$

Let

$$G_n = \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\} \cup \{x_i + 1 = x_k : i, k \in \{1, \ldots, n\}\}$$

Hypothesis 2. ([33] p. 109). If a system $S \subseteq G_n$ has only finitely many solutions in non-negative integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq h(2n)$.

Hypothesis 3. If a system $S \subseteq G_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq f(2n)$.

Observations 1 and 2 heuristically justify Hypothesis 3.

Observation 1. (cf. [33] p. 110, Observation 1). For every system $S \subseteq G_n$ which involves all the variables $x_1, \ldots, x_n$, the following new system

$$\left\{ \bigcup_{x_i \cdot x_j = x_k \in S} \{x_i \cdot x_j = x_k\} \bigcup \{x_i! = y_k : k \in \{1, \ldots, n\}\} \bigcup \{ \{1 \neq x_k, y_i \cdot x_k = y_k\} \right\}$$

is equivalent to $S$. If the system $S$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then the new system has only finitely many solutions in positive integers $x_1, \ldots, x_n, y_1, \ldots, y_n$.

Proof. It follows from Lemma 4.

Observation 2. The equation $x_1! = x_1$ has exactly two solutions in positive integers, namely $x_1 = 1$ and $x_1 = f(1)$. The system $\left\{ \begin{array}{l} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \end{array} \right.$ has exactly two solutions in positive integers, namely $(1, 1)$ and $(f(1), f(2))$. For every integer $n \geq 3$, the following system

$$\left\{ \begin{array}{l} x_1! = x_1 \\ x_1 \cdot x_1 = x_2 \\ \forall i \in \{2, \ldots, n - 1\} \ x_i! = x_{i+1} \end{array} \right.$$

has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(f(1), \ldots, f(n))$.

For a positive integer $n$, let $\Phi_n$ denote the following statement: if a system

$$S \subseteq \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\} \cup \{x_i! = x_k : i, k \in \{1, \ldots, n\}\} \cup \{1 \neq x_k : k \in \{1, \ldots, n\}\}$$

has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq f(n)$.

Theorem 8. The statement $\forall n \in \mathbb{N} \setminus \{0\} \Phi_n$ implies Hypothesis 3.

Proof. It follows from Lemma 4.

Let $\mathcal{R}_{\mathbb{N}}$ denote the class of all rings $K$ that extend $\mathbb{Z}$, and let

$$E_n = \{1 = x_k : k \in \{1, \ldots, n\}\} \cup \{x_i + x_j = x_k : i, j, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}$$

Th. Skolem proved that every Diophantine equation can be algorithmically transformed into an equivalent system of Diophantine equations of degree at most 2, see [25] pp. 2–3 and [15] pp. 3–4. The following result strengthens Skolem’s theorem.
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**Lemma 7.** (\cite{31} p. 720). Let $D(x_1, \ldots, x_p) \in \mathbb{Z}[x_1, \ldots, x_p]$. Assume that $\deg(D, x_i) \geq 1$ for each $i \in \{1, \ldots, p\}$. We can compute a positive integer $n > p$ and a system $T \subseteq E_n$ which satisfies the following two conditions:

1. If $K \in \mathcal{R}_n$ and $\{N, \mathbb{N} \setminus \{0\}\}$, then
   \[ \forall x_1, \ldots, x_p \in K \left( D(x_1, \ldots, x_p) = 0 \iff \exists x_{p+1}, \ldots, x_n \in K (x_1, \ldots, x_p, x_{p+1}, \ldots, x_n) \text{ solves } T \right) \]

2. If $K \in \mathcal{R}_n$ and $\{N, \mathbb{N} \setminus \{0\}\}$, then for each $x_1, \ldots, x_p \in K$ with $D(x_1, \ldots, x_p) = 0$, there exists a unique tuple $(x_{p+1}, \ldots, x_n) \in K^{n-p}$ such that the tuple $(x_1, \ldots, x_p, x_{p+1}, \ldots, x_n)$ solves $T$.

**Corollary 2.** We can express the equation $x + y = z$ as an equivalent system $\mathcal{F}$, where $\mathcal{F}$ involves $x, y, z$ and $9$ new variables, and where $\mathcal{F}$ consists of equations of the forms $\alpha + 1 = \gamma$ and $\alpha \cdot \beta = \gamma$. 

**Proof.** The new $9$ variables express the following polynomials:

\[ zx, \quad z\mathcal{x} + 1, \quad zy, \quad zy + 1, \quad z^2, \quad xy, \quad xy + 1, \quad z^2(xy + 1), \quad z^2(xy + 1) + 1 \]

\[ \square \]

**Lemma 8.** (\cite{23} p. 100) For each positive integers $x, y, z, x + y = z$ if and only if

\[ (zx + 1)(zy + 1) = z^2(xy + 1) + 1 \]

**Theorem 9.** Hypothesis\footnote{3} implies that there is an algorithm which takes as input a Diophantine equation, and returns an integer such that this integer is greater than the solutions in positive integers, if these solutions form a finite set.

**Proof.** It follows from Lemma\footnote{9} \[ \square \]

**Open Problem 1.** Is there an algorithm which takes as input a Diophantine equation, and returns an integer such that this integer is greater than the moduli of integer (non-negative integer, positive integer) solutions, if the solution set is finite?

Matiyasevich’s conjecture on finite-fold Diophantine representations (\cite{17}) implies a negative answer to Open Problem\footnote{1} see \cite{16} p. 42.

The statement $\forall n \in \mathbb{N} \setminus \{0\} \Phi_n$ implies that there is an algorithm which takes as input a factorial Diophantine equation, and returns an integer such that this integer is greater than the solutions in positive integers, if these solutions form a finite set. This conclusion is a bit strange because a computable upper bound on non-negative integer solutions does not exist for exponential Diophantine equations with a finite number of solutions, see \cite{14} p. 300.]
5 The Brocard-Ramanujan equation \( x! + 1 = y^2 \)

Let \( \mathcal{A} \) denote the following system of equations:

\[
\begin{align*}
x_1! &= x_2 \\
x_2! &= x_3 \\
x_3! &= x_6 \\
x_4 \cdot x_4 &= x_5 \\
x_3 \cdot x_5 &= x_6
\end{align*}
\]

Lemma 2 and the diagram in Figure 3 explain the construction of the system \( \mathcal{A} \).

\[
\begin{array}{c}
\xrightarrow{\text{x}_1!} & \xrightarrow{\text{or} \ x_2 = x_5 = 1} & \xrightarrow{\text{or} \ x_5!} & \xrightarrow{\text{squaring}} & \xrightarrow{\text{x}_2 \cdot x_5 = x_6} & \xrightarrow{\text{x}_6!} \\
\end{array}
\]

**Fig. 3** Construction of the system \( \mathcal{A} \)

**Lemma 10.** For every \( x_1, x_4 \in \mathbb{N} \setminus \{0, 1\} \), the system \( \mathcal{A} \) is solvable in positive integers \( x_2, x_3, x_5, x_6 \) if and only if \( x_1! + 1 = x_4^2 \). In this case, the integers \( x_2, x_3, x_5, x_6 \) are uniquely determined by the following equalities:

\[
\begin{align*}
x_2 &= x_1! \\
x_3 &= (x_1!)! \\
x_5 &= x_1! + 1 \\
x_6 &= (x_1! + 1)!
\end{align*}
\]

**Proof.** It follows from Lemma 2.

It is conjectured that \( x! + 1 \) is a perfect square only for \( x \in \{4, 5, 7\} \), see [34, p. 297]. A weak form of Szpiro’s conjecture implies that there are only finitely many solutions to the equation \( x! + 1 = y^2 \), see [20].

**Theorem 10.** If the equation \( x_1! + 1 = x_4^2 \) has only finitely many solutions in positive integers, then the statement \( \Psi_6 \) guarantees that each such solution \( (x_1, x_4) \) belongs to the set \( \{(4, 5), (5, 11), (7, 71)\} \).

**Proof.** Suppose that the antecedent holds. Let positive integers \( x_1 \) and \( x_4 \) satisfy \( x_1! + 1 = x_4^2 \). Then, \( x_1, x_4 \in \mathbb{N} \setminus \{0, 1\} \). By Lemma 10 the system \( \mathcal{A} \) is solvable in positive integers \( x_2, x_3, x_5, x_6 \). Since \( \mathcal{A} \subseteq \mathcal{B} \), the statement \( \Psi_6 \) implies that \( x_6 = (x_1! + 1)! \leq g(6) = g(5)! \). Hence, \( x_1! + 1 \leq g(5) = g(4)! \). Consequently, \( x_1 \leq g(4) = 24 \). If \( x_1 \in \{1, \ldots, 23\} \), then \( x_1! + 1 \) is a perfect square only for \( x_1 \in \{4, 5, 7\} \).

6 Are there infinitely many prime numbers of the form \( n^2 + 1 \)?

Let \( \mathcal{B} \) denote the following system of equations:

\[
\begin{align*}
x_2! &= x_3 \\
x_3! &= x_4 \\
x_5! &= x_6 \\
x_8! &= x_9 \\
x_1 \cdot x_1 &= x_2 \\
x_3 \cdot x_5 &= x_6 \\
x_4 \cdot x_8 &= x_9 \\
x_5 \cdot x_7 &= x_8
\end{align*}
\]

Lemma 2 and the diagram in Figure 4 explain the construction of the system \( \mathcal{B} \).
Lemma 11. For every integer $x_1 \geq 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:

\[
\begin{align*}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_2^2 + 1 \\
x_6 &= (x_3^2 + 1)! \\
x_7 &= (x_4^2)! + 1 \\
x_8 &= (x_5^2)! + 1 \\
x_9 &= ((x_6^2)! + 1)!
\end{align*}
\]

Proof. By Lemma 2, for every integer $x_1 \geq 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 11 follows from Lemma 6.

Lemma 12. There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system $\mathcal{B}$ and satisfy $x_1 = 1$.

Proof. If a tuple $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $\mathcal{B}$ and $x_1 = 1$, then $x_1, \ldots, x_9 \leq 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leq 2$.

Edmund Landau’s conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [18] pp. 37–38).

Theorem 11. The statement $\Psi_9$ proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $g(7)$, then there are infinitely many primes of the form $n^2 + 1$.

Proof. Suppose that the antecedent holds. By Lemma 11, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{B}$. Since $x_1^2 + 1 > g(7)$, we obtain that $x_1^2 \geq g(7)$. Hence, $(x_1^2)! \geq g(7)! = g(8)$. Consequently,

\[
x_9 = ((x_1^2)! + 1)! \geq (g(8) + 1)! > g(8)! = g(9)
\]

Since $\mathcal{B} \subseteq B_9$, the statement $\Psi_9$ and the inequality $x_9 > g(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 11 and 12, there are infinitely many primes of the form $n^2 + 1$. 

\[\square\]
7 Are there infinitely many prime numbers of the form \( n! + 1 \)?

It is conjectured that there are infinitely many primes of the form \( n! + 1 \), see [4, p. 443] and [26].

**Theorem 12.** (cf. Theorem 17). The statement \( \Psi \) proves the following implication: if there exists an integer \( x_1 \geq g(6) \) such that \( x_1! + 1 \) is prime, then there are infinitely many primes of the form \( n! + 1 \).

**Proof.** We leave the analogous proof to the reader. \( \square \)

8 The twin prime conjecture

Let \( C \) denote the following system of equations:

\[
\begin{align*}
  x_1! &= x_2 \\
  x_2! &= x_3 \\
  x_4! &= x_5 \\
  x_6! &= x_7 \\
  x_7! &= x_8 \\
  x_9! &= x_{10} \\
  x_{12}! &= x_{13} \\
  x_{15}! &= x_{16} \\
  x_2 \cdot x_4 &= x_5 \\
  x_5 \cdot x_6 &= x_7 \\
  x_7 \cdot x_9 &= x_{10} \\
  x_4 \cdot x_{11} &= x_{12} \\
  x_3 \cdot x_{12} &= x_{13} \\
  x_9 \cdot x_{14} &= x_{15} \\
  x_8 \cdot x_{15} &= x_{16}
\end{align*}
\]

Lemma[5] and the diagram in Figure 5 explain the construction of the system \( C \).

![Fig. 5 Construction of the system C](image-url)
Lemma 13. For every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system $C$ is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if $x_4$ and $x_9$ are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ are uniquely determined by the following equalities:

$$
\begin{align*}
    x_1 &= x_4 - 1 \\
    x_2 &= (x_4 - 1)! \\
    x_3 &= ((x_4 - 1)!)! \\
    x_5 &= x_4! \\
    x_6 &= x_9 - 1 \\
    x_7 &= (x_9 - 1)! \\
    x_8 &= ((x_9 - 1)!)! \\
    x_{10} &= x_9! \\
    x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
    x_{12} &= (x_4 - 1)! + 1 \\
    x_{13} &= ((x_4 - 1)! + 1)! \\
    x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
    x_{15} &= (x_9 - 1)! + 1 \\
    x_{16} &= ((x_9 - 1)! + 1)!
\end{align*}
$$

Proof. By Lemma 2 for every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system $C$ is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if

$$
(x_4 + 2 = x_9) \land (x_4((x_4 - 1)! + 1) \land (x_9((x_9 - 1)! + 1))
$$

Hence, the claim of Lemma 13 follows from Lemma 6.

Lemma 14. There are only finitely many tuples $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ which solve the system $C$ and satisfy

$$(x_4 \in \{1, 2\}) \lor (x_9 \in \{1, 2\})$$

Proof. If a tuple $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ solves the system $C$ and

$$(x_4 \in \{1, 2\}) \lor (x_9 \in \{1, 2\})$$

then $x_1, \ldots, x_{16} \leq 7!$. Indeed, for example, if $x_4 = 2$ then $x_6 = x_4 + 1 = 3$. Hence, $x_7 = x_6! = 6$. Therefore, $x_{15} = x_7 + 1 = 7$. Consequently, $x_{16} = x_{15}! = 7!$.

Theorem 13. The statement $\Psi_{16}$ proves the following implication: (*) if there exists a twin prime greater than $g(14)$, then there are infinitely many twin primes.

Proof. Suppose that the antecedent holds. Then, there exist prime numbers $x_4$ and $x_9$ such that $x_9 = x_4 + 2 > g(14)$. Hence, $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$. By Lemma 13 there exists a unique tuple $(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{14}$ such that the tuple $(x_1, \ldots, x_{16})$ solves the system $C$. Since $x_9 > g(14)$, we obtain that $x_9 - 1 \geq g(14)$. Therefore, $(x_9 - 1)! > g(14)! = g(15)$. Hence, $(x_9 - 1)! + 1 > g(15)$. Consequently,

$$
x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)
$$

Since $C \subseteq B_{16}$, the statement $\Psi_{16}$ and the inequality $x_{16} > g(16)$ imply that the system $C$ has infinitely many solutions in positive integers $x_1, \ldots, x_{16}$. According to Lemmas 13 and 14 there are infinitely many twin primes.

Let $P(x)$ denote the predicate "$x$ is a prime number". Dickson’s conjecture ([18, p. 36], [36, p. 109]) implies that the existential theory of $(\mathbb{N}, =, +, P)$ is decidable, see [36, Theorem 2, p. 109]. For a positive integer $n$, let $\Theta_n$ denote the following statement: for every system $S \subseteq \{x_i + 1 = x_k : i, k \in \{1, \ldots, n\} \cup \{P(x_i) : i \in \{1, \ldots, n\}\}$ the solvability of $S$ in non-negative integers is decidable.
Lemma 15. If the existential theory of $(\mathbb{N}, =, +, \mathbb{P})$ is decidable, then the statements $\Theta_n$ are true.

Proof. For every non-negative integers $x$ and $y$, $x + 1 = y$ if and only if
$$\exists u, v \in \mathbb{N} \ (u + u = v) \land \mathbb{P}(v) \land (x + u = y)$$
\[ \square \]

Theorem 14. The conjunction of the implication $(\ast)$ and the statement $\Theta_{g(14)+2}$ implies that the twin prime conjecture is decidable.

Proof. By the statement $\Theta_{g(14)+2}$, we can decide the truth of the sentence
$$\exists x_1 \ldots \exists x_{g(14)+2} \left( (\forall i \in \{1, \ldots, g(14)+1\} \ x_i + 1 = x_{i+1} \land \mathbb{P}(x_{g(14)}) \land \mathbb{P}(x_{g(14)+2}) \right)$$
(2)
If sentence (2) is false, then the twin prime conjecture is false. If sentence (2) is true, then there exists a twin prime greater than $g(14)$. In this case, the twin prime conjecture follows from Theorem 13. \[ \square \]

9 Hypothetical statements $\Delta_5, \ldots, \Delta_{14}$ about the Gamma function and their consequences

Let $\lambda(5) = \Gamma(25)$, and let $\lambda(n + 1) = \Gamma(\lambda(n))$ for every integer $n \geq 5$. For an integer $n \geq 5$, let $J_n$ denote the following system of equations:

$$\begin{cases} 
\forall i \in \{1, \ldots, n-1\} \setminus \{3\} \ \Gamma(x_i) & = \ x_{i+1} \\
\Gamma(x_1) \cdot x_1 & = \ x_4 \\
x_2 \cdot x_3 & = \ x_5 
\end{cases}$$

Lemma 3 and the diagram in Figure 6 explain the construction of the system $J_n$.

![Fig. 6 Construction of the system $J_n$](image)

Observation 3. For every integer $n \geq 5$, the system $J_n$ has exactly two solutions in positive integers, namely $(1, \ldots, 1)$ and $(5, 24, 23!, 25, \lambda(5), \ldots, \lambda(n))$.

For an integer $n \geq 5$, let $\Delta_n$ denote the following statement: if a system $S \subseteq \{\Gamma(x_i) = x_k : i, k \in \{1, \ldots, n\}\} \cup \{x_{i} \cdot x_{j} = x_{k} : i, j, k \in \{1, \ldots, n\}\}$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq \lambda(n)$.

Hypothesis 4. The statements $\Delta_5, \ldots, \Delta_{14}$ are true.

Lemmas 3 and 6 imply that the statements $\Delta_n$ have similar consequences as the statements $\Psi_n$.

Theorem 15. The statement $\Delta_6$ implies that any prime number $p \geq 25$ proves the infinitude of primes.

Proof. It follows from Lemmas 3 and 6. We leave the details to the reader. \[ \square \]
On sets $X \subseteq \mathbb{N}$ for which we know an algorithm ...

10 Hypothetical statements $\Sigma_3, \ldots, \Sigma_{16}$ about the Gamma function and their consequences

Let $\Gamma_n(k)$ denote $(k-1)!$, where $n \in \{3, \ldots, 16\}$ and $k \in \{2\} \cup \left\{2^{2^{n-3}} + 1, 2^{2^{n-3}} + 2, 2^{2^{n-3}} + 3, \ldots\right\}$. For an integer $n \in \{3, \ldots, 16\}$, let

$$Q_n = \{\Gamma_n(x_i) = x_k : i, k \in \{1, \ldots, n\}\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\}$$

For an integer $n \in \{3, \ldots, 16\}$, let $P_n$ denote the following system of equations:

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \Gamma_n(x_2) = x_1 \\ \forall i \in \{2, \ldots, n-1\} x_i \cdot x_i = x_{i+1} \end{cases}$$

Lemma 16. For every integer $n \in \{3, \ldots, 16\}$, $P_n \subseteq Q_n$ and the system $P_n$ has exactly one solution in positive integers $x_1, \ldots, x_n$, namely $(1, 2^{2^{0}}, 2^{2^{1}}, 2^{2^{2}}, \ldots, 2^{2^{n-2}})$.

For an integer $n \in \{3, \ldots, 16\}$, let $\Sigma_n$ denote the following statement: if a system of equations $S \subseteq Q_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq 2^{2^{n-2}}$.

Hypothesis 5. The statements $\Sigma_3, \ldots, \Sigma_{16}$ are true.

Lemma 17. (cf. Lemma 3). For every integer $n \in \{4, \ldots, 16\}$ and for every positive integers $x$ and $y$,

$$x \cdot \Gamma_n(x) = \Gamma_n(y) \text{ if and only if } (x + 1 = y) \land \left(x \geq 2^{2^{n-3}} + 1\right).$$

Let $Z_9 \subseteq Q_9$ be the system of equations in Figure 7.

![Fig. 7 Construction of the system $Z_9$](image)

Lemma 18. For every positive integer $x_1$, the system $Z_9$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1 > 2^{2^{9-4}}$ and $x_1^2 + 1$ is prime. In this case, positive integers $x_2, \ldots, x_9$ are uniquely determined by $x_1$.

Proof. It follows from Lemmas 6 and 17.
On sets $X \subseteq \mathbb{N}$ for which we know an algorithm ...

Lemma 19. ([29]). The number $(13!)^2 + 1 = 38775788043632640001$ is prime.

Lemma 20. $((13!)^2 \geq 2^{29-3} + 1 = 18446744073709551617) \land (13! > 2^{29-2})$.

Theorem 16. The statement $\Sigma_9$ implies the infinitude of primes of the form $n^2 + 1$.

Proof. It follows from Lemmas 18–20.

Theorem 17. (cf. Theorem [12]). The statement $\Sigma_9$ implies that any prime of the form $n! + 1$ with $n \geq 2^{29-3}$ proves the infinitude of primes of the form $n! + 1$.

Proof. We leave the proof to the reader.

Let $\mathcal{Z}_{14} \subseteq \mathcal{Q}_{14}$ be the system of equations in Figure 8.

![Fig. 8 Construction of the system $\mathcal{Z}_{14}$](image)

Lemma 21. For every positive integer $x_1$, the system $\mathcal{Z}_{14}$ is solvable in positive integers $x_2, \ldots, x_{14}$ if and only if $x_1$ and $x_1 + 2$ are prime and $x_1 \geq 2^{2^{14-3}} + 1$. In this case, positive integers $x_2, \ldots, x_{14}$ are uniquely determined by $x_1$.

Proof. It follows from Lemmas 6 and 17.

Lemma 22. ([37], p. 87). The numbers $459 \cdot 2^{8529} - 1$ and $459 \cdot 2^{8529} + 1$ are prime (Harvey Dubner).

Lemma 23. $459 \cdot 2^{8529} - 1 > 2^{2^{14-2}} = 2^{4096}$.

Theorem 18. The statement $\Sigma_{14}$ implies the infinitude of twin primes.

Proof. It follows from Lemmas 21–23.
A prime $p$ is said to be a Sophie Germain prime if both $p$ and $2p + 1$ are prime, see [35]. Let $Z_{16} \subseteq Q_{16}$ be the system of equations in Figure 9.

**Fig. 9** Construction of the system $Z_{16}$

**Lemma 24.** For every positive integer $x_1$, the system $Z_{16}$ is solvable in positive integers $x_2, \ldots, x_{16}$ if and only if $x_1$ is a Sophie Germain prime and $x_1 \geq 2^{16-3} + 1$. In this case, positive integers $x_2, \ldots, x_{16}$ are uniquely determined by $x_1$.

*Proof.* It follows from Lemmas [6] and [17].

**Lemma 25.** ([27] p. 330). $8069496435 \cdot 10^{5072} - 1$ is a Sophie Germain prime (Harvey Dubner).

**Lemma 26.** $8069496435 \cdot 10^{5072} - 1 > 2^{216-2}$.

**Theorem 19.** The statement $\Sigma_{16}$ implies the infinitude of Sophie Germain primes.

*Proof.* It follows from Lemmas [24]-[26].

**Theorem 20.** The statement $\Sigma_6$ proves the following implication: if the equation $x(x + 1) = y!$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x, y)$ belongs to the set $\{(1, 2), (2, 3)\}$.

*Proof.* We leave the proof to the reader.

The question of solving the equation $x(x + 1) = y!$ was posed by P. Erdös, see [2]. F. Luca proved that the abc conjecture implies that the equation $x(x + 1) = y!$ has only finitely many solutions in positive integers, see [13].

**Theorem 21.** The statement $\Sigma_6$ proves the following implication: if the equation $x! + 1 = y^2$ has only finitely many solutions in positive integers $x$ and $y$, then each such solution $(x, y)$ belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.

*Proof.* We leave the proof to the reader.
11 A hypothesis which implies the infinitude of Wilson primes

Let $\zeta(k)$ denote $(k-1)!$, where $k \in \{2\} \cup \{17, 18, 19, \ldots\}$. Let

$$V = \{ \zeta(x_i) = x_k : i, k \in \{1, \ldots, 7\} \} \cup \{ x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, 7\} \}$$

Let $I_7$ denote the following system of equations:

$$\begin{cases}
    x_1 \cdot x_1 &= x_1 \\
    \zeta(x_2) &= x_1 \\
    x_2 \cdot x_2 &= x_3 \\
    x_3 \cdot x_3 &= x_4 \\
    x_4 \cdot x_4 &= x_5 \\
    \zeta(x_5) &= x_6 \\
    \zeta(x_6) &= x_7 
\end{cases}$$

**Lemma 27.** $I_7 \subseteq V$ and the system $I_7$ has exactly one solution in positive integers $x_1, \ldots, x_7$, namely $(1, 2, 4, 16, 256, 255!, (255! - 1)!)$.

Let $\Xi_7$ denote the following statement: if a system of equations $S \subseteq V$ has only finitely many solutions in positive integers $x_1, \ldots, x_7$, then each such solution $(x_1, \ldots, x_7)$ satisfies $x_1, \ldots, x_7 \leq (255! - 1)!$.

**Hypothesis 6.** The statement $\Xi_7$ is true.

**Lemma 28.** (cf. Lemma 3). For every positive integers $x$ and $y$, $x \cdot \zeta(x) = \zeta(y)$ if and only if $(x + 1 = y) \land (x \geq 17)$.

A Wilson prime is a prime number $p$ such that $p^2$ divides $(p - 1)! + 1$, see [3], [21, p. 346], and [30]. It is conjectured that the set of Wilson primes is infinite, see [3]. Let $Z_7 \subseteq V$ be the system of equations in Figure 10.

**Fig. 10** Construction of the system $Z_7$

**Lemma 29.** For every positive integer $x_1$, the system $Z_7$ is solvable in positive integers $x_2, \ldots, x_7$ if and only if $x_1$ is a Wilson prime prime and $x_1 \geq 17$. In this case, positive integers $x_2, \ldots, x_7$ are uniquely determined by $x_1$.

**Proof.** It follows from Lemmas 6 and 28.

$\square$
On sets $X \subseteq \mathbb{N}$ for which we know an algorithm ...

Lemma 30. ([3], [21] p. 346), [30]). $563$ is a Wilson prime.

Lemma 31. $\zeta(563)(1) > (255! - 1)!$.

Theorem 22. The statement $\Xi_7$ implies the infinitude of Wilson primes.


12 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [12, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [12, p. 1].

Open Problem 2. ([12, p. 159]). Are there infinitely many composite numbers of the form $2^{2^n} + 1$?

Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geq 5$, see [11, p. 23].

Theorem 23. ([32]). An unproven inequality stated in [32] implies that $2^{2^n} + 1$ is composite for every integer $n \geq 5$.

Let

$H_n = \{ x_1 \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\} \cup \{2^{2^{x_i}} = x_k : i, k \in \{1, \ldots, n\}\}$

Lemma 32. The following subsystem of $H_n$

$$
\begin{align*}
\forall i \in \{1, \ldots, n-1\} & \quad 2^{2^{x_i}} = x_{i+1} \\
\end{align*}
$$

has exactly one solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(h(1), \ldots, h(n))$.

For a positive integer $n$, let $\Gamma_n$ denote the following statement: *if a system $S \subseteq H_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leq h(n)$.* The statement $\Gamma_n$ says that for subsystems of $H_n$ the largest known solution is indeed the largest possible.

Hypothesis 7. The statements $\Gamma_1, \ldots, \Gamma_{13}$ are true.

The truth of the statement $\forall n \in \mathbb{N} \setminus \{0\} \Gamma_n$ is doubtful because a computable upper bound on non-negative integer solutions does not exist for exponential Diophantine equations with a finite number of solutions, see [14, p. 300].

Theorem 24. Every statement $\Gamma_n$ is true with an unknown integer bound that depends on $n$.

Proof. For every positive integer $n$, the system $H_n$ has a finite number of subsystems. □

Theorem 25. The statement $\Gamma_{13}$ proves the following implication: *if $z \in \mathbb{N} \setminus \{0\}$ and $2^{2^z} + 1$ is composite and greater than $h(12)$, then $2^{2^z} + 1$ is composite for infinitely many positive integers $z$.*

Proof. Let us consider the equation

$$(x + 1)(y + 1) = 2^{2^z} + 1$$  \hspace{1cm} (3)

in positive integers. By Lemma 5, we can transform equation (3) into an equivalent system $G$ which has 13 variables $(x, y, z, \text{ and 10 other variables})$ and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^\alpha} = \gamma$, see the diagram in Figure 11.
On sets $X \subseteq \mathbb{N}$ for which we know an algorithm ...

Since $2^{2z} + 1 > h(12)$, we obtain that $2^{2^{2^z+1}} > h(13)$. By this, the statement $\Gamma_{13}$ implies that the system $G$ has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers.

13 Subsets of $\mathbb{N}$ whose infinitude is unconditionally equivalent to the halting of a Turing machine

The following lemma is known as Richert’s lemma.

Lemma 33. ([8], [22], [24, p. 152]). Let $(m_i)_{i=1}^{\infty}$ be an increasing sequence of positive integers such that for some positive integer $k$ the inequality $m_{i+1} \leq 2m_i$ holds for all $i \geq k$. Suppose there exists a non-negative integer $b$ such that the numbers $b+1$, $b+2$, $b+3$, ..., $b+m_{k+1}$ are all expressible as sums of one or more distinct elements of the set $\{m_1, \ldots, m_k\}$. Then every integer greater than $b$ is expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$.

Let $T$ denote the set of all positive integers $i$ such that every integer $j \geq i$ is expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$. Obviously, $T = \emptyset$ or $T = [d, \infty) \cap \mathbb{N}$ for some positive integer $d$.

Corollary 3. If the sequence $(m_i)_{i=1}^{\infty}$ is computable and the algorithm in Figure 12 terminates, then almost all positive integers are expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$. In particular, if the sequence $(m_i)_{i=1}^{\infty}$ is computable and the algorithm in Figure 12 terminates, then the set $T$ is infinite. In this case, the algorithm is Figure 12 prints all positive integers which are not expressible as a sum of one or more distinct elements of the set $\{m_1, m_2, m_3, \ldots\}$.
On sets \( X \subseteq \mathbb{N} \) for which we know an algorithm ...

**Theorem 26.** ([10], Theorem 2.3). If there exists \( \varepsilon > 0 \) such that the inequality \( m_{i+1} \leq (2 - \varepsilon) \cdot m_i \) holds for every sufficiently large \( i \), then the algorithm in Figure 12 terminates if and only if almost all positive integers are expressible as a sum of one or more distinct elements of the set \( \{m_1, m_2, m_3, \ldots\} \).

**Corollary 4.** If there exists \( \varepsilon > 0 \) such that the inequality \( m_{i+1} \leq (2 - \varepsilon) \cdot m_i \) holds for every sufficiently large \( i \), then the algorithm in Figure 12 terminates if and only if the set \( T \) is infinite.

We show how the algorithm in Figure 12 works for a concrete sequence \( \{m_i\}_{i=1}^{\infty} \). Let \([\cdot]\) denote the integer part function. For a positive integer \( i \), let \( t_i = \frac{(i + 19)^i + 19}{(i + 19)! \cdot 2^i + 19} \), and let \( m_i = [t_i] \).

**Lemma 34.** The inequality \( m_{i+1} \leq 2m_i \) holds for every positive integer \( i \).

**Proof.** For every positive integer \( i \),

\[
\frac{m_i}{m_{i+1}} = \frac{[t_i]}{[t_{i+1}]} > \frac{t_i - 1}{t_{i+1}} = \frac{t_i}{t_{i+1}} - \frac{1}{t_{i+1}} > \frac{t_i}{t_{i+1}} - \frac{1}{t_2} = 2 \cdot \frac{i + 20}{i + 19} \cdot \left(1 - \frac{1}{i + 20}\right)^{i+20} - 21! \cdot \frac{2^{21}}{2^{21}} > 2 \cdot \left(1 - \frac{1}{21}\right)^{21} - 21! \cdot \frac{2^{21}}{2^{21}} = \frac{4087158528442715204485120000}{5842587018385982521381124421}
\]

The last fraction was computed by MuPAD and is greater than \( \frac{1}{2} \). \( \square \)
On sets \( X \subseteq \mathbb{N} \) for which we know an algorithm ...

**Theorem 27.** The algorithm in Figure 12 terminates for the sequence \( \{m_i\}_{i=1}^{\infty} \).

**Proof.** By Lemma \( \square \) we take \( k = 2 \) as the initial value of \( k \). The following MuPAD code

\[
\begin{align*}
&k:=2: \\
&\text{repeat} \\
&A:=\{\text{floor}((i+19)^{(i+19})/(i+19)!*2^{(i+19)}) \mid i=1..k+1\}: \\
&B:=\{A[1]\}:
&\text{for } i \text{ from 2 to nops(A)-1 do} \\
&B:=B \cup \{A[i]\} \cup \{B[j]+A[i] \mid j=1..\text{nops}(B)\}: \\
&\text{end_for:} \\
&G:=\{y \mid y=B[1]-1..\text{B[nops(B)]+1}\} \setminus B: \\
&H:=\{G[n+1]-G[n] \mid n=1..\text{nops}(G)-1\}: \\
&k:=k+1: \\
&\text{until } H[\text{nops}(H)]>A[\text{nops}(A)] \text{ end_repeat:} \\
&b:=B[\text{nops}(B)]: \\
&k:=1: \\
&\text{while } \text{floor}((k+20)^{(k+20})/(k+20)!*2^{(k+20)}) \leq b \text{ do} \\
&k:=k+1: \\
&\text{end_while:} \\
&A:=\{\text{floor}((i+19)^{(i+19})/(i+19)!*2^{(i+19)}) \mid i=1..k\}: \\
&B:=\{A[1]\}:
&\text{for } i \text{ from 2 to nops(A)-1 do} \\
&B:=B \cup \{A[i]\} \cup \{B[j]+A[i] \mid j=1..\text{nops}(B)\}: \\
&\text{end_for:} \\
&\text{print(\{n \mid n=1..b\} \setminus B):}
\end{align*}
\]

implements the algorithm in Figure 12 because MuPAD automatically orders every finite set of integers and the inequality \( H[\text{nops}(H)]>A[\text{nops}(A)] \) holds true if and only if the set \( B \) contains \( m_{k+1} \) consecutive integers. The code returns the following output:

\[
\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20,
21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38,
39, 40, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 55, 56, 57, 58,
59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 73, 74, 75, 76, 77,
78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 97,
98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111,
112, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 127,
129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 140, 141, 142, 143,
144, 145, 146, 147, 148, 149, 151, 152, 153, 154, 155, 156, 157, 158,
159, 160, 161, 162, 163, 164, 165, 166, 171, 172, 173, 174, 175, 176,
177, 178, 179, 180, 181, 183, 184, 185, 186, 187, 188, 189, 190, 192,
193, 194, 195, 196, 197, 198, 199, 201, 202, 203, 204, 205, 206, 207,
\}
\]
On sets $X \subseteq \mathbb{N}$ for which we know an algorithm ...


\[\square\]

**Corollary 5.** $\mathcal{T} = [2762, \infty) \cap \mathbb{N}$.

*MuPAD* is a general-purpose computer algebra system. *MuPAD* is no longer available as a stand-alone computer program, but only as the *Symbolic Math Toolbox* of *MATLAB*. Fortunately, the presented code can be executed by *MuPAD Light*, which was offered for free for research and education until autumn 2005.

14 A hypothetical infinitude of various classes of primes via computer programs which halt for at most finitely many positive integers on the input

Let $\text{fact}^{-1}: \{1, 2, 6, 24, \ldots\} \to \mathbb{N} \setminus \{0\}$ denote the inverse function to the factorial function. For positive integers $x$ and $y$, let $\text{rem}(x, y)$ denote the remainder from dividing $x$ by $y$.

**Definition.** For a positive integer $n$, by a program of length $n$ we understand any sequence of terms $x_1, \ldots, x_n$ such that $x_1$ is defined as the variable $x$, and for every integer $i \in \{2, \ldots, n\}$, $x_i$ is defined as $\Gamma(x_i-1)$, or $\text{fact}^{-1}(x_i-1)$, or $\text{rem}(x_i-1, x_i-2)$ — but only if $i \geq 3$ and $x_i-1$ is defined as $\Gamma(x_i-2)$.

Let $\delta(4) = 3$, and let $\delta(n+1) = \delta(n)!$ for every integer $n \geq 4$. For an integer $n \geq 4$, let $\Omega_n$ denote the following statement: if a program of length $n$ returns positive integers $x_1, \ldots, x_n$ for at most finitely many positive integers $x_i$, then every such $x$ does not exceed $\delta(n)$.

**Theorem 28.** (cf. Theorem [5]). For every integer $n \geq 4$, the statement $\Omega_n$ is true with an unknown integer bound that depends on $n$. 

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\]

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On sets $X \subseteq \mathbb{N}$ for which we know an algorithm ...

Proof. For every positive integer $n$, there are only finitely many programs of length $n$. □

Lemma 35. ([24] pp. 214–215) . For every positive integer $x$, $\text{rem}(\Gamma(x), x) \in \mathbb{N} \setminus \{0\}$ if and only if $x \in \{4\} \cup \mathcal{P}$.

Theorem 29. For every integer $n \geq 4$ and for every positive integer $x$, the following program $\mathcal{H}_n$

$$
\begin{align*}
\forall i \in \{2, \ldots, n-3\} & \quad x_i := \text{fact}^{-1}(x_{i-1}) \\
x_{n-2} & := \Gamma(x_{n-3}) \\
x_{n-1} & := \Gamma(x_{n-2}) \\
x_n & := \text{rem}(x_{n-1}, x_{n-2})
\end{align*}
$$

returns positive integers $x_1, \ldots, x_n$ if and only if $x = \delta(n)$. Proof. We make three observations.

Observation 4. If $x_{n-3} = 3$, then $x_1, \ldots, x_{n-3} \in \mathbb{N} \setminus \{0\}$ and $x = x_1 = \delta(n)$. If $x = \delta(n)$, then $x_1, \ldots, x_{n-3} \in \mathbb{N} \setminus \{0\}$ and $x_{n-3} = 3$. Hence, $x_{n-2} = \Gamma(x_{n-3}) = 2$ and $x_{n-1} = \Gamma(x_{n-2}) = 1$. Therefore, $x_n = \text{rem}(x_{n-1}, x_{n-2}) = 1$.

Observation 5. If $x_{n-3} = 2$, then $x = x_1 = \ldots = x_{n-3} = 2$. If $x = 2$, then $x_1 = \ldots = x_{n-3} = 2$. Hence, $x_{n-2} = \Gamma(x_{n-3}) = 1$ and $x_{n-1} = \Gamma(x_{n-2}) = 1$. Therefore, $x_n = \text{rem}(x_{n-1}, x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$.

Observation 6. If $x_{n-3} = 1$, then $x_{n-2} = \Gamma(x_{n-3}) = 1$. Hence, $x_{n-1} = \Gamma(x_{n-2}) = 1$. Observations [4]–[6] cover the case when $x_{n-3} \in \{1, 2, 3\}$. If $x_{n-3} \geq 4$, then $x_{n-2} = \Gamma(x_{n-3})$ is greater than 4 and composite. By Lemma 35, $x_n = \text{rem}(x_{n-1}, x_{n-2}) = \text{rem}(\Gamma(x_{n-2}), x_{n-2}) = 0 \notin \mathbb{N} \setminus \{0\}$. □

Corollary 6. For every integer $n \geq 4$, the bound $\delta(n)$ in the statement $\Omega_n$ cannot be decreased.

Lemma 36. If $x \in \mathcal{P}$, then $\text{rem}(\Gamma(x), x) = x - 1$.

Proof. It follows from Lemma 6. □

Lemma 37. For every positive integer $x$, the following program $\mathcal{A}$

$$
\begin{align*}
x_1 & := x \\
x_2 & := \Gamma(x_1) \\
x_3 & := \text{rem}(x_2, x_1) \\
x_4 & := \text{fact}^{-1}(x_3)
\end{align*}
$$

returns positive integers $x_1, \ldots, x_4$ if and only if $x = 4$ or $x$ is a prime number of the form $n! + 1$.

Proof. For an integer $i \in \{1, \ldots, 4\}$, let $A_i$ denote the set of positive integers $x$ such that the first $i$ instructions of the program $\mathcal{A}$ returns positive integers $x_1, \ldots, x_i$. We show that

$$
A_4 = \{4\} \cup \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\} \cap \mathcal{P}
$$

(4)

For every positive integer $x$, the terms $x_1$ and $x_2$ belong to $\mathbb{N} \setminus \{0\}$. By Lemma 35, the term $x_3$ (which equals $\text{rem}(\Gamma(x), x)$) belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \{4\} \cup \mathcal{P}$. Hence, $A_3 = \{4\} \cup \mathcal{P}$. If $x = 4$, then $x_1, \ldots, x_4 \in \mathbb{N} \setminus \{0\}$. Hence, $4 \in A_4$. If $x \in \mathcal{P}$, then Lemma 36 implies that $x_3 = \text{rem}(\Gamma(x), x) = x - 1 \in \mathbb{N} \setminus \{0\}$. Therefore, for every $x \in \mathcal{P}$, the term $x_4 = \text{fact}^{-1}(x_3)$ belongs to $\mathbb{N} \setminus \{0\}$ if and only if $x \in \{n! + 1 : n \in \mathbb{N} \setminus \{0\}\}$. This proves equality (4). □

Theorem 30. The statement $\Omega_4$ implies that the set of primes of the form $n! + 1$ is infinite.
On sets $X \subseteq \mathbb{N}$ for which we know an algorithm ...

Proof. The number $3! + 1 = 7$ is prime. By Lemma 37, for $x = 7$ the program $\mathcal{A}$ returns positive integers $x_1, \ldots, x_4$. Since $x = 7 > 3 = \delta(4)$, the statement $\Omega_4$ guarantees that the program $\mathcal{A}$ returns positive integers $x_1, \ldots, x_4$ for infinitely many positive integers $x$. By Lemma 38 there are infinitely many primes of the form $n! + 1$. \hfill $\square$

Lemma 38. If $x \in \mathbb{N} \setminus \{0, 1\}$, then $\text{fact}^{-1}(\Gamma(x)) = x - 1$.

Theorem 31. If the set of primes of the form $n! + 1$ is infinite, then the statement $\Omega_4$ is true.

Proof. There exist exactly 10 programs of length 4 that differ from $\mathcal{H}_4$ and $\mathcal{A}$, see Figure 13. For every such program $\mathcal{F}_i$, we determine the set $S_i$ of all positive integers $x$ such that the program $\mathcal{F}_i$ outputs positive integers $x_1, \ldots, x_4$ on input $x$. We omit 10 easy proofs which use Lemmas 35 and 38. The sets $S_i$ are infinite, see Figure 13.

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|}
\hline
$\mathcal{F}_1$ & $x_1 := x$ & $x_2 := \Gamma(x_1)$ & $x_3 := \Gamma(x_2)$ & $x_4 := \Gamma(x_3)$ \\
\hline
$\mathcal{F}_2$ & $x_1 := x$ & $x_2 := \Gamma(x_1)$ & $x_3 := \Gamma(x_2)$ & $x_4 := \text{fact}^{-1}(x_3)$ \\
\hline
$\mathcal{H}_4$ & $x_1 := x$ & $x_2 := \Gamma(x_1)$ & $x_3 := \Gamma(x_2)$ & $x_4 := \text{rem}(x_3, x_2)$ \\
\hline
$\mathcal{F}_3$ & $x_1 := x$ & $x_2 := \Gamma(x_1)$ & $x_3 := \text{fact}^{-1}(x_2)$ & $x_4 := \Gamma(x_3)$ \\
\hline
$\mathcal{F}_4$ & $x_1 := x$ & $x_2 := \Gamma(x_1)$ & $x_3 := \text{fact}^{-1}(x_2)$ & $x_4 := \text{fact}^{-1}(x_3)$ \\
\hline
$\mathcal{F}_5$ & $x_1 := x$ & $x_2 := \Gamma(x_1)$ & $x_3 := \text{rem}(x_2, x_1)$ & $x_4 := \Gamma(x_3)$ \\
\hline
$\mathcal{A}$ & $x_1 := x$ & $x_2 := \Gamma(x_1)$ & $x_3 := \text{rem}(x_2, x_1)$ & $x_4 := \text{fact}^{-1}(x_3)$ \\
\hline
$\mathcal{F}_6$ & $x_1 := x$ & $x_2 := \text{fact}^{-1}(x_1)$ & $x_3 := \Gamma(x_2)$ & $x_4 := \Gamma(x_3)$ \\
\hline
$\mathcal{F}_7$ & $x_1 := x$ & $x_2 := \text{fact}^{-1}(x_1)$ & $x_3 := \Gamma(x_2)$ & $x_4 := \text{fact}^{-1}(x_3)$ \\
\hline
$\mathcal{F}_8$ & $x_1 := x$ & $x_2 := \text{fact}^{-1}(x_1)$ & $x_3 := \Gamma(x_2)$ & $x_4 := \text{rem}(x_3, x_2)$ \\
\hline
$\mathcal{F}_9$ & $x_1 := x$ & $x_2 := \text{fact}^{-1}(x_1)$ & $x_3 := \text{fact}^{-1}(x_2)$ & $x_4 := \Gamma(x_3)$ \\
\hline
$\mathcal{F}_{10}$ & $x_1 := x$ & $x_2 := \text{fact}^{-1}(x_1)$ & $x_3 := \text{fact}^{-1}(x_2)$ & $x_4 := \text{fact}^{-1}(x_3)$ \\
\hline
\end{tabular}
\end{table}

This completes the proof. \hfill $\square$

Hypothesis 8. The statements $\Omega_4, \ldots, \Omega_7$ are true.

Lemma 39. For every positive integer $x$, the following program $\mathcal{B}$

\[
\begin{align*}
x_1 & := x \\
x_2 & := \Gamma(x_1) \\
x_3 & := \text{rem}(x_2, x_1) \\
x_4 & := \text{fact}^{-1}(x_3) \\
x_5 & := \Gamma(x_4) \\
x_6 & := \text{rem}(x_5, x_4)
\end{align*}
\]

returns positive integers $x_1, \ldots, x_6$ if and only if $x \in \{4\} \cup \{n! + 1 : p \in \mathcal{P} \} \cap \mathcal{P}$

Proof. For an integer $i \in \{1, \ldots, 6\}$, let $B_i$ denote the set of positive integers $x$ such that the first $i$ instructions of the program $\mathcal{B}$ returns positive integers $x_1, \ldots, x_i$. Since the programs $\mathcal{A}$ and $\mathcal{B}$ have the same first four instructions, the equality $B_i = A_i$ holds for every $i \in \{1, \ldots, 4\}$. In particular,

\[
B_4 = \{4\} \cup \{n! + 1 : n \in \mathbb{N} \setminus \{0\} \} \cap \mathcal{P}
\]
On sets \( X \subseteq \mathbb{N} \) for which we know an algorithm ...

We show that

\[
B_6 = \{4\} \cup \{p! + 1 : p \in \mathcal{P}\} \cap \mathcal{P}
\]

(5)

If \( x = 4 \), then \( x_1, \ldots, x_6 \in \mathbb{N} \setminus \{0\} \). Hence, \( 4 \in B_6 \). Let \( x \in \mathcal{P} \), and let \( x = n! + 1 \), where \( n \in \mathbb{N} \setminus \{0\} \). Hence, \( n \neq 4 \). Lemma 36 implies that \( x_4 = \text{rem}(\Gamma(x), x) = x - 1 = n! \). Hence, \( x_4 = \text{fact}^{-1}(x_3) = n \) and \( x_5 = \Gamma(x_4) = \Gamma(n) \in \mathbb{N} \setminus \{0\} \). By Lemma 35 the term \( x_6 \) (which equals \( \text{rem}(\Gamma(n), n) \)) belongs to \( \mathbb{N} \setminus \{0\} \) if and only if \( n \in \{4\} \cup \mathcal{P} \). This proves equality (5) as \( n \neq 4 \).

\[\square\]

**Theorem 32.** The statement \( \Omega_6 \) implies that for infinitely many primes \( p \) the number \( p! + 1 \) is prime.

**Proof.** The numbers 11 and 11! + 1 are prime, see [4, p. 441] and [28]. By Lemma 39 for \( x = 11! + 1 \) the program \( B \) returns positive integers \( x_1, \ldots, x_6 \). Since \( x = 11! + 1 > 6! = \delta(6) \), the statement \( \Omega_6 \) guarantees that the program \( B \) returns positive integers \( x_1, \ldots, x_6 \) for infinitely many positive integers \( x \). By Lemma 39 for infinitely many primes \( p \) the number \( p! + 1 \) is prime.

\[\square\]

**Lemma 40.** For every positive integer \( x \), the following program \( C \)

\[
\begin{align*}
x_1 & := x \\
x_2 & := \Gamma(x_1) \\
x_3 & := \Gamma(x_2) \\
x_4 & := \text{fact}^{-1}(x_3) \\
x_5 & := \Gamma(x_4) \\
x_6 & := \text{rem}(x_5, x_4)
\end{align*}
\]

returns positive integers \( x_1, \ldots, x_6 \) if and only if \( (x - 1)! - 1 \) is prime.

**Proof.** For an integer \( i \in \{1, \ldots, 6\} \), let \( C_i \) denote the set of positive integers \( x \) such that the first \( i \) instructions of the program \( C \) returns positive integers \( x_1, \ldots, x_i \). If \( x \in \{1, 2, 3\} \), then \( x_6 = 0 \). Therefore, \( C_6 \subseteq \mathbb{N} \setminus \{0, 1, 2, 3\} \). By Lemma 38 for every integer \( x \geq 4 \), \( x_4 = (x - 1)! - 1 \), \( x_5 = \Gamma((x - 1)! - 1) \), and \( x_1, \ldots, x_5 \in \mathbb{N} \setminus \{0\} \). By Lemma 55 for every integer \( x \geq 4 \),

\[x_6 = \text{rem}(\Gamma((x - 1)! - 1), (x - 1)! - 1)\]

belongs to \( \mathbb{N} \setminus \{0\} \) if and only if \( (x - 1)! - 1 \in \{4\} \cup \mathcal{P} \). The last condition equivalently expresses that \( (x - 1)! - 1 \) is prime as \( (x - 1)! - 1 \geq 5 \) for every integer \( x \geq 4 \). Hence,

\[C_6 = (\mathbb{N} \setminus \{0, 1, 2, 3\}) \cap \{x \in \mathbb{N} \setminus \{0, 1, 2, 3\} : (x - 1)! - 1 \in \mathcal{P}\} = \{x \in \mathbb{N} \setminus \{0\} : (x - 1)! - 1 \in \mathcal{P}\}\]

\[\square\]

It is conjectured that there are infinitely many primes of the form \( n! - 1 \), see [4, p. 443] and [27].

**Theorem 33.** The statement \( \Omega_6 \) implies that there are infinitely many primes of the form \( x! - 1 \).

**Proof.** The number \( (975! - 1)! - 1 \) is prime, see [4, p. 441] and [27]. By Lemma 40 for \( x = 975 \) the program \( C \) returns positive integers \( x_1, \ldots, x_6 \). Since \( x = 975 > 720 = \delta(6) \), the statement \( \Omega_6 \) guarantees that the program \( C \) returns positive integers \( x_1, \ldots, x_6 \) for infinitely many positive integers \( x \). By Lemma 40 the set \( \{x \in \mathbb{N} \setminus \{0\} : (x - 1)! - 1 \in \mathcal{P}\} \) is infinite.

\[\square\]

**Lemma 41.** For every positive integer \( x \), the following program \( D \)

\[
\begin{align*}
x_1 & := x \\
x_2 & := \Gamma(x_1) \\
x_3 & := \text{rem}(x_2, x_1) \\
x_4 & := \Gamma(x_3) \\
x_5 & := \text{fact}^{-1}(x_4) \\
x_6 & := \Gamma(x_5) \\
x_7 & := \text{rem}(x_6, x_5)
\end{align*}
\]

returns positive integers \( x_1, \ldots, x_7 \) if and only if both \( x \) and \( x - 2 \) are prime.
We replace every instruction of the form
\[ \text{x}^{\text{th}} \] for infinitely many positive integers
x. Since x = 1, then x = 0. Hence, \( D_3 \subseteq D_2 \subseteq \mathbb{N} \setminus \{0, 1\} \).
If \( x \in \{2, 3, 4\} \), then \( x = 0 \). Therefore,
\[
D_7 \subseteq (\mathbb{N} \setminus \{0, 1\}) \cap (\mathbb{N} \setminus \{0, 2, 3, 4\}) = \mathbb{N} \setminus \{0, 1, 2, 3, 4\}
\]
By Lemma 35 for every integer \( x \geq 5 \), the term \( x \) (which equals \( \text{rem}(\Gamma(x), x) \)) belongs to \( \mathbb{N} \setminus \{0\} \) if and only if \( x \in \mathcal{P} \setminus \{2, 3\} \). By Lemma 36 for every \( x \in \mathcal{P} \setminus \{2, 3\}, x = x - 1 \in \mathbb{N} \setminus \{0, 1, 2, 3\} \). By Lemma 38 for every \( x \in \mathcal{P} \setminus \{2, 3\} \), the terms \( x_4 \) and \( x_5 \) belong to \( \mathbb{N} \setminus \{0\} \) and \( x_5 = x_3 - 1 = x - 2 \). By Lemma 35 for every \( x \in \mathcal{P} \setminus \{2, 3\} \), the term \( x_6 \) (which equals \( \text{rem}(\Gamma(x_3), x_3) \)) belongs to \( \mathbb{N} \setminus \{0\} \) if and only if \( x_5 = x - 2 \in \{4\} \cup \mathcal{P} \).
From these facts, we obtain that
\[
D_7 = (\mathbb{N} \setminus \{0, 1, 2, 3, 4\}) \cap (\mathcal{P} \setminus \{2, 3\}) \cap \{(6) \cup \{p + 2 : p \in \mathcal{P}\}\} = \{p \in \mathcal{P} : p - 2 \in \mathcal{P}\}
\]

\[ \square \]

**Theorem 34.** The statement \( \Omega_2 \) implies that there are infinitely many twin primes.

**Proof.** Harvey Dubner proved that the numbers \( 459 \cdot 2^{8529} - 1 \) and \( 459 \cdot 2^{8529} + 1 \) are prime, see [37, p. 87]. By Lemma 41 for \( x = 459 \cdot 2^{8529} + 1 \) the program \( \mathcal{D} \) returns positive integers \( x_1, \ldots, x_7 \). Since \( x > 720! = \delta(7) \), the statement \( \Omega_2 \) guarantees that the program \( \mathcal{D} \) returns positive integers \( x_1, \ldots, x_7 \) for infinitely many positive integers \( x \). By Lemma 41, there are infinitely many twin primes.

We can transform every program of length \( n \) into a computer program with \( n \) instructions which for every \( x \in \mathbb{N} \setminus \{0\} \) does the same if \( (x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n \), and never halts if \( (x_1, \ldots, x_n) \notin (\mathbb{N} \setminus \{0\})^n \) or the tuple \( (x_1, \ldots, x_n) \) is undefined. To do so, we perform the following steps:

a) We replace the instruction \( x_1 := x \) by the following instruction:
\[
x_1 := x \& \text{PRINT}(x_1)
\]
b) We replace every instruction of the form \( x_i = \Gamma(x_{i-1}) \) by the following instruction:
\[
x_i := \Gamma(x_{i-1}) \& \text{PRINT}(x_i)
\]
c) We replace every instruction of the form \( x_i := \text{fact}^{-1}(x_{i-1}) \) by the following instruction:
\[
\text{IF} \ \text{fact}^{-1}(x_{i-1}) \in \mathbb{N} \setminus \{0\} \text{ THEN } x_i := \text{fact}^{-1}(x_{i-1}) \& \text{PRINT}(x_i) \text{ ELSE GOTO Instruction 1}
\]
d) We replace every instruction of the form \( x_i := \text{rem}(x_{i-1}, x_{i-2}) \) by the following instruction:
\[
\text{IF} \ \text{rem}(x_{i-1}, x_{i-2}) \in \mathbb{N} \setminus \{0\} \text{ THEN } x_i := \text{rem}(x_{i-1}, x_{i-2}) \& \text{PRINT}(x_i) \text{ ELSE GOTO Instruction 1}
\]

**References**


On sets $X \subseteq \mathbb{N}$ for which we know an algorithm ...


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