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Complexity of some of Pyramid Graphs Created from a Gear Graph

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Abstract
In mathematics, one always aims to obtain new frameworks from specific ones. This also stratified to the regality of graphs, where one can produce numerous new graphs from a specific set of graphs. In this work we define some classes of pyramid graphs created by a gear graph and we derive straightforward formulas of the complexity of these graphs, using linear algebra matrix analysis techniques and employing knowledges of Chebyshev polynomials.

Keywords: Complexity; Chebyshev Polynomials; Gear graph; Pyramid graphs.

Mathematics Subject Classification: 05C05, 05C50.

1. **Introduction**

The graph theory is a theory gathering computer science and mathematics, permitting to solve considerable problems in several fields (telecom, social network, molecules, computer network, genetics,…) by designing them by graphs and facilitate them by idealistic cases such as the spanning trees See [1-10].

A spanning tree of a finite connected graph G is a maximal subset of the edges that contains no cycle, equivalently a minimal subset of the edges that connects all the vertices. The history of enumerating number of spanning trees $\tau(G)$ of a graph G dates back into year 1842 in which the physicist of Kirchhoff [11] gave the matrix tree theorem established on the determinants of a certain matrix gained from the Laplacian matrix L defined by the difference between the degree matrix D and adjacency matrix A of a graph G. That is

$$L_{ij} = \begin{cases} a_i & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j, \\ 0 & \text{otherwise} \end{cases}$$

where a_i denotes the degree of vertex i.

This method allows beneficial results for a graph comprising a small number of vertices, but it will be infeasible for large graph. There are one more methods for calculating $\tau(G)$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k = 0$ denote the eignvalues of the matrix L of a graph G with n vertices. Kelmans and Chelnokov [12] have derived that

$$\tau(G) = \frac{1}{k}\prod_{i=1}^{k-1} \lambda_k .$$

Many works have conceived techniques to derive the number of spanning tree of a graph. Now, we give the following Lemma:

Lemma 1.1[13]

$\tau(G) = \frac{1}{k^2}\det(kI - D^c + A^c)$ where A^c and D^c are the adjacency and degree matrices of G^c, the complement of G, respectively, and I is the $k \times k$ identity matrix.

The characteristic of this formula is to express $\tau(G)$ straightway as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

2. **Chebyshev Polynomial**

In this part we insert some relations regarding Chebyshev polynomials of the first and second types which we use it in our calculations.

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We start from their definitions, see Yuanping, et. al. [14].
Let $A_n(x)$ be $n\times n$ matrix such that

$$A_n(x)=\begin{pmatrix} 2x & -1 & 0 & \cdots & 0 \\ -1 & 2x & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2x \end{pmatrix}.$$

Furthermore, we rendering that the Chebyshev polynomials of the first type are defined by

$$T_n(x)=\cos(n\ \cos^{-1}x) \tag{1}$$

The Chebyshev polynomials of the second type are defined by

$$U_{n-1}(x)=\frac{1}{n}\frac{d}{dx}T_n(x)=\frac{\sin(n\ \cos^{-1}x)}{\sin(\cos^{-1}x)} \tag{2}$$

It is easily confirmed that

$$U_n(x)-2xU_{n-1}(x)+U_{n-2}(x)=0 \tag{3}$$

It can then be shown from this recursion that by expanding $\det A_n(x)$ one obtains

$$U_n(x)=\det(A_n(x)),n\geq 1 \tag{4}$$

Moreover by solving the recursion (3), one gets the straightforward formula

$$U_n(x)=\frac{(x+\sqrt{x^2-1})^{n+1}-(x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}},\ n\geq 1, \tag{5}$$

where the conformity is valid for all complex x (except at $x=\pm 1$, where the function can be taken as the limit).
The definition of $U_n(x)$ easily yields its zeros and it can therefore be confirmed that

$$U_{n-1}(x)=2^{n-1}\prod_{j=1}^{n-1}(x-\cos\frac{j\pi}{n}) \tag{6}$$

One further notes that

$$U_{n-1}(-x)=(-1)^{n-1}U_{n-1}(x) \tag{7}$$

From Eqs. (6) and (7) , we have:

$$U_{n-1}{}^2(x)=4^{n-1}\prod_{j=1}^{n-1}(x^2-\cos^2\frac{j\pi}{n}) \tag{8}$$

Finally, straightforward manipulation of the above formula gets the following formula (9), which is highly beneficial to us latter:

$$U_{n-1}{}^2(\sqrt{\frac{x+2}{4}})=\prod_{j=1}^{n-1}(x-2\cos\frac{2j\pi}{n}) \tag{9}$$

Moreover, one can see that

$$U_{n-1}{}^2(x)=\frac{1-T_{2n}}{2(1-x^2)}=\frac{1-T_n(2x^2-1)}{2(1-x^2)}, \tag{10}$$

$$T_n(x)=\frac{1}{2}x^2[(\sqrt{1-\frac{1}{x^2}}+1)^n+(\sqrt{1-\frac{1}{x^2}})^n]. \tag{11}$$

Now we introduce the following important two Lemmas.

Lemma 2.1 [15] Let $B_n(x)$ be $n\times n$ Circulant matrix such that

$$B_n(x)=\begin{pmatrix} x & 0 & 1 & \cdots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & x \end{pmatrix}.$$

Then for $n\geq 3, x\geq 4,$ one has

$$\det(B_n(x))=\frac{2(x+n-3)}{x-3}[T_n(\frac{x-1}{2})-1].$$

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Lemma 2.2 [16]
If $A \in F^{n\times n}$, $B \in F^{n\times m}$, $C \in F^{m\times n}$ and $D \in F^{m\times m}$. Suppose that A and D are nonsingular matrices, then:

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\det D = \det A \det(D - CA^{-1}B).$$

This Lemma give a type of symmetry for some matrices which simplify our calculations of the complexity of graphs studied in this paper.

3. Main Results

Definition 3.1 The pyramid graph $A_n^{(m)}$ is the graph created from the gear graph G_{m+1} with vertices $\{u_0; u_1, u_2, ..., u_m; w_1, w_2, ..., w_m\}$ and m sets of vertices, say, $\{v_1^1, v_2^1, ..., v_n^1\}, \{v_1^2, v_2^2, ..., v_n^2\}, ..., \{v_1^m, v_2^m, ..., v_n^m\}$, such that for all $i = 1, 2, ..., n$ the vertex v_i^j is adjacent to u_j and u_{j+1}, where $j = 1, 2, ..., m-1$, and v_i^m is adjacent to u_1 and u_m. See Fig. (1) .

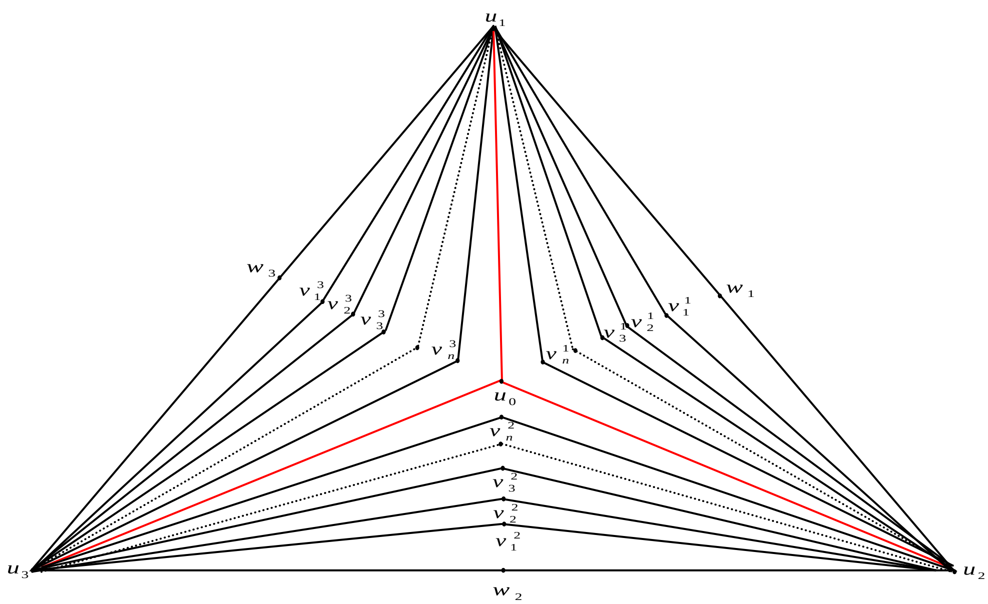

Fig. 1 The pyramid graph $A_n^{(3)}$

Theorem 1 For $n \geq 0$, $m \geq 3$, $\tau(A_n^{(m)}) = 2^{mn}[(n+2+\sqrt{2n+3})^m + (n+2-\sqrt{2n+3})^m - 2(n+1)^m]$.

Proof. Using Lemma 1.1, we have

$$\tau(A_n^{(m)}) = \frac{1}{(mn+2m+1)^2} \times \det((mn+2m+1)I - D^c + A^c) = \frac{1}{(mn+2m+1)^2} \times$$

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$$\det\begin{pmatrix}
(m+1) & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
0 & 2(n+2) & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 1 & \cdots & \cdots & \cdots & 1 & 2(n+2) & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 0 & 1 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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\vdots & 1 & 1 & 0 & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 1 & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
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\vdots & 1 & 1 & \vdots & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 1 & 1 & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
1 & 0 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 3
\end{pmatrix}$$

Let $j=(1\cdots1)$ be the $1\times n$ matrix with all one, and J_n be the $n\times n$ matrix with all one. Set $a=2n+4$ and $b=mn+2m+1$. Then we obtain:

$$\tau(A_n^{(m)})=\frac{1}{b^2}\times\det\begin{pmatrix}
m+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
0 & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}$$

$$
= \frac{1}{b^2} \times \det \begin{pmatrix}
b & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
b & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
b & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
b & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
b & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
bj^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} + J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
bj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b} \times \det \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
1 & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
1j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} + J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b} \times \det \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & a & 1 & \cdots & \cdots & \cdots & 1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 1 & \cdots & \cdots & \cdots & 1 & a & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 0 & 0 & 1 & \cdots & \cdots & 1 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & j^t & \cdots & \cdots & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & 0 & j^t & \cdots & \cdots & j^t & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}
$$

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$$=\frac{1}{b}\times\det\begin{pmatrix}
a & 1 & \cdots & \cdots & \cdots & 1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
1 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & \cdots & \cdots & \cdots & 1 & a & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 0 & 1 & \cdots & \cdots & 1 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & j^t & \cdots & \cdots & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & j^t & \cdots & \cdots & j^t & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}$$

Using Lemma 2.2, yields

$$\tau(A_n^{(m)})=\frac{1}{b}\times\det\begin{pmatrix} A & B \\ C & 2I_{mn}\end{pmatrix}=\frac{1}{b}\times\det(A-B\,\frac{1}{2I_{mn}}\,C)\times 2^{mn}$$

$$=\frac{1}{b}2^{mn}\times 2^{-2m}\times\det\begin{pmatrix}
2a & n+2 & 2(n+1) & \cdots & 2(n+1) & n+2 & -2 & 0 & \cdots & \cdots & 0 & -2 \\
n+2 & 2a & n+2 & 2(n+1) & \cdots & 2(n+1) & -2 & \ddots & \ddots & \ddots & \ddots & 0 \\
2(n+1) & n+2 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2(n+1) & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2(n+1) & \ddots & \ddots & \ddots & \ddots & n+2 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
n+2 & 2(n+1) & \cdots & 2(n+1) & n+2 & 2a & 0 & \cdots & \cdots & 0 & -2 & -2 \\
0 & 0 & 2 & \cdots & \cdots & 2 & 4 & 0 & \cdots & \cdots & \cdots & 0 \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 2 & \cdots & \cdots & 2 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 4
\end{pmatrix}$$

Using Lemma 2.2 again, yields

$$\tau(A_n^{(m)})=\frac{2^{mn-2m}}{b}\times\det\begin{pmatrix} D & E \\ F & 4I_m\end{pmatrix}=\frac{2^{mn}}{b}\times\det(D-E\,\frac{1}{4I_m}\,F)$$

$$\tau(A_n^{(m)})=\frac{2^{mn}}{b}\times\det\begin{pmatrix}
2a & (n+3) & 2(n+2) & \cdots & 2(n+2) & (n+3) \\
(n+3) & 2a & (n+3) & \ddots & \cdots & 2(n+2) \\
2(n+2) & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2(n+2) \\
2(n+2) & \ddots & \ddots & \ddots & \ddots & (n+3) \\
(n+3) & 2(n+2) & \cdots & 2(n+2) & (n+3) & 2a
\end{pmatrix}$$

Straightforward inducement using properties of determinants, one can obtain

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$$\tau(A_n^{(m)}) = \frac{2^{mn}}{b} \times \frac{2b}{m\,n + m + 2} \times \det \begin{pmatrix} (2a-n-3) & 0 & (n+1) & \cdots & (n+1) & 0 \\ 0 & (2a-n-3) & 0 & \ddots & \cdots & (n+1) \\ (n+1) & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (n+1) \\ (n+1) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (n+1) & \cdots & (n+1) & 0 & (2a-n-3) \end{pmatrix}$$

$$= \frac{2^{m\,n+1}(n+1)^m}{m\,n + m + 2} \times \det \begin{pmatrix} \frac{(2a-n-3)}{(n+1)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2a-n-3)}{(n+1)} & 0 & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2a-n-3)}{(n+1)} \end{pmatrix}$$

Using Lemma 2.1, yields

$$\tau(A_n^{(m)}) = 2^{m\,n+1} \times \frac{(n+1)^m}{mn + m + 2} \times \frac{2\left(\frac{2a-n-3}{n+1} + m - 3\right)}{\frac{2a-n-3}{n+1} - 3} \times \left[T_m\left(\frac{\frac{2a-n-3}{n+1} - 1}{2}\right) - 1\right]$$

$$= 2^{mn+1} \times (n+1)^m \times \left[T_m\left(\frac{n+2}{n+1}\right) - 1\right].$$

Using Equation (11), yields the result. □

Definition 3.2 The pyramid graph $B_n^{(m)}$ is the graph created from the gear graph G_{m+1} with vertices $\{u_0; u_1, u_2, \ldots, u_m; w_1, w_2, \ldots, w_m\}$ with double internal edges and m sets of vertices, say, $\{v_1^1, v_2^1, \ldots, v_n^1\}$, $\{v_1^2, v_2^2, \ldots, v_n^2\}, \ldots, \{v_1^m, v_2^m, \ldots, v_n^m\}$, such that for all $i = 1, 2, \ldots, n$ the vertex v_i^j is adjacent to u_j and u_{j+1}, where $j = 1, 2, \ldots, m-1$, and v_i^m is adjacent to u_1 and u_m. See Fig. 2 .

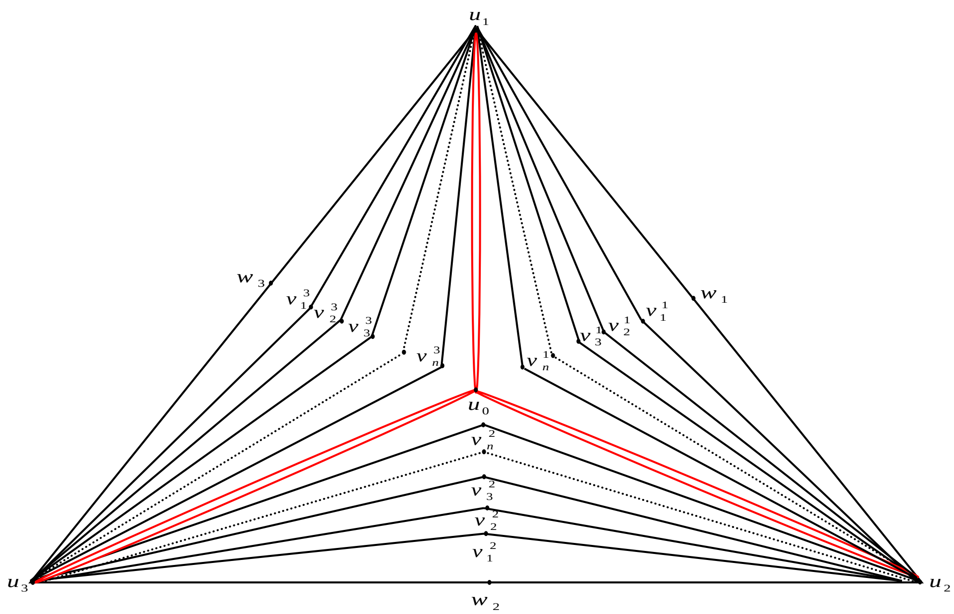

Fig. 2 The pyramid graph $B_n^{(3)}$

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Theorem 2 For $n \ge 0$, $m \ge 3$, $\tau(B_n^{(m)}) = 2^{mn}[(n+3+2\sqrt{n+2}\ )^m + (n+3-2\sqrt{n+2}\ )^m - 2(n+1)^m\,]$.

Proof. Using Lemma 1.1, we get:

$$\tau(B_n^{(m)}) = \frac{1}{(mn+2m+1)^2} \times \det((mn+2m+1)I - D^c + A^c) = \frac{1}{(mn+2m+1)^2} \times$$

$$\det\begin{pmatrix}
(2m+1) & -1 & -1 & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
-1 & (2n+5) & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
-1 & 1 & \cdots & \cdots & \cdots & 1 & (2n+5) & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 0 & 1 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 1 & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & \vdots & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 1 & 1 & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
1 & 0 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 3
\end{pmatrix}$$

Let $j = (1 \cdots 1)$ be the $1 \times n$ matrix with all one, and J_n be the $n \times n$ matrix with all one. Set $a = 2n+5$ and $b = mn+2m+1$. Then we get:

$$\tau(B_n^{(m)}) = \frac{1}{b^2} \times \det\begin{pmatrix}
2m+1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
-1 & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
-1 & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}$$

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$$
=\frac{1}{b^2}\times\det\left(\begin{array}{ccccccccccccccccccc}
b & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
b & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
b & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
b & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
b & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
bj^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
bj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{array}\right)
$$

$$
=\frac{1}{b}\times\det\left(\begin{array}{ccccccccccccccccccc}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & a & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
1 & 1 & \cdots & \cdots & \cdots & 1 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
1j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{array}\right)
$$

$$
=\frac{1}{b}\times\det\left(\begin{array}{ccccccccccccccccccc}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & (a+1) & 2 & \cdots & \cdots & \cdots & 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 2 & \cdots & \cdots & \cdots & 2 & (a+1) & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 1 & 1 & 2 & \cdots & \cdots & 2 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 2 & 1 & \ddots & \ddots & \ddots & 2 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & 2 & \cdots & \cdots & 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
\vdots & 2j^t & j^t & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & 2j^t & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{array}\right)
$$

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$$
=\frac{1}{b}\times\det\begin{pmatrix}
(a+1) & 2 & \cdots & \cdots & \cdots & 2 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & \cdots & \cdots & \cdots & 2 & (a+1) & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
1 & 1 & 2 & \cdots & \cdots & 2 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
2 & 1 & \ddots & \ddots & \ddots & 2 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 2 & \cdots & \cdots & 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
2j^t & j^t & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
2j^t & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}
$$

Using Lemma 2.2, yields

$$
\tau(B_n^{(m)})=\frac{1}{b}\times\det\begin{pmatrix} A & B \\ C & 2I_{mn}\end{pmatrix}=\frac{1}{b}\times\det(A-B\,\frac{1}{2I_{mn}}\,C)\times 2^{mn}
$$

$$
=\frac{1}{b}2^{mn}\times 2^{-2m}\times\det\begin{pmatrix}
(2a+2n+2) & 3n+4 & 4(n+1) & \cdots & 4(n+1) & 3n+4 & -2 & 0 & \cdots & \cdots & 0 & -2 \\
3n+4 & (2a+2n+2) & 3n+4 & 4(n+1) & \cdots & 4(n+1) & -2 & \ddots & \ddots & \ddots & \ddots & 0 \\
4(n+1) & 3n+4 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(n+1) & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
4(n+1) & \ddots & \ddots & \ddots & \ddots & 3n+4 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
3n+4 & 4(n+1) & \cdots & 4(n+1) & 3n+4 & (2a+2n+2) & 0 & \cdots & \cdots & 0 & -2 & -2 \\
2 & 2 & 4 & \cdots & \cdots & 4 & 4 & 0 & \cdots & \cdots & \cdots & 0 \\
4 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
4 & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & 4 & \cdots & \cdots & 4 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 4
\end{pmatrix}
$$

Using Lemma 2.2 again, yields

$$
\tau(B_n^{(m)})=\frac{2^{mn-2m}}{b}\times\det\begin{pmatrix} D & E \\ F & 4I_m\end{pmatrix}=\frac{2^{mn}}{b}\times\det(D-E\,\frac{1}{4I_m}\,F)
$$

$$
\tau(B_n^{(m)})=\frac{2^{mn}}{b}\times\det\begin{pmatrix}
(2a+2n+4) & (3n+7) & 4(n+2) & \cdots & 4(n+2) & (3n+7) \\
(3n+7) & (2a+2n+4) & (3n+7) & \ddots & \cdots & 4(n+2) \\
4(n+2) & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(n+2) \\
4(n+2) & \ddots & \ddots & \ddots & \ddots & (3n+7) \\
(3n+7) & 4(n+2) & \cdots & 4(n+2) & (3n+7) & (2a+2n+4)
\end{pmatrix}
$$

Straightforward inducement using properties of determinants, we obtain

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$$\tau(B_n^{(m)}) = \frac{2^{mn}}{b} \times \frac{4b}{m\,n+m+4} \times \det \begin{pmatrix} (2a-n-3) & 0 & (n+1) & \cdots & (n+1) & 0 \\ 0 & (2a-n-3) & 0 & \ddots & \cdots & (n+1) \\ (n+1) & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (n+1) \\ (n+1) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (n+1) & \cdots & (n+1) & 0 & (2a-n-3) \end{pmatrix}$$

$$= \frac{2^{m\,n+2} \times (n+1)^m}{m\,n+m+4} \times \det \begin{pmatrix} \frac{(2a-n-3)}{(n+1)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2a-n-3)}{(n+1)} & 0 & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2a-n-3)}{(n+1)} \end{pmatrix}$$

Using Lemma 2.1, yields

$$\tau(B_n^{(m)}) = 2^{m\,n+2} \times \frac{(n+1)^m}{mn+m+4} \times \frac{2(\frac{2a-n-3}{n+1}+m-3)}{\frac{2a-n-3}{n+1}-3} \times [T_m(\frac{\frac{2a-n-3}{n+1}-1}{2})-1]$$

$$= 2^{mn+1} \times (n+1)^m \times [T_m(\frac{n+3}{n+1})-1].$$

Using Equation (11), yields the result. □

Definition 3.3 The pyramid graph $C_n^{(m)}$ is the graph created from the gear graph G_{m+1} with vertices $\{u_0; u_1, u_2, \ldots, u_m; w_1, w_2, \ldots, w_m\}$ with double external edges and m sets of vertices, say, $\{v_1^1, v_2^1, \ldots, v_n^1\}$, $\{v_1^2, u_2^2, \ldots, v_n^2\}, \ldots, \{v_1^m, v_2^m, \ldots, v_n^m\}$, such that for all $i = 1, 2, \ldots, n$ the vertex v_i^j is adjacent to u_j and u_{j+1}, where $j = 1, 2, \ldots, m-1$, and v_i^m is adjacent to u_1 and u_m. See Fig. 3.

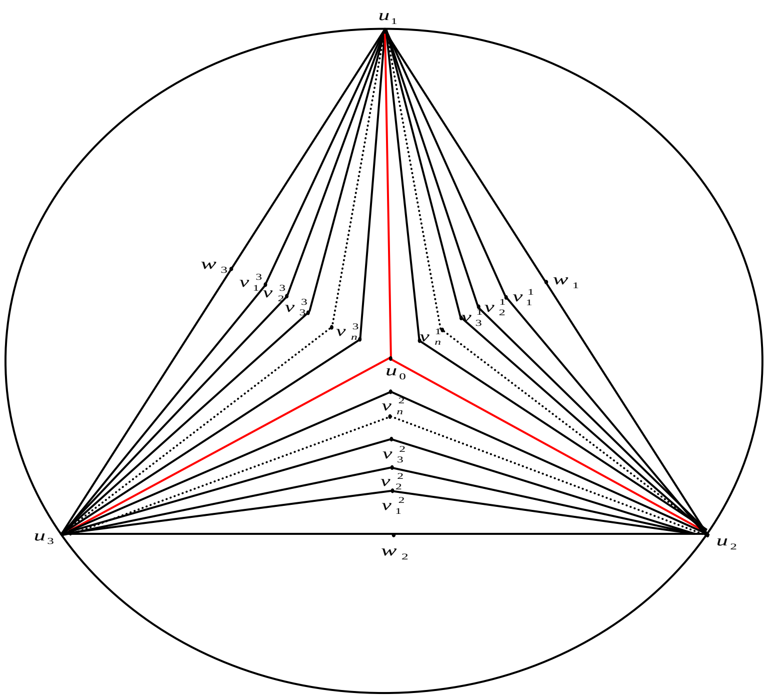

Fig. 3. The pyramid graph $C_n^{(3)}$

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Theorem 3 For $n \geq 0$, $m \geq 3$, $\tau(C_n^{(m)}) = 2^{mn}[(n+4+\sqrt{2n+7}\ )^m + (n+4-\sqrt{2n+7}\ )^m - 2(n+3)^m]$.

Proof: Using Lemma 1.1, we have:

$$\tau(C_n^{(m)}) = \frac{1}{(mn+2m+1)^2} \times \det((mn+2m+1)I - D^c + A^c) = \frac{1}{(mn+2m+1)^2} \times$$

$$\det\begin{pmatrix}
(m+1) & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
0 & 2(n+3) & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & 1 & \cdots & 1 & 0 & 2(n+3) & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 0 & 1 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 1 & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & \vdots & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 1 & 1 & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
1 & 0 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 3
\end{pmatrix}$$

Let $j = (1 \cdots 1)$ be the $1 \times n$ matrix with all one, and J_n be the $n \times n$ matrix with all one. Set $a = 2n+6$ and $b = mn + 2m + 1$. Then we have:

$$\tau(C_n^{(m)}) = \frac{1}{b^2} \times \det\begin{pmatrix}
m+1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
0 & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} + J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}$$

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$$
=\frac{1}{b^2}\times\det\begin{pmatrix}
b & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
b & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
b & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
b & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
b & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
bj^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
bj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
=\frac{1}{b}\times\det\begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
1 & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
1j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
=\frac{1}{b}\times\det\begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & a & 0 & 1 & \cdots & 1 & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 1 & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 1 & \cdots & 1 & 0 & a & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 0 & 0 & 1 & \cdots & \cdots & 1 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & j^t & \cdots & \cdots & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & 0 & j^t & \cdots & \cdots & j^t & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}
$$

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$$
=\frac{1}{b}\times\det\begin{pmatrix}
a & 0 & 1 & \cdots & 1 & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
0 & \ddots & \ddots & \ddots & \ddots & 1 & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 1 & \cdots & 1 & 0 & a & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 0 & 1 & \cdots & \cdots & 1 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
1 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 0 & j^{t} & \cdots & \cdots & j^{t} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
j^{t} & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & j^{t} & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^{t} & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & j^{t} & \cdots & \cdots & j^{t} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}
$$

Using Lemma 2.2, yields

$$
\tau(C_n^{(m)})=\frac{1}{b}\times\det\begin{pmatrix} A & B \\ C & 2I_{mn}\end{pmatrix}=\frac{1}{b}\times\det(A-B\,\frac{1}{2I_{mn}}\,C)\times 2^{mn}
$$

$$
=\frac{1}{b}\times 2^{mn}\times 2^{-2m}\times\det\begin{pmatrix}
2a & n & 2(n+1) & \cdots & 2(n+1) & n & -2 & 0 & \cdots & \cdots & 0 & -2 \\
n & 2a & n+2 & 2(n+1) & \cdots & 2(n+1) & -2 & \ddots & \ddots & \ddots & \ddots & 0 \\
2(n+1) & n & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2(n+1) & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2(n+1) & \ddots & \ddots & \ddots & \ddots & n & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
n & 2(n+1) & \cdots & 2(n+1) & n & 2a & 0 & \cdots & \cdots & 0 & -2 & -2 \\
0 & 0 & 2 & \cdots & \cdots & 2 & 4 & 0 & \cdots & \cdots & \cdots & 0 \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 2 & \cdots & \cdots & 2 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 4
\end{pmatrix}
$$

Using Lemma 2.2 again, yields

$$
\tau(C_n^{(m)})=\frac{2^{mn-2m}}{b}\times\det\begin{pmatrix} D & E \\ F & 4I_m\end{pmatrix}=\frac{2^{mn}}{b}\times\det(D-E\,\frac{1}{4I_m}\,F)
$$

$$
\tau(C_n^{(m)})=\frac{2^{mn}}{b}\times\det\begin{pmatrix}
2a & (n+1) & 2(n+2) & \cdots & 2(n+2) & (n+1) \\
(n+1) & 2a & (n+3) & \ddots & \cdots & 2(n+2) \\
2(n+2) & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2(n+2) \\
2(n+2) & \ddots & \ddots & \ddots & \ddots & (n+1) \\
(n+1) & 2(n+2) & \cdots & 2(n+2) & (n+1) & 2a
\end{pmatrix}
$$

Using properties of determinants, we have:

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$$\tau(C_n^{(m)}) = \frac{2^{mn}}{b} \times \frac{2b}{m\,n+3m+2} \times \det \begin{pmatrix} (2a-n-1) & 0 & (n+3) & \cdots & (n+3) & 0 \\ 0 & (2a-n-1) & 0 & \ddots & \cdots & (n+3) \\ (n+3) & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (n+3) \\ (n+3) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (n+3) & \cdots & (n+3) & 0 & (2a-n-1) \end{pmatrix}$$

$$= \frac{2^{m\,n+1}(n+3)^m}{m\,n+3m+2} \times \det \begin{pmatrix} \frac{(2a-n-1)}{(n+3)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2a-n-1)}{(n+3)} & 0 & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2a-n-1)}{(n+3)} \end{pmatrix}$$

Using Lemma 2.1, yields:

$$\tau(C_n^{(m)}) = 2^{m\,n+1} \times \frac{(n+3)^m}{mn+3m+2} \times \frac{2(\frac{2a-n-1}{n+3}+m-3)}{\frac{2a-n-1}{n+3}-3} \times [T_m(\frac{\frac{2a-n-1}{n+3}-1}{2})-1]$$

$$= 2^{mn+1} \times (n+3)^m \times [T_m(\frac{n+4}{n+3})-1].$$

Using Equation (11), yields the result. □

Definition 3.4 The pyramid graph $D_n^{(m)}$ is the graph created from the gear graph G_{m+1} with vertices $\{u_0; u_1, u_2, \ldots, u_m; w_1, w_2, \ldots, w_m\}$ with double internal and external edges and m sets of vertices, say, $\{v_1^1, v_2^1, \ldots, v_n^1\}$, $\{v_1^2, v_2^2, \ldots, v_n^2\}, \ldots, \{v_1^m, v_2^m, \ldots, v_n^m\}$, such that for all $i = 1, 2, \ldots, n$ the vertex v_i^j is adjacent to u_j and u_{j+1}, where $j = 1, 2, \ldots, m-1$, and v_i^m is adjacent to u_1 and u_m. See Fig.(4).

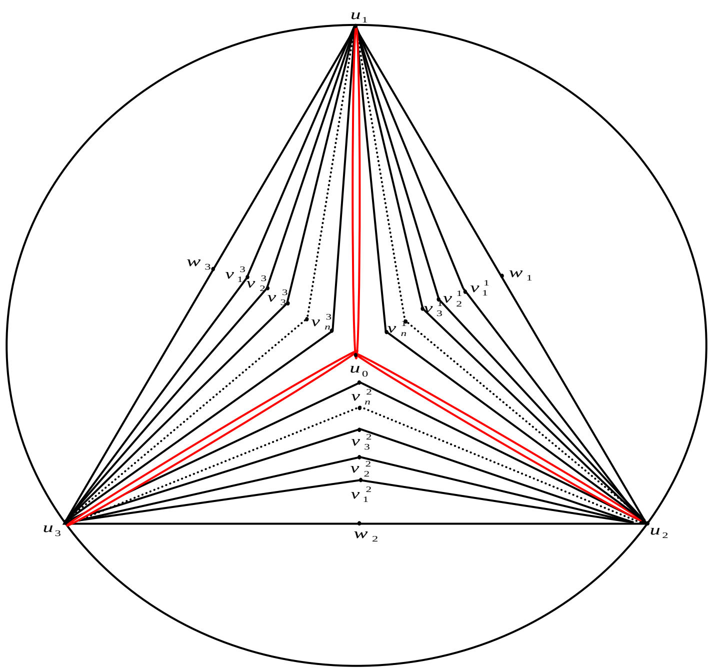

Fig. 4. The pyramid graph $D_n^{(3)}$

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Theorem 4 For $n \geq 0,\ m \geq 3,\ \tau(D_n^{(m)}) = 2^{mn}[(n+5+2\sqrt{n+4}\ )^m + (n+5-2\sqrt{n+4}\ )^m - 2(n+3)^m]$.

Proof: Applying Lemma 1.1, we have:

$$\tau(D_n^{(m)}) = \frac{1}{(mn+2m+1)^2} \times \det((mn+2m+1)I - D^c + A^c) = \frac{1}{(mn+2m+1)^2} \times$$

$$\det\begin{pmatrix}
(2m+1) & -1 & -1 & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
-1 & (2n+7) & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
-1 & 0 & 1 & \cdots & 1 & 0 & (2n+7) & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
1 & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\vdots & 0 & 0 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 3 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 0 & 1 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & 0 & \vdots & 1 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 0 & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & 1 & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 1 & \vdots & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 1 & 1 & \vdots & 0 & \vdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
1 & 0 & 1 & 1 & \cdots & 0 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 3
\end{pmatrix}$$

Let $j = (1 \cdots 1)$ be the $1 \times n$ matrix with all one, and J_n the $n \times n$ matrix with all one. Set $a = 2n+7$ and $b = mn + 2m + 1$. Then we have:

$$\tau(D_n^{(m)}) = \frac{1}{b^2}\det\begin{pmatrix}
2m+1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
-1 & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
-1 & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} + J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}$$

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$$
= \frac{1}{b^2} \det \begin{pmatrix}
b & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
b & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
b & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
b & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
b & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
bj^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
bj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b} \det \begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & a & 0 & 1 & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
1 & 0 & 1 & \cdots & 1 & 0 & a & 1 & \cdots & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 & 0 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\ddots\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & \cdots & 1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 3 & j & \cdots & \cdots & \cdots & \cdots & j \\
1j^t & 0 & 0 & j^t & \cdots & \cdots & j^t & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & & \\
\vdots & j^t & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn}+J_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & j^t & \cdots & \cdots & \cdots & \cdots & j^t & & & & & &
\end{pmatrix}
$$

$$
= \frac{1}{b} \det \begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & (a+1) & 1 & 2 & \cdots & 2 & 1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 2 & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 1 & 2 & \cdots & 2 & 1 & (a+1) & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & 1 & 1 & 2 & \cdots & \cdots & 2 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 2 & 1 & \ddots & \ddots & \ddots & 2 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & 2 & \cdots & \cdots & 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
\vdots & 2j^t & j^t & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & 2j^t & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
0 & j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}
$$

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$$
=\frac{1}{b}\det\begin{pmatrix}
(a+1) & 1 & 2 & \cdots & 2 & 1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\
1 & \ddots & \ddots & \ddots & \ddots & 2 & -1 & -1 & \ddots & \ddots & \ddots & 0 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & -1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & 2 & \cdots & 2 & 1 & (a+1) & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -j & -j \\
1 & 1 & 2 & \cdots & \cdots & 2 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
2 & 1 & \ddots & \ddots & \ddots & 2 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 2 & \cdots & \cdots & 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & & \\
2j^t & j^t & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{mn} & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
2j^t & \ddots & \ddots & \ddots & \ddots & j^t & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & 0 & \cdots & \cdots & \cdots & \cdots & 0 & & & & & &
\end{pmatrix}
$$

Using Lemma 2.2, yields

$$
\tau(D_n^{(m)})=\frac{1}{b}\times\det\begin{pmatrix} A & B \\ C & 2I_{mn}\end{pmatrix}=\frac{1}{b}\times\det(A-B\,\frac{1}{2I_{mn}}\,C)\times 2^{mn}
$$

$$
=\frac{1}{b}2^{mn}\times 2^{-2m}\times\det\begin{pmatrix}
(2a+2n+2) & 3n+2 & 4(n+1) & \cdots & 4(n+1) & 3n+2 & -2 & 0 & \cdots & \cdots & 0 & -2 \\
3n+2 & (2a+2n+2) & 3n+2 & 4(n+1) & \cdots & 4(n+1) & -2 & \ddots & \ddots & \ddots & \ddots & 0 \\
4(n+1) & 3n+4 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(n+1) & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
4(n+1) & \ddots & \ddots & \ddots & \ddots & 3n+2 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
3n+2 & 4(n+1) & \cdots & 4(n+1) & 3n+2 & (2a+2n+2) & 0 & \cdots & \cdots & 0 & -2 & -2 \\
2 & 2 & 4 & \cdots & \cdots & 4 & 4 & 0 & \cdots & \cdots & \cdots & 0 \\
4 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
4 & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & 4 & \cdots & \cdots & 4 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 4
\end{pmatrix}
$$

Using Lemma 2.2, yields

$$
\tau(D_n^{(m)})=\frac{2^{mn-2m}}{b}\times\det\begin{pmatrix} A & B \\ C & 4I_m\end{pmatrix}=\frac{2^{mn}}{b}\times\det(A-B\,\frac{1}{4I_m}\,C)
$$

$$
\tau(D_n^{(m)})=\frac{2^{mn}}{b}\times\det\begin{pmatrix}
(2a+2n+4) & (3n+5) & 4(n+2) & \cdots & 4(n+2) & (3n+5) \\
(3n+5) & (2a+2n+4) & (3n+5) & \ddots & \cdots & 4(n+2) \\
4(n+2) & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(n+2) \\
4(n+2) & \ddots & \ddots & \ddots & \ddots & (3n+5) \\
(3n+5) & 4(n+2) & \cdots & 4(n+2) & (3n+5) & (2a+2n+4)
\end{pmatrix}
$$

Straightforward inducement using properties of determinants, we get:

$$
\tau(D_n^{(m)})=\frac{2^{mn}}{b}\times\frac{4b}{mn+3m+4}\times\det\begin{pmatrix}
(2a-n-1) & 0 & (n+3) & \cdots & (n+3) & 0 \\
0 & (2a-n-1) & 0 & \ddots & \cdots & (n+3) \\
(n+3) & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & (n+3) \\
(n+3) & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & (n+3) & \cdots & (n+3) & 0 & (2a-n-1)
\end{pmatrix}
$$

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$$= \frac{2^{mn+2}(n+3)^m}{mn+3m+4} \times \det \begin{pmatrix} \frac{(2a-n-1)}{(n+3)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2a-n-1)}{(n+3)} & 0 & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2a-n-1)}{(n+3)} \end{pmatrix}$$

Using Lemma 2.1, yields:

$$\tau(D_n^{(m)}) = 2^{mn+2} \times \frac{(n+3)^m}{mn+3m+4} \times \frac{2(\frac{2a-n-1}{n+3}+m-3)}{\frac{2a-n-1}{n+3}-3} \times [T_m(\frac{\frac{2a-n-1}{n+3}-1}{2})-1] = 2^{mn+1} \times (n+3)^m \times [T_m(\frac{n+5}{n+3})-1].$$

Using Equation (11), yields the result. □

4. Numerical Results

The following table illustrate some values of number of spanning trees of studied pyramid graphs**.**

Table 4.1. Some values of the number of spanning trees of studied pyramid graphs.

m	n	$\tau(P_n^{(m)})$	$\tau(A_n^{(m)})$	$\tau(B_n^{(m)})$	$\tau(C_n^{(m)})$
3	0	50	196	242	676
3	1	1024	3200	3136	8192
3	2	15488	43264	36992	92416
3	3	200704	524288	409600	991232
3	4	2367488	5914624	4333568	10240000
3	5	26214400	63438848	44302336	102760448

m	n	$\tau(P_n^{(m)})$	$\tau(A_n^{(m)})$	$\tau(B_n^{(m)})$	$\tau(C_n^{(m)})$
4	0	192	1152	1792	6400
4	1	11520	49152	57600	184320
4	2	458752	1638400	1622016	4816896
4	3	14745600	47185920	41746432	117440512
4	4	415236096	1233125376	1006632960	2717908992
4	5	10687086592	30064771072	23102226432	60397977600

m	n	$\tau(P_n^{(m)})$	$\tau(A_n^{(m)})$	$\tau(B_n^{(m)})$	$\tau(C_n^{(m)})$
5	0	722	6724	12482	58564
5	1	123904	739328	984064	3964928
5	2	12781568	59969536	65619968	237899776
5	3	1007681536	4060086272	3901751296	13088325632
5	4	671948847232	243609370624	213408284672	674448277504
5	5	3995393327104	243609370624	10953240346624	33019708571648

5. Concluding remarks

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The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The computation of this number is not only pleasant from a mathematical (computational) standpoint, but also, it is an important measure of reliability of a network and electrical circuits layout. Some computationally laborious problems such as the travelling salesman problem can be resolved approximately by using spanning trees. Due to the high reliance of the network design and reliability of the network we gave the above important Theorems and their proofs.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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