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SYMMETRY IDENTITIES OF CHANGHEE POLYNOMIALS OF TYPE TWO

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ABSTRACT. In this paper we consider Changhee polynomials of type two, which are motivated from the recent work of Kim and Kim.

We investigate some symmetry identities for the Changhee polynomials of type two which are derived from the properties of symmtery for the fermionic p-adic integral on $\mathbb{Z}_p$.

0. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring fo p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$.

The p-adic norm $|\cdot|_p$ is normalized as $|\,p\,|_p = \frac{1}{p}$.

Let $f(x)$ be a continulus funciton on $\mathbb{Z}_p$. Then the fermionic p-adci integral on $\mathbb{Z}_p$ is defined by Kim as

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x) = \lim_{x\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{(see [9])} \tag{0.1}$$

For $n\in\mathbb{N}$, by (0.1), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x+n)d\mu_{-1}(x) &+ (-1)^{n-1}\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x)\\ &= 2\sum_{\ell=0}^{n-1} f(\ell)(-1)^{n-1-\ell}, \quad \text{(see [1,5,6,16])}\end{aligned} \tag{0.2}$$

In particular, if we take $n=1$, then we have

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2f(0), \quad \text{(see [10,11])} \tag{0.3}$$

We consider the Changhee polynomials $\widetilde{Ch}_n(x)$ of type two by the generating function

$$\sum_{n=0}^{\infty}\widetilde{Ch}_n(x)\frac{t^n}{n!} = \frac{2}{(1+t)+(1+t)^{-1}}(1+t)^x, \quad \text{(see [4])} \tag{0.4}$$

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For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, the Changhee polynomials of type two can be represented by the fermionic p-adic integrals of $\mathbb{Z}_p$:

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+t)^{2y+1+2x} d\mu_{-1}(y) &= \frac{2}{(1+t)^2+1}(1+t)^{2x+1} \\ &= \sum_{n=0}^{\infty} \widetilde{Ch}_n(x)\frac{t^n}{n!}\end{aligned} \tag{0.5}$$

When $x=0$, $\widetilde{Ch}_n = \widetilde{Ch}_n(0)$ are called the Changhee numbers of type two.

Observe

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{Ch}_n(x)\frac{t^n}{n!} &= \frac{2}{(1+t)+(1+t)^{-1}}(1+t)^{2x} \\ &= \sum_{m=0}^{\infty} \widetilde{Ch}_m \frac{t^m}{m!} \sum_{\ell=0}^{\infty} (2x)_\ell \frac{t^n}{\ell!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \widetilde{Ch}_m (2x)_{n-m} \right) \frac{t^n}{n!}\end{aligned}$$

Thus we have

$$\widetilde{Ch}_n(x) = \sum_{m=0}^{n} \binom{n}{m} (2x)_m \widetilde{Ch}_{n-m}, \tag{0.6}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$, $(x)_0 = 1$.

From (0.3), we can easily derive the following:

$$2\sum_{\ell=0}^{n} (-1)^\ell (1+t)^{2\ell} = \frac{2\{1+(-1)^{n+1}(1+t)^{2(n+1)}\}}{(1+t)^2+1} \tag{0.7}$$

The left hand side of (0.7) can be written as

$$2\sum_{\ell=0}^{n} (-1)^\ell (1+t)^{2\ell} = \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n-1} (-1)^\ell (2\ell)_n \right) \frac{t^n}{n!} \tag{0.8}$$

We use the notation of λ-falling factorial in [14, 17] for $\lambda \in \mathbb{R}$,

$$(\ell \,|\, \lambda)_n = \begin{cases} \ell(\ell-\lambda)\cdots(\ell-\lambda(n-1)), & (\text{if } n \geq 1) \\ 1, & (\text{if } n = 0). \end{cases}$$

Then the right hand side of (0.8) can be written as

$$2\sum_{\ell=0}^{n-1} (-1)^\ell (1+t)^{2\ell} = \sum_{n=0}^{\infty} T_m(n; (\ell \,|\, \tfrac{1}{2}))\tfrac{t^n}{n!}. \tag{0.9}$$

where we denote, for $\lambda \in \mathbb{R}$,

$$T_m(n; (\ell \,|\, \lambda)) = \sum_{\ell=0}^{n} (-1)^\ell (\ell \,|\, \lambda)_m.$$

For $n \in \mathbb{N}$, $n \equiv 1 \pmod 2$, $m \geq 0$ we have

$$\sum_{m=0}^{\infty} 2\left(\sum_{\ell=0}^{n} (-1)^\ell (-2\ell)_m \right) \frac{t^m}{m!} = \frac{2(1+(1+t)^{2(n+1)})}{(1+t)^2+1}. \tag{0.10}$$

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On the other hand, by (0.4) and (0.9), we have

$$\begin{aligned}\sum_{m=0}^{\infty}\left(\widetilde{Ch}_m+\widetilde{Ch}_m(n+1)\right)\frac{t^m}{m!} &= \frac{2(1+t)}{(1+t)^2+1}+\frac{2(1+t)^{2(n+1)}(1+t)}{(1+t)^2+1}\\ &= 2\sum_{\ell=0}^{n}(-1)^{\ell}(1+t)^{2\ell+1}\\ &= 2T_m(n;(\ell+\tfrac{1}{2}\mid\tfrac{1}{2})).\end{aligned}\tag{0.11}$$

The Stirling number $S_1(\ell,n)$ of the first kind is defined by the generating function

$$\big(\log(1+t)\big)^n = n!\sum_{\ell=n}^{\infty}S_1(\ell,n),\quad \text{see [1,5,6,16,18]}$$

and the Stirling number $S_2(m,n)$ of the second kind is given by the generating function

$$(e^t-1)^n = n!\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!},\quad \text{(see [6])}.$$

As is well known, the Euler polynomials $E_n(x)$ are defined by the generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \text{(see [3,20,21])}.\tag{0.12}$$

When $x=0$, $E_n=E_n(0)$, $(n\geq 0)$, are called the n-th Euler numbers, whereas the Euler numbers E_n^* of the second kind are given by the generating function

$$\operatorname{sech}(t)=\frac{2}{e^t+e^{-t}}=\sum_{n=0}^{\infty}E_n^*\frac{t^n}{n!},\quad \text{(see [3,19])}.\tag{0.13}$$

From (0.5) and (0.13), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{Ch}_n\frac{t^n}{n!} &= \frac{2}{(1+t)+(1+t)^{-1}}=\frac{2}{e^{\log(1+t)}+e^{-\log(1+t)}}\\ &= \operatorname{sech}(\log(1+t))\\ &= \sum_{n=0}^{\infty}E_n^*\frac{(\log(1+t))^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}E_k^*S_1(n,k)\right)\frac{t^n}{n!}.\end{aligned}\tag{0.14}$$

Thus we have

$$\widetilde{Ch}_n=\sum_{k=0}^{n}E_k^*S_1(n,k).\tag{0.15}$$

Using $E_0^*=1$, $E_1^*=0$, $E_2^*=-1$, $E_3^*=0$, $E_4^*=5$, $E_5^*=0$ and $S_1(n,n)=0$ for $n\geq 0$, $S_1(n,0)=0$ for $n\geq 1$, $S_1(2,1)=1$, $S_1(3,1)=2$, $S_1(4,1)=6$, $S_1(5,1)=24$, $S_1(3,2)=3$, $S_1(4,2)=11$, $S_1(5,2)=50$, $S_1(4,3)=6$, $S_1(5,3)=35$, $S_1(5,4)=10$, we get some values of $\widetilde{Ch}_n$'s as follows:

$$\widetilde{Ch}_0=E_0^*S_1(0,0)=1,$$

$$\widetilde{Ch}_1=E_0^*S_1(1,0)+E_1^*S_1(1,1)=0+0=0,$$

$$\widetilde{Ch}_2=E_0^*S_1(2,0)+E_1^*S_1(2,1)+E_2^*S_1(2,2)=0+0-1=-1.$$

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$$\begin{aligned}
\widetilde{Ch}_3 &= E_0^*S_1(3,0)+E_1^*S_1(3,1)+E_2^*S_1(3,2)+E_3^*S_1(3,3)\\
&= 0+0-3+0=-3,\\
\widetilde{Ch}_4 &= E_0^*S_1(4,0)+E_1^*S_1(4,1)+E_2^*S_1(4,2)+E_3^*S_1(4,3)+E_4^*S_1(4,4)\\
&= 0+0-11+0+5=-6,\\
\widetilde{Ch}_5 &= E_0^*S_1(5,0)+E_1^*S_1(5,1)+E_2^*S_1(5,2)+E_3^*S_1(5,3)+E_4^*S_1(5,4)\\
&\quad + E_5^*S_1(5,5)=0+0-50+0+50+0=0.
\end{aligned}$$

From (0.6) and (0.13), we get the following, by replacing t by e^t-1:

$$\begin{aligned}
\frac{2}{(1+t)^2+1}(1+t) &= \sum_{n=0}^{\infty}\widetilde{Ch}_n\frac{t^n}{n!}\\
\frac{2}{e^{2t}+1}e^t &= \sum_{k=0}^{\infty}\widetilde{Ch}_k\frac{1}{k!}(e^t-1)^k\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\widetilde{Ch}_kS_2(n,k)\right)\frac{t^n}{n!}\\
&= \frac{2}{e^t+e^{-t}} = \sum_{n=0}^{\infty}E_n^*\frac{t^n}{n!}.
\end{aligned} \tag{0.16}$$

Thus, from (0.16) we have

$$E_n^* = \sum_{k=0}^{n}\widetilde{Ch}_kS_2(n,k).$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{Z}_p}(1+t)^{2x+1}d\mu_{-1}(x) &= \int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}(2x+1)_n\frac{t^n}{n!}d\mu_{-1}(x)\\
&= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(2x+1)_nd\mu_{-1}\frac{t^n}{n!},
\end{aligned}$$

So we have the Witt's formula for Changhee numbers of type two as follows:

$$\widetilde{Ch}_n = \int_{\mathbb{Z}_p}(2x+1)_nd\mu_{-1}(x). \tag{0.17}$$

In this paper we consider Changhee polynomials of type two, which are motivated from the recent work of Kim and Kim. See [3, 4].

We investigate some symmetry identities for the Changhee polynomials of type two which are derived from the properties of symmtery for the fermionic p-adic integral on $\mathbb{Z}_p$. Many authors investigated symmetric properties of special polynomials and numbers. See [7, 8, 12, 14] and their references.

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1. Symmetry of Changhee polynomials of type two

Motivated from Kim and Kim [2], for $w \in \mathbb{N}$, we define w-torsion Changhee polynomials of type two by the following generating function

$$\frac{2}{(1+t)^{2w}+1}(1+t)^{2wx+1} = \sum_{n=0}^{\infty} \widetilde{Ch}_{n,w}(x)\frac{t^n}{n!}. \tag{1.1}$$

When $x = 0$, $\widetilde{Ch}_{n,w} = \widetilde{Ch}_{n,w}(0)$ are called the w-Changhee numbers of type two. When $w = 1$, $\widetilde{Ch}_{n,1}(x) = \widetilde{Ch}_n(x)$ are just the Changhee polynomials of type two in (0.6).

From (1.1) and (0.6), we present w-Changhee polynomials of type two by the well known Changhee polynomials of type two:

$$\widetilde{Ch}_{n,w}(x) = \sum_{\ell=0}^{n} \widetilde{Ch}_\ell(2wx). \tag{1.2}$$

On the other hand,

$$\begin{aligned}
\sum_{n=0}^{\infty} \widetilde{Ch}_{n,w}(x)\frac{t^n}{n!} &= \left(\sum_{\ell=0}^{\infty} \widetilde{Ch}_\ell \frac{t^\ell}{\ell!}\right)(1+t)^{2wx} \\
&= \left(\sum_{\ell=0}^{\infty} \widetilde{Ch}_\ell \frac{t^\ell}{\ell!}\right)\left(\sum_{m=0}^{\infty}(2wx)_m \frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n}\binom{n}{\ell}(2wx)_\ell \widetilde{Ch}_{n-\ell}\}\right)\frac{t^n}{n!}.
\end{aligned} \tag{1.3}$$

Thus we present w-Changhee polynomials of type two by Changhee numbers of type two;

$$\widetilde{Ch}_{n,w}(x) = \sum_{\ell=0}^{n}\binom{n}{\ell}(2wx)_\ell \widetilde{Ch}_{n-\ell}.$$

Now we consider a quotient of p-adic fermionic integrals on $\mathbb{Z}_p$,

$$\begin{aligned}
\frac{2\int_{\mathbb{Z}_p}(1+t)^{2w_2x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1+t)^{2w_1w_2x_1}d\mu_{-1}(x_1)} &= \sum_{\ell=0}^{w_1-1}(-1)(1+t)^{2w_2\ell} \\
&= \sum_{m=0}^{\infty}\sum_{\ell=0}^{w_1-1}(-1)^\ell(2w_2\ell)_m \\
&= \sum_{m=0}^{\infty}\sum_{\ell=0}^{w_1-1}(2w_2)^m(-1)^\ell\left(\ell\,|\,\tfrac{1}{2w_2}\right)_m \\
&= \sum_{m=0}^{\infty}(2w_2)^m T_m(w_1-1\,|\,\left(\ell\,|\,\tfrac{1}{2w_2}\right)),
\end{aligned} \tag{1.4}$$

where $T_m(n\,|\,(\ell\,|\,\lambda)) = \sum_{\ell=0}^{n}(-1)^\ell(\ell\,|\,\lambda)_m$ for $\lambda \in \mathbb{R}$.

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Consider the following

$$\begin{aligned}
T(w_1,w_2) &= \frac{2\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(1+t)^{2w_1x_1+2w_2x_2+2}d\mu_{-1}(x_1)\,d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1+t)^{2w_1w_2x_1+1}d\mu_{-1}(x_1)}(1+t)^{2w_1w_2x}\\
&= \int_{\mathbb{Z}_p}(1+t)^{2w_1x_1+1}d\mu_{-1}(x_1)(1+t)^{2w_1w_2x}\\
&\quad\times\frac{\int_{\mathbb{Z}_p}(1+t)^{2w_2x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1+t)^{2w_1w_2x_1}d\mu_{-1}(x_1)} \qquad (1.5)\\
&= \Big(\sum_{\ell=0}^{\infty}\widetilde{Ch}_{\ell,w_1}(w_2x)\frac{t^\ell}{\ell!}\Big)\Big(\sum_{k=0}^{\infty}(2w_2)^kT_k(w_1-1\,|\,(k\,|\,\tfrac{1}{2w_2}))\Big)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\widetilde{Ch}_{n-k,w_1}(w_2x)(2w_k)^kT_k(w_1-1\,|\,(k\,|\,\tfrac{1}{2w_2}))\right)\frac{t^n}{n!}.
\end{aligned}$$

Similarly we have the following identity for $T(w_1,w_2)$ because $T(w_1,w_2)$ is symmetric on w_1 and w_2.

$$T(w_1,w_2) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\widetilde{Ch}_{n-k,w_2}(w_1x)(2w_1)^kT_k(w_2-1\,|\,(k\,|\,\tfrac{1}{2w_1}))\right)\frac{t^n}{n!}. \qquad (1.6)$$

Thus, by (1.5) and (1.6), we have the following theorem.

Theorem 1. *For $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1 \pmod 2$, $w_2\equiv 1 \pmod 2$ and $n\geq 0$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\widetilde{Ch}_{n-k,w_2}(w_1x)(2w_1)^k\,T_k(w_2-1\,|\,(k\,|\,\tfrac{1}{2w_1}))\\
&= \sum_{k=0}^{n}\binom{n}{k}\widetilde{Ch}_{n-k,w_1}(w_2x)(2w_2)^k\,T_k(w_1-1\,|\,(k\,|\,\tfrac{1}{2w_2})).
\end{aligned}$$

Remark: If we take $w_2=1$ in Theorem 1, we have the following

Corollary 2. *For $w_1\in\mathbb{N}$ with $w_1\equiv 1 \pmod 2$ and $n\geq 0$, we have*

$$\widetilde{Ch}_n(w_1x) = \sum_{k=0}^{n}\binom{n}{k}\widetilde{Ch}_{n-k,w_1}(x)2^kT_k(w_1-1\,|\,(k\,|\,\tfrac{1}{2})).$$

From (1.5), we rewrite $T(w_1,w_2)$ as follows:

$$\begin{aligned}
T(w_1,w_2) &= \int_{\mathbb{Z}_p}(1+t)^{2w_1x_1}d\mu_{-1}(x_1)(1+t)^{2w_1w_2x}\\
&\quad\times\frac{2\int_{\mathbb{Z}_p}(1+t)^{2w_2x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1+t)^{2w_1w_2x_1}d\mu_{-1}(x_1)}\\
&= \int_{\mathbb{Z}_p}(1+t)^{2w_1x_1}d\mu_{-1}(x_1)(1+t)^{2w_1w_2x}\\
&\quad\times 2\sum_{\ell=0}^{w_1-1}(1+t)^{2w_2\ell}(-1)^\ell
\end{aligned}$$

$$\begin{aligned}
&= 2\sum_{\ell=0}^{w_1-1}(-1)^{\ell}\int_{\mathbb{Z}_p}(1+t)^{2w_1x_1+2w_1w_2x+2w_2\ell}d\mu_{-1}(x_1)\\
&= 2\sum_{\ell=0}^{w_1-1}(-1)^{\ell}\int_{\mathbb{Z}_p}(1+t)^{2w_1x_1+2w_1w_2x+\frac{w_2}{w_1}\ell}d\mu_{-1}(x_1)\\
&= 2\sum_{\ell=0}^{w_1-1}(-1)^{\ell}\sum_{k=0}^{\infty}\widetilde{Ch}_{k,w_1}\left(w_2x+\frac{w_2}{w_1}\ell\right)\frac{t^k}{k!}\\
&= \sum_{n=0}^{\infty}\left(2\sum_{\ell=0}^{w_1-1}(-1)^{\ell}\widetilde{Ch}_{n,w_1}\left(\frac{w_2}{w_1}\ell+w_2x\right)\right)\frac{t^n}{n!} \qquad (1.7)
\end{aligned}$$

Similarly, by the symmetry of $T(w_1, w_2)$, we have the following identity

$$T(w_1,w_2) = \sum_{n=0}^{\infty}\left(2\sum_{\ell=0}^{w_2-1}(-1)^{\ell}\widetilde{Ch}_{n,w_2}\left(\frac{w_1}{w_2}\ell+w_1x\right)\right)\frac{t^n}{n!}. \qquad (1.8)$$

Now from (1.7) and (1.8), we have the following theorem.

Theorem 3. *For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$ and $n \geq 0$, we have*

$$\sum_{\ell=0}^{w_1-1}(-1)^{\ell}\widetilde{Ch}_{n,w_1}\left(\frac{w_2}{w_1}\ell+w_2x\right) = \sum_{\ell=0}^{w_2-1}(-1)^{\ell}\widetilde{Ch}_{n,w_2}\left(\frac{w_1}{w_2}\ell+w_1x\right).$$

When we take $w_2 = 1$, we have

$$\widetilde{Ch}_n(w_1\ell+w_1x) = \sum_{\ell=0}^{w_1-1}(-1)^{\ell}\widetilde{Ch}_{n,w_1}\left(\frac{\ell}{w_1}+x\right).$$

2. Conclusion

The Changhee polynomials of type two are considered by D. Kim and T. Kim (see [4]) and various properties on their polynomials and numbers are investigated.

In this paper we investiate some symmetry identities for the Changhee polynomials of type two which are derived from the properties of symmetry for the fermionic p-adic integrals on $\mathbb{Z}_p$.

Especially we introduce w-Changhee polynomials of type two and investigate interesting symmetry identities.

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References

[1] D.S. Kim, *Identities associated with generalized twisted Euler polynomials twisted by ramified roots of unity*, Adv. Stud. Contemp. Math. (Kyungshang), **22** (2012), no. 3, 363–377
[2] D.S. Kim, T. Kim, *Triple symmetric identities for w-Caralan polynomials*, J. Korean Math. Soc., **54** (2017), no. 4, 1243–1264
[3] D.S. Kim, T. Kim, *Some p-adic integrals on $\mathbb{Z}_p$ associated with trigonometric functions*, Russ. J. Math. Phys., **25** (2018), no. 3, 300-308
[4] D.S. Kim, T. Kim, *A note on type 2 Changhee and Daehee polynomials*, arXiv:1809.05217 [math.NT]
[5] D.S. Kim, T. Kim, Y.H. Kim, S.H. Lee, *Some arithmetic properties of Bernoulli and Euler numbers*, Adv. Stud. Contemp. Math. (Kyungshang), **22** (2012), no. 4, 467–480
[6] D.S. Kim, T. Kim, J.J. Seo, *A note on Changhee polynomials and numbers*, Adv. Studies Theor. Phys., **7** (2013), no. 20, 993–1003
[7] D. S. Kim, N. Lee, H. Na, K. H. Park, *Abundant symmetry for higher-order Bernoulli polynomials (I)*, Adv. Stud. Contemp. Math., **23** (2013), no. 3, 461–482
[8] D. S. Kim, N. Lee, J. Na, K. H. Park, *Abundant symmetry for higher-order Bernoulli polynomials (II)*, Proc. Jangjeon Math. Soc., **16** (2013), no. 3, 359–378
[9] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., **9** (2002), no. 3, 288-299
[10] T. Kim, *Non Archmedean q-integrals associated with multiple Changhee q-Bernoulli polynomials*, Russ. Math. Phys., **10** (2003), no. 1, 91–98
[11] T. Kim, *p-adic q-integrals associated with the Changhee-Barnes' q-Bernoulli polynomials*, Integral Transforms Spec. Funct., **15** (2004), no. 5, 415–420
[12] T. Kim, *Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, J. Difference Equ. Appl., **14** (2008), no. 12, 1267-1277
[13] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys., **16** (2009), no. 1, 93–96
[14] T. Kim, D. Dolgy, *On the identities of symmetry for degenerate Bernoulli polynomials of order r*, Adv. Stud. Contemp. Math., **25** (2015), no. 4, 457–462
[15] T.Kim, S.-H. Rim, *On Changhee-Barnes' q-Euler numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **9** (2004), no. 2, 81–86
[16] T. Kim, S.-H. Rim, *New Changhee q-Euler numbers and polynomials associated with p-adic q-integerals*, Comput. Math. Appl., **54** (2007), no. 4, 484–489
[17] T. Kim, D.S. Kim, *Identities of symmetry for degenerate Euler polynomials and alternating generalized falling factorial sums*, Iran. J. Sci. Technol. Trans. A Sci., **41** (2017), no. 4, 939–949
[18] T. Kim, D.S. Kim, *Identities for degenerate Bernoulli polynomials and Korobov polynomials of the first kind*, Sci. China Math., (2018) https://doi.org/10.1007/s11425-018-9338-5
[19] D.E. Knuth, T.J. Buckholtz, *Computation of Tangent, Euler, and Bernoulli Numbers*, Math. Comput., **21** (1967), 663–688
[20] Y. Simsek, *Identities on Changhee numbers and Apostol-type Daehee polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **27** (2017), no. 2, 199–212
[21] W.P. Zhang, *Number of solutions to a congruence equations mod p. (Chinese)*, J. Northwest Univ. Natl. Sci., **46** (2016), no. 3, 313–316

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