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Nonlocal symmetries for time-dependent order differential equations

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Abstract

A new type of ordinary differential equation is introduced and discussed, namely, the time-dependent order ordinary differential equations. These equations can be solved via fractional calculus and are mapped into Volterra integral equations of second kind with singular integrable kernel. The solutions of the time-dependent order differential equations smoothly deforms solutions of the classical integer order ordinary differential equations into one-another, and can generate or remove singularities. An interesting symmetry of the solution in relation to the Riemann zeta function and Harmonic numbers was also proved.

1 Introduction

It is rather the exception than the rule when large space-time scale complex systems with their self-organization properties and competition-cooperation cycles, can be modeled with traditional, even nonlinear or stochastic, partial differential equations. The differential approach in modeling, very successful otherwise over a range of hundreds of years of science, is tributary to two strict features: dependence on given initial conditions, and evolution in a constant dimensional phase space. In the real world however, the range of interaction between neighbor sub-systems, and the amount of memory relevant for different phases of evolution, are in continuous change [1, 2]. In any brute force numerical simulation the ranges of interaction and the memory length, or the history dependence, are controlled by the number of neighbors or time steps considered. In the continuous limit these numbers determine the maximal orders of space and time derivatives in the continuous, differential model. The order of differentiation in a mathematical model determines the geometric structure of the differential equations, and the global structure of the solutions [3]. Thus it is reasonable to assume that the changing of the type of behavior of a complex system, [4, 5], can be related, among other things, to the variation of the order of derivatives in the mathematical model citeourcontrib. One needs to introduce a new type of derivative and corresponding differential equation with time dependent order of differentiation, which can be generically expressed in the form

$$\frac{d^{\alpha(t)}}{dt^{\alpha(t)}}Y(t,x) = L[Y], \tag{1}$$

where $\alpha(t)$ is real function taking integer values at the ends of the domain of definition of t, Y is a function dependent on time t, and other independent (space) coordinates x, and L is a differential operator in the x variables. The punctuated equilibrium in evolution of living systems, [6], or in the evolution of economy in some countries [7], the wide variability of time scales in transient population growth rates, [8, 9], memory dependent diffusion [10], networks with higher-order Markovian processes, [11], self-replicating clusters, [12], are examples of systems whose dynamics changes in time, and cannot be described by traditional differential approaches. For such systems one needs new mathematical approaches by considering time-dependent order of differentiation with respect to space or time. This variable order changes in time from an integer order to another integer order, as limiting cases. In present population dynamics models, for example, authors use piece-wise defined differential equations artificially predicting transition from exponential behavior to singular hyperbolic behavior [13–15]. Another typical example where the order of the leading derivative must change during the evolution of the system is given by the drag force upon an accelerated submerged object. The dynamics changes from inertia-less creep flow with force proportional to the first order time derivative (velocity), to Rayleigh drag with force proportional to the second order derivative (acceleration).

In the time-dependent order of differentiation the system can change its dynamics during its evolution. This approach has a great advantage over the traditional modeling approach of using artificial time-dependent coefficients management. The use of time variable coefficients has implications of existence and uniqueness of solutions. Moreover, critical behavior of complex systems is inherent to the system, and not controlled by, or uniquely dependent on the change of constants of material. Moreover, this time-dependent order of differentiation manifestly changes the dimension of the phase space of the system.

When the order of differentiation changes continuously with time, it takes non-integer values. The correct formalism to handle such non-integer operators is provided by fractional calculus and fractional derivatives[16–19]. In this formalism, introduced since Riemann and Liouville and developed to a great

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extend in the last decades, the order of differentiation is a real constant. In the following, we rely in our calculations on this concept of fractional derivative.

This recent trend of time-variable order differential equations, [20–22], is a possible candidate for modeling complex systems with complex unpredictable behavior. Time-dependent order of differentiation models can provide more realistic models for population growth in variable environments, [9], fractional derivatives model for the general laws of predator-prey biological population dynamics, [14], memory dependent diffusion, [10], stochastic processes and multiplex networks described by higher-order Markovian processes, [11], and boundary area and speed of action in self-replicating clusters [12]. A number of authors reached towards the same target of developing time-dependent or space-dependent orders of differentiation, but starting from the different direction of trying to generalize fractional differential equations, and/or to model the dynamics of systems with variable constants of material [23–29].

In our previous studies, [20–22], and in the present paper we underline the benefits of introduction of time-dependent order of differentiation from the fundamental physical necessity of explaining complex systems. This field of research (also known under the name VODE or DODE as in variable/dynamical order differential equations) is still in its stage of infancy, and a lot of caution should be considered in all hypotheses and conclusions. In the present paper we analyze a simple model in order to understand better the properties of the time-dependent order differential equation, its solutions and their symmetries. Namely, we investigate a one-dimensional linear differential equations, with respect to time, whose order of differentiation changes in time with one unit.

The paper is organized as follows: following the Introduction we present in section 2 we introduce the time-dependent order differential equation and its properties and briefly elaborate on the existence and uniqueness of the solutions. In the third section we analyze Frobenius types of series solution for this new equation, and also obtain interesting symmetries.

2 Time-dependent ordinary differential equation

In this section we introduce a time-dependent one-dimensional differential equation for the function $x(t) : (0,1) \to \mathbf{R}$ in the form

$$\frac{d^{\alpha(t)}x}{dt^{\alpha(t)}} = D^{\alpha(t)}x = f(t,x), \tag{2}$$

where D^{α} is the standard notation for fractional derivatives (to be defined below), the real function $\alpha(t)$ describes the time dependent order of differentiation, and f is the source term. As we mentioned above, we represent the variable order through the formalism of fractional derivatives [16–19, 30–37].

The fractional generalization of differential calculus can be defined in several types of fractional derivatives. For exemple we can enumerate the fractional derivatives in the sense of Riemann-Liouville, Caputo, Grünwald, Jumarie, or Weyl [31, 38]. These derivatives are non-local, and are able to model multiple

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scales systems, fractal differentiability, or nowhere differentiable functions [38]. The fractional derivatives and fractional integrals have applications in viscoelasticity, feedback amplifiers, electrical circuits, electro-analytical chemistry, fractional multipoles, neuron modeling and related areas in physics, chemistry, and biological sciences [18, 32].

In the following, we use the Riemann-Liouville form of fractional derivative in order to introduce the time-dependent derivative

$$ {}_{t_0}D_t^{\alpha(t)}\ x(t) = \frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-\alpha(t))}\int\limits_{t_0}^{t}\frac{x(s)}{(t-s)^{\alpha(t)-m+1}}ds\right), \tag{3}$$

where the order $\alpha(t) : \mathbf{R}_+ \to [m-1, m)$, and $x(t) : (0,1) \to \mathbf{R}$ are functions of class C^m, and m is a positive integer. In the following, we choose $t_0 = 0$, without any loss of generality [33, 34], so we can skip the subscripts from the expression of the fractional derivative. This derivative operator obeys the Leibnitz rule, chain rule and can be used in Taylor series [16, 17]. The fractional derivative converges uniformly towards the integer value at its bounds $\lim_{\alpha(t)->(m-1)^+} D^{\alpha(t)}x(t) = x^{(m-1)}$, and $\lim_{\alpha(t)->m^-} D^{\alpha(t)}x(t) = x^{(m)}$ [22].

We consider two initial value problems for a time-dependent ordinary differential equation. In the first case, we chose $\alpha(t) \in [0, 1]$, that is $m = 1$ in Eq. (3), and the differential equations has the form

$$ D^{\alpha(t)}(x - x_0) = f(t, x(t)), \quad x(0) = x_0, \tag{4}$$

where $t \in (0, 1), x_0 \in \mathbf{R}$, the order of differentiation function is a continuous function $\alpha : (0, 1) \to [0, 1]$, and the source term is a continuous function $f : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$.

In the second case we chose $\alpha(t) \in [1, 2]$, that is $m = 2$ in Eq. (3), and we have the form

$$ D^{\alpha(t)}(y - y_0 - y_1 t) = g(t, y(t)), \quad y(0) = y_0, \quad y'(0) = y_1, \tag{5}$$

where $t \in (0, 1), y_0, y_1 \in \mathbf{R}$, the order of differentiation function is a continuous function $\alpha : (0, 1) \to [1, 2]$, and the source term is a continuous function $g : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$.

The way initial data problem is formulated, even in the time-independent fractional differential equations case, is still in debate, and its physical meaning is not yet fully understood [16, 36]. Therefore, the incorporation of classical derivatives of the initial data in Eq. (4) was suggested by many authors, [10, 34, 36], as they are commonly used in initial value problems with integer-order equations.

It was proved, [20, 21], that the initial value problem in Eqs. (4,5) are reducible to Volterra integral equations of second kind with singular integrable kernel. In the case of Eq. (4) we have $k(t, \tau) = (t - \tau)^{\alpha(t)-1}$, as long as $\alpha(t) \in (0, 1)$

$$ x(t) = x_0 + \frac{1}{\Gamma(\alpha(t))}\int\limits_0^t \frac{f(\tau, x(\tau))d\tau}{(t-\tau)^{1-\alpha(t)}}. \tag{6}$$

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In a similar way, Eq. (5) can be mapped into a similar Volterra integral equation of second kind with singular integrable kernel

$$y(t) = y_0 + ty_1 + \frac{1}{\Gamma(\alpha(t)-1)} \int_{t_0}^{t} \frac{\int_{t_0}^{\tau} g(s, y(s))ds}{(t-\tau)^{2-\alpha(t)}} d\tau. \tag{7}$$

Any solution of Eqs. (6,7) is a solution of the initial value problem Eqs. (4,5), respectively. The solutions for Eqs. (6,7) smoothly approach the classical solutions of the corresponding classical differential equations of integer order in the limiting integer values of $\alpha = 0, 1, 2$. For the cases when the solutions are smooth and regular all over their domain of definition, the conditions of existence and uniqueness of smooth solutions of Eqs. (6,7) are covered in [20, 22].

In Fig. 1 we present an example of smooth solution for Eq. (5) with $\alpha \in (1, 2)$, for a nonhomogeneous term of the form $g(t, y) = -y$ and $y_0 = 1, y_1 = -1$. The solution was obtained numerically for the integral version, Eq. (7), of Eq. (5). In the limiting cases of integer order of differentiation it is easy to verify that the limiting solutions are

$$y(t) = \begin{cases} e^{-t} & \text{for } \alpha = 1, \\ \cos t & \text{for } \alpha = 2, \end{cases}$$

One can notice in Fig. 1 how the solution smoothly maps from negative exponential to periodic trigonometric function, with the change of the order of differentiation in the differential equation. More interesting situations occur in the case when the solutions have singularities. In this case we cannot apply the existence and uniqueness from [20, 22], and a different approach will be developed in the followings. We present such a transition from a smooth to a singular solution in Fig. 2. This is again an example of solution for Eq. (5) with $\alpha \in (1, 2)$, but for a nonhomogeneous term of the form $g(t, y) = -t^{-3/2}$. The solution was obtained numerically for the integral version, Eq. (7), of Eq. (5). In the limiting cases of integer order of differentiation it is easy to verify that the limiting solutions are

$$y(t) = \begin{cases} \frac{2}{\sqrt{t}} + C_0, & \text{for } \alpha = 1, \\ 4\sqrt{t} + C_0 + C_1 t, & \text{for } \alpha = 2. \end{cases}$$

One can notice that the numerical solution of the time-dependent order differential equation smoothly connects these two limiting cases, and makes a smooth transition from a solution with singularity at $t = 0$ for $\alpha = 1$ to a smooth solution on $t \in (0, 1)$ for $\alpha = 2$.

3 Initial value problem and non-local symmetries

In order to analyze the well-posedness of the initial value problem for time-variable order differential equations of type Eqs. (4,5) or equivalently the integral non-local forms Eqs. (4,5), we will expand the hypothetical solution $x(t)$

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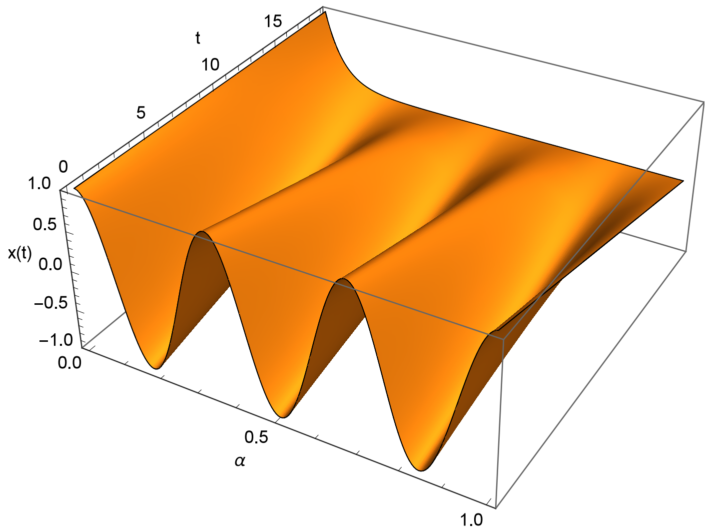

Figure 1: Plot of the solution for the time-dependent order initial problem for the differential equation $D^{\alpha(t)}y = -y$ for $\alpha \in (1,2), t \in (0,1)$. The solution smoothly deforms from exponential decay to trigonometric function, with the increase of α.

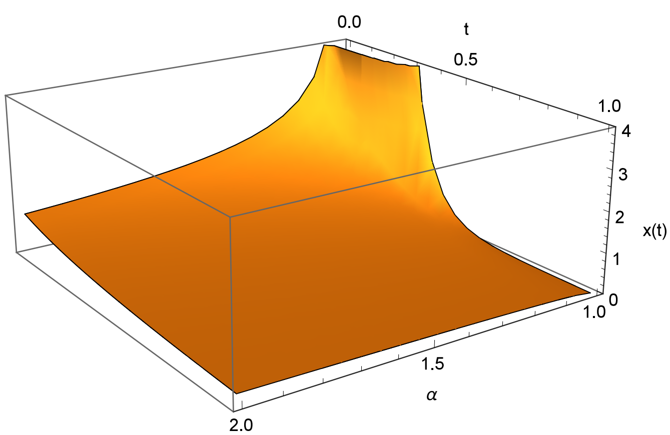

Figure 2: Plot of the solution for the time-dependent order differential equation $D^{\alpha(t)}y = -t^{-3/2}$ for $\alpha \in (1,2), t \in (0,1)$. The solution smoothly deforms from a singular hyperbolic dependence to a smooth power low with the increase of α. The initial value problem cannot be applied here in the traditional sense, because of the singularity at $t = 0$.

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or $y(t)$, respectively, in Frobenius formal series. We consider the transition $\alpha : 0 \to 1$ for $x(t)$ for Eq. (6) and we make the hypothesis that the solution has the form

$$x(t) = x_0 + t^r \sum_{k=0}^{\infty} c_k x^k, \tag{8}$$

for arbitrary scalar r. We plug the series Eq. (8) in Eq.(6) and we obtain

$$x(t) = x_0 + \frac{x_0 t^{\alpha(t)}}{\alpha(t)\Gamma(\alpha(t))}$$
$$+ \frac{t^{r+\alpha(t)}}{\Gamma(r+\alpha(t))} \sum_{k=1}^{\infty} \frac{t^k (k+r) c_k \Gamma(k+r)}{\prod_{j=0}^{k} (r+j+\alpha(t))} \tag{9}$$

In order to evaluate this result we chose a linear form for the time-dependent order, i.e. $\alpha(t) = t$ and calculate the first orders in Eq. (9)

$$x(t) = x_0 + t^r [t c_1 + t^2 (c_2 + c_1 \ln t - c_1 \psi(2+r))] + t^r \mathcal{O}_3, \tag{10}$$

where ψ is the digamma function. By comparison with Eq. (8) it is obvious that even in the simplest linear case for the time-dependence of the order of differentiation, the solution $x(t)$ is not holomorphic and has logarithmic singularity at $x = 0$, therefore one needs to use the second type of Frobenius solutions containing the logarithm $x(t) = C x_a(t) \ln x + x_b(t)$, where $x_{a,b}$ are holomorphic power series to be determined. It results that all solution of the time-dependent order of differentiation equations have irregular singularities at $x = 0$. By introducing the term $c_\infty \ln t$ in Eq. (8)

$$x(t) \to c_\infty \ln t + t^r \sum_{k=0}^{\infty} c_k t^k \tag{11}$$

we generate in the series expansion of the right hand side of Eq. (6) an extra term. That is an extra term added to the terms shown in Eq. (9), in the form

$$\frac{t^{\alpha(t)} (\ln t - H_{\alpha(t)})}{\alpha(t)}, \tag{12}$$

where $H_{\alpha(t)}$ is the fractional harmonic number given by

$$H_{\alpha(t)} = \alpha(t) \sum_{j=0}^{\infty} (-1)^j \alpha(t)^j \zeta(j+2),$$

where ζ is the Riemann zeta function. This term has an interesting harmonic symmetry and connection with other special functions. In the following, we present another interesting symmetry of the solutions of Eqs. (6,7) with respect

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to the variation of the order of differentiation. We prove the following result. For m positive integer and for any continuous functions $\alpha(t):(0,1)\to(m-1,m)$, and $g(t):(0,1)\to\mathbf{R}$, there are always two distinct values $\alpha_1\neq\alpha_2$ in $(m-1,m)$ such that the equality

$$\frac{1}{\Gamma(\alpha_1)}\int_0^t g(\tau)(t-\tau)^{\alpha_1}d\tau=\frac{1}{\Gamma(\alpha_2)}\int_0^t g(\tau)(t-\tau)^{\alpha_2}d\tau, \tag{13}$$

where $g(t)=f(t,x(t))$, is fulfilled for some $t\in(0,1)$. To prove this relation we re-write Eq. (13) in the form

$$\frac{1}{\Gamma(\alpha_2)}\int_0^t \frac{f(\tau,x(\tau))}{(t-\tau)^{1-\alpha_2}}\left(\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)(t-\tau)^{\alpha_2-\alpha_1}}-1\right)d\tau=0.$$

Without any loss of generality we assume $\alpha_2>\alpha_1$. The bottom limit $\tau\to 0^+$ of the integrand of the above integral is

$$\frac{f(0,x_0)}{t^{1-\alpha_2}}\left(\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)t^{\alpha_2-\alpha_1}}-1\right),$$

and can be always tuned to be an arbitrary large negative number. Indeed, by restricting $t\in(0,1)$ we have $t^{\alpha_2-\alpha_1}\in(0,1)$, and for any given α_2 we can find α_1 such that $\Gamma(\alpha_2)<\Gamma(\alpha_1)t^{\alpha_2-\alpha 1}$ since Γ is local monotonic, and we can chose $\alpha_{1,2}$ in the region where Γ is strictly decreasing. At the same time we have

$$\lim_{\tau\to t}\frac{f(\tau,x(\tau))}{(t-\tau)^{1-\alpha_2}}\left(\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)(t-\tau)^{\alpha_2-\alpha_1}}-1\right)=+\infty.$$

We obtained that the integrand oscillates between an arbitrary large negative value and plus infinity, while is guaranteed for the integral to be convergent by the theorem of existence of the solution [20]. It results we can always find a value for α_1 such that this integral is zero, and the affirmation in Eq. (13) is proved.

4 Conclusions

In this paper we introduce and discuss properties of time-dependent order ordinary differential equations. We show that such new types of differential equations can be represented in terms of generalizations of fractional derivatives with time-dependent order of differentiation. This approach allows us to map the time-dependent ordinary differential equation to a Volterra integral equation of second kind with singular integrable kernel, which is known to have unique solution for appropriately chosen initial conditions and smoothness of the parameters and solution. We demonstrate that the general solution of this time-dependent order differential equations can smoothly deform the corresponding limiting solutions of the classical integer order ordinary differential equations, one into another, and can even generate singularities from regular solutions, and conversely. We present an interesting symmetry of the solution, and its relation to the Riemann zeta function and Harmonic numbers.

Data Availability

The analytic and numerical calculations and plots data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

This paper is original, and it was not presented or published as preprint previously anywhere.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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