SYMMETRY IDENTITIES OF CHANGHEE POLYNOMIALS OF TYPE TWO

JOOHEE JEONG, DONG-JIN KANG, AND SEOG-HOON RIM∗

ABSTRACT. In this paper we consider Changhee polynomials of type two, which are motivated from the recent work of Kim and Kim.

We investigate some symmetry identities for the Changhee polynomials of type two which are derived from the properties of symmetry for the fermionic p-adic integral on \(Z_p\).

0. Introduction

Let \(p\) be a fixed odd prime number. Throughout this paper, \(Z_p\), \(Q_p\) and \(C_p\) will denote the ring of \(p\)-adic integers, the field of \(p\)-adic rational numbers and the completion of the algebraic closure of \(Q_p\).

The \(p\)-adic norm \(|·|_p\) is normalized as \(|p|_p = \frac{1}{p}\).

Let \(f(x)\) be a continuous function on \(Z_p\). Then the fermionic \(p\)-adic integral on \(Z_p\) is defined by Kim as

\[
\int_{Z_p} f(x)d\mu_1(x) = \lim_{N \to \infty} \sum_{x=0}^{pN-1} f(x)\mu_1(x) = \lim_{x \to \infty} \sum_{x=0}^{pN-1} f(x)(-1)^x, \quad (\text{see [9]}) \tag{0.1}
\]

For \(n \in \mathbb{N}\), by (0.1), we get

\[
\int_{Z_p} f(x+n)d\mu_1(x) + (-1)^{n-1} \int_{Z_p} f(x)d\mu_1(x) = 2 \sum_{\ell=0}^{n-1} f(\ell)(-1)^{n-1-\ell}, \quad (\text{see [1, 5, 6, 16]}) \tag{0.2}
\]

In particular, if we take \(n = 1\), then we have

\[
\int_{Z_p} f(x+1)d\mu_1(x) + \int_{Z_p} f(x)d\mu_1(x) = 2f(0), \quad (\text{see [10, 11]}) \tag{0.3}
\]

We consider the Changhee polynomials \(\tilde{Ch}_n(x)\) of type two by the generating function

\[
\sum_{n=0}^{\infty} \frac{\tilde{Ch}_n(x) t^n}{n!} = \frac{2}{(1+t)+(1+t)^{-1}(1+t)^x}, \quad (\text{see [4]}) \tag{0.4}
\]

∗Corresponding Author.

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For \( t \in \mathbb{C}_p \) with \(|t|_p < p^{-\frac{1}{p-1}}\), the Changhee polynomials of type two can be represented by the fermionic \( p \)-adic integrals of \( \mathbb{Z}_p \):

\[
\int_{\mathbb{Z}_p} (1 + t)^{2x+1} d\mu_{-1}(y) = \frac{2}{(1 + t)^2 + 1}(1 + t)^{2x+1}
\]

\[
= \sum_{n=0}^{\infty} \tilde{C}_n(x) \frac{t^n}{n!}
\]

When \( x = 0 \), \( \tilde{C}_n = \tilde{C}_n(0) \) are called the Changhee numbers of type two.

Observe

\[
\sum_{n=0}^{\infty} \tilde{C}_n(x) \frac{t^n}{n!} = \frac{2}{(1 + t) + (1 + t)^{-1}}(1 + t)^{2x}
\]

\[
= \sum_{m=0}^{\infty} \tilde{C}_m \frac{t^m}{m!} \sum_{\ell=0}^{\infty} (2x)_{\ell} \frac{t^n}{\ell!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} \tilde{C}_m (2x)_{n-m} \right) \frac{t^n}{n!}
\]

Thus we have

\[
\tilde{C}_n(x) = \sum_{m=0}^{n} \binom{n}{m} (2x)_m \tilde{C}_{n-m},
\]

where \((x)_n = x(x - 1) \cdots (x - n + 1), \ (n \geq 1), \ (x)_0 = 1\).

From (0.3), we can easily derive the following:

\[
2 \sum_{\ell=0}^{n} (-1)^{\ell} (1 + t)^{2\ell} = \frac{2\{1 + (-1)^{n+1} (1 + t)^{2(n+1)}\}}{(1 + t)^2 + 1}
\]

(0.7)

The left hand side of (0.7) can be written as

\[
2 \sum_{\ell=0}^{n} (-1)^{\ell} (1 + t)^{2\ell} = \sum_{n=0}^{\infty} \left( \sum_{\ell=0}^{n-1} (-1)^{\ell} (2\ell)_n \right) \frac{t^n}{n!}
\]

(0.8)

We use the notation of \( \lambda \)-falling factorial in [14,17] for \( \lambda \in \mathbb{R} \),

\[
(\ell | \lambda)_n = \left\{ \begin{array}{ll}
\ell(\ell - \lambda) \cdots (\ell - \lambda(n - 1)), & \text{if } n \geq 1 \\
1, & \text{if } n = 0
\end{array} \right.
\]

Then the right hand side of (0.8) can be written as

\[
2 \sum_{\ell=0}^{n-1} (-1)^{\ell} (1 + t)^{2\ell} = \sum_{n=0}^{\infty} T_m(n; (\ell | \frac{1}{2})) \frac{t^n}{n!}
\]

(0.9)

where we denote, for \( \lambda \in \mathbb{R} \),

\[
T_m(n; (\ell | \lambda)) = \sum_{\ell=0}^{n} (-1)^{\ell} (\ell | \lambda)_m.
\]

For \( n \in \mathbb{N}, \ n \equiv 1 \pmod{2}, \ m \geq 0 \) we have

\[
\sum_{m=0}^{\infty} 2 \left( \sum_{\ell=0}^{n} (-1)^{\ell} (2\ell)_m \right) \frac{t^m}{m!} = \frac{2(1 + (1 + t)^{2(n+1)})}{(1 + t)^2 + 1}.
\]

(0.10)
On the other hand, by (0.4) and (0.9), we have
\[
\sum_{m=0}^{\infty} \left( \widetilde{Ch}_m + \widetilde{Ch}_m(n + 1) \right) \frac{t^m}{m!} = \frac{2(1 + t)}{1 + t} + \frac{2(1 + t)^2(1 + t)}{(1 + t)^2 + 1} = 2 \sum_{\ell=0}^{n} (-1)^\ell (1 + t)^{2\ell+1} = 2T_m(n; (\ell + \frac{1}{2} | \frac{1}{2})).
\] (0.11)

The Stirling number $S_1(\ell, n)$ of the first kind is defined by the generating function
\[
\left( \log(1 + t) \right)^n = n! \sum_{\ell=n}^{\infty} S_1(\ell, n), \quad \text{see} \ [1, 5, 6, 16, 18]
\]
and the Stirling number $S_2(m, n)$ of the second kind is given by the generating function
\[
(e^t - 1)^n = n! \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!}, \quad \text{(see} \ [6])
\]

As is well known, the Euler polynomials $E_n(x)$ are defined by the generating function
\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see} \ [3, 20, 21]).
\] (0.12)

When $x = 0$, $E_n = E_n(0)$, $(n \geq 0)$, are called the $n$-th Euler numbers, whereas the Euler numbers $E_n^*$ of the second kind are given by the generating function
\[
\text{sech}(t) = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!}, \quad \text{(see} \ [3, 19]).
\] (0.13)

From (0.5) and (0.13), we have
\[
\sum_{n=0}^{\infty} \widetilde{Ch}_n \frac{t^n}{n!} = \frac{2}{(1 + t) + (1 + t)^{-1}} = \frac{2}{e^\text{log}(1+t) + e^{-\text{log}(1+t)}} = \text{sech}(\text{log}(1 + t)) = \sum_{n=0}^{\infty} E_n^* \frac{(\log(1 + t))^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} E_k^* S_1(n, k) \right) \frac{t^n}{n!}.
\] (0.14)

Thus we have
\[
\widetilde{Ch}_n = \sum_{k=0}^{n} E_k^* S_1(n, k).
\] (0.15)

Using $E_0^* = 1$, $E_1^* = 0$, $E_2^* = -1$, $E_3^* = 0$, $E_4^* = 5$, $E_5^* = 0$ and $S_1(n, n) = 0$ for $n \geq 0$, $S_1(n, 0) = 0$ for $n \geq 1$, $S_1(2, 1) = 1$, $S_1(3, 1) = 2$, $S_1(4, 1) = 6$, $S_1(5, 1) = 24$, $S_1(3, 2) = 3$, $S_1(4, 2) = 11$, $S_1(5, 2) = 50$, $S_1(4, 3) = 6$, $S_1(5, 3) = 35$, $S_1(5, 4) = 10$, we get some values of $\widetilde{Ch}_n$'s as follows:

\[
\begin{align*}
\widetilde{Ch}_0 &= E_0^* S_1(0, 0) = 1, \\
\widetilde{Ch}_1 &= E_0^* S_1(1, 0) + E_1^* S_1(1, 1) = 0 + 0 = 0, \\
\widetilde{Ch}_2 &= E_0^* S_1(2, 0) + E_1^* S_1(2, 1) + E_2^* S_1(2, 2) = 0 + 0 - 1 = -1.
\end{align*}
\]
\[
\begin{align*}
\tilde{Ch}_3 &= E^*_0 S_1(3,0) + E^*_1 S_1(3,1) + E^*_2 S_1(3,2) + E^*_3 S_1(3,3) \\
&= 0 + 0 - 3 + 0 = -3, \\
\tilde{Ch}_4 &= E^*_0 S_1(4,0) + E^*_1 S_1(4,1) + E^*_2 S_1(4,2) + E^*_3 S_1(4,3) + E^*_4 S_1(4,4) \\
&= 0 + 0 - 11 + 0 + 5 = -6, \\
\tilde{Ch}_5 &= E^*_0 S_1(5,0) + E^*_1 S_1(5,1) + E^*_2 S_1(5,2) + E^*_3 S_1(5,3) + E^*_4 S_1(5,4) \\
&\quad + E^*_5 S_1(5,5) = 0 + 0 - 50 + 0 + 50 + 0 = 0.
\end{align*}
\]

From (0.6) and (0.13), we get the following, by replacing \( t \) by \( e^t - 1 \):

\[
\frac{2}{(1 + t)^2} + 1 = \sum_{n=0}^{\infty} \tilde{Ch}_n \frac{t^n}{n!}.
\]

\[
\frac{2}{e^{2t} + 1} e^t = \sum_{k=0}^{\infty} \tilde{Ch}_k \frac{1}{k!} (e^t - 1)^k
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \tilde{Ch}_k S_2(n,k) \right) \frac{t^n}{n!}
\]

\[
= \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E^*_n \frac{t^n}{n!}.
\]

Thus, from (0.16) we have

\[
E^*_n = \sum_{k=0}^{n} \tilde{Ch}_k S_2(n,k).
\]

On the other hand,

\[
\int_{\mathbb{Z}_p} (1 + t)^{2x+1} d\mu_{-1}(x) = \sum_{n=0}^{\infty} (2x + 1)_n \frac{t^n}{n!} d\mu_{-1}(x)
\]

\[
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (2x + 1)_n d\mu_{-1} \frac{t^n}{n!},
\]

So we have the Witt’s formula for Changhee numbers of type two as follows:

\[
\tilde{Ch}_n = \int_{\mathbb{Z}_p} (2x + 1)_n d\mu_{-1}(x).
\]

In this paper we consider Changhee polynomials of type two, which are motivated from the recent work of Kim and Kim. See [3, 4].

We investigate some symmetry identities for the Changhee polynomials of type two which are derived from the properties of symmetry for the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \). Many authors investigated symmetric properties of special polynomials and numbers. See [7, 8, 12, 14] and their references.
1. Symmetry of Changhee polynomials of type two

Motivated from Kim and Kim [2], for \( w \in \mathbb{N} \), we define \( w \)-torsion Changhee polynomials of type two by the following generating function

\[
\frac{2}{(1 + t)^{2w} + 1} (1 + t)^{2wx + 1} = \sum_{n=0}^{\infty} \tilde{C}_n(w)(x) \frac{t^n}{n!}.
\]  

(1.1)

When \( x = 0 \), \( \tilde{C}_n(w)(0) \) are called the \( w \)-Changhee numbers of type two. When \( w = 1 \), \( \tilde{C}_n,1(x) = \tilde{C}_n(x) \) are just the Changhee polynomials of type two in (0.6).

From (1.1) and (0.6), we present \( w \)-Changhee polynomials of type two by the well known Changhee polynomials of type two:

\[
\tilde{C}_n(w)(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} (2wx)^{\ell} \tilde{Ch}_{n-\ell}.
\]  

(1.2)

On the other hand,

\[
\sum_{n=0}^{\infty} \tilde{C}_n(w)(x) \frac{t^n}{n!} = \left( \sum_{\ell=0}^{\infty} \tilde{Ch}_{\ell} \frac{t^\ell}{\ell!} \right) (1 + t)^{2wx} \sum_{m=0}^{\infty} (2wx)_m \frac{t^m}{m!} = \sum_{n=0}^{\infty} \left( \sum_{\ell=0}^{n} \binom{n}{\ell} (2wx)^{\ell} \tilde{Ch}_{n-\ell} \right) \frac{t^n}{n!}.
\]  

(1.3)

Thus we present \( w \)-Changhee polynomials of type two by Changhee numbers of type two;

\[
\tilde{C}_n(w)(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} (2wx)^{\ell} \tilde{Ch}_{n-\ell}.
\]

Now we consider a quotient of \( p \)-adic fermionic integrals on \( \mathbb{Z}_p \),

\[
\frac{2 \int_{\mathbb{Z}_p} (1 + t)^{2w_2 x_2} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} (1 + t)^{2w_1 w_2 x_1} d\mu_{-1}(x_1)} = \sum_{\ell=0}^{w_1-1} (-1)^{\ell} (1 + t)^{2w_2 \ell}
\]

\[
= \sum_{m=0}^{\infty} \sum_{\ell=0}^{w_1-1} (-1)^{\ell} (2w_2 \ell)_m
\]

\[
= \sum_{m=0}^{\infty} \sum_{\ell=0}^{w_1-1} (2w_2)_m (-1)^{\ell} \left( \ell \mid \frac{1}{2w_2} \right) m
\]

\[
= \sum_{m=0}^{\infty} (2w_2)^m T_m(w_1 - 1 \mid \left( \ell \mid \frac{1}{2w_2} \right)),
\]  

(1.4)

where \( T_m(n \mid (\ell \mid \lambda)) = \sum_{\ell=0}^{n} (-1)^{\ell} (\ell \mid \lambda)_m \) for \( \lambda \in \mathbb{R} \).
Consider the following
\[
T(w_1, w_2) = \frac{2 \int_{Z_p} \int_{Z_p} (1 + t)^{2w_1x_1 + 2w_2x_2} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_{Z_p} (1 + t)^{2w_1x_1 + 1} d\mu_{-1}(x_1)} (1 + t)^{2w_1w_2x}
\]
\[
= \int_{Z_p} (1 + t)^{2w_1x_1 + 1} d\mu_{-1}(x_1) (1 + t)^{2w_1w_2x} \\
\times \frac{\int_{Z_p} (1 + t)^{2w_2x_2} d\mu_{-1}(x_2)}{\int_{Z_p} (1 + t)^{2w_1w_2x_1} d\mu_{-1}(x_1)} 
\]
\[
= \left( \sum_{\ell=0}^{\infty} \frac{\tilde{C}_{\ell,w_1}(w_2x)^{\ell}}{\ell!} \left( \sum_{k=0}^{\infty} (2w_2)^k T_k(w_1 - 1 | (k | \frac{1}{2w_2})) \right) \right) \\
\times \left( \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \tilde{C}_{n-k,w_2}(w_1)^k T_k(w_2 - 1 | (k | \frac{1}{2w_2})) \right) \right) \\ \\
\quad \left( \sum_{k=0}^{n} \binom{n}{k} \tilde{C}_{n-k,w_2}(w_1)^k T_k(w_2 - 1 | (k | \frac{1}{2w_2})) \right). 
\]

Similarly, we have the following identity for \(T(w_1, w_2)\) because \(T(w_1, w_2)\) is symmetric on \(w_1\) and \(w_2\).
\[
T(w_1, w_2) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \tilde{C}_{n-k,w_2}(w_1)^k T_k(w_2 - 1 | (k | \frac{1}{2w_2})) \right) \frac{t^n}{n!}. 
\]

Thus, by (1.5) and (1.6), we have the following theorem.

**Theorem 1.** For \(w_1, w_2 \in \mathbb{N}\) with \(w_1 \equiv 1 \pmod{2}\), \(w_2 \equiv 1 \pmod{2}\) and \(n \geq 0\), we have
\[
\sum_{k=0}^{n} \binom{n}{k} \tilde{C}_{n-k,w_2}(w_1)^k T_k(w_2 - 1 | (k | \frac{1}{2w_2})) \\
= \sum_{k=0}^{n} \binom{n}{k} \tilde{C}_{n-k,w_2}(w_1)^k T_k(w_1 - 1 | (k | \frac{1}{2w_2})). 
\]

Remark: If we take \(w_2 = 1\) in Theorem 1, we have the following

**Corollary 2.** For \(w_1 \in \mathbb{N}\) with \(w_1 \equiv 1 \pmod{2}\) and \(n \geq 0\), we have
\[
\tilde{C}_{n}(w_1) = \sum_{k=0}^{n} \binom{n}{k} \tilde{C}_{n-k,w_1}(w_1)^k T_k(w_1 - 1 | (k | \frac{1}{2})). 
\]

From (1.5), we rewrite \(T(w_1, w_2)\) as follows:
\[
T(w_1, w_2) = \int_{Z_p} (1 + t)^{2w_1x_1} d\mu_{-1}(x_1) (1 + t)^{2w_1w_2x} \\
\times 2 \frac{\int_{Z_p} (1 + t)^{2w_2x_2} d\mu_{-1}(x_2)}{\int_{Z_p} (1 + t)^{2w_1w_2x_1} d\mu_{-1}(x_1)} \\
= \int_{Z_p} (1 + t)^{2w_1x_1} d\mu_{-1}(x_1) (1 + t)^{2w_1w_2x} \\
\times 2 \sum_{\ell=0}^{w_1-1} (1 + t)^{2w_2\ell} (-1)^{\ell}. 
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\[
\begin{align*}
= 2 \sum_{\ell=0}^{w_1-1} (-1)^\ell \int_{\mathbb{Z}_p} (1 + t)^{2w_1x_1 + 2w_1w_2x + 2w_2\ell} d\mu_{-1}(x_1) \\
= 2 \sum_{\ell=0}^{w_1-1} (-1)^\ell \int_{\mathbb{Z}_p} (1 + t)^{2w_1x_1 + 2w_1w_2x + \frac{w_2\ell}{w_1}} d\mu_{-1}(x_1) \\
= 2 \sum_{\ell=0}^{w_1-1} (-1)^\ell \sum_{k=0}^{\infty} \tilde{Ch}_{k,w_1}(w_2x + \frac{w_2\ell}{w_1}) \frac{t^k}{k!} \\
= \sum_{n=0}^{\infty} \left( 2 \sum_{\ell=0}^{w_1-1} (-1)^\ell \tilde{Ch}_{n,w_1}(\frac{w_2}{w_1}x + w_2x) \right) \frac{t^n}{n!}.
\end{align*}
\]  

(1.7)

Similarly, by the symmetry of \( T(w_1, w_2) \), we have the following identity

\[
T(w_1, w_2) = \sum_{n=0}^{\infty} \left( 2 \sum_{\ell=0}^{w_2-1} (-1)^\ell \tilde{Ch}_{n,w_2}(\frac{w_1}{w_2}x + w_1x) \right) \frac{t^n}{n!}.
\]  

(1.8)

Now from (1.7) and (1.8), we have the following theorem.

**Theorem 3.** For \( w_1, w_2 \in \mathbb{N} \) with \( w_1 \equiv 1 \pmod{2} \), \( w_2 \equiv 1 \pmod{2} \) and \( n \geq 0 \), we have

\[
\sum_{\ell=0}^{w_1-1} (-1)^\ell \tilde{Ch}_{n,w_1}(\frac{w_2}{w_1}x + w_2x) = \sum_{\ell=0}^{w_2-1} (-1)^\ell \tilde{Ch}_{n,w_2}(\frac{w_1}{w_2}x + w_1x).
\]

When we take \( w_2 = 1 \), we have

\[
\tilde{Ch}_{n}(w_1x + w_1x) = \sum_{\ell=0}^{w_1-1} (-1)^\ell \tilde{Ch}_{n,w_1}(\frac{\ell}{w_1} + x).
\]

2. **Conclusion**

The Changhee polynomials of type two are considered by D. Kim and T. Kim (see [4]) and various properties on their polynomials and numbers are investigated.

In this paper we investigate some symmetry identities for the Changhee polynomials of type two which are derived from the properties of symmetry for the fermionic \( p \)-adic integrals on \( \mathbb{Z}_p \).

Especially we introduce \( w \)-Changhee polynomials of type two and investigate interesting symmetry identities.

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References